Brownian Paths Homogeneously Distributed in Space: Percolation Phase Transition and Uniqueness of the Unbounded Cluster

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Abstract

We consider a continuum percolation model on $\mathbb{R}^d$, $d \geq 1$. For $t, \lambda \in (0, \infty)$ and $d \in \{1, 2, 3\}$, the occupied set is given by the union of independent Brownian paths running up to time t whose initial points form a Poisson point process with intensity $\lambda > 0$. When $d \geq 4$, the Brownian paths are replaced by Wiener sausages with radius $r > 0$.

We establish that, for $d = 1$ and all choices of t, no percolation occurs, whereas for $d \geq 2$, there is a non-trivial percolation transition in t, provided λ and r are chosen properly. The last statement means that λ has to be chosen to be strictly smaller than the critical percolation parameter for the occupied set at time zero (which is infinite when $d \in \{2, 3\}$, but finite and dependent on r when $d \geq 4$). We further show that for all $d \geq 2$, the unbounded cluster in the supercritical phase is unique.

Along the line a finite box criterion for non-percolation in the Boolean model is extended to radius distributions with an exponential tails. This may be of independent interest.

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1 Introduction

Notation. For every $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$. $||\cdot||$ and $||\cdot||_\infty$ stand for the Euclidean norm and supremum norm on $\mathbb{R}^d$, respectively. For any set A, the symbol $A^\complement$ refers to the complement set of A. The open ball with center z and radius r with respect to the Euclidean norm is denoted by $\mathcal{B}(z,r)$, whereas $\mathcal{B}_\infty(z,r)$ stands for the same ball with respect to the supremum norm. Furthermore, for every $0 < r < r'$, we denote by $\mathcal{A}(r,r') = \mathcal{B}(0,r') \backslash \overline{\mathcal{B}}(0,r)$ and $\mathcal{A}_\infty(r,r') = \mathcal{B}_\infty(0,r') \backslash \overline{\mathcal{B}}_\infty(0,r)$ the annulus delimited by the balls of radii r and r' with respect to the Euclidean norm and supremum norm, respectively. Given a d-dimensional Brownian motion $(B_t)_{t\geq 0}$, we denote its i-th component by $(B_{t,i})_{t\geq 0}$, for $i \in \{1, 2, \ldots, d\}$. For all $I \subseteq \mathbb{R}^+$, we denote by B_I the set $\{B_t,\, t \in I\}$. The symbol $\mathbb{P}^a$ denotes the law of a Brownian motion starting in a. Finally, $\mathbb{P}^{a_1,a_2}$ denotes the law of two independent Brownian motions starting in a_1 and a_2, respectively.

1.1 Overview and motivation

For $\lambda > 0$, let $(\Omega_p, \mathcal{A}_p, \mathrm{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \text{Leb}_d$ is defined. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B^x_t)_{t\geq 0},\, x \in \mathcal{E}\}$ such that for each $x \in \mathcal{E}$, $B^x_0 = x$ and such that $(B^x_t - x)_{t\geq 0}$ is independent of $\mathcal{E}$. A more rigorous definition is provided in Section 1.3 below, where ergodic properties are obtained along. We study for $t, r \geq 0$ the *occupied* set (see Figure 1 below):

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B^x_s, r). \tag{1.1}$$

In the rest of the paper, we write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.

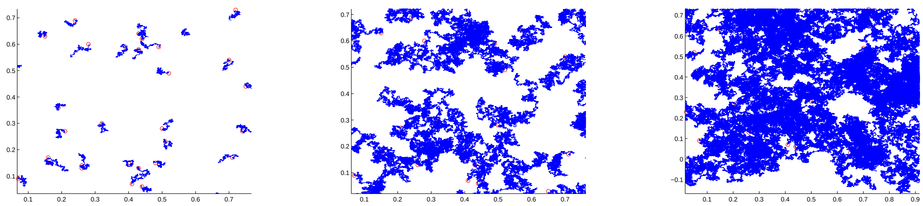

Figure 1: Simulations of $\mathcal{O}_t$ in the case $d = 2$, at a small time, intermediate and large time.

Two points x and y of $\mathbb{R}^d$ are said to be *connected* in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a component. A component $\mathcal{C}$ is bounded if there exists $R > 0$ such that $\mathcal{C} \subseteq \mathcal{B}(0,R)$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component which is maximal in the sense that it is not strictly contained in another connected component.

We are interested in the percolative properties of the occupied set: is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: is there a percolation transition in t?

1.2 Results

We fix $\lambda > 0$.

Theorem 1.1. **[No percolation for** $d = 1$**]** *Let $d = 1$. Then, for all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*

Theorem 1.2. **[Percolation phase transition and uniqueness for** $d \in \{2, 3\}$**]** *Suppose that $d \in \{2, 3\}$. There exists $t_c = t_c(\lambda, d) > 0$ such that for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster, whereas for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*

Let $d \geq 4$, $r > 0$ and let δ_r be the Dirac measure concentrated on r. We denote by $\lambda_c(\delta_r)$ the critical value for $\mathcal{O}_{0,r}$ such that for all $\lambda < \lambda_c(\delta_r)$ the set $\mathcal{O}_{0,r}$ almost surely does not contain an unbounded cluster, and such that for $\lambda > \lambda_c(\delta_r)$ it does, see also (2.5). It follows from Theorem 2.1, that $\lambda_c(\delta_r) > 0$ and $\lim_{r \to 0} \lambda_c(\delta_r) = \infty$.

Theorem 1.3. **[Percolation phase transition and uniqueness for** $d \geq 4$**]** *Suppose that $d \geq 4$ and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then, there exists $t_c = t_c(\lambda, d, r) > 0$ such that for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster, whereas for $t > t_c$, it has almost surely a unique unbounded cluster.*

1.3 Construction and an ergodic property.

In this section we briefly outline how to construct the model described in Section 1.1 and we state an ergodic theorem. The construction is very close to the construction of the Boolean percolation model, in which balls of random radii are placed around each point of a Poisson point process. We refer the reader to Section 1.4 of [MR96], where a more detailed description of the Boolean percolation model is given (see also Section 2 in the present work).

Construction. Let $\mathcal{E}$ be a Poisson point process with intensity $\lambda \times \mathrm{Leb}_d$ defined on $(\Omega_p, \mathcal{A}_p, \mathrm{P}_\lambda)$. Consider the family of binary cubes

$$K(n, z) = \prod_{i=1}^{d} (z_i 2^{-n}, (z_i + 1)2^{-n}], \qquad \forall n \in \mathbb{N},\ z = (z_i)_{1 \leq i \leq d} \in \mathbb{Z}^d, \tag{1.2}$$

so that for each $n \in \mathbb{N}$, $\{K(n, z), z \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. In particular, for each $x \in \mathcal{E}$ and $n \in \mathbb{N}$, there exists a unique $z(n, x)$ such that $x \in K(n, z(n, x))$. Consequently, P_λ-a.s., for each $x \in \mathcal{E}$,

$$n_0(x) := \inf\{n \geq 1 \ :\ K(n, z(n, x)) \cap \mathcal{E} = \{x\}\} \tag{1.3}$$

is well defined. Let $\mathcal{B}(C([0, \infty), \mathbb{R}^d))$ be the Borel σ-algebra on $C([0, \infty), \mathbb{R}^d)$ with respect to the supremum norm. To continue define $\Omega_B = C([0, \infty), \mathbb{R}^d)^{\mathbb{N} \times \mathbb{Z}^d}$, equip Ω_B with the product σ-algebra $\mathcal{A}_B = \mathcal{B}(C([0, \infty), \mathbb{R}^d))^{\mathbb{N} \times \mathbb{Z}^d}$ and let $\mathrm{P}_B = W_B^{\otimes \mathbb{N} \times \mathbb{Z}^d}$, where W_B is the Wiener measure on $C([0, \infty), \mathbb{R}^d)$. The Brownian path associated to $x \in \mathcal{E}$ is defined to be

$$w_B(n_0(x), z(n_0(x), x)), \quad w_B \in \Omega_B. \tag{1.4}$$

Finally, we set $\Omega = \Omega_p \times \Omega_B$, $\mathcal{A} = \mathcal{A}_p \times \mathcal{A}_B$ and $\mathrm{P} = \mathrm{P}_\lambda \times \mathrm{P}_B$, so that the probability space $(\Omega, \mathcal{A}, \mathrm{P})$ corresponds to the model described in Section 1.1.

Ergodicity. For $x \in \mathbb{Z}^d$ let $T_x : \mathbb{R}^d \to \mathbb{R}^d$ be the translation defined by $T_x(y) = y + x$, $y \in \mathbb{R}^d$. This induces a translation S_x on Ω_p via the equation $(S_x\omega_p)(A) = \omega_p(T_x^{-1}A)$, $A \in \mathcal{A}_p$. A translation on Ω_B is given by the formula $(U_x\omega_B)(n, z) = \omega_B(n, z - x)$, so that we finally can define the translation $\widetilde{T}_x$ on the product space Ω as $\widetilde{T}_x\omega = (S_x\omega_p, U_x\omega_B)$. A simple adaption of the proof of Proposition 2.8 in [MR96] yields the following result.

Proposition 1.4. *For all $t, r \geq 0$ the set $\mathcal{O}_{t,r}$ defined in (1.1) is ergodic with respect to the family of translations $\{\widetilde{T}_x,\ x \in \mathbb{Z}^d\}$.*

1.4 Discussion

Motivation and related models. Our model fits into the class of continuum percolation models, which have been studied by both mathematicians and physicists. Their first appearance can be traced back (at least) to Gilbert [G61] under the name of random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $R > 0$. He did this by connecting each two points of a Poisson point process on $\mathbb{R}^2$, whenever their distance is less than R. Another application, which is mentioned in his work is the modeling of a contagious infection. Here, each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: first throw a point process (e.g. Poisson point process) and attach to each of its points a geometric object, like a disk of random radius (Boolean model) or a segment of random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: we attach to each point of a Poisson point process a Brownian path (a path of a Wiener sausage when $d \geq 4$). It could actually be seen as a model of defects randomly distributed in a material and propagating at random (see also Menshikov, Molchanov and Sidorenko [MMS88] for other physical motivations of continuum percolation). One can think for example of an (infinite) piece of wood containing (homogeneously distributed) worms, where each worm tunnels through the piece of wood at random, and we wonder when the latter "breaks". The informal description above is reminiscent of (and actually, borrowed from) the problem of the disconnection of a cylinder by a random walk, which itself is linked to interlacement percolation [Szn10]. The latter is given by the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, when started from the uniform distribution and running up to time uN^d, as $N \uparrow \infty$. Here u plays the role of an intensity parameter for the interlacements set. However, even though the model of random interlacements and our model seem to share some similarities, there is an important difference: in the interlacement model, the number of trajectories which enter a ball of radius R scales like cR^{d-2} for some $c > 0$, whereas in our case it is at least of order R^d.

Another motivation for studying such a model is that it should arise as the scaling limit of a certain class of discrete dependent percolation models. More precisely, percolation models for a system of independent finite-time random walks homogeneously distributed on $\mathbb{Z}^d$. This could also be seen as a system of non-interacting ideal polymer chains.

Comments on the results. First of all notice that we investigated a phase transition in t. It would also be possible to play with the intensity λ instead. Indeed, multiplying the intensity λ by a factor η changes the typical distance between two Poisson points by a factor $\eta^{-1/d}$. Thus, by scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead. Hence, tuning λ boils down to tuning t.

Moreover, it is worthwhile mentioning that Theorem 1.2 is stated only in the case $r = 0$, which is the case of interest to us. The result is the same when $r > 0$, up to minor modifications. However, if $d \geq 4$ the paths of two independent d-dimensional Brownian motions starting at different points do not intersect. Hence, in this case r has to be chosen positive, otherwise no percolation phase transition occurs.

Besides, we draw the reader's attention to Lemma 2.3, which is useful in proving the continuity result in Proposition 2.2. This lemma provides a finite-box criterion for non-percolation for the Boolean model. It is stated in the case of radius distributions with exponential tail. To our knowledge such a criterion was only proved for bounded radii.

To sum up, the results proven in this article answer the first questions typically asked when studying a new percolation model. However, there are still many challenges left open. One may wonder for instance how fast is the decay of the probability (in the supercritical regime) that there is a ball of a certain size, centered in the origin, which is contained in the vacant set. Moreover, it would be interesting to investigate the scaling behaviour of t_c in dimension $d \geq 4$ as r tends to zero. In the same line one could ask for sharp upper and lower bounds for t_c. Finally, it is not clear whether percolation occurs at t_c.

1.5 Outline of the paper

We shortly describe the organization of the article. In Section 2 we introduce the Boolean percolation model and list and prove some of its properties. In Section 3 we prove Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Sections 4–6. Section 4 (resp. 5) deals with the existence of a non-percolation (resp. percolation) phase. In Section 6 the uniqueness of the unbounded cluster is established. The appendix provides proofs of technical lemmas, which are needed in Sections 2 and 6.

2 Preliminaries on Boolean percolation

The model of Boolean percolation has been discussed in great detail in Meester and Roy [MR96] and we refer to this source for a discussion which goes beyond the description we are giving here.

2.1 Introduction of the model

Let ϱ be a probability measure on $[0, \infty)$ and let χ be the Poisson point process on $\mathbb{R}^d \times [0, \infty)$ with intensity $(\lambda \times \mathrm{Leb}_d) \otimes \varrho$. We denote the corresponding probability measure by $\mathrm{P}_{\lambda,\varrho}$. A point $(x, r(x)) \in \chi$ is interpreted to be the open ball in $\mathbb{R}^d$ with center x and radius $r(x)$. Furthermore, we let $\mathcal{E}$ be the projection of χ onto $\mathbb{R}^d$. Boolean percolation deals with properties of the random set

$$\Sigma = \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r(x)). \tag{2.1}$$

Moreover, $C(y)$, $y \in \mathbb{R}^d$, denotes the cluster of Σ which contains y. If $y \notin \Sigma$, then $C(y) = \emptyset$.

