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Abstract

The Mumford process X is a stochastic distribution modulo constant and can not be defined as a stochastic distribution invariant in law by dilations. We present two expansions of X -using wavelet bases- in $X = X_0 + X_1$ which allow us to confine the divergence on the "small term" X_1 and which respect the invariance in law by dyadic dilations of the process.

Key words: Mumford process, infrared divergence, wavelet bases, self similarity

1 Introduction

The Mumford process has been introduced by Mumford and Gidas in [8] as the simplest process which can generate images. In this paper, Mumford and Gidas define the axioms that a stochastic process shall verify to generate images. Let us cite, for example,

(1) the scaling invariance which express the fact that an object seems bigger but does not change of form when one approaches it,
(2) the infinite divisibility which means that an image can be seen as the superposition of (less complex) independant images.

The Mumford process satisfies these axioms since it is a gaussian stochastic process with stationary increments and invariant by dilations. Nevertheless, since it is gaussian, it can only simulate clouds and not complex images.

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This process is defined as a stochastic distribution modulo constants almost everywhere. It is known that it can be defined as a stochastic distribution, but with this definition, the property of scaling invariance is lost. Our point of view is to conserve this scaling invariance. We will see (in Section 2) that it is then impossible to define the Mumford process as a stochastic distribution, invariant in law by dilations. In particular, any expansion on a wavelet basis of the Sobolev space $\dot{H}^1(\mathbb{R}^2)$ leads to the phenomenon of infrared divergence and does not converge in the distributional sense.

Similarly to what has been done in [13] for the confinement of the infrared divergence of the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$, with $s - \frac{n}{2} \in \mathbb{N}$, our goal is to divide the Mumford process X in $X = X_0 + X_1$ where X_0 can be defined as a stochastic distribution and X_1 is "as small as possible". Moreover, we are looking for solutions which can be rapidly and robustly implemented (the robustness will be given by the unconditionality of the basis on which the processes are expanded).

We present in this paper two explicit solutions. The first one consists in writing, in the frequency domain, $\widehat{X}(\xi, \omega)$ as the sum of a radial term and an anti-radial term. Expanding the terms on a suitable orthonormal basis, the infrared divergence is carried by the radial term (Section 3).

The second solution is based on the construction of an adapted basis, the wavelet basis with pseudo-constant (Section 4). It allows us to confine the infrared divergence on a smaller term than with the previous solution but the terms are now correlated (Section 5).

Let us just mention that there exists an orthonormal basis which provides us a confinement of the same order than the one given by the wavelet basis with pseudo-constant, but with decorrelated terms. But this 'ideal' solution is not constructive (the result can be found in [11]).

Notations. We will denote by $\mathcal{S}_0(\mathbb{R}^2)$ the subspace of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ formed by the functions u satisfying

$$\int x^\alpha u(x) dx = 0 \quad \forall \alpha \in \mathbb{N}^2,$$

and by $\mathcal{S}_0'(\mathbb{R}^2)$ its dual. This space is identified with $\mathcal{S}'(\mathbb{R}^2)/\mathcal{P}$.

Let us denote by $\dot{\mathcal{H}}^1(\mathbb{R}^2)$ the subspace of distributions f such that

$$R(f) := \left(\|\partial_{x_1} f\|_{L^2}^2 + \|\partial_{x_2} f\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty.$$

The homogeneous Sobolev space denoted by $\dot{H}^1(\mathbb{R}^2)$ is the quotient of $\dot{\mathcal{H}}(\mathbb{R}^2)$ with $\mathbb{C}$ and is equipped with the norm $\| \cdot \|_{\dot{H}^1} = R(\cdot)$. Its dual space, for the L^2-scalar product, is the homogeneous Sobolev space $\dot{H}^{-1}(\mathbb{R}^2)$.

2 Definition of the Mumford Process

We will present the definitions of the complex and the real Mumford processes.

The complex Mumford process $X(x,\omega)$ is formally defined from the complex white noise $Z(x,\omega)$, for $x \in \mathbb{R}^2$, by

$$X(x,\omega) = \Lambda^{-1} Z(x,\omega), \tag{1}$$

where the operator Λ^{-1} is defined, in the frequency domain for $\xi \in \mathbb{R}^2$, by

$$\widehat{\Lambda^{-1} f}(\xi) = \frac{\widehat{f}(\xi)}{|\xi|}.$$

Applying the Fourier transform, the definition (1) becomes

$$\widehat{X}(\xi,\omega) = \frac{1}{|\xi|} Z(\xi,\omega) \tag{2}$$

since the white noise is invariant on the unitary action of the Fourier transform.

For any orthonormal basis $\{\psi_i, i \in I\}$ of $L^2(\mathbb{R}^2)$, one has

$$Z(x,\omega) = \sum_{i\in I} g_i(\omega)\psi_i(x)$$

where the complex random variables g_i are independant and identically distributed (*i.i.d.*) of law $\mathcal{N}(0,1)$. It turns out that (1) and (2) can be written as

$$X(x,\omega) = \sum_{i\in I} g_i(\omega)\Lambda^{-1}(\psi_i)(x) \tag{3}$$

and

$$\widehat{X}(\xi,\omega) = \sum_{i\in I} g_i(\omega)\frac{\widehat{\psi_i}(\xi)}{|\xi|} \tag{4}$$

respectively.

Observe that the operator Λ^{-1} may be defined as the convolution with the Riesz potential $\frac{c}{|x|}$ on $\mathbb{R}^2$. Hence, Λ^{-1} preserves the real-valued functions. Then Eq (1) can be applied to the real white noise and provides us a definition of the real Mumford process. In this case, formula (3) is applied with real-valued variables g_i and real-valued functions ψ_i.

Consequently, the real Mumford process is the real part of the complex Mumford process. Nevertheless, it is not determined by its Fourier transform.

Lemma 1 *The Mumford process belongs $\omega - a.e.$ to $\mathcal{S}_0'(\mathbb{R}^2)$.*

PROOF. It is sufficient to consider the complex case. We know that the white noise is $\omega - a.e.$ in $\mathcal{S}'(\mathbb{R}^2)$. Choose $\varphi_n \in \mathcal{S}_0(\mathbb{R}^2)$, $n \in \mathbb{N}$, and $\varphi \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in $\mathcal{S}(\mathbb{R}^2)$. One has $\partial^\alpha \widehat{\varphi_n}(0) = \partial^\alpha \widehat{\varphi}(0) = 0$ for all $\alpha \in \mathbb{N}^2$. It follows that the functions u_n and u, defined for $\xi \in \mathbb{R}^2$ by

$$u_n(\xi) = \frac{\widehat{\varphi_n}(\xi)}{|\xi|}, \quad u(\xi) = \frac{\widehat{\varphi}(\xi)}{|\xi|},$$

belong to $\mathcal{S}(\mathbb{R}^2)$ and $\lim_{n\to\infty} u_n = u$ in $\mathcal{S}(\mathbb{R}^2)$. Finally, one obtains almost everywhere,

$$\lim_{n\to\infty} \langle \widehat{X}(\cdot,\omega), \varphi_n \rangle = \lim_{n\to\infty} \langle Z(\cdot,\omega), u_n \rangle = \langle Z(\cdot,\omega), u \rangle = \langle \widehat{X}(\cdot,\omega), \varphi \rangle,$$

which finishes the proof.

The Mumford process has stationary increments, is invariant by dilations and isotropic. That means that

(1) for all $y \in \mathbb{R}^2$, $X(\cdot + y, \omega) - X(\cdot, \omega)$ is a stationary process,
(2) for all $\lambda > 0$, $X(\lambda \cdot, \omega) \stackrel{L}{=} X(\cdot, \omega)$,
(3) for all $\rho \in SO(2, \mathbb{R})$, $X(\rho \cdot, \omega) \stackrel{L}{=} X(\cdot, \omega)$,

where "$\stackrel{L}{=}$" means that the laws of the processes are identical.

It is possible to give a meaning to $X(\cdot,\omega)$ as a random tempered distribution, *i.e.* to find an expansion of $X(\cdot,\omega)$ which belongs ω-a.e. to $\mathcal{S}'(\mathbb{R}^n)$. Let us see how. We consider the orthonormal Meyer wavelet basis of $L^2(\mathbb{R}^n)$, $\{\psi^\varepsilon_{j,k}(\cdot) = 2^j \psi^\varepsilon(2^j \cdot - k),\ j \in \mathbb{Z}, k \in \mathbb{Z}^2,\ \varepsilon \in \{1,2,3\}\}$ with $\psi^\varepsilon \in \mathcal{S}_0(\mathbb{R}^2)$ (cf. [5] for precise definition and properties of this basis) . The series (4) becomes

$$X(\cdot,\omega) = \sum_{\varepsilon\in\{1,2,3\}} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2} g_{j,k,\varepsilon}(\omega) \psi^\varepsilon(2^j \cdot - k), \tag{5}$$

where the random variables $g_{j,k,\varepsilon}$ are *i.i.d.* of law $\mathcal{N}(0,1)$. This series does not converge $\omega - a.e.$ in $\mathcal{S}(\mathbb{R}^2)$ since an infrared divergence appears, which means that the low-frequency term ($j \le 0$) diverges in the distributional sense. To settle this divergence, we can make an additive renormalization of the low-frequency part in the wavelet expansion.

Proposition 2 *The expansion*

$$\begin{aligned} &\sum_{\varepsilon\in\{1,2,3\}} \sum_{j\le 0} \sum_{k\in\mathbb{Z}^2} g_{j,k,\varepsilon}(\omega)[(\Lambda^{-1}\psi^\varepsilon)(2^j \cdot - k) - (\Lambda^{-1}\psi^\varepsilon)(-k)] \\ &+ \sum_{\varepsilon\in\{1,2,3\}} \sum_{j>0} \sum_{k\in\mathbb{Z}^2} g_{j,k,\varepsilon}(\omega)(\Lambda^{-1}\psi^\varepsilon)(2^j \cdot - k) \end{aligned} \tag{6}$$

is convergent $\omega - a.e.$ *in* $\mathcal{S}'(\mathbb{R}^2)$.

