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# Upper $k$ -tuple domination in graphs

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For a positive integer  $k$ , a  $k$ -tuple dominating set of a graph  $G$  is a subset  $S$  of  $V(G)$  such that  $|N[v] \cap S| \geq k$  for every vertex  $v$ , where  $N[v] = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$ . The upper  $k$ -tuple domination number of  $G$ , denoted by  $\Gamma_{\times k}(G)$ , is the maximum cardinality of a minimal  $k$ -tuple dominating set of  $G$ . In this paper we present an upper bound on  $\Gamma_{\times k}(G)$  for  $r$ -regular graphs  $G$  with  $r \geq k$ , and characterize extremal graphs achieving the upper bound. We also establish an upper bound on  $\Gamma_{\times 2}(G)$  for claw-free  $r$ -regular graphs. For the algorithmic aspect, we show that the upper  $k$ -tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.

**Keywords:** Upper  $k$ -tuple domination,  $r$ -regular graph, bipartite graph, split graph, chordal graph, NP-completeness.

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## 1 Introduction

All graphs considered in this paper are finite, simple and undirected. In a graph  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ , the *open neighborhood* of a vertex  $v$  is  $N(v) = \{u \in V(G) : uv \in E(G)\}$  and the *closed neighborhood* is  $N[v] = \{v\} \cup N(v)$ . The *degree* of  $v$ , denoted by  $d(v)$ , is the cardinality of  $N(v)$ . Denote by  $\delta(G)$  the minimum degree of a vertex in  $G$ . A graph is  $r$ -regular if  $d(v) = r$  for all  $v \in V$ . A *stable set* (respectively, *clique*) of  $G$  is a subset  $S$  of  $V(G)$  in which every two vertices are not adjacent

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(respectively, are adjacent). For two disjoint subsets  $A$  and  $B$  of  $V(G)$ , let  $e[A, B]$  denote the number of edges between  $A$  and  $B$ .

For  $S \subseteq V(G)$ , the *subgraph induced* by  $S$  is the graph  $G[S]$  with vertex set  $S$  and edge set  $\{uv \in E(G) : u, v \in S\}$ . A *bipartite* graph is a graph whose vertex set can be partitioned into two sets such that every two distinct vertices may be adjacent only if they are in different sets. A *split* graph is a graph whose vertex set can be partitioned into a stable set and a clique. A *chord* of a cycle is an edge joining two vertices on the cycle that are not adjacent on the cycle. A *chordal graph* is a graph in which every cycle of length at least four has a chord. Split graphs are chordal. A graph  $G$  is called *claw-free* if it does not contain the bipartite complete graph  $K_{1,3}$  as an induced subgraph.

For positive integer  $k$ , a  *$k$ -tuple dominating set* of  $G$  is a subset  $S$  of  $V(G)$  such that  $|N[v] \cap S| \geq k$  for all  $v \in V(G)$ . For a  $k$ -tuple dominating set  $S$ , any vertex in  $N[v] \cap S$  is said to *dominate*  $v$ . Notice that a graph has a  $k$ -tuple dominating set if and only if  $\delta(G) \geq k - 1$ . The  *$k$ -tuple domination number*  $\gamma_{\times k}(G)$  of  $G$  is the minimum cardinality of a  $k$ -tuple dominating set of  $G$ , while the *upper  $k$ -tuple domination number*  $\Gamma_{\times k}(G)$  is the maximum cardinality of a minimal  $k$ -tuple dominating set. A  $\Gamma_{\times k}(G)$ -*set* of  $G$  is a minimal  $k$ -tuple dominating set of  $G$  of cardinality  $\Gamma_{\times k}(G)$ . An application of  $k$ -tuple domination for fault tolerance networks is presented in [9, 12]. For more results on  $k$ -tuple domination, we refer to [1, 2, 3, 4, 5, 13, 14, 15, 16, 17, 18, 19].

In this paper we first give an upper bound on  $\Gamma_{\times k}$  for  $r$ -regular graphs, and characterize the extremal graphs achieving the upper bound. We also establish a sharp upper bound on  $\Gamma_{\times 2}(G)$  for claw-free  $r$ -regular graphs. Finally, we show that the upper  $k$ -tuple domination problem is NP-complete for bipartite graphs and chordal graphs.

## 2 Upper $k$ -tuple domination for $r$ -regular graphs

This section establishes a sharp upper bound for upper  $k$ -tuple domination on  $r$ -regular graphs.

First, a  $k$ -tuple dominating set  $S$  is minimal if and only if every vertex in  $S$  is not *avoidable*, that is, it has a closed neighbor that is dominated by exactly  $k$  vertices in  $S$ . Hence, we have the following property.