Theorem 2.1 (Gouéré, [Gou08], Theorem 2.1)**.** *For all probability measures ϱ on $(0,\infty)$ the following assertions are equivalent:*
(a)

$$\int_0^\infty x^d \, \varrho(dx) < \infty. \tag{2.2}$$

(b) *There exists $\lambda_0 \in (0,\infty)$ such that for all $\lambda < \lambda_0$,*

$$\mathrm{P}_{\lambda,\varrho}(C(0) \textit{ is unbounded}) = 0. \tag{2.3}$$

Moreover, if (a) *holds, then, for some $C = C(d) > 0$, (2.3) is satisfied for all*

$$\lambda < C\Big(\int_0^\infty x^d \varrho(dx)\Big)^{-1}. \tag{2.4}$$

It is immediate from Theorem 2.1, that

$$\lambda_c(\varrho) := \inf\{\lambda > 0 : \mathrm{P}_{\lambda,\varrho}(C(0) \text{ is unbounded}) > 0\} > 0. \tag{2.5}$$

Moreover, from the remark on page 52 of [MR96] it also follows that $\lambda_c(\varrho) < \infty$ if $\varrho((0,\infty)) > 0$. A more geometric fashion to characterize (2.5) is via crossing probabilities. For that fix $N_1, N_2, \dots, N_d > 0$ and let $\mathrm{CROSS}(N_1, N_2, \dots, N_d)$ be the event that the set $[0,N_1] \times [0,N_2] \times \cdots \times [0,N_d]$ contains a component $\mathcal{C}$ such that $\mathcal{C} \cap \{0\} \times [0,N_2] \times \cdots \times [0,N_d] \neq \emptyset$ and $\mathcal{C} \cap \{N_1\} \times [0,N_2] \times \cdots \times [0,N_d] \neq \emptyset$. The critical value λ_{CROSS} with respect to this event is defined by

$$\lambda_{\mathrm{CROSS}}(\varrho) = \inf\Big\{\lambda > 0 : \limsup_{N\to\infty} \mathrm{P}_{\lambda,\varrho}\left(\mathrm{CROSS}(N, 3N, \dots, 3N)\right) > 0\Big\}. \tag{2.6}$$

Assuming that ϱ has compact support, Menshikov, Molchanov and Sidorenko [MMS88] proved that

$$\lambda_c(\varrho) = \lambda_{\mathrm{CROSS}}(\varrho). \tag{2.7}$$

2.2 Continuity of $\lambda_c(\varrho)$

Given two probability measures ν and μ on a predefined probability space we write $\nu \preceq \mu$, if μ stochastically dominates ν.

Proposition 2.2. *Let ϱ be a probability measure on $[0,\infty)$ with bounded support and let $(\varrho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[0,\infty)$ such that $\varrho_n \to \varrho$ weakly as $n \to \infty$ and such that $\varrho \preceq \varrho_n$ for each $n \in \mathbb{N}$. Moreover, assume that*

- *there are $C > 0$ and $R_0 > 0$ such that for all $n \in \mathbb{N}$, $\varrho_n([R,\infty)) \le e^{-CR}$ for all $R \ge R_0$;*

- *there is a probability measure ϱ' on $[0,\infty)$ with finite moments of order d such that $\varrho_n \preceq \varrho'$ for all $n \in \mathbb{N}$.*

Then,

$$\lim_{n\to\infty} \lambda_c(\varrho_n) = \lambda_c(\varrho). \tag{2.8}$$

The proof of Proposition 2.2 relies on the following two lemmas whose proofs are given in the appendix and at the end of this section, respectively.

Lemma 2.3. *Let $N \in \mathbb{N}$, $\lambda > 0$ and let ϱ be a probability measure on $[0,\infty)$ such that there are constants $C = C(\varrho) > 0$ and $R_0 > 0$ such that $\varrho([R,\infty)) \le e^{-CR}$ for all $R \ge R_0$. There is an $\varepsilon = \varepsilon(C,d) > 0$ such that if*

$$\mathrm{P}_{\lambda,\varrho}(\mathrm{CROSS}(N, 3N, \dots, 3N)) \le \varepsilon, \tag{2.9}$$

then $\mathrm{P}_{\lambda,\varrho}(\exists\, y \in \mathbb{R}^d \, : \, \mathrm{Leb}_d(C(y)) = \infty) = 0.$

Lemma 2.4. *Choose $\eta > 0$ and ϱ' according to Proposition 2.2, then for all $N \in \mathbb{N}$*

$$\lim_{M\to\infty} \mathrm{P}_{\lambda,\varrho'}\Big(\exists\, y \in \mathcal{B}_\infty(0,M)^{\complement} \cap \mathcal{E} \text{ s.t. } \mathcal{B}(y, r(y)) \cap [0,N] \times [0,3N]^{d-1} \neq \emptyset\Big) = 0. \tag{2.10}$$

Remark 2.5. *We expect that our proof of Lemma 2.3 still works when ϱ has a polynomial tail (of sufficiently large order) instead of an exponential tail. However, since we do not need Lemma 2.3 in this stronger version, we did not verify all the details needed for that.*

We start with the proof of Proposition 2.2 subject to Lemmas 2.3–2.4.

Proof of Proposition 2.2. The idea of the proof is due to Penrose [Pen95]. First, note that

$$\limsup_{n\to\infty} \lambda_c(\varrho_n) \le \lambda_c(\varrho), \tag{2.11}$$

since $\varrho \preceq \varrho_n$ for all $n \in \mathbb{N}$. Thus, we may focus on the reversed direction in (2.11). Second, fix $\lambda < \lambda_c(\varrho)$ and let $\varepsilon > 0$ be chosen according to Lemma 2.3. By (2.7) there is a $N \in \mathbb{N}$ such that

$$\mathrm{P}_{\lambda,\varrho}\left(\mathrm{CROSS}(N, 3N, \dots, 3N)\right) \le \varepsilon/3. \tag{2.12}$$

We consider $(\hat{\Omega}, \hat{\mathrm{P}})$ the following coupling of $\{\mathrm{P}_{\lambda,\varrho_n}\}_{n\in\mathbb{N}}$ and $\mathrm{P}_{\lambda,\varrho}$:

- the points of $\mathcal{E}$ are sampled according to P_λ;
- for each point $x \in \mathcal{E}$, by Skorokhod's embedding theorem, the radii $\{r_n(x)\}_{n\in\mathbb{N}}$ and $r(x)$ can be chosen such that they have respective distributions $\{\varrho_n\}_{n\in\mathbb{N}}$ and ϱ and are coupled such that $r_n(x) \xrightarrow[n\to\infty]{} r(x)$ a.s.

The configurations obtained via this coupling are denoted by

$$\Sigma_n := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r_n(x)),\ n \in \mathbb{N}, \quad \text{and} \quad \Sigma_\infty := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r(x)). \tag{2.13}$$

Let $M > 0$ and consider the events

$$E_n = \{\hat{\Sigma} := (\Sigma_k)_{k\in\mathbb{N}\cup\{\infty\}} \, : \, \Sigma_n \in \mathrm{CROSS}^M\}, \quad n \in \mathbb{N} \cup \{\infty\},$$

where

$$\mathrm{CROSS}^M = \left\{ \begin{array}{c} \mathrm{CROSS}(N, 3N, \dots, 3N) \text{ happens by open balls} \\ \text{whose centers are in } \mathcal{B}_\infty(0, M) \end{array} \right\}.$$

Since the number of points in $\mathcal{B}_\infty(0, M) \cap \mathcal{E}$ is finite a.s., we may conclude that

$$\lim_{n\to\infty} \mathbb{1}_{E_n} = \mathbb{1}_{E_\infty} \qquad a.s. \tag{2.14}$$

(Note that the convergence in (2.14) is not true for every possible realization, but indeed on a set of probability one.) Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \hat{\mathrm{P}}(E_n) = \hat{\mathrm{P}}(E_\infty).$$

Therefore,

$$\lim_{n\to\infty} \mathrm{P}_{\lambda,\varrho_n}(\mathrm{CROSS}^M) = \mathrm{P}_{\lambda,\varrho}(\mathrm{CROSS}^M),$$

so that for all $n \in \mathbb{N}$ large enough,

$$\mathrm{P}_{\lambda,\varrho_n}(\mathrm{CROSS}^M) \le 2\varepsilon/3. \tag{2.15}$$

Whence, Lemma 2.4 and the fact that $\varrho_n \preceq \varrho'$ for all $n \in \mathbb{N}$, yields that there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\mathrm{P}_{\lambda,\varrho_n}(\mathrm{CROSS}(N, 3N, \dots, 3N)) \le \varepsilon. \tag{2.16}$$

Thus, an application of Lemma 2.3 yields that there is no unbounded component under $\mathrm{P}_{\lambda,\varrho_n}$ for all $n \ge n_0$. Consequently, $\lambda < \lambda_c(\varrho_n)$ for all $n \ge n_0$, from which Proposition 2.2 follows. $\square$

The proof of Lemma 2.3 is given in Appendix A.

Proof of Lemma 2.4. Recall that $\mathcal{A}_\infty(K, K+1)$ denotes the annulus $\mathcal{B}_\infty(0, K+1) \backslash \overline{\mathcal{B}}_\infty(0, K)$. Then, by summing over the positions of all Poisson points,

$$\begin{aligned}
&\mathrm{P}_{\lambda,\varrho'}\Big(\exists\, y \in \mathcal{B}_\infty(0, M)^{\complement} \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0, N] \times [0, 3N]^{d-1} \neq \emptyset\Big) \\
&= \sum_{K=M}^{\infty} \mathrm{P}_{\lambda,\varrho'}\Big(\exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0, N] \times [0, 3N]^{d-1} \neq \emptyset\Big) \\
&\le \sum_{K=M}^{\infty} \mathrm{P}_{\lambda,\varrho'}\Big(\exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E}\colon\ r(y) \ge K - 3N\Big) \\
&= \sum_{K=M}^{\infty} \sum_{\ell=1}^{\infty} \mathrm{P}_{\lambda,\varrho'}\Big(|\mathcal{A}_\infty(K, K+1) \cap \mathcal{E}| = \ell,\ \exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E} : r(y) \ge K - 3N\Big).
\end{aligned} \tag{2.17}$$

Using that for some constant $c = c(d) > 0$ and all $K \in \mathbb{N}$, $\mathrm{Leb}_d(\mathcal{A}_\infty(K+1, K)) = cK^{d-1}$, the last term in (2.17) may be estimated from above by

$$\sum_{K=M}^{\infty} \sum_{\ell=1}^{\infty} e^{-\lambda c K^{d-1}} \frac{(\lambda c K^{d-1})^\ell}{\ell!} \ell \varrho'([K - 3N, \infty)) \le \mathrm{Cst} \sum_{K=M-3N}^{\infty} K^{d-1} \varrho'([K, \infty)), \tag{2.18}$$

which goes to 0 as M goes to infinity since ϱ' has moments of order d. $\square$

3 Proof of Theorem 1.1

Let $t > 0$. Note that

$$\Sigma_t := \bigcup_{x\in\mathcal{E}} \mathcal{B}\Big(x, \sup_{0\leq s\leq t} \|B_s^x - x\|\Big) \tag{3.1}$$

has the same law as the occupied set in the Boolean percolation model with radius distribution

$$\varrho_t([L,\infty)) = \mathbb{P}^0\Big(\sup_{0\leq s\leq t} \|B_s\| \geq L\Big). \tag{3.2}$$

Note that ϱ_t has finite moments of order d. Indeed, for all $L > 0$,

$$\varrho_t([L,\infty)) \leq 2\mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq L\Big) \leq 4\mathbb{P}^0\Big(B_t \geq L\Big) \leq \frac{4}{L}\sqrt{\frac{t}{2\pi}}e^{-L^2/2t}, \tag{3.3}$$

where we used the reflexion principle in the second inequality. Thus, by Theorem 3.1 in [MR96], almost-surely, the set Σ_t does not contain an unbounded cluster. Finally, the relation $\mathcal{O}_t \subseteq \Sigma_t$ yields the result.

4 Theorems 1.2-1.3: no percolation for small times

In this section we show that there is a $t_c = t_c(\lambda, d) > 0$ ($t_c = t_c(\lambda, d, r) > 0$ when $d \geq 4$) such that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \geq 4$) does not percolate when $t < t_c$. The proof for $d \in \{2, 3\}$ comes in Section 4.1, whereas the proof for $d \geq 4$ comes in Section 4.2. Both proofs heavily rely on the results of Section 2.

4.1 No percolation for $d \in \{2, 3\}$

Let $t > 0$ and define Σ_t and ϱ_t as in Section 3, but with the one-dimensional Brownian motions of Section 3 replaced by its d-dimensional counterparts. As in Section 3 it is sufficient to show the existence of a $t_c > 0$ such that for all $t < t_c$ the set Σ_t almost surely does not have an unbounded component. For that we intend to apply Theorem 2.1. For all $\varepsilon > 0$,

$$\int_0^\infty x^d \varrho_t(dx) \leq \varepsilon^d \int_0^\varepsilon \varrho_t(dx) + \int_\varepsilon^\infty x^d \varrho_t(dx) = \varepsilon^d + \int_{\varepsilon^d}^\infty \varrho_t([y^{1/d},\infty))dy. \tag{4.1}$$

A calculation similar to the one in (3.3) shows that the second term on the right-hand side of (4.1) is bounded by

$$4d\sqrt{\frac{td}{2\pi}} \int_{\varepsilon^d}^\infty \frac{1}{y^{1/d}} e^{-y^{2/d}/2td}\, dy, \tag{4.2}$$

which tends towards zero, as $t \to 0$. Thus, by (4.1)–(4.2) we see that

$$\lim_{t\to 0} \int_0^\infty x^d\, \varrho_t(dx) = 0. \tag{4.3}$$

An application of equation (2.4) in Theorem 2.1 yields the claim.

4.2 No percolation for $d \geq 4$

Let $t > 0$ and let $\varrho_{r,t}$ be the probability measure on $[r, \infty)$ defined via

$$\varrho_{t,r}([a,b]) = \mathbb{P}^0\left(\sup_{0 \leq s \leq t} \|B_s\| \in [a-r, b-r] \right), \qquad r \leq a \leq b. \tag{4.4}$$

Note that $\varrho_{t,r} \to \delta_r$ weakly as $t \to 0$. Thus, by similar calculations as in (3.3) and Proposition 2.2 (with $\varrho' = \varrho_{1,r}$), $\lambda_c(\varrho_{t,r}) \to \lambda_c(\delta_r)$ as $t \to 0$. Hence, there is a $t_0 > 0$ such that $\lambda < \lambda_c(\varrho_{t,r})$ holds for all $t < t_0$. Finally, observe that the set

$$\Sigma_{t,r} = \bigcup_{x \in \mathcal{E}} \mathcal{B}\left(x, \sup_{0 \leq s \leq t} \|B_s^x - x\| + r\right), \qquad \forall t \geq 0, \tag{4.5}$$

is generated by the Poisson point process with intensity measure $(\lambda \times \mathrm{Leb}_d) \otimes \varrho_{t,r}$ and contains $\mathcal{O}_{t,r}$, see (1.1). This is enough to conclude the claim.