We could then decide to define $X(\cdot,\omega)$ as the sum of the series (6) since this definition coincides with (5) on $\mathcal{S}_0'(\mathbb{R}^2)$.

Remark To make this renormalization, we have to introduce an arbitrary reference scale ($j = 0$) and the expansion does not preserve the property of dilation invariance of the Mumford process. We still have $X(2^j x,\omega) \stackrel{L}{=} X(x,\omega)$ in $\mathcal{S}_0'(\mathbb{R}^2)$ but it is not true for the renormalization given by (6) in $\mathcal{S}'(\mathbb{R}^2)$.

Since we want to preserve the dilation invariance, we will not use this definition in this paper.

It would be interesting to be able to give a meaning to $X(\cdot,\omega)$ as a stochastic distribution which preserves the homogeneity. Unfortunately, that is not possible, which is shown in the following proposition.

Proposition 3 *Let us denote by* $\langle\cdot,\cdot\rangle$ *the duality product* $\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)$. *There is no stochastic distribution* $M(x,\omega)$ *satisfying* $M(x,\omega) = X(x,\omega)$ *in law in* $\mathcal{S}_0'(\mathbb{R}^2)$, $M(\lambda x,\omega) = M(x,\omega)$ *in law in* $\mathcal{S}'(\mathbb{R}^2)$ *and, for all* $\theta \in \mathcal{S}(\mathbb{R}^2)$, $\mathbb{E}(|\langle M(\cdot,\omega),\theta\rangle|^2) < +\infty$.

PROOF. Suppose, contrary to our claim, that there exists such a stochastic distribution. For $\theta \in \mathcal{S}_0(\mathbb{R}^2)$, one has

$$\mathbb{E}(|\langle M(\cdot,\omega),\theta\rangle|^2) = \mathbb{E}(|\langle X(\cdot,\omega),\theta\rangle|^2) = \mathbb{E}(|\langle Z(\cdot,\omega),\Lambda^{-1}\theta\rangle|^2) = \|\Lambda^{-1}\theta\|_{L^2}^2.$$

Consequently, the map T defined for $\theta \in \mathcal{S}_0(\mathbb{R}^2)$ by

$$T(\theta) = \|\Lambda^{-1}\theta\|_{L^2},$$

can be extended to a map F defined on $\mathcal{S}(\mathbb{R}^2)$ by

$$F(\varphi) = \sqrt{\mathbb{E}(\langle M(\cdot,\omega),\varphi\rangle)^2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Therefore, F is sublinear on $\mathcal{S}(\mathbb{R}^2)$ ($F(u+v) \le F(u)+F(v)$) and is homogeneous of degree -2 (that is $F(f[\lambda.]) = \lambda^{-2}F(f)$ for all $\lambda > 0$ and $f \in \mathcal{S}(\mathbb{R}^2)$).

Let us now consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $Supp\,\widehat{\varphi} \subset \{\xi \in \mathbb{R}^2 : |\xi| \le 2\}$ and $\widehat{\varphi}(\xi) = 1$ if $|\xi| \le 1$.

For $l \ge 0$, one has

$$F(\varphi - 2^{2l}\varphi(2^l.)) \le F(\varphi) + F(\varphi(2^l.)) \le 2F(\varphi).$$

But, writing $\varphi - 2^{2l}\varphi(2^l\cdot) = \sum_{j=0}^{l-1} 2^{2j}\varphi(2^j\cdot) - 2^{2(j+1)}\varphi(2^{j+1}\cdot)$, it comes that

$$F(\varphi - 2^{2l}\varphi(2^l.)) = \| \sum_{j=0}^{l-1} \Lambda^{-1}[\varphi(2^j\cdot) - 2^{2(j+1)}\varphi(2^{j+1}\cdot)]\|_2 \simeq l,$$

and we obtain a contradiction.

As it is the case of the space $\dot{H}^1(\mathbb{R}^2)$, the only hope is to confine the infrared divergence to a "small" term.

Definition 4 *A couple of stochastic processes (X_0, X_1) is a confinement of the infrared divergence of order m $(0 \leq m \leq +\infty)$ of the Mumford process if there exists an unconditional basis $\{\phi_{j,k} = \phi_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_0\} \cup \{\theta_{j,k} = \theta_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_1\}$ with $Card\, K_1 = m$, such that*

(1) $X = X_0 + X_1$.
(2) For all $j \in \mathbb{Z}$, $X_0(2^j\cdot, \omega) = X_0(\cdot, \omega)$ in law, and $X_1(2^j\cdot, \omega) = X_1(\cdot, \omega)$ in law (as stochastic distributions modulo polynomials).
(3) $X_0(x, \omega) = \sum_{j\in\mathbb{Z}} \sum_{k\in K_0} h_{j,k}(\omega)\phi_{j,k}(x)$, where $h_{j,k}$ are some random variables.
(4) $X_1(x, \omega) = \sum_{j\in\mathbb{Z}} \sum_{k\in K_1} H_{j,k}(\omega)\theta_{j,k}(x)$, where $H_{j,k}$ are some random variables.
(5) The expansion $X_0(x, \omega) = \sum_{j\in\mathbb{Z}} \sum_{k\in K_0} h_{j,k}(\omega)\phi_{j,k}(x)$ converges $\omega - a.e.$ in $\mathcal{S}'(\mathbb{R}^2)$ and, for all $j \in \mathbb{Z}$, $X_0(2^j\cdot, \omega)$ and $X_0(\cdot, \omega)$ have the same law (as stochastic distributions).

Proposition 5 *In Definition 4, the order of the confinement m depends only on (X_0, X_1) and not on the choice of the unconditional basis $\{\phi_{j,k} = \phi_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_0\} \cup \{\theta_{j,k} = \theta_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_1\}$.*

To prove Proposition 5 we will use the following lemma (shown in [13]).

Lemma 6 *Let B be a Banach space, U an automorphism of B and $n \in \mathbb{N}^*$, such that there exist n vectors $e_1, \ldots, e_n \in B$ for which the collection*

$$\{U^k(e_i)\,;\, k \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$$

is an unconditional basis of B. Let us assume that there exist some vectors $f_j \in B$, indexed by a set E, such that the collection

$$\{U^k(f_j)\,;\, k \in \mathbb{Z}, j \in E\}$$

is also an unconditional basis of B. Then E is finite of cardinality n.

PROOF. [of Proposition 5] Let (X_0, X_1) be a confinement of the infrared divergence and $\Phi = \{\phi_{j,k} = \phi_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_0\} \cup \{\theta_{j,k} = \theta_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in K_1\}$ (with $K_0 \cap K_1 = \emptyset$) and $\Psi = \{\psi_{j,k} = \psi_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in L_0\} \cup \{\tau_{j,k} = \tau_k(2^j\cdot),\ j \in \mathbb{Z},\ k \in L_1\}$ (with $L_0 \cap L_1 = \emptyset$), two unconditional bases of $\dot{H}^1(\mathbb{R}^2)$ such that

$$X_0(\cdot,\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in K_0} h_{j,k}(\omega)\phi_{j,k}(\cdot) = \sum_{j\in\mathbb{Z}}\sum_{k\in L_0} f_{j,k}(\omega)\psi_{j,k}(\cdot) \tag{7}$$

and

$$X_1(\cdot,\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in K_1} h_{j,k}(\omega)\theta_{j,k}(\cdot) = \sum_{j\in\mathbb{Z}}\sum_{k\in L_1} f_{j,k}(\omega)\tau_{j,k}(\cdot) \tag{8}$$

where $h_{j,k}$ and $f_{j,k}$, $j \in \mathbb{Z}$, $k \in K_0 \cup K_1$, are some random variables.

We denote by $\{\phi^*_{j,k},\ j \in \mathbb{Z}, k \in K_0\} \cup \{\theta^*_{j,k},\ j \in \mathbb{Z},\ k \in K_1\}$ (resp. $\{\psi^*_{j,k},\ j \in \mathbb{Z}, k \in L_0\} \cup \{\tau^*_{j,k},\ j \in \mathbb{Z},\ k \in L_1\}$) the dual basis of Φ (resp. Ψ) for the H^1-scalar product given, for f and g in $\dot{H}^1(\mathbb{R}^2)$, by

$$\langle f, g\rangle = \int \partial_{x_1} f(x)\overline{\partial_{x_1} g(x)}dx + \int \partial_{x_2} f(x)\overline{\partial_{x_2} g(x)}dx.$$

Let us denote F_1 and G_1 the closed subspaces of $\dot{H}^1(\mathbb{R}^2)$, given by

$$F_1 = Span\,\{\theta_{j,k},\ j \in \mathbb{Z},\ k \in K_1\} \quad G_1 = Span\,\{\tau_{j,k},\ j \in \mathbb{Z},\ k \in L_1\}.$$

On account of Lemma 6, it is sufficient to prove that $F_1 = G_1$. For that purpose, let us consider an orthonormal basis $\{e_l,\ l \in \mathbb{Z}\}$ of $\dot{H}^1(\mathbb{R}^2)$. We have

$$X(x,\omega) = \sum_{l\in\mathbb{Z}} g_l(\omega)e_l(x),$$

where g_l, $l \in \mathbb{Z}$, are *i.i.d.* of law $\mathcal{N}(0,1)$. Moreover, by putting

$$e_l = \sum_{j\in\mathbb{Z}}\sum_{k\in K_0} \langle e_l, \phi^*_{j,k}\rangle\phi_{j,k} + \sum_{j\in\mathbb{Z}}\sum_{k\in K_1} \langle e_l, \theta^*_{j,k}\rangle\theta_{j,k},$$

we get

$$X(\cdot,\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in K_0}\sum_{l\in\mathbb{Z}}\langle e_l, \phi^*_{j,k}\rangle g_l(\omega)\phi_{j,k} + \sum_{j\in\mathbb{Z}}\sum_{k\in K_1}\sum_{l\in\mathbb{Z}}\langle e_l, \theta^*_{j,k}\rangle g_l(\omega)\theta_{j,k}. \tag{9}$$

