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Supplementary material to: Back-pressure traffic signal control with unknown routing rates

Jean Gregoire* Emilio Frazzoli† Arnaud de La Fortelle** Tichakorn Wongpiromsarn•

Abstract—This is the supplementary material to the paper: back-pressure traffic signal control with unknown routing rates. It details proofs not included in the paper due to space limitations. The characterization of the capacity region, the optimality of BP* and the behaviour of the Lyapunov drift under BP control are proved.

I. CHARACTERIZATION OF THE CAPACITY REGION

Theorem (Capacity region characterisation). *Given a routing matrix r , the capacity region Λ_r is the set of arrival vectors λ such that there exists $g \in \Gamma$ satisfying:*

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + g_a^{\text{in}}) \leq g_{ab} \quad (1)$$

where Γ is the set of feasible long-term endogenous service rates, defined below:

$$\Gamma = \text{Convex_Hull}\{\mu(p) : p \in \mathcal{P}\} \quad (2)$$

Moreover,

- $\lambda \in \Lambda_r$ is a necessary condition for network stability, considering all possible controls (including those that have perfect knowledge of future events)
- $\lambda \in \text{int}(\Lambda_r)$ is a sufficient condition for the network to be stabilized by a control that does not have knowledge of future events.

Proof. The proof is a slightly modified version of the characterization of the capacity region of [1]. Let $\tilde{\Lambda}_r$ denote the set of arrival rates vectors such that there exists $g \in \Gamma$ satisfying:

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + g_a^{\text{in}}) \leq g_{ab} \quad (3)$$

Let prove that $\lambda \in \tilde{\Lambda}_r$ is a necessary condition for network stability, considering all possible controls (including those that have perfect knowledge of future events). Suppose that the network is empty at $t = 0$, using the equation of the dynamics of the network, we obtain:

$$Q_{ab}(T) = \sum_{t=0}^{T-1} R_{ab}^{(t)} (A_a(t) + f_a^{\text{in}}(t)) - \sum_{t=0}^{T-1} f_{ab}(t) \quad (4)$$

Suppose the network is stabilized and fix an arbitrary small value $\epsilon > 0$. By the network stability necessary condition

of [1], there must exist some finite value V such that at arbitrary large times T , the queues lengths are simultaneously less than V with probability at least $1/2$. Hence, there exists a time T such that with probability at least $1/2$, the following inequalities hold for all $a, b \in \mathcal{N}$:

$$Q_{ab}(T) \leq V \quad (5)$$

$$\frac{V}{T} \leq \epsilon \quad (6)$$

$$\frac{\sum_{t=0}^{T-1} R_{ab}^{(t)} A_a(t)}{T} \geq r_{ab} \lambda_a - \epsilon \quad (7)$$

$$\frac{\sum_{t=0}^{T-1} R_{ab}^{(t)} f_a^{\text{in}}(t)}{T} \geq r_{ab} f_a^{\text{in}}(t) - \epsilon \quad (8)$$

$$(9)$$

Define variables $g_{ab} = \sum_{t=0}^{T-1} f_{ab}(t)/T$. Using the above inequalities together with Equation 4 provides:

$$r_{ab} (\lambda_a + g_a^{\text{in}}) \leq g_{ab} + 3\epsilon \quad (10)$$

Hence, with probability greater than $1/2$, the flows g_{ab} come arbitrary close to satisfying Inequality 3. As a result, there must exist sample paths $f_{ab}(t)$ from which flow variables g_{ab} are defined that satisfy Inequality 10. This proves that λ is a limit point of $\tilde{\Lambda}_r$. As $\tilde{\Lambda}_r$ is a compact, it contains its limit points, and we finally obtain: $\Lambda_r \subset \tilde{\Lambda}_r$.

Now, assume that λ is interior to $\tilde{\Lambda}_r$. Then, as proved in Section II, one can build a randomized control that stabilizes the network under arrival rates vector λ . Hence, $\lambda \in \text{int}(\tilde{\Lambda}_r)$ is a sufficient condition for the network to be stabilized by some control. Combining the two above results proves that $\tilde{\Lambda}_r = \Lambda_r$. \square

II. EXISTENCE OF A STABILIZING RANDOMIZED POLICY

The following theorem proves the existence of a stabilizing stationary randomized policy for all arrival rates vectors interior to the capacity region.