5 Theorems 1.2–1.3: percolation for large times

In this section we establish that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \geq 4$) percolates, when t is sufficiently large. The proof for $d \in \{2, 3\}$ comes in Section 5.1, whereas the proof for $d \geq 4$ comes in Section 5.2.

5.1 Proof of the percolation phase in $d \in \{2, 3\}$

We use a coarse-graining argument to prove existence of a percolation phase. More precisely, we divide $\mathbb{R}^d$ into boxes which are indexed by $\mathbb{Z}^d$ and we consider an edge percolation model on the coarse-grained graph whose vertices are identified with the centers of the boxes and the edges connect nearest-neighbours. An edge connecting nearest-neighbours, say x and x', in $\mathbb{Z}^d$, is said to be open if (i) both boxes associated to x and x' contain at least one point of the Poisson point process, and (ii) the Brownian motions which correspond to the point of the Poisson point process which are the closest to the centers of their respective boxes, intersect each other. Some technical computations and a domination result by Liggett, Schonmann and Stacey [LSS97] finally show that percolation in that coarse-grained model occurs if one suitably chooses the size of the boxes and let time run long enough. This implies percolation of our original model.

We now define this coarse-grained model more rigorously. Let $R > 0$ and $t > 0$ to be chosen later. For $x \in \mathbb{Z}^d$, we define

$$\mathcal{B}_x^{(R)} := \mathcal{B}_\infty(2Rx, R) \tag{5.1}$$

and the random variable

$$N^{(R)}(x) := |\, \mathcal{E} \cap \mathcal{B}_x^{(R)} \,| \, . \tag{5.2}$$

When $N^{(R)}(x) \geq 1$, we define the point $z^{(R,x)}$, which is almost surely uniquely determined, via

$$\|z^{(R,x)} - 2Rx\| = \inf_{z \in \mathcal{E} \cap \mathcal{B}_x^{(R)}} \|z - 2Rx\|. \tag{5.3}$$

Note that $z^{(R,x)}$ is the point which is the closest to the center of the box $\mathcal{B}_x^{(R)}$ among all Poisson points of $\mathcal{B}_x^{(R)}$. We denote by $B^{(R,x)}$ the Brownian motion starting from $z^{(R,x)}$. For all couples of nearest-neighbours $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we say that the edge (x, y), which connects x and y, is open if

$$\text{(i)} \quad N^{(R)}(x) \geq 1, \tag{5.4}$$

$$\text{(ii)} \quad N^{(R)}(y) \geq 1, \tag{5.5}$$

$$\text{(iii)} \quad B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset. \tag{5.6}$$

We let $X^{R,t}_{(x,y)}$ be the random variable which takes value 1 if the edge (x, y) is open, and 0 otherwise. In what follows, to not burden the notation, we write $X_{(x,y)}$ instead of $X^{R,t}_{(x,y)}$.

Lemma 5.1. *Let $\varepsilon > 0$. There exists $R > 0$ and $t > 0$ such that for any couple of nearest-neigbours $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $\mathrm{P}(X_{(x,y)} = 1) \geq 1 - \varepsilon$.*

The proof of Lemma 5.1 is deferred to the end of this section. We first show how one may deduce the existence of a percolation phase from it.

Proof of the existence of a percolation phase. Note that if (x, x') and (y, y') is a couple of nearest-neighbour points in $\mathbb{Z}^d$ such that $\{x, x'\} \cap \{y, y'\} = \emptyset$, then $X_{(x,x')}$ and $X_{(y,y')}$ are independent. Therefore, the coarse-grained percolation model is a 2-dependent percolation model. Thus, Theorem 0.0 of Liggett, Schonmann and Stacey [LSS97] yields that we may bound the coarse-grained percolation model from below by Bernoulli bond percolation, whose parameter, say p^*, can be chosen to be arbitrarily close to 1, when $\mathrm{P}(X_{(x,y)} = 1)$ is sufficiently close to 1. Let $p_c(\mathbb{Z}^d)$ be the critical percolation parameter for Bernoulli bond percolation. Then, by Lemma 5.1, there are $R_0 > 0$ and $t_0 > 0$ such that $p^* > p_c(\mathbb{Z}^d)$ for all $R \geq R_0$ and $t \geq t_0$. In that case, the coarse-grained model percolates, and so does $\mathcal{O}_t$. ☐

Consequently, it remains to prove Lemma 5.1. For that we need an additional lemma. It states that the probability that two independent Brownian motions, starting at points $x, y \in \mathbb{R}^d$ have a non-empty intersection up to time t increases, when we move the starting points towards each other.

Lemma 5.2. *Let $t > 0$. Then,*

$$(x, y) \mapsto \mathbb{P}^{x,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big), \qquad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{5.7}$$

is a non-increasing function of $\|x - y\|$.

We first prove Lemma 5.1 subject to Lemma 5.2. The proof of Lemma 5.2 comes afterwards.

Proof of Lemma 5.1. By independence of the events in (i)–(iii), we have

$$\mathrm{P}(X_{(x,y)} = 1) = \mathrm{P}(N^{(R)}(x) \geq 1)^2 \times \mathrm{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset\Big). \tag{5.8}$$

To proceed, we fix $R > 0$ large enough such that

$$\mathrm{P}(N^{(R)}(x) \geq 1) = 1 - e^{-\lambda(2R)^d} \geq 1 - \varepsilon. \tag{5.9}$$

Furthermore, by Lemma 5.2, $\mathrm{P}(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset)$ decreases, when $\|z^{(R,x)} - z^{(R,y)}\|$ increases. However, note that $\|z^{(R,x)} - z^{(R,y)}\| \leq R\sqrt{4(d-1)+16}$, when $\|x - y\| = 1$. Thus,

$$\mathrm{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset\Big) \geq \mathrm{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset \big| \|z^{(R,x)} - z^{(R,y)}\| = R\sqrt{4(d-1)+16}\Big) \tag{5.10}$$

$$= \mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big), \tag{5.11}$$

for any choice of z_1 and z_2 such that $\|z_1 - z_2\| = R\sqrt{4(d-1)+16}$. Using Theorem 9.1 (b) in Mörters and Peres [MP10], there exists t large enough such that for all such choices of z_1 and z_2,

$$\mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) \geq 1 - \varepsilon, \tag{5.12}$$

which is enough to deduce the claim. □

We now prove Lemma 5.2.

Proof of Lemma 5.2. Note that it is enough to prove the claim for the function

$$y \mapsto \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big). \tag{5.13}$$

We fix $R' > R > 0$ and $y, y' \in \mathbb{R}^d$ such that $||y|| = R$ and $||y'|| = R'$, respectively. Using rotational invariance of Brownian motion in the first equality and scale invariance of Brownian motion in the last equality, we may write

$$\mathbb{P}^{0,y'}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) = \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) \tag{5.14}$$

$$\leq \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,(R'/R)^2 t]} \cap B^{(2)}_{[0,(R'/R)^2 t]} \neq \emptyset\Big) \tag{5.15}$$

$$= \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big). \tag{5.16}$$

This yields the claim. □

5.2 Proof of the percolation phase for $d \geq 4$

Throughout the proof, z always denotes the d-th coordinate of $x = (\xi, z) \in \mathbb{R}^d$. We further define

$$\mathcal{H}_0 = \{(\xi, z) \in \mathbb{R}^d : z = 0\}. \tag{5.17}$$

The main idea is to reduce the problem to a Boolean percolation problem on $\mathcal{H}_0$. More precisely, we use that for each $x \in \mathcal{E}$, B^x will eventually hit $\mathcal{H}_0$. From this we deduce that for t large enough, the traces of the Wiener sausages which hit $\mathcal{H}_0$ dominate a supercritical $(d-1)$-dimensional Boolean percolation model, and therefore percolate.

We now formalize this strategy. For each $k \in \mathbb{N}$, let

$$\mathcal{S}_k := \{(\xi, z) \in \mathbb{R}^d \ : \ k-1 < z \leq k\}, \tag{5.18}$$

so that $(\mathcal{S}_k)_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^{d-1} \times (0, \infty)$. We fix $k \in \mathbb{N}$ and consider

$$\mathcal{E}_k = \{\xi : \exists\ z \in \mathbb{R} \text{ s.t. } (\xi, z) \in \mathcal{S}_k \cap \mathcal{E}\}. \tag{5.19}$$

Note that $(\mathcal{E}_k)_{k\geq 0}$ are i.i.d. Poisson point processes with parameter $\lambda \times \mathrm{Leb}_{d-1}$. Given $\mathcal{E}_k$, we construct a random set C_t^k in the following way:

- Thinning: each $\xi \in \mathcal{E}_k$ is kept if $\tau_0(z^\xi) \leq t$, where z^ξ is such that $(\xi, z^\xi) \in \mathcal{S}_k \cap \mathcal{E}$ (there is almost-surely only one choice), and $\tau_0(z)$ is the first hitting time of the origin of an one-dimensional Brownian motion starting at z. We choose all Brownian motions, which are associated to some $\xi \in \mathcal{E}_k$, to be independent. Otherwise ξ is discarded.

- Translation: each $\xi \in \mathcal{E}_k$ that was not removed after the previous step is translated by $\bar{B}(\tau_0(z^\xi))$, where $\bar{B}$ is $(d-1)$-dimensional Brownian motion starting at the origin, which is independent of all the previous variables.

Note that z^ξ is uniformly distributed in $(k-1, k)$. Moreover, z^ξ, $\tau_0(z^\xi)$ and $\bar{B}$ are independent of ξ. Thus, C_t^k is the result of a thinning and a translation of $\mathcal{E}_k$, and both operations depend on random variables, which are independent of $\mathcal{E}_k$. Therefore, $(C_t^k)_{k\geq 0}$ is a collection of i.i.d. Poisson point processes with parameter $\lambda p_t^k \times \mathrm{Leb}_{d-1}$, where

$$p_t^k = \int_{k-1}^{k} \mathbb{P}^z\Big(\inf_{0\leq s\leq t} B_s \leq 0\Big)\, dz \geq \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq k\Big). \tag{5.20}$$

By independence of the C_t^k's, the set $\mathcal{C}_t := \bigcup_{k=1}^{\infty} C_t^k$ is thus a Poisson point process with parameter $\lambda \sum_{k\geq 1} p_t^k \times \mathrm{Leb}_{d-1}$.

Let us now consider the Boolean model generated by $\mathcal{C}_t$ with deterministic radius r. Observe that,

$$\sum_{k=1}^{\infty} p_t^k \geq \sum_{k=0}^{\infty} \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq k\Big) - \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq 0\Big) \geq \mathbb{E}^0\Big[\sup_{0\leq s\leq t} B_s\Big] - 1. \tag{5.21}$$

Note that the right-hand side of (5.21) tends to infinity as $t \to \infty$. Thus, by the remark on page 52 in [MR96], there exists $t_0 > 0$ large enough such that the Boolean model generated by $\mathcal{C}_t$ percolates for all $t \geq t_0$. Finally, note that $\mathcal{C}_t$ is stochastically dominated by $\mathcal{O}_t \cap \mathcal{H}_0$, in the sense that $\mathcal{C}_t$ has the same distribution as a subset of $\mathcal{O}_t \cap \mathcal{H}_0$. This completes the proof.

6 Theorems 1.2–1.3: uniqueness of the unbounded cluster

We fix $t, r, \lambda \geq 0$ such that $t > t_c(\lambda, d, r)$. In the following we denote by N_∞ the number of unbounded clusters in $\mathcal{O}_{t,r}$, which is almost-surely a constant as a consequence of Proposition 1.4. For all $d \geq 2$, the proof of uniqueness consists of (i) excluding the case $N_\infty = k$ with $k \in \mathbb{N}\backslash\{1\}$ and of (ii) excluding the case $N_\infty = \infty$. This section is organized as follows. In Section 6.1, we give a short heuristic of (i) in the case $d = 2$, which we use as a guideline for the proofs in all other cases. Section 6.2 contains the proof of uniqueness for Wiener sausages ($r > 0$) in $d \geq 4$, which is also on a technical level close to the heuristics in Section 6.1. This is not true anymore in dimension $d = 3$, which is due to the fact that there is no simple way under which the paths of two independent three-dimensional Brownian motions intersect each other. Therefore, when $d = 3$, the strategy described in Section 6.1 needs to be adapted, which requires a certain number of technical steps. Since the proof for $d = 3$ works for $d = 2$ as well, we decided to give a unified proof for both cases in Section 6.3.

6.1 Heuristics

Let $d = 2$ and $r = 0$. We proceed by contradiction and assume that almost-surely, $N_\infty = k$ with $k \in \mathbb{N}\backslash\{1\}$. For $R_2 > R_1 > 0$, we introduce the event (see Fig. 2 below):

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects all } k \text{ unbounded clusters} \\ \text{without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \tag{6.1}$$

We fix $R_1 > 0$. First, note that by monotonicity in R_2,

$$\mathrm{P}(E_{R_1,R_2}) \geq \mathrm{P}(E_{R_1,R_2} \cap \{\mathcal{E} \cap \mathcal{B}(0, R_1) = \emptyset\}) \overset{R_2\to\infty}{\longrightarrow} \mathrm{P}(\mathcal{E} \cap \mathcal{B}(0, R_1) = \emptyset) > 0. \tag{6.2}$$

Therefore, we can find a $R_2 > 0$ such that $\mathrm{P}(E_{R_1,R_2}) > 0$. Let us fix such a R_2 and observe that E_{R_1,R_2} is independent from the points in $\mathcal{E} \cap \mathcal{B}(0, R_1)$ and the Brownian motions starting from them. Next, one can show that the event

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{E} \cap \mathcal{B}(0, R_1), \\ B^x_{[0,t]} \text{ contains a “loop” in } \mathcal{A}(R_2, R_2 + 1) \end{array} \right\} \tag{6.3}$$

has positive probability. Finally, the contradiction is a consequence of

$$\mathrm{P}(N_\infty = 1) \geq \mathrm{P}(E_{R_1,R_2} \cap L_{R_1,R_2}) = \mathrm{P}(E_{R_1,R_2})\mathrm{P}(L_{R_1,R_2}) > 0, \tag{6.4}$$

since we assumed that $\mathrm{P}(N_\infty = k) = 1$, $k \in \mathbb{N} \setminus \{1\}$.