Comparing $\langle X, \theta^*_{j,k}\rangle$ from (7), (8) and (9), we obtain, for all $j \in \mathbb{Z}$ and $k \in K_1$,

$$h_{j,k}(\omega) = \sum_{l\in\mathbb{Z}}\langle e_l, \theta^*_{j,k}\rangle g_l(\omega)$$

and

$$X_1(\cdot,\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in K_1}\sum_{l\in\mathbb{Z}}\langle e_l, \theta^*_{j,k}\rangle g_l(\omega)\theta_{j,k}. \tag{10}$$

By the same argument, we have also

$$X_1(x,\omega) = \sum_{j\in\mathbb{Z}} \sum_{k\in L_1} \sum_{l\in\mathbb{Z}} \langle e_l, \tau^*_{j,k}\rangle g_l(\omega)\tau_{j,k}. \tag{11}$$

Combining (11) with the fact that $\langle \tau_{j,k}, \psi^*_{j_0,k_0}\rangle = 0$, we get $\mathbb{E}|\langle X_1(\cdot,\omega), \psi^*_{j_0,k_0}\rangle|^2 = 0$. But, by (10), we obtain

$$\begin{aligned}
\mathbb{E}|\langle X_1(\cdot,\omega), \psi^*_{j_0,k_0}\rangle|^2 &= \mathbb{E}|\sum_{j\in\mathbb{Z}} \sum_{k\in K_1} \sum_{l\in\mathbb{Z}} \langle e_l, \theta^*_{j,k}\rangle \langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle g_l(\omega)|^2 \\
&= \sum_{l\in\mathbb{Z}} |\sum_{j\in\mathbb{Z}} \sum_{k\in K_1} \langle e_l, \theta^*_{j,k}\rangle \langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle|^2 \\
&= \left\| \sum_{j\in\mathbb{Z}} \sum_{k\in K_1} \langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle \theta^*_{j,k} \right\|^2_{L^2}.
\end{aligned}$$

It follows that $\sum_{j\in\mathbb{Z}} \sum_{k\in K_1} \langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle \theta^*_{j,k} = 0$ and for all $j, j_0 \in \mathbb{Z}$, $k \in K_1$ and $k_0 \in L_0$, we get $\langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle = 0$.

Now, let us suppose than $F_1 \not\subset G_1$ and let $f = \sum_{j\in\mathbb{Z}} \sum_{k\in K_1} c_{j,k}\theta_{j,k} \in F_1$, $f \notin G_1$. Then

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in L_0} \langle f, \psi^*_{j,k}\rangle \psi_{j,k} + \sum_{j\in\mathbb{Z}} \sum_{k\in L_1} \langle f, \tau^*_{j,k}\rangle \tau_{j,k}$$

with $\sum_{j\in\mathbb{Z}} \sum_{k\in L_0} \langle f, \psi^*_{j,k}\rangle \psi_{j,k} \not\equiv 0$. But for $j_0 \in \mathbb{Z}$ and $k_0 \in L_0$, one has

$$\langle f, \psi^*_{j_0,k_0}\rangle = \sum_{j\in\mathbb{Z}} \sum_{k\in K_1} c_{j,k} \langle \theta_{j,k}, \psi^*_{j_0,k_0}\rangle = 0,$$

which gives us a contradiction and $F_1 \subset G_1$. Interchanging the role of the unconditional bases Φ and Ψ, we obtain also $G_1 \subset F_1$, which completes the proof.

3 Confinement of the infrared divergence on the radial term

The Mumford process is defined in the frequency plan by $Z(\xi,\omega) = |\xi|\widehat{X}(\xi,\omega)$. It is then natural to search an expansion of $\widehat{X}$ in polar coordinates. We have the following result.

Theorem 7 *In the Fourier domain, the Mumford process $\widehat{X}(\xi,\omega)$ can be written as $\widehat{X}(\xi,\omega) = \widehat{X_0}(\xi,\omega) + \widehat{X_1}(\xi,\omega)$, where*

(1) $\widehat{X_0}(\xi,\omega)$ is a radial distribution

(2) $\widehat{X_1}(\xi,\omega)$ *is automatically renormalized. In other words,* $\widehat{X_1}(\xi,\omega)$ *is* $\omega-a.e.$ *in* $\mathcal{D}'(\mathbb{R}^2)$ *and this embedding commutes with the dyadic dilations (for all* $j\in\mathbb{Z}$, $\widehat{X_1}(2^j\xi,\omega)$ *and* $2^{2j}\widehat{X_1}(\xi,\omega)$ *have the same law).*

To prove this result, we begin by expanding the white noise $Z(\xi,\omega)$ on an orthonormal basis of $L^2(\mathbb{R}^2)$. This basis is given in polar coordinates (ρ,θ) by the tensorial product $w_{j,k}(\rho)e_n(\theta)$, where $\{w_{j,k}\}$ is an orthonormal basis of the Hilbert space $L^2(]0,+\infty[,\rho d\rho)$ and $\{e_n\}$ is the usual trigonemetric system on $[0,2\pi]$ ($e_n(\theta)=\frac{1}{\sqrt{2\pi}}e^{in\theta}$, $n\in\mathbb{Z}$).

It is sufficient to choose the basis $\{w_{j,k}\}$ in a suitable way. In the following subsections we describe two choices. The first is based on the Haar system and the second on the Malvar-Wilson basis.

3.1 Construction of $\{w_{j,k}\}$ from the Haar system

We divide the interval $]0,+\infty[$ into dyadic intervals $I_j=[2^j,2^{j+1}[$, $j\in\mathbb{Z}$. On each dyadic interval I_j, we use the usual Haar system of $L^2(I_j,d\rho)$ adapted to this interval. It is given by the functions $h_{j,k}=2^{-\frac{j}{2}}h(2^{-j}\cdot)$, $k\in\mathbb{N}$, where $\{h_k,\ k\in\mathbb{N}\}$ is the Haar system on $L^2([1,2[)$. More precisely, $h_0=\mathbf{1}_{[1,2[}$ and for $k=2^l+p$ with $l\geq 0$ and $1\leq p\leq 2^l$,

$$h_k=2^{\frac{l}{2}}(\mathbf{1}_{[1+2^{-l}(p-1),1+2^{-l}(p-\frac{1}{2})[}-\mathbf{1}_{[1+2^{-l}(p-\frac{1}{2}),1+2^{-l}p[}). \tag{12}$$

The functions $w_{j,k}$, $j\in\mathbb{Z}$, $k\in\mathbb{N}$, are defined on $]0,+\infty[$ by

$$w_{j,k}(\rho)=\rho^{-\frac{1}{2}}h_{j,k}(\rho)=\rho^{-\frac{1}{2}}2^{-\frac{j}{2}}h_k(2^{-j}\rho).$$

The white noise $Z(\xi,\omega)$ is then written, with $\xi=\rho e^{i\theta}$, as

$$Z(\xi,\omega)=\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}g_{j,k,n}(\omega)w_{j,k}(\rho)e_n(\theta),$$

where the complex random variables $g_{j,k,n}$, $j\in\mathbb{Z}$, $k\in\mathbb{N}$, $n\in\mathbb{Z}$, are *i.i.d.* of law $\mathcal{N}(0,1)$, and the Mumford process is written as

$$\begin{aligned}\widehat{X}(\xi,\omega)&=\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}g_{j,k,n}(\omega)\rho^{-1}w_{j,k}(\rho)e_n(\theta)\\&=\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}g_{j,k,n}(\omega)\rho^{-\frac{3}{2}}2^{-\frac{j}{2}}h_k(2^{-j}\rho)e_n(\theta).\end{aligned} \tag{13}$$

We define $\widehat{X_0}$ and $\widehat{X_1}$ by

$$\widehat{X_0}(\xi,\omega)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}g_{j,k,0}(\omega)\rho^{-\frac{3}{2}}2^{-\frac{j}{2}}h_k(2^{-j}\rho) \tag{14}$$

and $\widehat{X_1}(\xi,\omega)=\widehat{X}(\xi,\omega)-\widehat{X_0}(\xi,\omega)$.

The expansion (13) of $\widehat{X}(\xi,\omega)$ satisfies the conclusion of Theorem 7. Indeed, we have

Proposition 8 *The oscillatory part $\widehat{X_1}(\xi,\omega)$ is automatically renormalized since the series*

$$\widehat{X_1}(\xi,\omega)=\sum_{n\in\mathbb{Z},\,n\neq 0}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}} g_{j,k,n}(\omega)\rho^{-\frac{3}{2}}2^{-\frac{j}{2}}h_k(2^{-j}\rho)e_n(\theta) \tag{15}$$

converges $\omega-a.e.$ in $\mathcal{D}'(\mathbb{R}^2)$ and, for all $j\in\mathbb{Z}$, $\widehat{X_1}(2^j\xi,\omega)$ and $2^{2j}\widehat{X_1}(\xi,\omega)$ have the same laws. Hence, the couple (X_0,X_1) (given by (14) and (15)) is a confinement of the Mumford process of order ∞.

To prove the previous result, we need the following classical lemma.

Lemma 9 *Let $\{g_m,\, m\geq 2\}$ be a sequence of i.i.d. random variables of law $\mathcal{N}(0,1)$. Then, the estimate*

$$|g_m(\omega)|\leq C(\omega)\sqrt{\ln m},$$

holds $\omega-a.e.$, for all $m\geq 2$.