Theorem (Existence of a stabilizing randomized policy). *Suppose there exists $\epsilon > 0$ such that $\lambda + \epsilon \in \text{int}(\Lambda_r)$, i.e. λ and $\lambda + \epsilon$ interior to the capacity region. Then, there exists a stationary randomized control $\tilde{p}(t)$ such that:*

$$\mathbb{E} \{ \tilde{\mu}_{ab}(p(t)) - r_{ab} (\tilde{\mu}_a^{\text{in}}(p(t)) + \lambda_a) \} \geq r_{ab} \epsilon_a \quad (11)$$

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Proof. For the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X(t))\} = r_{ab}\mathbb{E}\{X(t)\}$ for all $X(t)$ independent from $R_{ab}^{(t)}$. It is not true in the general case (it is the particular case of i.i.d. arrivals and routing), and the reader can refer to [1] for the principle of an extension to the general case using a K-steps Lyapunov drift.

Suppose λ is interior to the capacity region, i.e. there exists a positive vector ϵ such that $\lambda + \epsilon \in \Lambda$. By Theorem I, there exists $g \in \Gamma$ such that:

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + \epsilon_a + g_a^{\text{in}}) \leq g_{ab} \quad (12)$$

Since $g \in \Gamma$, it can be expressed as a weighted sum as follows:

$$g = \sum_{p \in \mathcal{P}} w_p \mu(p) \quad (13)$$

where weights w_p sum to 1. Let define the randomized policy \tilde{p} that selects randomly the phase to apply at every time slot according to probabilities $(w_p)_{p \in \mathcal{P}}$.

It is direct that it will result in a randomized stationary service matrix $\mu(\tilde{p}(t))$ verifying:

$$\mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} = g_{ab} \quad (14)$$

As a result,

$$\mathbb{E}\{\mu_{ab}(\tilde{p}(t)) - r_{ab}(\tilde{\mu}_a^{\text{in}}(\tilde{p}(t)) + \lambda_a)\} \geq r_{ab}\epsilon_a \quad (15)$$

Now, assume that $\tilde{p}(t)$ is applied to the queuing network. Then, using the equation of the dynamics of the network:

$$Q_{ab}(t+1) \leq \max(Q_{ab}(t) - \mu_{ab}(\tilde{p}(t)), 0) + R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t))) \quad (16)$$

An inequality holds instead of an equality because the number of vehicles transferred is less or equal to the transmission rate offered by servers.

Squaring both sides and using $\max^2(x, 0) \leq x^2$, we obtain:

$$Q_{ab}(t+1)^2 - Q_{ab}(t)^2 \leq \left(R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t)))\right)^2 + \mu_{ab}(\tilde{p}(t))^2 - 2Q_{ab}(t) \left(\mu_{ab}(\tilde{p}(t)) - R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t)))\right) \quad (17)$$

Define the Lyapunov function $\mathcal{V}(Q(t)) = V(t) = \sum_{a,b} Q_{ab}(t)^2$. Taking expectations, summing over all $a, b \in \mathcal{N}$, using independences and noting that $\mathbb{E}\{A\} \leq \sqrt{\mathbb{E}\{A^2\}}$, we obtain:

$$\mathbb{E}\{V(t+1) - V(t)|Q(t)\} \leq B - 2 \sum_{a,b} Q_{ab}(t) \mathbb{E}\{\mu_{ab}(\tilde{p}(t)) - r_{ab}(\lambda_a + \mu_a^{\text{in}}(\tilde{p}(t)))\} \quad (18)$$

Using Inequality 15, we obtain:

$$\mathbb{E}\{V(t+1) - V(t)|Q(t)\} \leq B - 2 \sum_{a,b} r_{ab}\epsilon_a Q_{ab}(t) \quad (19)$$

Let define $\eta = 2 \min_{a,b} r_{ab}\epsilon_a > 0$, we finally obtain:

$$\mathbb{E}\{V(t+1) - V(t)|Q(t)\} \leq B - \eta \sum_{a,b} Q_{ab}(t) \quad (20)$$

The sufficient condition using Lyapunov drift proved in [1] enables to conclude stability of the queuing network. \square

III. OPTIMALITY OF BP*

Theorem (Back-pressure optimality). *Assume that the phase selection policy ϕ in case of equality always privileges phases p such that for all $a, b \in \mathcal{P}$, $\mu_{ab}(p) = 0$ if $W_{ab} = 0$ and pressure functions are linear with strictly positive slopes. Then, BP* is stability-optimal.*

Proof. Again, for the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X(t))\} = r_{ab}\mathbb{E}\{X(t)\}$ for all $X(t)$ independent from $R_{ab}^{(t)}$. The reader can refer to [1] for the principle of an extension to the general case using a K-steps Lyapunov drift.