Remark 6.1. *The above heuristics also shows how to create trifurcation points. In combination with Lemma 6.3, the strategy alluded to above will be used to exclude the possibility of having infinitely many unbounded clusters.*

6.2 Uniqueness in $d \geq 4$

6.2.1 Excluding $2 \leq N_\infty < \infty$

Again we proceed by contradiction. Let us assume that N_∞ is almost-surely equal to a constant $k \in \mathbb{N} \setminus \{1\}$. For simplicity, we further assume that $k = 2$, the extension of the argument to other values of k being straightforward.

For $R_2 > R_1 > 0$, let us define E_{R_1,R_2} as follows

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects at least one path of each of the two} \\ \text{unbounded clusters, without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \tag{6.5}$$

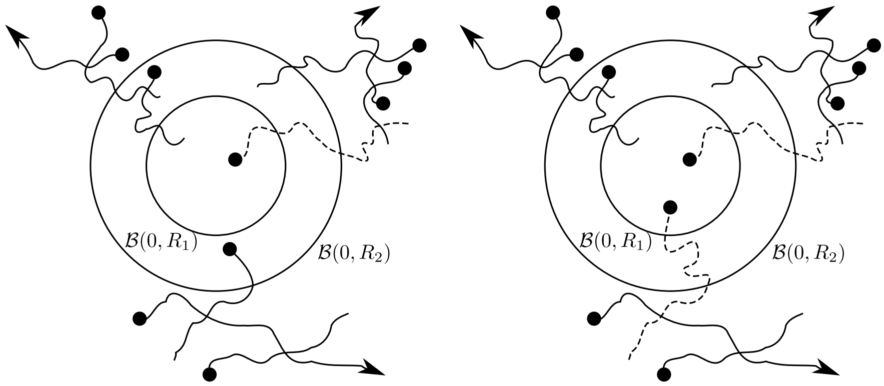

Figure 2: The plot on the left hand side represents a configuration of the event E_{R_1,R_2} with $k = 3$. The symbol $\bullet$ represents the points of $\mathcal{E}$, whereas $\blacktriangleright$ represents connectivity with infinity. Finally, the dashed line emphasizes the fact that points starting inside $\mathcal{B}(0, R_1)$ are not considered for the intersection condition in (6.1). Because of that, the configuration represented on the right hand side does not belong to E_{R_1,R_2}.

First, we note that there exist R_1 and R_2 such that

$$\mathrm{P}(E_{R_1,R_2}) > 0, \tag{6.6}$$

which can be seen as in the lines following (6.2). Next, we consider the event analogous to (6.3),

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{B}(0, R_1) \cap \mathcal{E}, \\ \overline{\mathcal{A}}(R_2 - 3r/2, R_2 - r/2) \subset \cup_{0 \le s \le t} \mathcal{B}(B^x_s, r) \subset \mathcal{B}(0, R_2) \end{array} \right\}, \tag{6.7}$$

which is independent of E_{R_1,R_2} and has positive probability, see Remark 6.2 below. The independence is due to the fact that E_{R_1,R_2} and L_{R_1,R_2} depend on different points of $\mathcal{E}$ and on different Brownian paths. Note that on $E_{R_1,R_2} \cap L_{R_1,R_2}$ the two unbounded clusters, are only connected inside $\mathcal{B}(0, R_2)$.
The contradiction now follows as in (6.4).

Remark 6.2. *A sketch of the proof that L_{R_1,R_2} has positive probability goes as follows. Let $\varepsilon \in (0, r/8)$. By compactness, $\overline{\mathcal{A}}(R - 3r/2, R_2 - 3r/2 + \varepsilon)$ can be covered by a finite number of balls of radius ε. Moreover, a Brownian motion starting in $\mathcal{B}(0, R_1)$ has a positive probability of visiting all these balls before time t and before leaving $\mathcal{B}(0, R_2 - r)$. Consequently, on the aforementioned event, L_{R_1,R_2} is satisfied.*

6.2.2 Excluding $N_\infty = \infty$

We assume that $N_\infty = \infty$. We show that this assumption leads to a contradiction. The proof is based on ideas of Meester and Roy [MR94, Theorem 2.1], who extended a technique developed by Burton and Keane [BK89] to a continuous percolation model. In the proof we use the following counting lemma, which is due to Gandolfi, Keane and Newman [GKN92]. It will yield a contradiction to the existence of trifurcation points, which will be constructed in the first step of the proof.

Lemma 6.3 (Lemma 4.2 in [GKN92]). *Let S be a set, R be a non-empty finite subset of S and $K > 0$. Suppose that*
(a) *for all $z \in R$, there is a family $(C_z^1, C_z^2, \ldots, C_z^{n_z})$, $n_z \geq 3$, of disjoint non-empty subsets of S, which do not contain z and are such that $|C_z^i| \geq K$, for all z and for all $i \in \{1, 2, \ldots, n_z\}$,*
(b) *for all $z, z' \in R$ one of the following cases occurs (where we abbreviate $C_z = \cup_{i=1}^{n_z} C_z^i$ for all $z \in R$):*
(i) $(\{z\} \cup C_z) \cap (\{z'\} \cup C_{z'}) = \emptyset$;
(ii) *there are $i, j \in \{1, 2, \ldots, n_z\}$ such that $\{z'\} \cup C_{z'} \setminus C_{z'}^j \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^j$;*
(iii) *there is $i \in \{1, 2, \ldots, n_z\}$ such that $\{z'\} \cup C_{z'} \subseteq C_z^i$;*
(iv) *there is $j \in \{1, 2, \ldots, n_{z'}\}$ such that $\{z\} \cup C_z \subseteq C_{z'}^j$.*
Then $|S| \geq K(|R| + 2)$.

STEP 1. Balls containing a trifurcation point. Again, we define $E_{R_1,R_2}(0)$ and L_{R_1,R_2} as in (6.5) ,(6.7), respectively. By means of these events, in the same manner as in Subsection 6.2.1, one can show that there are $\delta > 0$ and $R \in \mathbb{N}$ such that the event

$$E_R(0) := \left\{ \begin{array}{l} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0, R)^\complement \text{ contains at} \\ \text{least three unbounded clusters, } |C \cap \mathcal{B}_\infty(0, R) \cap \mathcal{E}| \geq 1 \text{ and each} \\ \text{cluster which intersects } \mathcal{B}_\infty(0, R) \text{ belongs to } C. \end{array} \right\}, \quad (6.8)$$

has probability at least δ. Note that $E_R(0)$ implies that each $x \in \mathcal{B}_\infty(0, R)$ which belongs to an infinite cluster also belongs to C. We call each unbounded cluster in $C \cap \mathcal{B}_\infty(0, R)^\complement$ a branch. To proceed, we fix $K > 0$ and choose $M > 0$ such that the event

$$E_{R,M}(0) = E_R(0) \cap \left\{ \begin{array}{l} \text{there are at least three different branches of } \mathcal{B}_\infty(0, R) \text{ which} \\ \text{contain at least } K \text{ points in } \mathcal{E} \cap (\mathcal{B}_\infty(0, RM) \setminus \mathcal{B}_\infty(0, R)) \end{array} \right\}, \quad (6.9)$$

has probability at least $\delta/2$ (see Fig. 3 below). For $z \in \mathbb{Z}^d$, the events $E_{R,M}(2Rz)$ and $E_R(2Rz)$ are defined in a similar manner as $E_{R,M}(0)$ and $E_R(0)$, except that the balls in the definitions are centered around $2Rz$.
Let $L > M + 2$ and define the set

$$\mathcal{R} = \{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR),\ E_{R,M}(2Rz) \text{ occurs}\}. \quad (6.10)$$

Note that

$$|\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR)\}| \geq (L - M - 2)^d, \quad (6.11)$$

so that we obtain by stationarity

$$\mathrm{E}(|\mathcal{R}|) \geq \frac{(L - M - 2)^d \delta}{2}. \quad (6.12)$$

STEP 2. Application of Lemma 6.3 and contradiction. We identify each $z \in \mathcal{R}$ with a Poisson point in $\mathcal{B}_\infty(2Rz, R) \cap C$, which is contained in the corresponding infinite cluster. In what follows we write Λ_z instead of $\mathcal{B}_\infty(2Rz, R)$, for simplicity of notation. Let n_z be the total number of branches of Λ_z, which contain at least K Poisson points in

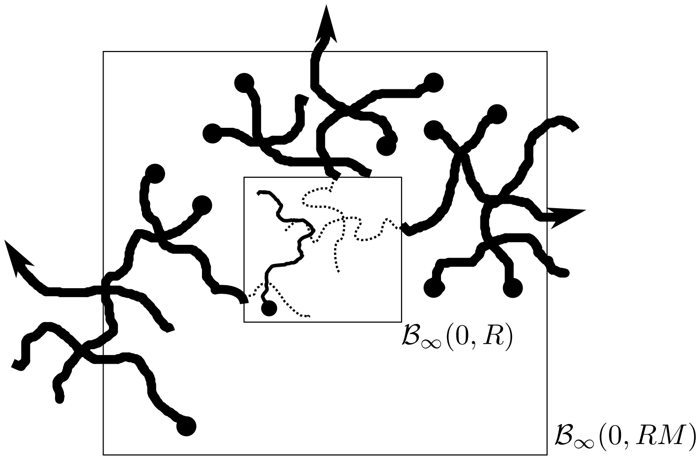

Figure 3: The plot represents a configuration in $E_{R,M}(0)$ with $K=3$ (see (6.8)-(6.9)). The thick lines belong to the branches. As in the previous figure, ▶ represents connection to infinity.

$\mathcal{B}_\infty(2Rz,R)$. For $i \in \{1,\dots,n_z\}$, let $\mathbf{B}_z^i$ be the branch which is the ith-closest to $2Rz$ among all branches of $\mathcal{B}_\infty(2Rz,R)$, see Equation (6.9).

A point x is said to be connected to a set A *through* the set Λ if there exists a continuous function $\gamma:[0,1]\mapsto \Lambda\cap\mathcal{O}_{t,r}$ such that $\gamma(0)=x$ and $\gamma(1)\in A$. We denote it briefly by $x \overset{\Lambda}{\longleftrightarrow} A$. Finally, we define

$$C_z^i = \mathcal{E}\cap\mathcal{B}(0,LR)\cap\mathbf{B}_z^i = \left\{x\in\mathcal{E}\cap\mathcal{B}_\infty(0,LR)\colon\ x \overset{\Lambda_z^c}{\longleftrightarrow} \mathbf{B}_z^i\right\} \qquad \forall i\in\{1,\dots,n_z\}. \quad (6.13)$$

Now we proceed to check that the conditions of Lemma 6.3 are fulfilled. Here $S=\mathcal{B}_\infty(0,LR)\cap\mathcal{E}$. First note that, by definition of a branch, we have that for all $z\in\mathcal{R}$:

- $|C_z^i|\geq K$,
- $C_z^i\cap C_z^j=\emptyset$ for all $i,j\in\{1,\dots,n_z\}$ with $i\neq j$,
- $z\notin C_z$.

Hence, assumption (a) of Lemma 6.3 is met.

We now claim that the collection $\{C_z^i\}_{z\in R, i\in\{1,\dots,n_z\}}$ satisfies also assumption (b) of Lemma 6.3. At this point we would like to stress some facts to be used later:

a. Due to (6.8), $z \overset{\Lambda_z}{\longleftrightarrow} C_z^i$ for all $i\in\{1,\dots,n_z\}$.

b. If $\widetilde{C}$ is an unbounded cluster such that $\widetilde{C}\cap\Lambda_z\neq\emptyset$, then $z \overset{\Lambda_z}{\longleftrightarrow} \widetilde{C}$.

Suppose that $(\{z\}\cup C_z)\cap(\{z'\}\cup C_{z'})\neq\emptyset$. We consider three different cases:

(1) If $z'\in C_z$ then there exists a unique $i\in\{1,\dots,n_z\}$ such that $z'\in C_z^i$. We consider two sub-cases:

- If $z \in C_{z'}$, then there exists a unique $i' \in \{1, \dots, n_{z'}\}$ such that $z \in C_{z'}^{i'}$, and we claim that $\{z'\} \cup C_{z'} \setminus C_{z'}^{i'} \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^{i'}$. Indeed, pick $x' \in C_{z'} \setminus C_{z'}^{i'}$. Then there exists a unique $j' \neq i'$ such that $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$. It is crucial to note that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ since otherwise, due to **b.**, $z \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$ (by first connecting z to x' in $\Lambda_{z'}^c$ and then x' to $C_{z'}^{j'}$ in $\Lambda_{z'}^c$), which contradicts the uniqueness of i'.

  Finally, we have that $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$, $z' \overset{\Lambda_{z'} \subset \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$, $z' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$. A concatenation of all these paths gives $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$, that is $x' \in C_z^i$. This proves the first inclusion that we claimed. The second inclusion follows by symmetry.

- If $z \notin C_{z'}$, then we claim: $\{z'\} \cup C_{z'} \subseteq C_z^i$.

  Indeed, take $x' \in C_{z'}$, then there exists a unique j' such that $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$. As before we have that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ (this time the contradiction follows from $z \notin C_{z'}$). The conclusion follows in the same way as in the previous case.

(2) If $z \in C_{z'}$, then one may conclude as in (1).

(3) Suppose that there exist i, i'such that $C_z^i \cap C_{z'}^{i'} \neq \emptyset$. Take $x' \in C_z^i \cap C_{z'}^{i'}$, then we have that $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ and $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$. There are two cases:

- The path $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ intersects $\Lambda_{z'}$: Due to **b.** we have that $z' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$. Hence $z' \in C_z$, which reduces this case to a previous one.

- In the second case, $x' \overset{\Lambda_z^c \cap \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$: Due to **a.**, we have $z \overset{\Lambda_z \subset \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$. Finally, a concatenation of the previous two paths with $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$ yields that $z \in C_{z'}$, which reduces this case again to a previous one.

Hence, by Lemma 6.3

$$\mathrm{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K(\mathrm{E}(|\mathcal{R}|) + 2), \tag{6.14}$$

so that, by (6.12),

$$\mathrm{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K((L - M - 2)^d \delta/2 + 2). \tag{6.15}$$

On the other hand, since $\mathcal{E}$ is a Poisson point process with intensity measure $\lambda \times \mathrm{Leb}_d$,

$$\mathrm{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) = \lambda(2LR)^d. \tag{6.16}$$

Thus, combining (6.15) and (6.16), yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \leq \lambda(2LR)^d. \tag{6.17}$$

Note that M depends on K, so in order to get a contradiction one can choose $L = 2M$ and let K go to ∞ in the inequality above.