PROOF. [of Proposition 8] We will show that the series (15) converges uniformly on any bounded subset B of $\mathcal{C}_0^\infty(\mathbb{R}^2)$. So let $B\subset\mathcal{C}_0^\infty(\mathbb{R}^2)$ be a set for which the support of the test functions $\varphi\in B$ are embedded in a same ball $B(0,N)$ and such that $\|\partial^\alpha\varphi\|_\infty\leq C(\alpha)$ where $C(\alpha)$ does not depend of $\varphi\in B$. Let $\varphi\in B$ one of these test functions. Since the functions $w_{j,k}$, $k\in\mathbb{Z}$, are supported on the dyadic interval I_j, only the scales $j\leq C(N)$ have to be taken into account. We then have to estimate

$$\sum_{n\in\mathbb{Z},\,n\neq 0}\sum_{j\leq C(N)}\sum_{k\in\mathbb{N}}|g_{j,k,n}(\omega)I(j,k,n)|,$$

where

$$\begin{aligned}I(j,k,n)&=\int\int\rho^{-\frac{1}{2}}2^{-\frac{j}{2}}h_k(2^{-j}\rho)e_n(\theta)\varphi(\rho\cos\theta,\rho\sin\theta)d\rho d\theta\\&=\int\int 2^{-j}H_k(2^{-j}\rho)e_n(\theta)\varphi(\rho\cos\theta,\rho\sin\theta)d\rho d\theta,\end{aligned}$$

with $H_k(\rho)=\rho^{-\frac{1}{2}}h_k(\rho)$. Using the change of variable $t=2^{-j}\rho$, we get

$$I(j,k,n)=\int\int H_k(t)\varphi(2^jt\cos\theta,2^jt\sin\theta)e_n(\theta)dtd\theta.$$

But, for all $j \leq C(N)$, one has

$$H_k(|\xi|)\varphi(2^j\xi) = H_k(|\xi|)\varphi(0) + 2^j h_k(|\xi|)R_j(\xi),$$

where R_j belongs to a set $\overline{B} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$, bounded and independent of j. Moreover, R_j is supported on the annulus $\{\xi \in \mathbb{R}^2;\ \frac{1}{2} \leq |\xi| \leq \frac{5}{2}\}$.

Indeed, we have

$$\begin{aligned} H_k(|\xi|)\varphi(2^j\xi) - H_k(|\xi|)\varphi(0) &= H_k(|\xi|)[\varphi(2^j\xi) - \varphi(0)] \\ &= h_k(|\xi|)\frac{[\varphi(2^j\xi) - \varphi(0)]}{|\xi|^{\frac{1}{2}}} \\ &= 2^j h_k(|\xi|)R_j(\xi), \end{aligned}$$

with R_j defined for $j \leq C(N)$ by

$$R_j(\xi) = \left(\frac{[\varphi(2^j\xi) - \varphi(0)]}{2^j|\xi|^{\frac{1}{2}}}\right)\theta(\xi)$$

where $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is supported on the annulus $\{\xi \in \mathbb{R}^2; \frac{1}{2} \leq |\xi| \leq \frac{5}{2}\}$ and is identically equal to 1 on $\{\xi \in \mathbb{R}^2; 1 \leq |\xi| \leq 2\}$. Since $j \leq C(N)$, it follows that the functions R_j belong to a same set $\overline{B} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$, bounded and independent of j.

Returning to the calculus of $I(j,k,n)$, since $n \neq 0$, we get

$$\int\int H_k(t)\varphi(0)e_n(\theta)dtd\theta = 0$$

and

$$I(j,k,n) = 2^j \int\int h_k(t)R_j(t\cos\theta, t\sin\theta)e_n(\theta)dtd\theta.$$

The integrals $I(j,k,n)$ are uniformly bounded by $K(N)2^j(1+k)^{-\frac{3}{2}}(1+n)^{-2}$, where the constant $K(N)$ does not depend on φ, but only on B. This estimation is obtained by using the fact that for $k \neq 0$, h_k is given by (12). In particular, h_k has one vanishing moment ($\int h_k = 0$) and its support has a length of order k^{-1}.

Uniformly for $\varphi \in B$, we thus obtain the following estimation

$$\begin{aligned} &\sum_{n\in\mathbb{Z},\, n\neq 0}\ \sum_{j\leq C(N)}\ \sum_{k\in\mathbb{N}} |g_{j,k,n}(\omega)||I(j,k,n)| \\ &\leq C \sum_{n\in\mathbb{Z},\, n\neq 0}\ \sum_{j\leq C(N)}\ \sum_{k\in\mathbb{N}} C(\omega)\sqrt{\ln(2+|j|+k+|n|)}K(N)2^j(1+k)^{-\frac{3}{2}}|n|^{-2} \\ &\leq C(\omega)K(N). \end{aligned}$$

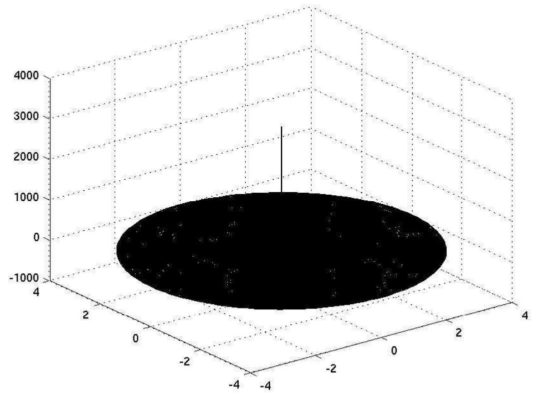

Figure 1. Simulation of the real part of the radial term in frequency.

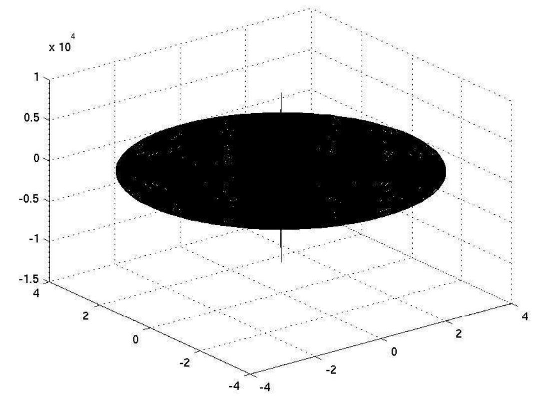

Figure 2. Simulation of the anti-radial term in frequency.

Finally, the series $\widehat{X_1}$ converges uniformly $\omega - a.e.$ on B. Hence, the term $\widehat{X_1}$ converges $\omega - a.e.$ in the distributional sense.

Remark The constant $K(N)$ has a polynomial growth as a function of N, which implies the convergence $\omega - a.e.$ of $\widehat{X_1}$ in $\mathcal{S}'(\mathbb{R}^2)$.

Remark Figures 1 and 2 have been realized with MATLAB. Since the processes do not converge ponctually, they do not give the behaviour of the processes but show the first terms of the expansion in the wavelet basis (10 scales). Nevertheless, with these few terms, the infrared divergence ($\xi = 0$) can already be seen.

3.2 Construction of $\{w_{j,k}\}$ from the Malvar-Wilson basis

We now use the orthonormal Malvar-Wilson basis of $L^2([0,\infty[)$. It is the sequence $2^{\frac{j}{2}}w(2^jx)\cos[\pi(k+\frac{1}{2})2^jx]$, $j\in\mathbb{Z}$, $k\in\mathbb{N}$, where $w\in C_0^\infty([\frac{1}{3},3])$ satisfies

(1) $\quad 0\le w\le 1,$

(2) $\quad w^2(x)+w^2(2-x)=1 \qquad$ for all $x\in[\frac{2}{3},\frac{4}{3}]$,

(3) $\quad w^2(x)+w^2(\frac{x}{2})=1 \qquad$ for all $x\in[\frac{4}{3},\frac{8}{3}]$.

This construction can be found in [1].

For $j\in\mathbb{Z}$, $k\in\mathbb{N}$ and $m\in\mathbb{Z}$, we define in polar coordinates, the functions $w_{j,k,m}$ by

$$w_{j,k,m}(\rho,\theta)=\rho^{-\frac{1}{2}}2^{\frac{j}{2}}w(2^j\rho)\cos[\pi(k+\frac{1}{2})2^j\rho]\frac{e^{im\theta}}{\sqrt{2\pi}}.$$

The system $\{w_{j,k,m},\ j\in\mathbb{Z},\ k\in\mathbb{N},\ m\in\mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{R}^2)$. The white noise can then be expanded in, with $\xi=\rho e^{i\theta}$,

$$Z(\xi,\omega)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{Z}}g_{j,k,m}(\omega)w_{j,k,m}(\rho,\theta),$$

where the random variables $g_{j,k,m}$ are *i.i.d.* of law $\mathcal{N}(0,1)$, and the Fourier transform of the Mumford process is given by

$$\widehat{X}(\xi,\omega)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{Z}}g_{j,k,m}(\omega)\rho^{-1}w_{j,k,m}(\rho,\theta).$$

Again, this expansion does not converge in $\mathcal{D}'(\mathbb{R}^2)$. Nevertheless, we get

Proposition 10 *The oscillatory part is automatically renormalized, in the sense that the series*

$$\widehat{X_1}(\xi,\omega)=\sum_{m\in\mathbb{Z},\,m\neq 0}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}}g_{j,k,m}(\omega)\rho^{-1}w_{j,k,m}(\rho,\theta)$$

converges $\omega-a.e.$ in the (tempered) distributional sense.

The proof is similar to the proof of Proposition 8.

4 Wavelet basis with pseudo-constant

4.1 Construction

The wavelet basis with pseudo-constant is a modification of the Meyer adapted wavelet basis (constructed in [7]).

Proposition 11 (Meyer adapted wavelet basis) *Let $N \in \mathbb{N}$ be an odd integer greater than 3. There exists an orthonormal basis of $L^2(\mathbb{R}^2)$ of real valued functions, formed by*

(1) the Daubechies wavelets $\psi^\varepsilon_{j,k} = 2^j \psi^\varepsilon(2^j \cdot -k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2 \setminus K, \varepsilon \in \mathcal{E} = \{1, 2, 3\}$ (where K is a finite set), of class $\mathcal{C}^r$ with $r = r(N) > 0$, such that
i) $Supp\, \psi^\varepsilon = [0, N]^2$,
ii) $k \notin K$ if and only if $Supp\, \psi^\varepsilon(\cdot - k) \cap] - N, N[^2 = \emptyset$,
iii) $\int x^\alpha \psi^\varepsilon(x)dx = 0$ for all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$ (and $m = m(N) \geq r$),
(2) some functions $\psi^\varepsilon_{j,k} = \psi^\varepsilon_k(2^j \cdot)$, $j \in \mathbb{Z}$, $k \in K$, $\varepsilon \in \mathcal{E}$, of class $\mathcal{C}^r$, such that
i) $Supp\, \psi^\varepsilon_k \subset [-3N, 3N]^2 \setminus] - N, N[^2$,
ii) there exists a couple $(k, \varepsilon) \in K \times \mathcal{E}$ such that $\int \psi^\varepsilon_k(x)dx \neq 0$.