Let $\theta_{ab} > 0$ denote the slope of linear pressure function P_{ab} and $\Pi_{ab}(t) = P_{ab}(Q_{ab}(t))$ the evolution of pressures over time. Define the Lyapunov function $\mathcal{V}(Q(t)) = V(t) = \sum_{a,b} \theta_{ab} Q_{ab}(t)^2$ and let $p(t)$ denote the control applied to the queuing network. With the same manipulations as for the proof of Theorem I, we obtain:

$$\begin{aligned} V(t+1) - V(t) &= \sum_{a,b} \Pi_{ab}(t+1)^2 - \Pi_{ab}(t)^2 = \\ &= \sum_{a,b} \theta_{ab} (Q_{ab}(t+1)^2 - Q_{ab}(t)^2) \leq \\ &= \sum_{a,b} \theta_{ab} \left[\left(R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(p(t))) \right)^2 + \mu_{ab}(p(t))^2 \right] \\ &\quad - 2 \sum_{a,b} \theta_{ab} Q_{ab}(t) \left(\mu_{ab}(p(t)) - R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(p(t))) \right) \leq \\ &= B(t) - 2 \sum_{a,b} \theta_{ab} Q_{ab}(t) \left(\mu_{ab}(p(t)) - R_{ab}^{(t)}(A_a(t) + \mu_a^{\text{in}}(p(t))) \right) \end{aligned} \quad (21)$$

with the upper-bound $B(t)$ defined below:

$$B(t) = \sum_{a,b} \theta_{ab} \left[R_{ab}^{(t)} \left(A_a(t) + \sup_{p \in \mathcal{P}} \mu_a^{\text{in}}(p) \right) \right]^2 + \left(\sup_{p \in \mathcal{P}} \mu_{ab}(p) \right)^2 \quad (22)$$

Taking expectation and using independences, we get :

$$\begin{aligned} \mathbb{E}\{V(t+1) - V(t)|Q(t)\} &\leq B \\ &- 2 \sum_{a,b} \theta_{ab} Q_{ab}(t) \mathbb{E}\{\mu_{ab}(p(t)) - r_{ab} \mu_a^{\text{in}}(p(t))|Q(t)\} \\ &\quad + 2 \sum_{a,b} \theta_{ab} Q_{ab}(t) r_{ab} \lambda_a \end{aligned} \quad (23)$$

The upper-bound B is obtained using $\mathbb{E}\{A_a\} \leq \sqrt{\mathbb{E}\{A_a^2\}}$:

$$\begin{aligned} B &= \sum_{a,b} \theta_{ab} r_{ab}^2 \left(A_a^{\text{max}} + \sup_{p \in \mathcal{P}} \mu_a^{\text{in}}(p) \right)^2 \\ &\quad + \theta_{ab} \left(\sup_{p \in \mathcal{P}} \mu_{ab}(p) \right)^2 \end{aligned} \quad (24)$$

By simple sum manipulation, the following identity is obtained:

$$\sum_{a,b} M_{ab} (g_{ab} - r_{ab} g_a^{\text{in}}) = \sum_{a,b} (M_{ab} - \sum_c r_{bc} M_{bc}) g_{ab} \quad (25)$$

Using identity 25, Equation 23 becomes:

$$\begin{aligned} \mathbb{E}\{V(t+1) - V(t)|Q(t)\} &\leq B \\ &- 2 \sum_{a,b} \left(\theta_{ab} Q_{ab}(t) - \sum_c r_{bc} \theta_{bc} Q_{bc}(t) \right) \mathbb{E}\{f_{ab}(t)|Q(t)\} \\ &\quad + 2 \sum_{a,b} \theta_{ab} Q_{ab}(t) r_{ab} \lambda_a = B \\ &- 2 \sum_{a,b} \left(\Pi_{ab}(t) - \sum_c r_{bc} \Pi_{bc}(t) \right) \mathbb{E}\{f_{ab}(t)|Q(t)\} \\ &\quad + 2 \sum_{a,b} \Pi_{ab}(t) r_{ab} \lambda_a \end{aligned} \quad (26)$$

Now, assume that BP* control $p^*(t)$ is applied and let $V^*(t)$ denote the Lyapunov function under $p^*(t)$. Due to service rates independence assumption and because the phase selection policy ϕ in case of equality always privileges phases p such that $\mu_{ab}(p) = 0$ if $W_{ab} = 0$, it is clear that the returned phase to apply is such that $\mu_{ab}(p^*(t)) = 0$ if $W_{ab}(t) = 0$.