6.3 Uniqueness in $d \in \{2,3\}$

6.3.1 Excluding $\{2 \le N_\infty < \infty\}$

As in the heuristic of Section 6.1, we proceed by contradiction: we assume that $\mathrm{P}(N_\infty = k) = 1$ for some $k \in \mathbb{N} \setminus \{1\}$ and prove that $\mathrm{P}(N_\infty = 1) > 0$, which is absurd. To make the proof more accessible, we assume that $k = 2$ (see Remark 6.7 below).

Remark: The previous heuristic does not work verbatim for $d = 3$ because of clear geometrical reasons: a three-dimensional Brownian motion travelling around an annulus, which is crossed by the two unbounded clusters, does not necessarily connect them. Let us first briefly describe how we adapt this strategy. For R large enough and ε small enough, we show that with positive probability, both unbounded clusters intersect $\mathcal{B}(0,R)$ in such a way, such that each of them contains a Brownian path crossing $\mathcal{A}(R-\varepsilon, R+\varepsilon)$. Afterwards, we show that, still with positive probability, we can reroute the (let us say first) excursions inside $\mathcal{A}(R-\varepsilon, R+\varepsilon)$ of each of these two Brownian paths such that they intersect each other and, as a consequence, merge the two unbounded clusters into a single one. This leads to the desired contradiction, since our construction provides a set of configurations of positive probability on which $N_\infty = 1$.

We now give the proof in full detail. Let $R > 0$ and denote by N_∞^R the number of unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$. In the case that N_∞^R is not empty, we denote those by $\{\mathcal{C}_i(R), 1 \le i \le N_\infty^R\}$ (though it has little relevance, let us agree that clusters are indexed according to the order in which one finds them by radially exploring the occupied set from 0). We also consider the 'extended' clusters, defined by

$$\mathcal{C}_i^{\mathrm{ext}}(R) = \bigcup_{x \in \mathcal{E}\,:\, B^x_{[0,t]} \cap \mathcal{C}_i(R) \neq \emptyset} B^x_{[0,t]}, \tag{6.18}$$

i.e. $\mathcal{C}_i^{\mathrm{ext}}(R)$ is the union of all Brownian paths up to time t, which have a non-empty intersection with $\mathcal{C}_i(R)$ (see Fig. 4 below).

We further define in five steps a notion of good extended clusters and prove that those occur with positive probability.

Good extended clusters in five steps.

STEP 1. Intersection with a large ball. We use the abbreviations $\mathcal{C}_1^{\mathrm{ext}} := \mathcal{C}_1^{\mathrm{ext}}(R)$ and $\mathcal{C}_2^{\mathrm{ext}} := \mathcal{C}_2^{\mathrm{ext}}(R)$ for the two extended unbounded clusters and define

$$E_R := \{N_\infty^R = 2\} \cap \{\mathcal{C}_1^{\mathrm{ext}} \cap \mathcal{B}(0,R) \neq \emptyset\} \cap \{\mathcal{C}_2^{\mathrm{ext}} \cap \mathcal{B}(0,R) \neq \emptyset\}. \tag{6.19}$$

One way of having exactly two unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$ is to have exactly two unbounded clusters in total (i.e. on the whole configuration), hence

$$\mathrm{P}(E_R) \ge \mathrm{P}(N_\infty = 2,\, \mathcal{C}_1^{\mathrm{ext}} \cap \mathcal{B}(0,R) \neq \emptyset,\, \mathcal{C}_2^{\mathrm{ext}} \cap \mathcal{B}(0,R) \neq \emptyset). \tag{6.20}$$

Since the event on the right-hand side of (6.20) is increasing in R, its probability converges, as R tends to ∞, to $\mathrm{P}(N_\infty = 2)$, which equals 1 by our initial assumption. Therefore, we may choose R large enough such that $\mathrm{P}(E_R) \ge 1/2$.

STEP 2. Choice of a path in each cluster. For $i \in \{1,2\}$, define

$$\mathrm{Cross}(i) = \{x \in \mathcal{E} \cap \mathcal{C}_i^{\mathrm{ext}} : \exists s \in [0,t],\ (\|x\| - R)(\|B_s^x\| - R) < 0\}, \tag{6.21}$$

that is the set of points in $\mathcal{E} \cap \mathcal{C}_i^{\mathrm{ext}}$, whose associated Brownian motion crosses $\partial\mathcal{B}(0,R)$. Note that $\mathrm{Cross}(i) \neq \emptyset$ on E_R. For $i \in \{1,2\}$ we denote by x_i the almost-surely uniquely defined $x_i \in \mathrm{Cross}(i)$, such that

$$\|x_i\| = \inf_{y \in \mathrm{Cross}(i)} \|y\|. \tag{6.22}$$

Note that this way of picking x_i is arbitrary. Any other way would serve our purpose as well.

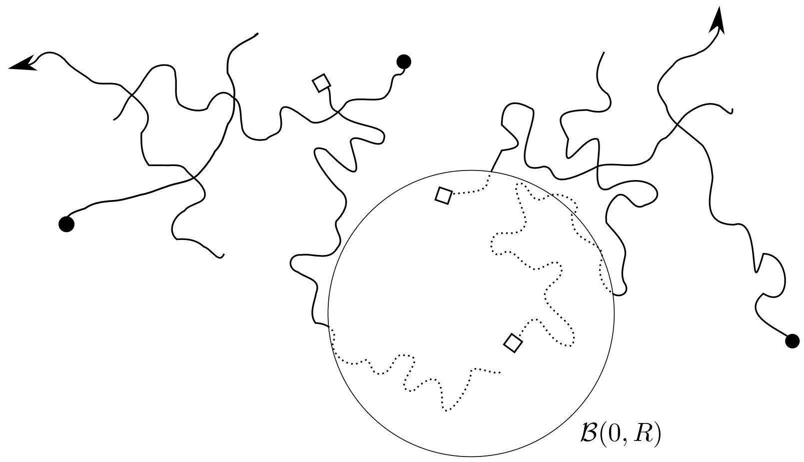

Figure 4: The regular lines are a realization of $\mathcal{C}_i,\ i = 1,2$. In addition with the dotted lines they form the extended clusters $\mathcal{C}_i^{ext},\ i = 1,2$. The points marked with $\square$ are the ones in $\mathrm{Cross}(i),\ i = 1,2$.

STEP 3. First excursion through an annulus centered around $\mathcal{B}(0,R)$. For some $\varepsilon > 0$ to be determined, let us consider the annulus $\mathcal{A}_{R,\varepsilon} := \mathcal{A}(R-\varepsilon, R+\varepsilon)$. Further, define for each $x \in \mathcal{E}$,

$$I(x) := \mathbb{1}\{\inf\{s \geq 0 : \|B_s^x\| = R+\varepsilon\} < \inf\{s \geq 0 : \|B_s^x\| = R-\varepsilon\}\}, \tag{6.23}$$

in the case when at least one of the infima is finite. Otherwise, we set $I(x) = 0$. We will see later that the latter case is of no importance. For $i \in \{1,2\}$, we introduce the following entrance and exit times:

$$\begin{aligned} \sigma_i^{\mathrm{out}} &= \inf\{s \geq 0 : \|B_s^{x_i}\| = R + (-1)^{I(x_i)}\varepsilon\}, \\ \sigma_i^{\mathrm{in}} &= \sup\{s \leq \sigma_i^{\mathrm{out}} : \|B_s^{x_i}\| = R - (-1)^{I(x_i)}\varepsilon\}, \end{aligned} \tag{6.24}$$

i.e. $B^{x_i}_{[\sigma_i^{\mathrm{in}},\sigma_i^{\mathrm{out}}]}$ is the first excursion through $\mathcal{A}_{R,\varepsilon}$ of B^{x_i} (see Fig. 5 below). The reason for this at a first glance strange definition is, that we do not want to exclude the possibility that x_1 or x_2 is located inside $\mathcal{B}(0,R)$. By choosing ε small enough we guarantee that the Brownian motions started at x_1 and x_2 cross $\mathcal{A}_{R,\varepsilon}$, that is, $\sigma_i^{\mathrm{in}} \leq \sigma_i^{\mathrm{out}} \leq t$ for $i \in \{1,2\}$. Later in the proof we will merge $\mathcal{C}_1^{\text{ext}}$ and $\mathcal{C}_2^{\text{ext}}$ into a single unbounded cluster by "replacing" $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ with suitable excursions. However, this operation should not disconnect $B^{x_i}_{[0,t]}$ from $\mathcal{C}_i^{\text{ext}}$. For that reason, we consider the event on which $B^{x_i}_{[0,\sigma_i^{\text{in}})}$ or $B^{x_i}_{(\sigma_i^{\text{out}},t]}$ is already connected to $\mathcal{C}_i^{\text{ext}}$, i.e. we introduce for $i \in \{1,2\}$

$$E_{\varepsilon,i}^{\text{conn}} := \left\{ \left(B^{x_i}_{[0,\sigma_i^{\text{in}})} \cup B^{x_i}_{(\sigma_i^{\text{out}},t]} \right) \cap \mathcal{C}_i^{\text{ext}} \neq \emptyset \right\}. \tag{6.25}$$

Summing everything up, we restrict ourselves to configurations in the set

$$E_{R,\varepsilon} = E_R \bigcap_{i=1,2} \{\sigma_i^{\text{in}} \le \sigma_i^{\text{out}} \le t\} \cap E_{\varepsilon,i}^{\text{conn}}. \tag{6.26}$$

By monotonicity in ε, $\mathrm{P}(E_{R,\varepsilon})$ converges to $\mathrm{P}(E_R) > 1/2$ as ε tends to 0. Therefore, we may fix for the rest of the proof $\varepsilon > 0$ such that $\mathrm{P}(E_{R,\varepsilon}) \ge 1/4$.

STEP 4. Restriction on the time spent to cross the annulus. As has been explained above, our goal is to restrict ourselves to some specific excursions of $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$. The probability of those turn out to be easier to control when we have a deterministic lower bound on the random time lengths $\sigma_i^{\text{out}} - \sigma_i^{\text{in}}$. Therefore, we introduce for $T \in (0,t)$ the following event:

$$E_{R,\varepsilon,T} = E_{R,\varepsilon} \bigcap_{i=1,2} \{\sigma_i^{\text{out}} - \sigma_i^{\text{in}} \ge T\}. \tag{6.27}$$

Again, by monotonicity in T, we can choose the latter small enough such that $\mathrm{P}(E_{R,\varepsilon,T}) \ge \mathrm{P}(E_{R,\varepsilon})/2 \ge 1/8$.

STEP 5. Staying away from the boundary of the annulus during the excursion. To obtain a configuration with a unique unbounded cluster, we restrict ourselves to configurations in the set $E_{R,\varepsilon,T}$ and we reroute $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ such that they intersect each other. Since σ_i^{in} is not a stopping time, the law of $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ is not the one of a Brownian motion. Conditioned on both endpoints, $(B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]})$, $i \in \{1,2\}$, are instead Brownian excursions, the law of which is not absolutely continuous with respect to the one of a Brownian motion. As a consequence, we cannot directly use our knowledge on the intersection probabilities of two Brownian motions. This is why we will work with $(B^{x_i}_{[\sigma_i^{\text{in}}+\delta,\sigma_i^{\text{out}}-\delta]})$, $i \in \{1,2\}$, for some $\delta \in (0, T/8)$ instead (the restriction to consider the Brownian motions only up to time $\sigma_i^{\text{out}} - \delta$ is just for esthetic reasons) . These subpaths, when conditioned on both endpoints, are Brownian bridges conditioned to stay in $\mathcal{A}_{R,\varepsilon}$, and indeed the density of a Brownian bridge with respect to a Brownian motion is explicit and tractable. To be more precise, the latter property holds only on time intervals excluding neighbourhoods of the endpoints, so we need to work with $B^{x_i}_{[\sigma_i^{\text{in}}+2\delta,\sigma_i^{\text{out}}-2\delta]}$ instead. To get a uniform lower bound on the intersection probability (see (6.37)), we consider for some

$\overline{\varepsilon} \in (0,\varepsilon)$ in addition the events

$$\widetilde{E}_{R,\varepsilon,T,\overline{\varepsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{ B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{in}}+2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\varepsilon}} \right\}, \quad \text{and} \tag{6.28}$$

$$E_{R,\varepsilon,T,\overline{\varepsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{ B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\varepsilon}} \right\}. \tag{6.29}$$

Again, by monotonicity of $E_{R,\varepsilon,T,\overline{\varepsilon}}$ w.r.t. $\overline{\varepsilon}$, as $\overline{\varepsilon}$ converges to ε, $\mathrm{P}(E_{R,\varepsilon,T,\overline{\varepsilon}})$ converges to $\mathrm{P}(E_{R,\varepsilon,T}) = 1/8$. Hence, we may choose $\overline{\varepsilon}$ such that $\mathrm{P}(E_{R,\varepsilon,T,\overline{\varepsilon}}) \geq 1/16 > 0$. Finally, we call a configuration which lies in $E_{R,\varepsilon,T,\overline{\varepsilon}}$ a configuration of good extended clusters.

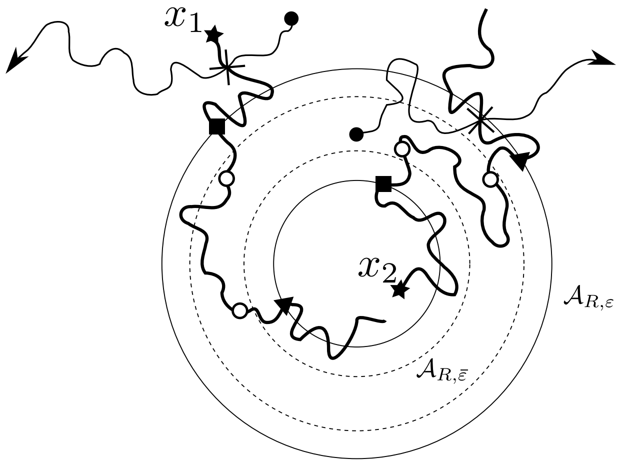

Figure 5: In this picture the points marked with ★ are $x_i,\ i = 1,2$. The symbols ■, ▲ refer to the times σ^{in} and σ^{out}, respectively. The symbol ∘ represents the times $\sigma^{\text{in}}+\delta$ and $\sigma^{\text{out}}-\delta$, respectively. Finally, the symbol × stresses the fact that condition (6.25) is fulfilled.