Taking finite linear combinations of $\{\psi^\varepsilon_k,\, k \in K, \varepsilon \in \mathcal{E}\}$, we can reorganized these wavelets into an orthonormal system $\{\phi\} \cup \{\phi^\varepsilon_k, (k, \varepsilon) \in \Lambda = K \times \mathcal{E} \setminus (0, 1)\}$ such that

- For all $(k, \varepsilon) \in \Lambda$, $\int \phi^\varepsilon_k = 0$,
- $\int \phi \neq 0$.

For $j \in \mathbb{Z}$, $(k, \varepsilon) \in \Lambda$, we put $\phi_j = 2^j \phi(2^j \cdot)$ and $\phi^\varepsilon_{j,k} = 2^j \phi^\varepsilon_k(2^j \cdot)$.

Lemma 12 *There is a constant $C \neq 0$ such that, for all $x \in \mathbb{R}^2$*

$$\sum_{j \in \mathbb{Z}} \phi(2^j x) = C. \tag{16}$$

PROOF. We consider the function $f = \mathbf{1}_{[-4N,4N]^2}$. Expanding it on the wavelet basis, we get

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2 \setminus K} \sum_{\varepsilon \in \mathcal{E}} c(j, k, \varepsilon)\psi^\varepsilon_{j,k} + \sum_{j \in \mathbb{Z}} \sum_{(k,\varepsilon) \in \Lambda} c(j, k, \varepsilon)\phi^\varepsilon_{j,k} + \sum_{j \in \mathbb{Z}} c(j)\phi_j$$

where $c(j, k, \varepsilon) = \int f(x)\psi^\varepsilon_{j,k}(x)dx$ if $k \notin K$, $c(j, k, \varepsilon) = \int f(x)\phi^\varepsilon_{j,k}(x)dx$ if $(k, \varepsilon) \in \Lambda$ and $c(j) = \int f(x)\phi_j(x)dx$.

Let $g \in L^2(\mathbb{R}^2)$ be given by

$$g = \sum_{j>0} c(j)\phi_j.$$

We have $f \equiv g$ on $[-N, N]^2$. Indeed, if $j \leq 0$, then

$$Supp\, \psi_{j,k}^{\varepsilon} \cap]-N, N[^2 = Supp\, \phi_{j,k}^{\varepsilon} \cap]-N, N[^2 = Supp\phi \cap]-N, N[^2 = \emptyset.$$

For $j > 0$, we have to divide the proof into two cases. If the support of the wavelet does not intersect $]-2N, 2N[^2$, the wavelet is identically equal to 0 on $]-N, N[^2$. If its support intersects $]-2N, 2N[^2$, then the wavelet is supported on $]-4N, 4N[^2$. Since the corresponding coefficient $c(j, k, \varepsilon)$ is the integral of the wavelet, it is equal to 0.

Finally, we get

$$\sum_{j>0} c(j)\phi_j(x) = 1$$

for $x \in [-N, N]^2$. In addition, one has, for $j > 0$, $c(j) = \int \phi_j = c(0)2^{-j}$. Thus, by dilation, we obtain (16), and Lemma 12 follows.

We are now in position to introduce the pseudo-constant θ defined by

$$\theta \stackrel{L^2}{=} \sum_{j \leq 0} \phi(2^j \cdot)$$

and, for $j \in \mathbb{Z}$, we put $\theta_j = 2^j \theta(2^j \cdot)$.

Lemma 13 *The function θ is of class $\mathcal{C}^r$, is supported on the ball $B(0, 3N\sqrt{2})$ and, for all $x \in B(0, N)$,*

$$\theta(x) = 1.$$

In particular, for all $\gamma \in \mathbb{N}^2 \setminus \{(0,0)\}$ with $|\gamma| \leq r$, one has

$$Supp\, \partial^{\gamma}\theta \subset \{x \in \mathbb{R}^2;\ N \leq |x| \leq 3N\sqrt{2}\}.$$

This result is an obvious corollary of Lemma 12.

Proposition 14 *The system*

$$\{\psi_{j,k}^{\varepsilon},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^2 \setminus K,\ \varepsilon \in \mathcal{E}\} \cup \{\phi_{j,k}^{\varepsilon},\ j \in \mathbb{Z},\ (k, \varepsilon) \in \Lambda\} \cup \{\theta_j,\ j \in \mathbb{Z}\} \quad (17)$$

is a Riesz basis of $L^2(\mathbb{R}^2)$ called "wavelet basis with pseudo-constant".

PROOF. It is easier to prove that the dual system is a Riesz basis of $L^2(\mathbb{R}^2)$. So, we introduce the functions θ^* and θ_j^* given by

$$\theta^* = \phi - \frac{1}{4}\phi(\frac{\cdot}{2})$$

and $\theta_j^* = 2^j\theta^*(2^j\cdot)$. We denote by E and F the closed subspaces of $L^2(\mathbb{R}^2)$ defined by

$$E = Span\,\{\phi_j,\, j \in \mathbb{Z}\}$$

and

$$F = Span\,\{\psi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^2 \setminus K,\, \varepsilon \in \mathcal{E}\} \cup \{\phi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, (k,\varepsilon) \in \Lambda\}.$$

For $h = e + f \in L^2(\mathbb{R}^2)$ with $e \in E$ and $f \in F$ we define the operator S by

$$S(h)(\cdot) = \frac{1}{4}e(\frac{\cdot}{2}).$$

Finally, the operator T is defined on $L^2(\mathbb{R}^2)$ by $T = Id - S$, where Id is the identity operator. Since $L^2(\mathbb{R}^2) = E \overset{\perp}{\oplus} F$, one has, for $h = e + f$,

$$\|S(h)\|_{L^2} = \|\frac{1}{4}e(\frac{\cdot}{2})\|_{L^2} = \frac{1}{2}\|e\|_{L^2} \leq \frac{1}{2}\|f\|_{L^2}.$$

Hence, $\|S\|_{L^2} < 1$ and T is an isomorphism on $L^2(\mathbb{R}^2)$, which maps the basis $\{\psi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^2 \setminus K,\, \varepsilon \in \mathcal{E}\} \cup \{\phi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, (k,\varepsilon) \in \Lambda\} \cup \{\phi_j,\, j \in \mathbb{Z}\}$ to the system

$$\{\psi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^2 \setminus K,\, \varepsilon \in \mathcal{E}\} \cup \{\phi_{j,k}^\varepsilon,\, j \in \mathbb{Z},\, (k,\varepsilon) \in \Lambda\} \cup \{\theta_j^*,\, j \in \mathbb{Z}\}. \quad (18)$$

It follows that the system (18) is a Riesz basis of $L^2(\mathbb{R}^2)$. An easy computation shows that

$$T^{-1*}(\phi_j) = \sum_{l \geq 0} \phi_j(2^l\cdot) = \theta_j.$$

Consequently, the dual basis of (18) is the system (17) which proves Proposition 14.

Theorem 15 *The wavelet basis with pseudo-constant chosen with wavelets of class $\mathcal{C}^r$ with $r > 1$ is an unconditional basis of $\dot{H}^1(\mathbb{R}^2)$.*

4.2 Proof of Theorem 15

We refer to [10] for characterizations and properties of unconditional bi-orthogonal bases.

It is sufficient to show that

(i) The systems (17) and (18) are biorthogonal.

(ii) There exists a constant $C_0 > 0$ such that, for any finite sums, one has

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}} c_{j,k,\varepsilon}\psi^{\varepsilon}_{j,k}\|^2_{\dot{H}^1} \leq C_0 \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}} |c_{j,k,\varepsilon}|^2 2^{2j},$$

$$\|\sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda} c_{j,k,\varepsilon}\phi^{\varepsilon}_{j,k}\|^2_{\dot{H}^1} \leq C_0 \sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda} |c_{j,k,\varepsilon}|^2 2^{2j} \tag{19}$$

and

$$\|\sum_{j\in\mathbb{Z}} c_j\theta_j\|^2_{\dot{H}^1} \leq C_0 \sum_{j\in\mathbb{Z}} |c_j|^2 2^{2j}. \tag{20}$$

(iii) There exists a constant $C_1 > 0$ such that, for any finite sums, one has

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}} c_{j,k,\varepsilon}\psi^{\varepsilon}_{j,k}\|^2_{\dot{H}^{-1}} \leq C_1 \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}} |c_{j,k,\varepsilon}|^2 2^{-2j},$$

$$\|\sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda} c_{j,k,\varepsilon}\phi^{\varepsilon}_{j,k}\|^2_{\dot{H}^{-1}} \leq C_1 \sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda} |c_{j,k,\varepsilon}|^2 2^{-2j} \tag{21}$$

and

$$\|\sum_{j\in\mathbb{Z}} c_j\theta^*_j\|^2_{\dot{H}^{-1}} \leq C_1 \sum_{j\in\mathbb{Z}} |c_j|^2 2^{-2j}. \tag{22}$$

(iv) The wavelet basis with pseudo-constant is a total system in $\dot{H}^1(\mathbb{R}^2)$.

By construction of the systems, (i) is satisfied. The estimations (ii) and (iii) are already known for the functions $\psi^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2 \setminus K$, $\varepsilon \in \mathcal{E}$, since they are classical Daubechies wavelets (cf. [4] or [5] for characterizations of Sobolev spaces by wavelets).

To obtain the estimations (ii) and (iii) for the other terms, we will use the properties of localization of functions, which lead to the property of quasi-orthogonality.

Definition 16 *Let H be a Hilbert space equipped with the scalar product $\langle\cdot,\cdot\rangle$. A system $\{f_k,\ k \in \mathbb{Z}\}$ of H is said to be quasi-orthogonal if there exists $l \in \mathbb{N}$ such that, for any fixed $k \in \mathbb{Z}$,*

$$\textit{for all } k' \in \mathbb{Z},\ |k'-k| \geq l, \quad \langle f_k, f_{k'}\rangle = 0. \tag{23}$$

Before proving the estimations (19) and (20), let us give the following lemma.