As a result, we obtain:

$$\begin{aligned} \mathbb{E}\{V^*(t+1) - V^*(t)|Q(t)\} &\leq B \\ &- 2 \sum_{a,b} W_{ab}(t) \mathbb{E}\{\mu_{ab}(p^*(t))|Q(t)\} \\ &\quad + 2 \sum_{a,b} \Pi_{ab}(t) r_{ab} \lambda_a \end{aligned} \quad (27)$$

By construction of back-pressure control $p^*(t)$ (see the arg max in the algorithm), $p^*(t)$ maximizes $\sum_{a,b} W_{ab}(t) \mu_{ab}(p(t))$ over all possible alternative controls $p(t)$.

Now, suppose that the arrival rates vector is interior to the capacity region Λ_r , i.e. there exists $\epsilon > 0$ such that $\lambda + \epsilon \in \Lambda$.

Then, as proved in the supplementary material not provided in this paper due to space limitations, there exists a stabilizing stationary randomized control $\tilde{p}(t)$ such that for all $a, b \in \mathcal{N}$:

$$\mathbb{E}\{\mu_{ab}(\tilde{p}(t)) - r_{ab} (\mu_a^{\text{in}}(\tilde{p}(t)) + \lambda_a)\} \geq r_{ab} \epsilon_a \quad (28)$$

Combining the two above statements, taking expectations, and noting that the control $\tilde{p}(t)$ is stationary result in:

$$\begin{aligned} \sum_{a,b} W_{ab}(t) \mathbb{E}\{\mu_{ab}(p^*(t))|Q(t)\} &\geq \\ \sum_{a,b} W_{ab}(t) \mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} &\geq \\ \sum_{a,b} \left(\Pi_{ab}(t) - \sum_c r_{bc} \Pi_{bc}(t) \right) \mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} &= \\ \sum_{a,b} \Pi_{ab}(t) \mathbb{E}\{\mu_{ab}(\tilde{p}(t)) - r_{ab} \mu_a^{\text{in}}(\tilde{p}(t))\} &\geq \\ \sum_{a,b} \Pi_{ab}(t) (r_{ab} \lambda_a + r_{ab} \epsilon_a) &\quad (29) \end{aligned}$$

Injecting the above result in the Lyapunov drift inequality results in:

$$\mathbb{E}\{V^*(t+1) - V^*(t)|Q(t)\} \leq B - 2 \sum_{a,b} \theta_{ab} r_{ab} \epsilon_a Q_{ab}(t) \quad (30)$$

Let $\eta = 2 \min_{a,b} \theta_{ab} r_{ab} \epsilon_a > 0$. We finally obtain:

$$\mathbb{E}\{V^*(t+1) - V^*(t)|Q(t)\} \leq B - \eta \sum_{a,b} Q_{ab}(t) \quad (31)$$

The sufficient condition using Lyapunov drift proved in [1] enables to conclude stability of the queuing network. \square

IV. BEHAVIOUR OF THE LYAPUNOV DRIFT IN HEAVY LOAD CONDITIONS

Theorem (Lyapunov drift under heavy load conditions). *Assume $\lambda + \epsilon \in \Lambda_r$, BP control is applied and the network is in heavy load conditions, then there exists $B, \eta > 0$ such that :*

$$\mathbb{E}\{V(t+1) - V(t) | Q(t)\} \leq B - \eta \sum_a Q_a(t) \quad (32)$$

for sufficiently large ϵ .

Proof. Again, for the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X)\} = r_{ab} X$ for all $X \in \mathbb{N}$.