Additional notation. At this point we would like to introduce some notation in order to avoid repetitions of complicated expressions.
First, let us introduce the events of interest. Let $s > r \geq 0$. For a set $D \subset \mathbb{R}^d$, we denote by

$$\mathcal{S}_{[r,s]}(D) := \{\Pi \in C([0,\infty), \mathbb{R}^d) : \Pi_{[r,s]} \subseteq D\}, \tag{6.30}$$

the set of all continuous paths, which *stay in the set* D during the whole time interval $[r,s]$, and by

$$\mathcal{L}_{r,s}(D) := \{\Pi \in C([0,\infty), \mathbb{R}^d) : \Pi_r, \Pi_s \in D\}, \tag{6.31}$$

the set of all continuous paths , which *belong to the set* D at times r,s.
In the same fashion we also define for $s_1 > r_1 \geq 0$ and $s_2 > r_2 \geq 0$

$$\mathcal{I}_{[s_1,r_1],[s_2,r_2]} := \{\Pi^{(1)}, \Pi^{(2)} \in C([0,\infty), \mathbb{R}^d) : \Pi^{(1)}_{[s_1,r_1]} \bigcap \Pi^{(2)}_{[s_2,r_2]} \neq \emptyset\}, \tag{6.32}$$

the set of all pairs of continuous paths $\Pi^{(1)}$ and $\Pi^{(2)}$ whose traces, when restricted to the time intervals $[r_1,s_1]$ and $[r_2,s_2]$, respectively, have a non-empty intersection.
Secondly, we modify our previous notation a bit: $\mathbb{P}^a_t$ now denotes the law of Brownian

motion starting at a and running from time 0 up to time t. If we consider Brownian bridges instead of Brownian motions we substitute the letter a by $\mathbf{a} = (\underline{a}; \overline{a})$ containing the starting and ending position of the Brownian bridge. In case of considering two independent copies of a Brownian motion (Brownian bridge) we will add a superscript/subscript, ie. $\mathbb{P}^{a_1,a_2}_{t_1,t_2}$ ($\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{t_1,t_2}$). Finally, we will refer to a Brownian bridge as W.

Connecting $\mathcal{C}_1^{\rm ext}$ and $\mathcal{C}_2^{\rm ext}$ inside the annulus. Step 1–Step 5 translates into the following lower bound:

$$\mathrm{P}(N_\infty = 1) \geq \mathrm{P}(\widetilde{E}_{R,\varepsilon,T,\overline{\varepsilon}} \bigcap \{B^{x_1}_{[\sigma_1^{\rm in}+\delta, \sigma_1^{\rm out}-\delta]} \cap B^{x_2}_{[\sigma_2^{\rm in}+\delta, \sigma_2^{\rm out}-\delta]} \neq \emptyset\}), \tag{6.33}$$

which equals

$$\begin{aligned}&\mathrm{P}(E_{R,\varepsilon,T,\overline{\varepsilon}}) \\ &\quad\times \mathrm{P}\Big(\Big\{B^{x_1}_{[\sigma_1^{\rm in}+\delta, \sigma_1^{\rm out}-\delta]} \cap B^{x_2}_{[\sigma_2^{\rm in}+\delta, \sigma_2^{\rm out}-\delta]} \neq \emptyset\Big\} \bigcap_{i=1,2} \Big\{B^{x_i}_{\sigma_i^{\rm in}+2\delta}, B^{x_i}_{\sigma_i^{\rm out}-2\delta} \in \mathcal{A}_{R,\overline{\varepsilon}}\Big\} \mid E_{R,\varepsilon,T,\overline{\varepsilon}}\Big).\end{aligned} \tag{6.34}$$

Observation: For $i \in \{1,2\}$, conditionally on $T_i := \sigma_i^{\rm out} - \sigma_i^{\rm in}$ and the endpoints $(B^{x_i}_{\sigma_i^{\rm in}+\delta}, B^{x_i}_{\sigma_i^{\rm out}-\delta}) = (a_i, b_i)$, $B^{x_i}_{[\sigma_i^{\rm in}+\delta, \sigma_i^{\rm out}-\delta]}$ is a Brownian bridge running from a_i to b_i in a time interval of length $\tau_i := T_i - 2\delta \geq \frac{3T}{4}$, conditioned to stay in $\mathcal{A}_{R,\varepsilon}$ (recall the definitions of $\sigma_i^{\rm in}$ and $\sigma_i^{\rm out}$, $i \in \{1,2\}$).

The observation above yields,

$$\mathrm{P}(N_\infty = 1) \geq \mathrm{P}(E_{R,\varepsilon,T,\overline{\varepsilon}}) \inf_{\substack{\tau_1,\tau_2 \geq 3T/4 \\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\varepsilon}}}} \mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2), \tag{6.35}$$

where

$$\mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2) := \mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta, \tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ , \ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1],[0,\tau_2]}\Big) \tag{6.36}$$

and the superscript i, $i \in \{1,2\}$, on the events in (6.36) refers to the i-th copy of the corresponding processes. Since $\mathrm{P}(E_{R,\varepsilon,T,\overline{\varepsilon}}) > 0$, by Step 1–Step 5, it is enough to prove that

$$\inf_{\substack{\tau_1,\tau_2 \geq 3T/4 \\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\varepsilon}}}} \mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2) > 0. \tag{6.37}$$

Proof of Equation (6.37)**.** We fix $\mathbf{a_1}, \mathbf{a_2} \in \mathcal{A}_{R,\overline{\varepsilon}}$ and $\tau_1, \tau_2 \geq 3T/4$. The right-hand side of (6.36) may be bounded from below by

$$\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta, \tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ , \ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{6.38}$$

which equals, by the Markov property applied at time $\tau_i - \delta$, $i \in \{1,2\}$,

$$\mathbb{E}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\prod_{i=1,2} 1\!\!1\{\mathcal{L}^i_{\delta, \tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ , \ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\} \ . \ 1\!\!1\{\mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\} \Phi_\delta((W^{(i)}_{\tau_i-\delta}; \overline{a}_i))\Big) \tag{6.39}$$

where

$$\Phi_\delta(\mathbf{a}) := \mathbb{P}^{\mathbf{a}}_\delta(\mathcal{S}_{[0,\delta]}(\mathcal{A}_{R,\varepsilon})), \qquad \mathbf{a} = (\underline{a}, \overline{a}) \in (\mathbb{R}^d)^2. \tag{6.40}$$

is the probability that a Brownian bridge, going from $\underline{a}$ to $\overline{a}$ within the time interval $[0,\delta]$, stays in $\mathcal{A}_{R,\varepsilon}$. To bound (6.39) from below we use the following three lemmas, whose proofs are postponed to the appendix.

Lemma 6.4. [Positive probability for a Brownian bridge to stay inside the annulus] *There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}^2_{R,\overline{\varepsilon}}$, $\Phi_\delta(\mathbf{a}) \geq c$.*

Lemma 6.5. [Substitution of the Brownian bridge by a Brownian motion] *Let $\tau > 0$ and $\delta \in (0,\tau)$. There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}^2_{R,\overline{\varepsilon}}$, $\mathbf{a} = (\underline{a}, \overline{a})$,*

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_\tau(W_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\varepsilon}}))}{\mathrm{d}\mathbb{P}^{\underline{a}}_\tau(B_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\varepsilon}}))} \geq c. \tag{6.41}$$

Lemma 6.6. [Two Brownian motions restricted to be inside the annulus do intersect] *Let $\tau_1, \tau_2 > 0$ and $0 < \delta < \frac{\tau_1 \wedge \tau_2}{2}$. There exists $c > 0$ such that for all $a_1, a_2 \in \mathcal{A}_{R,\overline{\varepsilon}}$*

$$\mathbb{P}^{a_1,a_2}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big) \geq c. \tag{6.42}$$

We now explain how to get (6.37) by applying Lemmas 6.4–6.6 to (6.39). Since the $W_{\tau_i-\delta}$, $i \in \{1,2\}$, appearing in (6.39) are in $\mathcal{A}_{R,\overline{\varepsilon}}$, Lemma 6.4 yields that, for some $c > 0$, (6.39) is not smaller than

$$c^2\ .\ \mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big). \tag{6.43}$$

Next, a change of measure argument together with the bound on the Radon-Nikodym derivative as provided in Lemma 6.5 yields, for a possibly different constant $c > 0$, that (6.43) is at least

$$c\ .\ \mathbb{P}^{\underline{a_1},\underline{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{6.44}$$

which is positive by Lemma 6.6. To deduce Equation (6.37) from it, it is enough to note that all the previous estimates were uniform in $\mathbf{a_1}, \mathbf{a_2} \in \mathcal{A}_{R,\overline{\varepsilon}}$. This finally yields the claim.

Remark 6.7. *If $k > 2$, then one follows the same scheme. The notion of good extended clusters is easily generalized and one ends up connecting k excursions in an annulus. Using the same proof as for two excursions, one can connect $B^{x_1}_{[\sigma_1^{\mathrm{in}},\sigma_1^{\mathrm{out}}]}$ to $B^{x_i}_{[\sigma_i^{\mathrm{in}},\sigma_i^{\mathrm{out}}]}$ during the time interval $[\sigma_1^{\mathrm{in}} + (i-1)\delta/k, \sigma_1^{\mathrm{in}} + i\delta/k]$, where $\delta \in (0,T)$, for all $1 < i \leq k$.*

6.3.2 Excluding $N_\infty = \infty$

Let us assume that the number N_∞ of unbounded clusters in $\mathcal{O}_t$ is almost-surely equal to infinity. In the same fashion as in Subsection 6.2.2 we show that this leads to a contradiction. We define the event

$$E_R(0) := \left\{ \begin{matrix} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0,R)^\complement \text{ contains at} \\ \text{least three unbounded clusters and each unbounded cluster which} \\ \text{has a non-empty intersection with } \mathcal{B}_\infty(0,R) \text{ equals } C. \end{matrix} \right\}, \quad (6.45)$$

The fact that there is R large enough such that $E_R(0)$ has positive probability can be seen as follows. First, note that for R large enough, with positive probability the event

$$E_R^1(0) = \bigcup_{k\geq 3} \left\{ \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^\complement \text{ which intersect } \mathcal{B}(0,R) \right\} \quad (6.46)$$

happens. As a consequence, there is $k^* \geq 3$ such that the event inside the union in (6.46) occurs for $k = k^*$ with positive probability. Moreover, we may write

$$\begin{aligned} E_R(0) &= \bigcup_{k\geq 3} \left\{ \begin{matrix} \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^\complement, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{matrix} \right\} \\ &\supseteq \left\{ \begin{matrix} \exists\, k^* \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^\complement, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{matrix} \right\}. \end{aligned} \quad (6.47)$$

Remark 6.7 and the lines preceding (6.47) yield that the last event in (6.47) has positive probability and consequently, so does $E_R(0)$. From now on, the proof works similarly as the proof in Section 6.2.2. Thus, to avoid repetitions we just point out the differences with the proof in Section 6.2.2.

The identification done in **STEP 2.** of Section 6.2.2 has to be changed. For each $z \in \mathbb{Z}^d$, we replace the Poisson point inside $\mathcal{B}_\infty(2Rz, R)$ that was used to connect the "external" clusters by what we call an *intersection point*, which is just an arbitrarily chosen point $\tilde{z} \in \mathcal{B}_\infty(2Rz, R)$ contained in all the clusters. Finally, at the moment of applying Lemma 6.3, we consider

$$C_z^i = \left\{ x \in \{\mathcal{E} \cap \mathcal{B}_\infty(0, LR)\} \cup \{\text{intersection points}\} \colon\ x \stackrel{\Lambda_z^c}{\longleftrightarrow} \mathbf{B}_z^i \right\} \qquad i = 1, \dots, n_z,$$

and

$$S = \mathcal{B}_\infty(0, LR) \cap (\mathcal{E} \cup \{\text{intersection points}\}).$$

This choice generates a minor difference at the moment of getting the contradiction in (6.17). Indeed, we have that

$$\mathrm{E}(|S|) \geq K((L - M - 2)^d \delta/2 + 2) \quad (6.48)$$

but, taking into account the intersection points we have that,

$$\mathrm{E}(|S|) \leq \mathrm{E}(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|) + \mathrm{E}(|\mathcal{R}|) \leq \lambda(2LR)^d + (L - M + 2)^d. \quad (6.49)$$

In the last inequality we used that

$$|\mathcal{R}| \leq |\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, R) \subseteq \mathcal{B}_\infty(0, LR)\}| \leq (L - M + 2)^d. \tag{6.50}$$

Thus, combining (6.48) and (6.49) yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \leq \lambda(2LR)^d + (L - M + 2)^d, \tag{6.51}$$

from which we obtain the desired contradiction in the same way as in the case $d \geq 4$.

A Proof of Lemma 2.3

The proof consists of two steps. In the first step a coarse-graining procedure is introduced, which reduces the problem of showing subcriticality of a continuous percolation model to showing subcriticality of an infinite range site percolation model on $\mathbb{Z}^d$. This coarse-graining was essentially already introduced in [MR96, Lemma 3.3], where ϱ was supposed to have a compact support. To overcome the additional difficulties arising from the long range dependencies in the coarse-grained model we use a renormalization scheme, which is very similar to the one in Sznitman [Szn10, Theorem 3.5].

STEP 1. Coarse-graining.
We fix $N \in \mathbb{N}$. For $n \in \mathbb{N}$, a sequence of vertices $z_0, z_1, \ldots, z_{n-1}$ in $\mathbb{Z}^d$ is called a $*$-path, when $\|z_i - z_{i-1}\|_\infty = 1$ for all $i \in \{1, 2, \ldots, n-1\}$. Furthermore, a site $z = (z(j), 1 \leq j \leq d) \in \mathbb{Z}^d$ is called open when there is an occupied cluster Λ of Σ such that

$$(i)\ \Lambda \cap \prod_{j=1}^d [z(j)N, (z(j)+1)N) \neq \emptyset \text{ and } (ii)\ \Lambda \cap \left(\prod_{j=1}^d [(z(j)-1)N, (z(j)+2)N) \right)^{\mathsf{C}} \neq \emptyset. \tag{A.1}$$

Otherwise z is called closed. It was shown in [MR96, Lemma 3.3] that to obtain Lemma 2.3 it suffices to show that

$$\mathrm{P}_{\lambda,\varrho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) = 0. \tag{A.2}$$

To prove (A.2) we introduce a renormalization scheme.