Lemma 17 *Let H be a Hilbert space equipped with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $\{f_j,\ j \in \mathbb{Z}\}$ be a quasi-orthogonal system of H and $l \in \mathbb{N}$ satisfying (23). Then,*

$$\|\sum_{j\in\mathbb{Z}} f_j\|^2 \leq (2l+1)\sum_{j\in\mathbb{Z}}\|f_j\|^2.$$

We now give the proof of estimations (19) and (20). By Lemma 12, the functions $\partial_{x_1}\theta$ and $\partial_{x_2}\theta$ are supported on the annulus $\{x \in \mathbb{R}^2;\ N \le |x| \le 3N\sqrt{2}\}$. Then, the systems $\{\partial_{x_1}\theta_j\}$ and $\{\partial_{x_2}\theta_j\}$ are quasi-orthonormal in $L^2(\mathbb{R}^2)$ and we get,

$$\|\sum_{j\in\mathbb{Z}} c_j\theta_j\|^2_{\dot{H}^1} \le C(\|\sum_{j\in\mathbb{Z}} c_j\partial_{x_1}\theta_j\|^2_{L^2} + \|\sum_{j\in\mathbb{Z}} c_j\partial_{x_2}\theta_j\|^2_{L^2}) \le C\sum_{j\in\mathbb{Z}}|c_j|^2 2^{2j}.$$

The same argument can be applied to the functions $\phi^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $(k,\varepsilon) \in \Lambda$, to obtain (19) since they are supported on dyadic annula.

To obtain the dual estimation, we will use the following lemma (a proof can be found in [9]).

Lemma 18 *Let Ω a connected bounded open set of $\mathbb{R}^n$, which is strongly Lipschitz and let $f \in L^2(\Omega)$. One has $\int_\Omega f(x)dx = 0$ if and only if there exist n functions $f_1, ..., f_n \in H^1_0(\Omega)$ such that*

$$f = \partial_{x_1} f_1 + ... + \partial_{x_n} f_n.$$

Consequently, since the function θ^* is supported on the annulus $\Gamma = \{N \le |x| \le 6N\sqrt{2}\}$, there exist two functions Θ_1 and Θ_2 in $L^2(\mathbb{R}^2)$, supported on Γ, such that

$$\theta^* = \partial_{x_1}\Theta_1 + \partial_{x_2}\Theta_2.$$

It follows that

$$\sum_{j\in\mathbb{Z}}\|c_j\theta^*_j\|^2_{\dot{H}^{-1}} \le C(\|\sum_{j\in\mathbb{Z}} c_j\Theta_1(2^j\cdot)\|^2_{L^2} + \|\sum_{j\in\mathbb{Z}} c_j\Theta_2(2^j\cdot)\|^2_{L^2} \le C\sum_j |c_j|^2 2^{-2j},$$

where the last majoration is obtained by using the property of quasi-orthogonality of the systems $\{2^j\Theta_1(2^j\cdot), j \in \mathbb{Z}\}$ and $\{2^j\Theta_2(2^j\cdot),\ j \in \mathbb{Z}\}$.

The functions ϕ^{ε}_k, $(k,\varepsilon) \in \Lambda$, have also a vanishing moment and are supported on an annulus. Then, the same argument can be applied to obtain (22).

To prove (iv), since $\mathcal{S}(\mathbb{R}^2)$ (modulo constant) is dense in $\dot{H}^1(\mathbb{R})$, it is sufficient to show that any function $f \in \mathcal{S}(\mathbb{R}^2)$ can be written as

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}}\langle f, \psi^{\varepsilon}_{j,k}\rangle\psi^{\varepsilon}_{j,k} + \sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda}\langle f, \phi^{\varepsilon}_{j,k}\rangle\phi^{\varepsilon}_{j,k} + \sum_{j\in\mathbb{Z}}\langle f, \theta^*_j\rangle\theta_j \tag{24}$$

with

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}}|\langle f, \psi^{\varepsilon}_{j,k}\rangle|^2 2^{2j} + \sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda}|\langle f, \phi^{\varepsilon}_{j,k}\rangle|^2 2^{2j} + \sum_{j\in\mathbb{Z}}|\langle f, \theta^*_j\rangle|^2 2^{2j} < \infty. \tag{25}$$

So let $f \in \mathcal{S}(\mathbb{R}^2)$. Since the two systems are dual bases in $L^2(\mathbb{R}^2)$, the equality (24) is true in $L^2(\mathbb{R}^2)$. We then use the following lemma.

Lemma 19 *There exists a constant $C > 0$ such that for any $g \in \dot{H}^1(\mathbb{R}^2)$,*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2\setminus K}\sum_{\varepsilon\in\mathcal{E}} 2^{2j}|\langle g, \psi^{\varepsilon}_{j,k}\rangle|^2 + \sum_{j\in\mathbb{Z}}\sum_{(k,\varepsilon)\in\Lambda} 2^{2j}|\langle g, \phi^{\varepsilon}_{j,k}\rangle|^2 + \sum_{j\in\mathbb{Z}} 2^{2j}|\langle g, \theta^*_j\rangle|^2 \le C\|g\|^2_{\dot{H}^1}.$$

PROOF. The inequality in question is known for the terms in $\langle g, \psi^{\varepsilon}_{j,k}\rangle$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2 \setminus K$, $\varepsilon \in \mathcal{E}$, since the $\psi^{\varepsilon}_{j,k}$ are classical Daubechies wavelets (cf. [4] for a proof). For the other terms, since the wavelets have a vanishing moment, we can make an integration by part and then use the property of quasi-orthogonality to obtain the estimation.

Returning to the function $f \in \mathcal{S}(\mathbb{R}^2)$, since it belongs to $\dot{H}^1(\mathbb{R}^2)$ by the previous lemma, we get (25). Using the majoration (ii), it turns out that the series in (24) converges in $\dot{H}^1(\mathbb{R}^2)$ and by uniqueness of the limit (in the space of distributions modulo constants), its sums is f. This finishes the proof of Theorem 15.

Remark In the PhD thesis [11], the construction is generalized to wavelet bases with pseudo-polynomials of the order m ($m \in N$) in $\mathbb{R}^n$. These bases allow us to obtain results of confinement for the Sobolev spaces $H^{m+\frac{n}{2}}(\mathbb{R}^n)$. This study will be presented in a future publication.

5 A more accurate confinement

In this section, the results are given for the real and the complex Mumford processes (we do not use the Fourier transform of the process).

We consider the orthonormal Meyer wavelet basis $\{2^l g(2^l \cdot -k),\ l \in \mathbb{Z},\ k \in \mathbb{Z}^2,\ g \in \{g_1, g_2, g_3\}\}$, where $g_i \in \mathcal{S}_0(\mathbb{R}^2)$ is real and $\widehat{g_i}$ is supported in an annulus, which does not contains 0 (cf. [5] for more details). Thus, the functions $f_i = \Lambda^{-1} g_i$ belong to $\mathcal{S}_0(\mathbb{R}^2)$ and by definition, one has

$$X(x,\omega) = \Lambda^{-1}Z(x,\omega) = \sum_{l\in\mathbb{Z}}\sum_{p\in\mathbb{Z}^2}\sum_{i=1}^{3} g_{i,l,p}(\omega) f_{i,l,p}(x), \tag{26}$$

where the random variables $g_{i,l,p}$ are $i.i.d.$ of law $\mathcal{N}(0,1)$ and $f_{i,l,p}(x) = f_i(2^l x - p)$. The convergence of the series (26) holds $\omega - a.e.$ modulo constants.

The idea is to make a change of basis of $\dot{H}^1(\mathbb{R}^2)$ in order to obtain an expansion of $X(x,\omega)$ in the (non-othonormal) wavelet basis with pseudo-constants (with the regularity of the wavelets $r \geq 1$).

Expanding the functions $f_{i,l,p}$ in the wavelet basis with pseudo-constants, we formally obtain

$$X(x,\omega) = X_0(x,\omega) + X_1(x,\omega)$$

where

$$X_0(x,\omega) = \sum_{j\in\mathbb{Z}} h_j(\omega) 2^{-j}\theta_j(x) \tag{27}$$

with

$$h_j(\omega) = 2^j \sum_{l\in\mathbb{Z}} \sum_{p\in\mathbb{Z}^2} \sum_{i=1}^{3} \langle f_{i,l,p},\, \theta_j^* \rangle g_{i,l,p}(\omega),$$

and

$$X_1(x,\omega) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus K} \sum_{\varepsilon\in\mathcal{E}} h_{\varepsilon,j,k}(\omega) 2^{-j}\psi^{\varepsilon}_{j,k}(x) + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda} h_{\varepsilon,j,k}(\omega) 2^{-j}\phi^{\varepsilon}_{j,k}(x) \tag{28}$$

with

$$h_{\varepsilon,j,k}(\omega) = \begin{cases} 2^j \sum_{l\in\mathbb{Z}} \sum_{p\in\mathbb{Z}^2} \sum_{i=1}^{3} \langle f_{i,l,p},\, \psi^{\varepsilon}_{j,k} \rangle g_{i,l,p}(\omega) \text{ if } j\in\mathbb{Z},\, k\in\mathbb{Z}^2\setminus K \text{ and } \varepsilon\in\mathcal{E}, \\ 2^j \sum_{l\in\mathbb{Z}} \sum_{p\in\mathbb{Z}^2} \sum_{i=1}^{3} \langle f_{i,l,p},\, \phi^{\varepsilon}_{j,k} \rangle g_{i,l,p}(\omega) \text{ if } j\in\mathbb{Z} \text{ and } (k,\varepsilon)\in\Lambda\ . \end{cases}$$

Here $\langle\cdot,\,\cdot\rangle$ denotes the L^2-scalar product. Random variables $h_{\varepsilon,j,k}$ and h_j are not *i.i.d.* of law $\mathcal{N}(0,1)$, since the basis is not orthonormal. Nevertheless, they are linear combinations of gaussians variables and we have the following estimation.