Let $\Pi_a(t)$ denote the evolution of $P_a(Q_a(t))$ over time and $p(t)$ the control applied to the queuing network. By simple manipulations, we get:

$$\begin{aligned}
V(t+1) - V(t) = & \\
& \sum_a \theta_a (Q_a(t+1) - Q_a(t))^2 \\
& + 2 \sum_a \theta_a Q_a(t) (Q_a(t+1) - Q_a(t)) \quad (33)
\end{aligned}$$

As a result, Inequality 33 becomes:

$$\begin{aligned}
V(t+1) - V(t) \leq & B(t) - 2 \sum_a \theta_a Q_a(t) \\
& \sum_b \left(f_{ab}(t) - R_{ab}^{(t)} (A_a(t) + f_a^{\text{in}}(t)) \right) \quad (34)
\end{aligned}$$

with the upper-bound $B(t)$ defined below:

$$\begin{aligned}
B(t) = & \sum_a \theta_a \sum_b \left[R_{ab}^{(t)} \left(A_a(t) + \sup_{p \in \mathcal{P}} \mu_a^{\text{in}}(p) \right) \right]^2 \\
& + \sum_a \theta_a \sum_b \left(\sup_{p \in \mathcal{P}} \mu_{ab}(p) \right)^2 \quad (35)
\end{aligned}$$

Taking expectations and using independences, we obtain :

$$\begin{aligned}
\mathbb{E} \{ V(t+1) - V(t) \mid Q(t) \} \leq & B+2 \sum_a \theta_a Q_a(t) \lambda_a \sum_b r_{ab} \\
& - 2 \mathbb{E} \left\{ \sum_{a,b} \theta_a Q_a(t) (f_{ab}(t) - r_{ab} f_a^{\text{in}}(t)) \mid Q(t) \right\} \quad (36)
\end{aligned}$$

Moreover, by simple sum manipulations, we get the below identity:

$$\sum_{a,b} P_a (f_{ab} - r_{ab} f_a^{\text{in}}) = \sum_{a,b} \left(P_a - \sum_c r_{bc} P_b \right) f_{ab} \quad (37)$$

Using identity 37, Equation 36 becomes:

$$\begin{aligned}
\mathbb{E} \{ V(t+1) - V(t) \mid Q(t) \} \leq & B + 2 \sum_a \theta_a Q_a(t) \lambda_a \\
& - 2 \sum_{a,b} \left(\Pi_a(t) - \left(\sum_c r_{bc} \right) \Pi_b(t) \right) \mathbb{E} \{ f_{ab}(t) \mid Q(t) \} \quad (38)
\end{aligned}$$

Since $\sum_c r_{bc} \leq 1$, we obtain:

$$\begin{aligned}
\mathbb{E} \{ V(t+1) - V(t) \mid Q(t) \} \leq & B + 2 \sum_a \theta_a Q_a(t) \lambda_a \\
& - 2 \sum_{a,b} (\Pi_a(t) - \Pi_b(t)) \mathbb{E} \{ f_{ab}(t) \mid Q(t) \} \quad (39)
\end{aligned}$$

Now, assume that BP control $p^{\text{BP}}(t)$ is applied and let $V^{\text{BP}}(t)$ denote the Lyapunov function under $p^{\text{BP}}(t)$. Due to service rates independence and because the phase selection policy ϕ in case of equality always privileges phases p such

that $\mu_{ab}(p) = 0$ if $W_{ab} = 0$, it is clear that the returned phase to apply is such that $\mu_{ab}(p^{\text{BP}}(t)) = 0$ if $W_{ab} = 0$. Moreover, by definition of $d_{ab}(t)$, under infinite capacities, $f_{ab}(t) = d_{ab}(t) \mu_{ab}(p(t))$. Hence, we obtain:

$$\begin{aligned}
\mathbb{E} \{ V^{\text{BP}}(t+1) - V^{\text{BP}}(t) \mid Q(t) \} \leq & B + 2 \sum_a \theta_a Q_a(t) \lambda_a \\
& - 2 \sum_{a,b} W_{ab}(t) \mathbb{E} \{ \mu_{ab}(p^{\text{BP}}(t)) \mid Q(t) \} \quad (40)
\end{aligned}$$

By construction of back-pressure control $p^{\text{BP}}(t)$ (see $\arg \max$ in the algorithm), $p^{\text{BP}}(t)$ maximizes $\sum_{a,b} W_{ab}(t) \mu_{ab}(p(t))$ over all possible alternative controls $p(t)$.