STEP 2. Renormalization.
• **New notation and a first bound.** We start by introducing a fair amount of new notation. We fix integers $R > 1$ and $L_0 > 1$, both to be determined and we introduce an increasing sequence of scales via

$$\forall n \in \mathbb{N}_0, \quad L_{n+1} = R^{n+1} L_n. \tag{A.3}$$

Moreover, for $i \in \mathbb{Z}^d$, we introduce a sequence of increasing boxes via

$$\begin{aligned} C_n(i) &= \prod_{j=1}^d [i(j)L_n, (i(j)+1)L_n) \cap \mathbb{Z}^d \quad \text{and} \\ \widetilde{C}_n(i) &= \prod_{j=1}^d [(i(j)-1)L_n, (i(j)+2)L_n) \cap \mathbb{Z}^d. \end{aligned} \tag{A.4}$$

We further abbreviate $C_n = C_n(0)$ and $\widetilde{C}_n = \widetilde{C}_n(0)$. Thus, $\widetilde{C}_n(i)$ is the union of boxes $C_n(j)$ such that $\|j-i\|_\infty \leq 1$. Moreover, for $n \in \mathbb{N}$, we introduce the events

$$A_n(i) = \Big\{\text{There is a } *\text{-path of open sites from } C_n(i) \text{ to } \partial_{\text{int}}\widetilde{C}_n(i).\Big\}, \tag{A.5}$$

and we write A_n instead of $A_n(0)$. Here, $\partial_{\text{int}}B$ refers to the inner boundary of a set $B \subseteq \mathbb{Z}^d$ with respect to the $\|\cdot\|_\infty$-norm. The idea of the renormalization scheme is to bound the probability of A_{n+1} in terms of the probability of the intersection of events $A_n(i)$ and $A_n(k)$, where $i \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$ are far apart. By our assumption on the radius distribution ϱ, the events $A_n(i)$ and $A_n(k)$ can then be treated as being basically independent. This will result in a recursion inequality, which relates the events A_n, $n \in \mathbb{N}$, at different scales to each other. For that, we fix $n \in \mathbb{N}$ and let

$$\begin{aligned} \mathcal{H}_1 &= \Big\{i \in \mathbb{Z}^d \,:\, C_n(i) \subseteq C_{n+1}, C_n(i) \cap \partial_{\text{int}} C_{n+1} \neq \emptyset\Big\} \quad \text{and} \\ \mathcal{H}_2 &= \Big\{k \in \mathbb{Z}^d \,:\, C_n(k) \cap \Big\{z \in \mathbb{Z}^d \,:\, \text{dist}(z, C_{n+1}) = \frac{L_{n+1}}{2}\Big\} \neq \emptyset\Big\}. \end{aligned} \tag{A.6}$$

Here, $\text{dist}(z, C_{n+1})$ denotes the distance of z from the set C_{n+1} with respect to the supremum norm. Note that here and in the rest of the proof, for notational convenience, we pretend that expressions like $L_{n+1}/2$ are integers. Observe that if A_{n+1} occurs, then there are $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$ such that both $A_n(i)$ and $A_n(k)$ occur. Hence,

$$\begin{aligned} \mathrm{P}_{\lambda,\varrho}(A_{n+1}) &\leq \sum_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathrm{P}_{\lambda,\varrho}(A_n(i) \cap A_n(k)) \\ &\leq c_1 R^{2(d-1)(n+1)} \sup_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathrm{P}_{\lambda,\varrho}(A_n(i) \cap A_n(k)), \end{aligned} \tag{A.7}$$

where $c_1 = c_1(d) > 0$ is a constant which only depends on the dimension.

•Partition of $A_n(i) \cap A_n(k)$. We fix $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$. Let $z \in \widetilde{C}_n(i)$ and note that to decide if z is open, it suffices to look at the trace of the Boolean percolation model on

$$\prod_{j=1}^{d} [(z(j)-1)N, (z(j)+2)N). \tag{A.8}$$

In a similar fashion one sees that the area which determines if $A_n(i)$ occurs is given by

$$\begin{aligned} &\prod_{j=1}^{d} [((i(j)-1)L_n - 1)N, ((i(j)+2)L_n + 2)N] \\ &\subseteq \prod_{j=1}^{d} [(i(j)-2)L_n N, (i(j)+3)L_n N) \stackrel{\text{def}}{=} \text{DET}(\widetilde{C}_n(i)) \end{aligned} \tag{A.9}$$

and likewise for $A_n(k)$ with i replaced by k. Here, we used that by our choice of R and L_0 the relation $L_n \geq 2$ holds for all $n \in \mathbb{N}$. We introduce

$$\mathcal{D}(x, r(x)) := \{\mathcal{B}(x, r(x)) \cap \text{DET}(\widetilde{C}_n(i)) \neq \emptyset,\, \mathcal{B}(x, r(x)) \cap \text{DET}(\widetilde{C}_n(k)) \neq \emptyset\} \tag{A.10}$$

and

$$B_n(i,k) := \bigcup_{x\in\mathcal{E}} \mathcal{D}(x, r(x)) \tag{A.11}$$

so that,

$$\begin{aligned} \mathrm{P}_{\lambda,\varrho}(A_n(i)\cap A_n(k)) = \mathrm{P}_{\lambda,\varrho}(A_n(i)\cap A_n(k) | B_n(i,k)^{\complement}) \times \mathrm{P}_{\lambda,\varrho}(B_n(i,k)^{\complement}) \\ + \mathrm{P}_{\lambda,\varrho}(A_n(i)\cap A_n(k) | B_n(i,k)) \times \mathrm{P}_{\lambda,\varrho}(B_n(i,k)). \end{aligned} \tag{A.12}$$

•Analysis of the first term on the right hand side of (A.12). We claim that under $\mathrm{P}_{\lambda,\varrho}(\cdot|B_n(i,k)^{\complement})$ the events $A_n(i)$ and $A_n(k)$ are independent. To see that, note that the Poisson point process χ on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $\nu = (\lambda \times \mathrm{Leb}_d) \otimes \varrho$ (see Section 2.1) is a Poisson point process under $\mathrm{P}_{\lambda,\varrho}(\cdot|B_n(i,k)^{\complement})$, with intensity measure

$$\mathbb{1}\{\text{there is no } (x, r(x)) \in \chi \text{ such that } \mathcal{D}(x, r(x)) \text{ occurs}\} \times \nu. \tag{A.13}$$

However, on $B_n(i,k)^{\complement}$, the events $A_n(i)$ and $A_n(k)$ depend on disjoint subsets of $\mathbb{R}^d \times [0,\infty)$. Consequently, they are independent under $\mathrm{P}_{\lambda,\varrho}(\cdot|B_n(i,k)^{\complement})$. Hence,

$$\begin{aligned} &\mathrm{P}_{\lambda,\varrho}(A_n(i)\cap A_n(k)|B_n(i,k)^{\complement}) \times \mathrm{P}_{\lambda,\varrho}(B_n(i,k)^{\complement}) \\ &\quad = \mathrm{P}_{\lambda,\varrho}(A_n(i)|B_n(i,k)^{\complement})\mathrm{P}_{\lambda,\varrho}(A_n(k)|B_n(i,k)^{\complement}) \times \mathrm{P}_{\lambda,\varrho}(B_n(i,k)^{\complement}) \\ &\quad \le \mathrm{P}_{\lambda,\varrho}(A_n)^2 \times \mathrm{P}_{\lambda,\varrho}(B_n(i,k)^{\complement})^{-1}. \end{aligned} \tag{A.14}$$

For the last inequality in (A.14) we also used the fact that $\mathrm{P}_{\lambda,\varrho}(A_n(i))$ does not depend on $i \in \mathbb{Z}^d$.

•Analysis of the second term on the right hand side of (A.12). To bound the second term on the right hand side of (A.12) it will be enough to bound $\mathrm{P}_{\lambda,\varrho}(B_n(i,k))$, since the other term may be bounded by one. Note that

$$\mathrm{P}_{\lambda,\varrho}(B_n(i,k)) \le \sum_{\ell\in 3\mathbb{Z}^d} \mathrm{P}_{\lambda,\varrho}\begin{pmatrix} \exists x \in \mathcal{E}\cap \widetilde{C}_{n+1}(\ell)N: & \mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(i)) \neq \emptyset \\ \text{and} & \mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(k)) \neq \emptyset \end{pmatrix}. \tag{A.15}$$

Here, the set $\widetilde{C}_{n+1}(\ell)N$ is the set $\{x \in \mathbb{R}^d : x = zN, z \in \widetilde{C}_{n+1}(\ell)\}$. To warm up, we first treat the term $\ell = 0$ in the sum (A.15). Note that,

$$\mathrm{dist}(\mathrm{DET}(\widetilde{C}_n(i)), \mathrm{DET}(\widetilde{C}_n(k)) \ge \Big(\frac{L_{n+1}}{2} - 8L_n\Big)N \ge \frac{L_{n+1}}{3}N, \tag{A.16}$$

where the last inequality holds for all $n \in \mathbb{N}$, provided R and L_0 are chosen accordingly. Thus, if there is a Poisson point whose corresponding ball intersects $\mathrm{DET}(\widetilde{C}_n(i))$ and $\mathrm{DET}(\widetilde{C}_n(k))$, then its radius is at least $L_{n+1}N/6$. This yields

$$\begin{aligned} &\mathrm{P}_{\lambda,\varrho}\begin{pmatrix} \exists x \in \mathcal{E}\cap \widetilde{C}_{n+1}N: & \mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(i)) \neq \emptyset \\ \text{and} & \mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(k)) \neq \emptyset \end{pmatrix} \\ &\quad \le \mathrm{P}_{\lambda,\varrho}\Big(\exists x \in \mathcal{E}\cap \widetilde{C}_{n+1}N : r(x) \ge L_{n+1}N/6\Big). \end{aligned} \tag{A.17}$$

We may rewrite (A.17) as

$$
\begin{aligned}
&1-\sum_{m=0}^{\infty} \mathrm{P}_{\lambda,\varrho}\Big(\forall x\in\mathcal{E}\cap\widetilde{C}_{n+1}, r(x)<L_{n+1}N/6\Big||\mathcal{E}\cap\widetilde{C}_{n+1}N|=m\Big)\times \mathrm{P}_{\lambda,\varrho}(|\mathcal{E}\cap\widetilde{C}_{n+1}N|=m)\\
&=1-\sum_{m=0}^{\infty}[1-\varrho([L_{n+1}N/6,\infty))]^m\times\frac{(\lambda\mathrm{Leb}_d(\widetilde{C}_{n+1}N))^m}{m!}\times e^{-\lambda\mathrm{Leb}_d(\widetilde{C}_{n+1}N)}\\
&=1-\exp\big\{-\lambda\mathrm{Leb}_d(\widetilde{C}_{n+1}N)\varrho([L_{n+1}N/6,\infty))\big\},
\end{aligned}
\tag{A.18}
$$

which is at most $\lambda\mathrm{Leb}_d(\widetilde{C}_{n+1}N)\varrho([L_{n+1}N/6,\infty))$. By our assumption on the radius distribution, for R and L_0 large enough, there is a constant $c_2=c_2(\varrho)>0$ such that the last term may be bounded by $\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}$. Similar arguments show that the left hand side of (A.15) is at most

$$
\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}+\sum_{m=1}^{\infty}\sum_{\substack{\ell\in3\mathbb{Z}^d\\ \|\ell\|_\infty=m}}\lambda(3L_{n+1}N)^d\times e^{-c_2(3(m-1)+1/2)L_{n+1}N}. \tag{A.19}
$$

This may be bounded by

$$
c_3\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}, \tag{A.20}
$$

for some constant $c_3>0$ which is independent of R, L_0 and N. Hence, we have bounded the second term on the right hand side of (A.12). In particular, we may deduce that for all $n\in\mathbb{N}$, again for a suitable choice of R and L_0, $\mathrm{P}_{\lambda,\varrho}(B_n(i,k)^\complement)\geq 1/2$.

•**Analysis of the recursion scheme.** Equation (A.7) in combination with (A.12) and the arguments following it show that

$$
\mathrm{P}_{\lambda,\varrho}(A_{n+1})\leq 2c_1R^{2(d-1)(n+1)}\Big(\mathrm{P}_{\lambda,\varrho}(A_n)^2+c_3\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}\Big). \tag{A.21}
$$

To deduce the desired result, we first show with the help of (A.21) that $\mathrm{P}_{\lambda,\varrho}(A_n)$ being small implies that $\mathrm{P}_{\lambda,\varrho}(A_{n+1})$ is small as well. As a final step it then remains to show that $\mathrm{P}_{\lambda,\varrho}(A_0)$ is already small. We now make this idea more precise. We put

$$
\forall n\in\mathbb{N},\quad a_n=2c_1R^{2(d-1)n}\mathrm{P}_{\lambda,\varrho}(A_n). \tag{A.22}
$$

Claim A.1. *For R large enough, for all $n\in\mathbb{N}$ and for all $L_0\geq 2R^{4(d-1)+1}$, the inequality $a_n\leq L_n^{-1}$ implies that $a_{n+1}\leq L_{n+1}^{-1}$.*

Proof. To prove the claim, let $n\in\mathbb{N}$ and assume that $a_n\leq L_n^{-1}$. Then,

$$
\begin{aligned}
a_{n+1}&=2c_1R^{2(d-1)(n+1)}\mathrm{P}_{\lambda,\varrho}(A_{n+1})\\
&\leq 4c_1^2R^{4(d-1)(n+1)}\Big[\mathrm{P}_{\lambda,\varrho}(A_n)^2+c_3\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}\Big]\\
&=a_n^2R^{4(d-1)}+4c_1^2c_3R^{4(d-1)(n+1)}\lambda(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}.
\end{aligned}
\tag{A.23}
$$

Thus, it is enough to show that

$$
a_n^2R^{4(d-1)}\leq(2L_{n+1})^{-1}\quad\text{and}\quad 4c_1^2c_3R^{4(d-1)(n+1)}(3L_{n+1}N)^d e^{-c_2L_{n+1}N/6}\leq(2L_{n+1})^{-1}. \tag{A.24}
$$

For that, note that by our assumption on a_n

$$a_n^2 R^{4(d-1)} 2L_{n+1} \le 2L_n^{-2} R^{4(d-1)} L_{n+1} = 2R^{4(d-1)} \frac{R^{n+1}}{R^n L_{n-1}} \le 2R^{4(d-1)+1} L_0^{-1}. \tag{A.25}$$

Thus, choosing $L_0 \ge 2R^{4(d-1)+1}$ yields the first desired inequality. The second term on the right hand side of (A.23) may be bounded using similar considerations. This yields Claim A.1. □

Hence, to use the claim, we need that $\mathrm{P}_{\lambda,\varrho}(A_0) \le L_0^{-1}$. For that observe that

$$\begin{aligned} \mathrm{P}_{\lambda,\varrho}(A_0) &= \mathrm{P}_{\lambda,\varrho}\Big(\text{There is a } *\text{-path of open sites from } [0, L_0)^d \text{ to } \partial_{\text{int}}[-L_0, 2L_0)^d.\Big) \\ &\le \mathrm{P}_{\lambda,\varrho}\Big(\text{There is } z \in \partial_{\text{int}}[-L_0, 2L_0)^d, \text{ which is open.}\Big) \\ &\le c_4 L_0^{d-1} \mathrm{P}_{\lambda,\varrho}(0 \text{ is open}), \end{aligned} \tag{A.26}$$

where $c_4 = c_4(d) > 0$ does only depend on the dimension. Equation (3.64) of [MR96] shows that

$$\mathrm{P}_{\lambda,\varrho}(0 \text{ is open}) \le 2d\mathrm{P}_{\lambda,\varrho}(\mathrm{CROSS}(N, 3N, \dots, 3N)). \tag{A.27}$$

Therefore, if the right hand side of (A.27) is smaller than $(4dc_1c_4L_0^d)^{-1}$, we get from (A.26) that $\mathrm{P}_{\lambda,\varrho}(A_0) \le (2c_1L_0)^{-1}$, which is the same as saying that $a_0 \le L_0^{-1}$. This, in combination with Claim A.1 and the observation that an infinite $*$-path of open sites containing zero implies the events A_n for all $n \in \mathbb{N}$, finally yields

$$\mathrm{P}_{\lambda,\varrho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) \le \lim_{n\to\infty} \mathrm{P}_{\lambda,\varrho}(A_n) = 0. \tag{A.28}$$

Consequently, we have shown that Lemma 2.3 is satisfied for $\varepsilon \le (4dc_1c_4L_0^{d+1})^{-1}$.