Lemma 20 *For all $j\in\mathbb{Z}$, $k\in\mathbb{Z}^2\setminus K$ and $\varepsilon\in\mathcal{E}$, we obtain, $\omega - a.e.$,*

$$|h_{\varepsilon,j,k}(\omega)| \leq C(\omega)\sqrt{\ln(2+|j|+|k|)}.$$

For all $j\in\mathbb{Z}$ and for all $(k,\varepsilon)\in\Lambda$, we get $\omega - a.e.$,

$$|h_{\varepsilon,j,k}(\omega)| \leq C(\omega)\sqrt{\ln(2+|j|)}$$

and

$$|h_j(\omega)| \leq C(\omega)\sqrt{\ln(2+|j|)}.$$

As we will show it, these estimations are due to the fact that the coefficients of the "matrix" of change of basis are the scalar products between two "wavelets". Let us admit Lemma 20 for the moment and show

Theorem 21 *The series (28) converges $\omega - a.e.$ in $\mathcal{S}(\mathbb{R}^2)$ and is invariant in law by dyadic dilations. It follows that the couple (X_0, X_1) (defined by (27) and (28)) is a confinement of the Mumford process of order 1.*

PROOF. Let B be a bounded set of $\mathcal{C}_0^\infty(\mathbb{R}^2)$. There exists a constant $R > 0$ such that, for any $\varphi \in B$, $Supp\,\varphi \subset B(0,R)$. We will estimate, for $\varphi \in B$,

$$I_1 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus K} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)}|\langle 2^{-j}\psi_{j,k}^\varepsilon, \varphi\rangle|$$

and

$$I_2 = \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda} \sqrt{\ln(2+|j|)}|\langle 2^{-j}\phi_{j,k}^\varepsilon, \varphi\rangle|$$

by $C(R)N(\varphi)$, where $N(\varphi)$ is a semi-norm on $\mathcal{C}_0^\infty(\mathbb{R}^2)$ which does not depend on the choice of φ. If these majorations hold, then by Lemma 9, $\langle X_1(\cdot,\omega),\varphi\rangle$ is defined $\omega - a.e.$ by

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus K} \sum_{\varepsilon\in\mathcal{E}} h_{\varepsilon,j,k}(\omega)2^{-j}\langle\psi_{j,k}^\varepsilon,\varphi\rangle + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda} h_{\varepsilon,j,k}(\omega)2^{-j}\langle\phi_{j,k}^\varepsilon,\varphi\rangle.$$

Let us first consider I_1. We have $2^{-j}\psi_{j,k}^\varepsilon = \psi^\varepsilon(2^j\cdot - k)$ and $Supp\,\psi^\varepsilon(\cdot - k)\cap]-N,N[^2 = \emptyset$ (cf. (1) of Proposition 11). Thus, there exists a constant $C(R)$ such that $\langle\psi_{j,k}^\varepsilon,\varphi\rangle = 0$, except eventually for $j \geq C(R)$ and $|k| \leq C(R)2^j$. Then,

$$I_1 \leq \sum_{j\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ |k|\leq C(R)2^j}} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)}|\langle 2^{-j}\psi_{j,k}^\varepsilon, \varphi\rangle|.$$

Using the first vanishing moment of ψ^ε, $\varepsilon \in \mathcal{E}$, we can write $\psi^\varepsilon = \partial_{x_1} g_1^\varepsilon + \partial_{x_2} g_2^\varepsilon$ with $g_1^\varepsilon, g_2^\varepsilon \in L^2(Supp\,\psi^\varepsilon) = L^2([0,N]^2)$. We get

$$\begin{aligned}
|\langle 2^{-j}\psi_{j,k}^\varepsilon,\varphi\rangle| &\leq C2^{-j}\left(\|g_1^\varepsilon(2^j\cdot - k)\|_1 + \|g_2^\varepsilon(2^j\cdot - k)\|_1\right)(\|\partial_{x_1}\varphi\|_\infty + \partial_{x_2}\varphi\|_\infty) \\
&\leq C2^{-3j}\left(\|\partial_{x_1}\varphi\|_\infty + \|\partial_{x_2}\varphi\|_\infty\right).
\end{aligned}$$

Hence, $I_1 \leq K(R)(\|\partial_{x_1}\varphi\|_\infty + \|\partial_{x_2}\varphi\|_\infty)$. The case of I_2 is easier. The functions $\phi_{j,k}^\varepsilon$, for $j \in \mathbb{Z}$ and $(k,\varepsilon) \in \Lambda$, are supported in the annulus $\{x \in \mathbb{R}^2; N \leq |x| \leq 3N\sqrt{2}\}$. Since a finite number of couples of indices (k,ε) appears, it is not necessary to make an integration by parts after having reduced the summation on j to $j \geq C(R)$. So,

$$\begin{aligned}
I_1 &\leq \sum_{j\geq C(R)} \sum_{(k,\varepsilon)\in\Lambda} \sqrt{\ln(2+|j|)}2^{-2j}\left(\sup_{(k,\varepsilon)\in\Lambda}\|\phi_k^\varepsilon\|_1\right)\|\varphi\|_\infty \\
&\leq K(R)\|\varphi\|_\infty.
\end{aligned}$$

The property of invariance by dyadic dilations can be obtained by observing that to change $X_1(\cdot,\omega)$ in $X_1(2^m\cdot,\omega)$ consists to replace $h_{\varepsilon,j,k}(\omega)$ with $h_{\varepsilon,j-m,k}(\omega)$, *i.e.* $g_{i,l,p}(\omega)$ with $g_{i,l+m,p}(\omega)$. Since the variables $g_{i,l,p}$ are *i.i.d.* of law $\mathcal{N}(0,1)$, it follows that $X_1(\cdot,\omega)$ and $X_1(2^m\cdot,\omega)$ have the same law.

Remark The proof of Theorem 21 shows that $X_1(\cdot,\omega)$ is the sum $\omega - a.e.$ of a distribution of order 0 (terms on $\phi^{\varepsilon}_{j,k}$) and of a distribution of order less than or equal to 1 (terms on $\psi^{\varepsilon}_{j,k}$).

We still have to prove Lemma 20.

PROOF. Let us begin with $h_{\varepsilon,j,k}(\omega)$ for $k \in \mathbb{Z}^2 \setminus K$. In this case, we have the scalar product between the two wavelets $f_i(2^l \cdot -k)$ and $\psi^{\varepsilon}(2^j \cdot -k)$, where $f_i \in \mathcal{S}_0(\mathbb{R}^2)$ and $\psi \in \mathcal{C}^r$ with $r > 1$ has a compact support and at least one vanishing moment. Therefore, we have (cf. [6]) for $M > 2$ and $0 < \delta < r$,

$$|\langle f_{i,l,p}, \psi^{\varepsilon}_{j,k}\rangle| \leq C 2^{-j} 2^{-l} 2^{-|j-l|(1+\delta)} \left(\frac{2^{-j}+2^{-l}}{2^{-j}+2^{-l}+|p2^{-l}-k2^{-j}|}\right)^M . \qquad (29)$$

To obtain estimation 29, for $l \geq j$, we integrate $f_{i,l,p}$ and differenciate $\psi^{\varepsilon}_{j,k}$. For $j > l$, we differenciate $f_{i,l,p}$ and integrate $\psi^{\varepsilon}_{j,k}$. It follows from 29 that $|h_{\varepsilon,j,k}(\omega)| \leq C(\omega) A$ with

$$A = \sum_{l\in\mathbb{Z}} \sum_{p\in\mathbb{Z}^2} \sum_{i=1}^{3} 2^{(j-l)} 2^{-|j-l|(1+\delta)} \left(\frac{2^{-j}+2^{-l}}{2^{-j}+2^{-l}+|p2^{-l}-k2^{-j}|}\right)^M \sqrt{\ln(2+|l|+|p|)}.$$

We divide A into $A = A_1 + A_2$ where A_1 corresponds to the summation on the indices $l \leq j$ and A_2 to the indices $l > j$. For the term A_1, since $M > 2$, we get

$$\begin{aligned}
A_1 &\leq C \sum_{l\leq j} \sum_{p\in\mathbb{Z}^2} 2^{-(j-l)\delta} \left(\frac{1}{1+|p-k2^{l-j}|}\right)^M \sqrt{\ln(2+|l|+|p|)} \\
&\leq C \sum_{l\leq j} 2^{-(j-l)\delta} \sqrt{\ln(2+|l|+|k|2^{l-j})} \\
&\leq C \sum_{l\leq j} 2^{-(j-l)\delta} \sqrt{\ln(2+|l|+|k|)} \\
&\leq C \sqrt{\ln(2+|j|+|k|)}.
\end{aligned}$$

For A_2 ($l > j$), since $M > 2$, we get

$$
\begin{aligned}
A_2 \leq & C \sum_{l>j} \sum_{p\in\mathbb{Z}^2} 2^{(j-l)(\delta+2)} \left(\frac{1}{1+|p2^{j-l}-k|} \right)^M \sqrt{\ln(2+|l|+|p|)} \\
\leq & C \sum_{l>j} 2^{(j-l)(\delta+2)} \sqrt{\ln(2+|l|+|k|2^{l-j})} 2^{2(l-j)} \\
\leq & C \sum_{l>j} 2^{(j-l)\delta} \sqrt{\ln(2+|l|+|k|2^{l-j})} \\
\leq & C \sqrt{\ln(2+|j|+|k|)},
\end{aligned}
$$

and the first estimation is proved.

Let us continue with $h_{\varepsilon,j,k}$ for $(k,\varepsilon)\in\Lambda$. In this case, we have $2^{-j}\phi^\varepsilon_{j,k} = \phi^\varepsilon_k(2^j\cdot)$, where $\phi^\varepsilon_k \in \mathcal{C}^r$, $Supp\,\phi^\varepsilon_k \subset B(0,R)$ for a $R>0$ and $\int \phi^\varepsilon_k(x)dx = 0$. We obtain the estimation (29) by replacing $k2^{-j}$ with 0 and the calculations for $h^\varepsilon_{j,k}$ are similar.