Now, assume $\lambda + \epsilon \in \Lambda_r$ where ϵ is a positive vector. Then, as proved in Section II, there exists a stabilizing stationary randomized control $\tilde{p}(t)$ such that:

$$\exists g \in \Gamma : \forall a, b \in \mathcal{N}, \mathbb{E} \{ \mu_{ab}(\tilde{p}(t)) \} = g_{ab} \quad (41)$$

Combining the two above statements and taking expectations results in:

$$\begin{aligned}
\sum_{a,b} W_{ab}(t) \mathbb{E} \{ \mu_{ab}(p^{\text{BP}}(t)) \mid Q(t) \} \geq & \\
\sum_{a,b} W_{ab}(t) \mathbb{E} \{ \mu_{ab}(\tilde{p}(t)) \mid Q(t) \} = & \\
\sum_{a,b} d_{ab}(t) (\Pi_a(t) - \Pi_b(t)) g_{ab} & \quad (42)
\end{aligned}$$

Now, assume that the network in heavy load conditions, then $d_{ab}(t) = 1$ and we obtain:

$$\begin{aligned}
\sum_{a,b} W_{ab}(t) \mathbb{E} \{ \mu_{ab}(p^{\text{BP}}(t)) \mid Q(t) \} \geq & \\
\sum_{a,b} (\Pi_a(t) - \Pi_b(t)) g_{ab} & \quad (43)
\end{aligned}$$

By simple manipulation, we get the following identity:

$$\sum_{a,b} (\Pi_a - \Pi_b) g_{ab} = \sum_a \Pi_a (g_a^{\text{out}} - g_a^{\text{in}}) \quad (44)$$

Hence,

$$\begin{aligned}
\sum_{a,b} W_{ab}(t) \mathbb{E} \{ \mu_{ab}(p^{\text{BP}}(t)) \mid Q(t) \} \geq & \\
\sum_a \Pi_a(t) (g_a^{\text{out}} - g_a^{\text{in}}) = \sum_a \theta_a Q_a(t) (g_a^{\text{out}} - g_a^{\text{in}}) & \quad (45)
\end{aligned}$$

Moreover, by definition of the input/output flow,

$$\begin{aligned}
g_a^{\text{out}} - g_a^{\text{in}} = \sum_b g_{ab} - g_a^{\text{in}} = \sum_b (g_{ab} - r_{ab} g_a^{\text{in}}) \\
- (1 - \sum_b r_{ab}) g_a^{\text{in}} \quad (46)
\end{aligned}$$

As a result, using the inequalities verified by $g \in \Gamma$, we obtain:

$$\begin{aligned} & \sum_{a,b} W_{ab}(t) \mathbb{E}\{\mu_{ab}(p^{\text{BP}}(t)) \mid Q(t)\} \geq \\ & \sum_a \theta_a Q_a(t) \left[\sum_b (r_{ab} \lambda_a + r_{ab} \epsilon_a) - \left(1 - \sum_b r_{ab}\right) g_a^{\text{in}} \right] \end{aligned} \quad (47)$$

Injecting the latter result in Inequality 40 provides:

$$\begin{aligned} & \mathbb{E}\{V^{\text{BP}}(t+1) - V^{\text{BP}}(t) \mid Q(t)\} \leq B \\ & - 2 \sum_a \theta_a Q_a(t) \left[\sum_b r_{ab} \epsilon_a - \left(1 - \sum_b r_{ab}\right) g_a^{\text{in}} \right] \end{aligned} \quad (48)$$

Let define η as follows:

$$\eta = 2 \min_a \theta_a \left(\sum_b r_{ab} \epsilon_a - \left(1 - \sum_b r_{ab}\right) g_a^{\text{in}} \right) \quad (49)$$

We finally obtain:

$$\mathbb{E}\{V^{\text{BP}}(t+1) - V^{\text{BP}}(t) \mid Q(t)\} \leq B - \eta \sum_a Q_a(t) \quad (50)$$

If for all $a \in \mathcal{N}$,

$$\sum_b r_{ab} \epsilon_a > \left(1 - \sum_b r_{ab}\right) g_a^{\text{in}} \quad (51)$$

which can be verified for sufficiently large ϵ then $\eta > 0$ and the inequality of the sufficient condition for network stability using Lyapunov drift of [1] is verified in heavy load conditions. However, it does not imply that the network is stable under BP control since the heavy load assumption is not necessarily verified at any time.

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□