B Proofs of Lemmas 6.4–6.6

B.1 Proof of Lemma 6.4

Proof. Let $\mathbf{a} \in \overline{\mathcal{A}}^2_{R,\bar{\varepsilon}}$. First, note that

$$\Phi_\delta(\mathbf{a}) > 0. \tag{B.1}$$

Indeed, since for all $\bar{\delta} < \delta$ the path of a Brownian motion $B_{[0,\bar{\delta}]}$ starting in $\underline{a}$ is absolutely continuous with respect to that of the Brownian bridge $W_{[0,\bar{\delta}]}$,

$$\mathbb{P}^{\mathbf{a}}_\delta(W_{[0,\delta/2]} \subseteq \mathcal{A}_{R,\varepsilon},\, W_{\delta/2} \in \mathcal{B}(\overline{a}, \varepsilon')) > 0, \tag{B.2}$$

where $\varepsilon' > 0$ is chosen so small that $\mathcal{B}(\overline{a}, \varepsilon') \subseteq \mathcal{A}_{R,\varepsilon}$. From the representation

$$\forall s \in [0, \delta], \quad W_s = B_s - \frac{s}{\delta}(B_\delta - \overline{a}) \tag{B.3}$$

and the fact that a Brownian motion stays with a positive probability in an arbitrary small ball around its starting point within finite time intervals, we have the following:

$$\forall a' \in \mathcal{B}(\overline{a}, \varepsilon'), \quad \mathbb{P}^{\mathbf{a}}_\delta(W_{[\delta/2,\delta]} \subseteq \mathcal{A}_{R,\varepsilon} \mid W_{\delta/2} = a') > 0. \tag{B.4}$$

Equation (B.1) then follows from (B.2), the Markov property applied at time $\delta/2$ and (B.4). Second, the representation in (B.3) shows that the map

$$\mathbf{a} \mapsto \mathbb{P}_\delta^{\mathbf{a}}\Big(W_{[0,\delta]} \in \cdot\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\varepsilon}}, \tag{B.5}$$

is weakly continuous. Moreover, the probability for a Brownian bridge to hit the boundary of $\mathcal{A}_{R,\varepsilon}$ but to stay inside $\mathcal{A}_{R,\varepsilon}$ is zero. Thus, an application of the Portemanteau Theorem yields that the function

$$\mathbf{a} \mapsto \mathbb{P}_\delta^{\mathbf{a}}\Big(W_{[0,\delta]} \in \mathcal{A}_{R,\varepsilon}\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\varepsilon}}, \tag{B.6}$$

is continuous. This fact together with (B.1) is enough to conclude the claim. □

B.2 Proof of Lemma 6.5

Proof. First, for $\mathbf{a} = (\underline{a}, \overline{a}) \in \mathcal{A}_{R,\overline{\varepsilon}}$ we have that (see Exercise 1.5 in [MP10])

$$\frac{\mathrm{d}\mathbb{P}_\tau^{\mathbf{a}}(W_{[0,\tau-\delta]} \in \cdot\,)}{\mathrm{d}\mathbb{P}_\tau^{\underline{a}}(B_{[0,\tau-\delta]} \in \cdot\,)} = \frac{p(\delta, W_{\tau-\delta}, \overline{a})}{p(\tau, \underline{a}, \overline{a})}, \tag{B.7}$$

where

$$p(s,x,y) := \frac{1}{(2\pi s)^{d/2}} \exp\left(-\frac{\|x-y\|^2}{2s}\right). \tag{B.8}$$

Moreover, there exist constants c_1 and c_2 such that

$$0 < c_1 \le \inf_{\substack{\delta \le s \le \tau \\ x,y \in \mathcal{A}_{R,\overline{\varepsilon}}}} p(s,x,y) \le \sup_{\substack{\delta \le s \le \tau \\ x,y \in \mathcal{A}_{R,\overline{\varepsilon}}}} p(s,x,y) \le c_2 < \infty. \tag{B.9}$$

Therefore,

$$\frac{\mathrm{d}\mathbb{P}_\tau^{\mathbf{a}}(W_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\varepsilon}}))}{\mathrm{d}\mathbb{P}_\tau^{\underline{a}}(B_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\varepsilon}}))} \ge \left(\frac{c_1}{c_2}\right) > 0. \tag{B.10}$$

□

B.3 Proof of Lemma 6.6

Proof. To achieve the intersection event, we use the following strategy:

- before time δ, both paths enter a ball inside $\mathcal{A}_{R,\overline{\varepsilon}}$, and from this moment, stay in a slightly bigger ball;
- the two paths intersect each other between time δ and $\tau_1 \wedge \tau_2 - \delta$, while staying in a larger ball contained in $\mathcal{A}_{R,\overline{\varepsilon}}$.

More precisely, let us choose arbitrarily $d \in \mathcal{A}_{R,\overline{\varepsilon}}$. Let $\varepsilon_4 > \varepsilon_3 > \varepsilon_2 > \varepsilon_1 > 0$ to be determined later. For the moment we only assume that $\mathcal{B}(d,\varepsilon_4) \subset \mathcal{A}_{R,\overline{\varepsilon}}$. For $i \in \{1,2\}$, let us define

$$\sigma_1^{(i)} = \inf\{s \ge 0 : B_s^{a_i} \in \overline{\mathcal{B}}(d,\varepsilon_1)\} \tag{B.11}$$

$$\sigma_2^{(i)} = \inf\{s \ge \sigma_1^{(i)} : B_s^{a_i} \notin \mathcal{B}(d,\varepsilon_2)\}. \tag{B.12}$$

First note that with $\hat\tau := \tau_1 \wedge \tau_2$ and $\check\tau := \tau_1 \vee \tau_2$

$$\Big\{ \bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar\varepsilon}) \ , \ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]} \Big\} \supseteq \tag{B.13}$$

$$\left\{ \begin{array}{c} \sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta, \ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta, \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\varepsilon}), \\ \bigcap_{i=1,2} \Big\{ \mathcal{S}^i_{[\delta,\hat\tau-\delta]}(\mathcal{B}(d,\varepsilon_3)), \mathcal{S}^i_{[\hat\tau-\delta,\check\tau-\delta]}(\mathcal{B}(d,\varepsilon_4)) \Big\}, \ \mathcal{I}_{[\delta,\hat\tau-\delta],[\delta,\hat\tau-\delta]} \end{array} \right\}.$$

An application of the Markov property at time δ shows that it is enough to establish that

$$\inf_{a_1,a_2\in\overline{\mathcal{A}}_{R,\overline\varepsilon}} \mathbb{P}^{a_1,a_2}_{\delta,\delta}\Big(\sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta, \ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta, \ \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\varepsilon})\Big) > 0, \tag{B.14}$$

and

$$\inf_{x,y\in\mathcal{B}(d,\varepsilon_2)} \mathbb{P}^{x,y}_{\check\tau-2\delta,\check\tau-2\delta}\Big(\mathcal{I}_{[0,\hat\tau-2\delta],[0,\hat\tau-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat\tau-2\delta]}(\mathcal{B}(d,\varepsilon_3)), \mathcal{S}^i_{[\hat\tau-2\delta,\check\tau-2\delta]}(\mathcal{B}(d,\varepsilon_4))\Big) > 0. \tag{B.15}$$

Let us first prove (B.14). The probability in the infimum is clearly positive for all a_1, a_2 in the compact set $\overline{\mathcal{A}}_{R,\overline\varepsilon}$. Furthermore, one can use the same arguments as in the proof of Lemma 6.4 to show that it is continuous in (a_1, a_2) on $\overline{\mathcal{A}}_{R,\overline\varepsilon} \times \overline{\mathcal{A}}_{R,\overline\varepsilon}$, hence the infimum is also positive.

Now we proceed to prove (B.15). Again, an application of the Markov property at time $\hat\tau - 2\delta$ shows that it is enough to prove that

$$\inf_{x,y\in\mathcal{B}(d,\varepsilon_2)} \mathbb{P}^{x,y}_{\hat\tau-2\delta,\hat\tau-2\delta}\Big(\mathcal{I}_{[0,\hat\tau-2\delta],[0,\hat\tau-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat\tau-2\delta]}(\mathcal{B}(d,\varepsilon_3))\Big) > 0, \tag{B.16}$$

and

$$\inf_{x,y\in\mathcal{B}(d,\varepsilon_3)} \mathbb{P}^{x,y}_{\check\tau-\hat\tau,\check\tau-\hat\tau}\Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\check\tau-\hat\tau]}(\mathcal{B}(d,\varepsilon_4))\Big) > 0. \tag{B.17}$$

Now we focus on (B.16). For all $\tau_0 > 0$ and $R_0 > 1$, let us consider

$$\begin{aligned} \varrho(\tau_0, R_0) &:= \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\tau_0]}(\mathcal{B}(0,R_0))\Big) \\ &\geq \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]}\Big) - 2 \sup_{x\in\mathcal{B}(0,1)} \mathbb{P}^x_{\tau_0}\Big(\sup_{s\in[0,\tau_0]} \|B_s\| > R_0\Big). \end{aligned} \tag{B.18}$$

By using the monotonicity argument in Lemma 5.2 and Theorem 9.1 in [MP10], the last infimum can be made arbitrarily close to 1 by choosing τ_0 large enough, whereas standard estimates yield that the supremum goes to 0 as R_0 goes to infinity. Therefore, there is a choice of τ_0 and R_0 leading to $\varrho(\tau_0, R_0) > 0$. By the scale invariance of Brownian motion,

$$\forall u > 0, \quad \inf_{x,y\in\mathcal{B}(0,u)} \mathbb{P}^{x,y}_{u^2\tau_0,u^2\tau_0}\Big(\mathcal{I}_{[0,u^2\tau_0],[0,u^2\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,u^2\tau_0]}(\mathcal{B}(0,uR_0))\Big) = \varrho(\tau_0, R_0) > 0. \tag{B.19}$$

We may now choose $u_0 > 0$ such that

$$\tau' := u_0^2 \tau_0 < \hat{\tau} - 2\delta, \quad 2u_0 R_0 < \mathrm{dist}(d, \mathcal{A}_{R,\varepsilon}), \tag{B.20}$$

and we set

$$\varepsilon_2 := u_0, \quad \varepsilon_3 := 2u_0 R_0. \tag{B.21}$$

Note that we may choose R_0 such that $\varepsilon_3/2 > \varepsilon_2$. Hence, an application of the Markov property at time τ' to the left hand side of (B.16) yields,

$$\text{l.h.s. of (B.16)} \geq \varrho(\tau_0, R_0) \inf_{x,y \in \mathcal{B}(0,\varepsilon_3/2)} \mathbb{P}^{x,y}_{\hat{\tau}-2\delta-\tau', \hat{\tau}-2\delta-\tau'} \Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat{\tau}-2\delta-\tau']}(\mathcal{B}(0,\varepsilon_3)) \Big) > 0. \tag{B.22}$$

The positivity of the second factor of (B.22) and of (B.17) may be shown by using similar arguments as in the proof of Lemma 6.4. This finally yields the claim. □

References

[BK89] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[GKN92] A. Gandolfi, M. S. Keane, and C. M. Newman. Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses. *Probab. Theory Related Fields*, 92(4):511–527, 1992.

[G61] E. N. Gilbert. Random Plane Networks. *Journal of the Society for Industrial and Applied Mathematics,* 9(4): 533–543, 1961.

[Gou08] J.-B. Gouéré. Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab.*, 36(4):1209–1220, 2008.

[Gri00] G. R. Grimmett. Percolation. In *Development of mathematics 1950–2000*, pages 547–575. Birkhäuser, Basel, 2000.

[KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[LSS97] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1):71–95, 1997.

[MMS88] M. V. Men′shikov, S. A. Molchanov, and A. F. Sidorenko. Percolation theory and some applications. In *Probability theory. Mathematical statistics. Theoretical cybernetics, Vol. 24 (Russian)*, Itogi Nauki i Tekhniki, pages 53–110, i. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1986. Translated in J. Soviet Math. **42** (1988), no. 4, 1766–1810.

[MP10] P. Mörters and Y. Peres. *Brownian motion.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[MR94] R. Meester and R. Roy. Uniqueness of unbounded occupied and vacant components in Boolean models. *Ann. Appl. Probab.*, 4(3):933–951, 1994.

[MR96] R. Meester and R. Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[MRS94] R. Meester, R. Roy, and A. Sarkar. Nonuniversality and continuity of the critical covered volume fraction in continuum percolation. *J. Statist. Phys.*, 75(1-2):123–134, 1994.

[Pen95] M. D. Penrose. Continuity of critical density in a boolean model. unpublished notes, 1995.

[Szn10] A.-S. Sznitman. Vacant set of random interlacements and percolation. *Ann. of Math. (2)*, 171(3):2039–2087, 2010.