Let us finish with $h_j(\omega)$. We have to estimate $\langle f_{i,l,p}, \theta^*(2^j.)\rangle$, with $\theta^* \in \mathcal{C}^r$, $Supp\,\theta^* \subset B(0,R')$ for a $R'>0$ and $\int \theta^*(x)dx = 0$. The result is then given by the previous case.

Proposition 22 *Let $0<s<2$. The expansion (28) of $X_1(x,\omega)$ satisfies that $\frac{X_1(x,\omega)}{|x|^s}$ converges $\omega - a.e.$ in $\mathcal{D}'(\mathbb{R}^2)$.*

Remark The result shows that the convergence of the term X_1 is stronger than in the distributional sense. It satisfies indeed a "weak Hardy inequality". It can be compared to the deterministic case of the Sobolev $\dot{H}^1(\mathbb{R}^2)$. We have indeed given in [13] a confinement of $\dot{H}^1(\mathbb{R}^2)$ in $\dot{H}^1(\mathbb{R}^2) = X \oplus Y$ where the distributions f of the realized part Y (included in $\mathcal{S}'(\mathbb{R}^2)$) satisfy the Hardy inequality $\int |f(x)|^2/|x|^2 dx < +\infty$.

PROOF. Again, let B be a bounded subset of $\mathcal{C}^\infty_0(\mathbb{R}^2)$ and $R>0$ such that for $\varphi \in B(0,R)$, $Supp\,\varphi \subset B(0,R)$. Using the estimations for the random variables given in Lemma 20, it is sufficient to show that, for $\varphi \in B$,

$$
S_1 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus K} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)} |\langle 2^{-j} \frac{\psi^\varepsilon_{j,k}}{|x|^s}, \varphi\rangle|
$$

and

$$
S_2 = \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda} \sqrt{\ln(2+|j|)} |\langle 2^{-j} \frac{\phi^\varepsilon_{j,k}}{|x|^s}, \varphi\rangle|
$$

can be majorate by $C(R)N(\varphi)$, where $N(\varphi)$ is a semi-norm on $\mathcal{C}^\infty_0(\mathbb{R}^2)$ which does not depend on the choice of φ. Let us first estimate S_2. Again, there exists a constant $C(R)$ such that the summation can be reduced to $j \geq C(R)$. Since $Supp\,\phi^\varepsilon_k \cap]-N,N[^2 = \emptyset$, one has $|x|^s \geq 2^{-j}N$ on $Supp\,\phi^\varepsilon_{j,k}$ and we get

$$
\begin{aligned}
S_2 &\leq K(R) \sum_{j\geq C(R)} \sum_{(k,\varepsilon)\in\Lambda} \sqrt{\ln(2+|j|)}2^{js} \int |\phi_k^\varepsilon(2^j x)\varphi(x)|dx \\
&\leq K(R) \sum_{j\geq C(R)} \sum_{(k,\varepsilon)\in\Lambda} \sqrt{\ln(2+|j|)}2^{j(s-2)} \|\phi_k^\varepsilon\|_1 \|\varphi\|_\infty \\
&\leq K(R)\|\varphi\|_\infty
\end{aligned}
$$

since $s<2$. Let us now estimate S_1. We divide S_1 in $S_1 = S_a + S_b$, where

$$
S_a = \sum_{j\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ |k|\leq 2^{j/2}}} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)} |\langle 2^{-j}\frac{\psi_{j,k}^\varepsilon}{|x|^s}, \varphi\rangle|
$$

and

$$
S_b = \sum_{j\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ 2^{j/2}\leq |k|\leq C(R)2^j}} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)} |\langle 2^{-j}\frac{\psi_{j,k}^\varepsilon}{|x|^s}, \varphi\rangle|. \tag{30}
$$

To majorate S_a, we use the fact that $|x|^s \geq C(|k|2^{-j})^s$ on $Supp\, \psi_{j,k}^\varepsilon = [k2^{-j}, (k+N)2^{-j}]^2$ for $k\notin K$ (and we have $|k|\neq 0$ if $k\notin K$). Thus,

$$
\begin{aligned}
|S_a| &\leq C \sum_{j\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ |k|\leq 2^{j/2}}} \sum_{\varepsilon\in\mathcal{E}} \sqrt{\ln(2+|j|+|k|)} 2^{js}|k|^{-s} \int |\psi^\varepsilon(2^j x-k)\varphi(x)|dx \\
&\leq C \sum_{j\in\mathbb{Z}} \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ |k|\leq 2^{j/2}}} |k|^s \sqrt{|j|+2}\, 2^{j(s-2)} \left(\sup_{\varepsilon\in\mathcal{E}} \|\psi^\varepsilon\|_1\right) \|\varphi\|_\infty \\
&\leq K \sum_{j\geq C(R)} \sqrt{|j|+2}\, 2^{-\frac{j}{2}(s-2)} 2^{j(s-2)} \|\varphi\|_\infty \\
&\leq K(R)\|\varphi\|_\infty
\end{aligned}
$$

for $s<2$. Let us now consider S_b. We use the fact that ψ has at least two vanishing moments (since it is a Daubechies wavelet of regularity $r>1$). Then we can write

$$
\psi^\varepsilon = \partial_{x_1}^2 g_1^\varepsilon + \partial_{x_2}^2 g_2^\varepsilon + \partial_{x_1x_2}^2 g_3^\varepsilon
$$

where the functions $g_1^\varepsilon, g_2^\varepsilon, g_3^\varepsilon$ are of class $\mathcal{C}^{r+2}$ and are supported on $[0,N]^2$. Since the function $\frac{\varphi}{|x|^s}$ is $\mathcal{C}^\infty$ on $Supp\,\psi_{j,k}^\varepsilon$, we can make an integration by parts, and we obtain

$$
\begin{aligned}
\int \psi^\varepsilon(2^j x-k)\,\frac{\varphi(x)}{|x|^s}dx = 2^{-2j} \int \Bigg(& g_1^\varepsilon(2^j x-k)\partial_{x_1}^2\left(\frac{\varphi}{|x|^s}\right) \\
&+ g_2^\varepsilon(2^j x-k)\partial_{x_2}^2\left(\frac{\varphi}{|x|^s}\right) + g_3^\varepsilon(2^j x-k)\partial_{x_1x_2}^2\left(\frac{\varphi}{|x|^s}\right)\Bigg) dx
\end{aligned}
$$

and

$$\left|\int \frac{\psi^\varepsilon(2^jx-k)\varphi(x)}{|x|^s}dx\right| \le C2^{-4j}\left(\left\|\partial^2_{x_1}\left(\frac{\varphi}{|x|^s}\right)\right\|_{L^\infty(\Gamma_{j,k})} + \left\|\partial^2_{x_2}\left(\frac{\varphi}{|x|^s}\right)\right\|_{L^\infty(\Gamma_{j,k})} + \left\|\partial^2_{x_1x_2}\left(\frac{\varphi}{|x|^s}\right)\right\|_{L^\infty(\Gamma_{j,k})}\right)$$

with $C = \|g_1^\varepsilon\|_1 + \|g_2^\varepsilon\|_1 + \|g_3^\varepsilon\|_1$ and $\Gamma_{j,k} = Supp\,\psi^\varepsilon_{j,k} = [k2^{-j},(k+N)2^{-j}]^2$. Moreover,

$$\begin{aligned}
&\left\|\partial^2_{x_1}\left(\frac{\varphi(x)}{|x|^s}\right)\right\|_{L^\infty(\Gamma_{j,k})} \\
\le\ & \left\|\frac{\partial^2_{x_1}\varphi(x)}{|x|^s}\right\|_{L^\infty(\Gamma_{j,k})} + 2\left\|\partial_{x_1}\varphi(x)\partial_{x_1}\frac{1}{|x|^s}\right\|_{L^\infty(\Gamma_{j,k})} + \left\|\varphi(x)\partial^2_{x_1}\frac{1}{|x|^s}\right\|_{L^\infty(\Gamma_{j,k})} \\
&\le C\left(\|\partial^2_{x_1}\varphi\|_\infty \left\|\frac{1}{|x|^s}\right\|_{L^\infty(\Gamma_{j,k})} + \|\partial_{x_1}\varphi\|_\infty\left\|\frac{1}{|x|^{s+1}}\right\|_{L^\infty(\Gamma_{j,k})} + \right. \\
&\qquad \left. + \|\varphi\|_\infty\left\|\frac{1}{|x|^{s+2}}\right\|_{L^\infty(\Gamma_{j,k})}\right) \\
&\le C\left(\|\partial^2_{x_1}\varphi\|_\infty + \|\partial_{x_1}\varphi\|_\infty + \|\varphi\|_\infty\right)2^{\frac{j}{2}(s+2)},
\end{aligned}$$

since $|x| \ge C2^{\frac{j}{2}}$ on $\Gamma_{j,k}$ for $|k| \ge 2^{\frac{j}{2}}$. By the same kind of estimation for the other terms, we obtain

$$\left|\int \frac{\psi^\varepsilon(2^jx\cdot k)\varphi(x)}{|x|^s}\right| \le K2^{-3j}2^{j\frac{s}{2}}\sum_{|\alpha|\le 2}\|\partial^\alpha\varphi\|_\infty.$$

Using this majoration in (30), we get

$$\begin{aligned}
S_b &\le \sum_{j\ge C(R)}\ \sum_{\substack{k\in\mathbb{Z}^2\setminus K \\ 2^{j/2}\le|k|\le C(R)2^j}}\ \sum_{\varepsilon\in\mathcal{E}}\sqrt{\ln(2+|j|+|k|)}2^{-3j}2^{j\frac{s}{2}}\sum_{|\alpha|\le 2}\|\partial^\alpha\varphi\|_\infty \\
&\le K(R)\sum_{|\alpha|\le 2}\|\partial^\alpha\varphi\|_\infty.
\end{aligned}$$

Remark We do not know if the result can be extended to the case $s = 2$. Because of the logarithmic estimation we have for the random variables, the answer seems to be negative.

This paper is mainly a part of the PhD thesis [11] and the results -without proofs- have been presented in the Note [12].

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