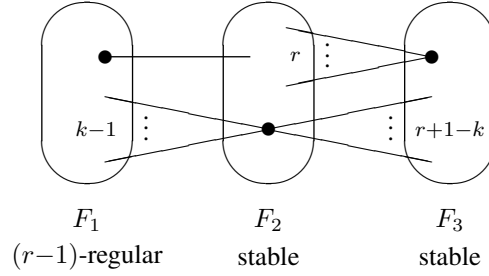
**Lemma 1** *In a graph  $G$  with  $\delta(G) \geq k - 1$ , a  $k$ -tuple dominating set  $S$  is minimal if and only if each vertex in  $S$  has some closed neighbor  $u$  with  $|N[u] \cap S| = k$ .*

For integers  $r \geq k \geq 1$ , let  $\mathcal{H}_{r,k}$  be the family of  $r$ -regular graphs  $H$  whose vertex set is the disjoint union  $F_1 \cup F_2 \cup F_3$ , where  $F_1$  induces an  $(r - 1)$ -regular graph of which each vertex has exactly one neighbor in  $F_2$ ,  $F_2$  is a stable set of which each vertex has exactly  $k - 1$  neighbors in  $F_1$  and exactly  $r + 1 - k$  neighbors in  $F_3$ , and  $F_3$  is a stable set of which each vertex has exactly  $r$  neighbors in  $F_2$ , see Figure 1. Since  $(r + 1 - k)|F_2| = e[F_2, F_3] = r|F_3|$ , there is some integer  $m \geq 1$  such that  $|F_2| = rm/g$  and  $|F_3| = (r + 1 - k)m/g$ , where  $g = \gcd(r + 1 - k, r) = \gcd(r, k - 1)$ . And then  $|F_1| = (k - 1)rm/g$ . The total number of vertices in  $H$  is  $n = (kr + r + 1 - k)m/g$ . According to Lemma 1,  $F_1 \cup F_2$  is a minimal  $k$ -upper dominating set of  $H$  and so  $\Gamma_{\times k}(H) \geq \frac{krn}{kr+r+1-k}$ .

**Theorem 2** *If  $G$  is a  $r$ -regular graph of order  $n$  with  $r \geq k \geq 2$ , then  $\Gamma_{\times k}(G) \leq \frac{krn}{kr+r+1-k}$  with equality if and only if  $G \in \mathcal{H}_{r,k}$ .*

**Proof:** Let  $S$  be a  $\Gamma_{\times k}(G)$ -set of  $G$  and  $S' = V(G) \setminus S$ . For  $k \leq i \leq r + 1$ , we define

$$S_i = \{u \in S : |N[u] \cap S| = i\} \quad \text{and} \quad S'_i = \{u \in S' : |N[u] \cap S| = i\}.$$



**Fig. 1:** An  $r$ -regular graph  $H$  in  $\mathcal{H}_{r,k}$ .

Notice that  $S'_{r+1} = \emptyset$  and  $S'_r$  is stable as every vertex of  $S'_r$  has neighbors only in  $S$ . Since every vertex has at least  $k$  closed neighbors in  $S$ , it is the case that  $S = \bigcup_{i=k}^{r+1} S_i$  and  $S' = \bigcup_{i=k}^r S'_i$  are disjoint unions. Therefore,  $|S| = \sum_{i=k}^{r+1} |S_i|$  and  $|S'| = \sum_{i=k}^r |S'_i|$ .

According to Lemma 1, every vertex in  $S_{r+1}$  has at least one neighbor in  $S_k$ , while every vertex in  $S_k$  has at most  $k - 1$  neighbors in  $S_{r+1}$ . Therefore,

$$|S_{r+1}| \leq e[S_{r+1}, S_k] \leq (k - 1)|S_k| \tag{1}$$

or equivalently

$$|S_{r+1}| - (k - 1)|S_k| \leq 0. \tag{2}$$

By Lemma 1 again, every vertex in  $X = \bigcup_{i=k+1}^{r+1} S_i$  has at least one neighbor in  $S_k \cup S'_k$  and every vertex in  $S_k$  (respectively,  $S'_k$ ) has at most  $k - 1$  (respectively,  $k$ ) neighbors in  $X$ . Therefore,

$$\sum_{i=k+1}^{r+1} |S_i| \leq e[X, S_k] + e[X, S'_k] \leq (k - 1)|S_k| + k|S'_k| \tag{3}$$

or equivalently

$$\sum_{i=k+1}^{r+1} |S_i| - (k - 1)|S_k| \leq k|S'_k|. \tag{4}$$

We then use a double counting of  $e[S, S']$  to get

$$\sum_{i=k}^r (r + 1 - i)|S_i| = e[S, S'] = \sum_{i=k}^r i|S'_i|. \tag{5}$$

Let  $p = \min(\frac{r+1-k}{k}, 1)$  and  $q = \max(\frac{r+1-k}{k}, 1) - 1$ . Then,  $p > 0, q \geq 0$  and  $q + 1 \geq \frac{r+1-k}{k} = p + q$ . Consequently,  $(r + 1 - k) - (p + q)(k - 1) = p + q$  and  $q + r + 1 - i \geq q + 1 \geq p + q$  for  $k + 1 \leq i \leq r$ , which gives (6). Adding  $p$  times (2),  $q$  times (4) and (5) gives the first inequality in (7). And  $(q + 1)k = \max\{r + 1 - k, k\} \leq r$  with equality only when  $r = k$ , which gives the second inequality in (7).

$$(p + q)|S| \leq ((r + 1 - k) - (p + q)(k - 1))|S_k| + (p + q)|S_{r+1}| + \sum_{i=k+1}^r (q + r + 1 - i)|S_i| \tag{6}$$

$$\leq (q + 1)k|S'_k| + \sum_{i=k+1}^r i|S'_i| \leq r|S'| = rn - r|S|. \tag{7}$$

Hence,  $\frac{kr+r+1-k}{k}|S| = (r + p + q)|S| \leq rn$  and so  $\Gamma_{\times k}(G) \leq \frac{krn}{kr+r+1-k}$ , which proves the first part of the theorem.

Suppose  $\Gamma_{\times k}(G) = \frac{krn}{kr+r+1-k}$ . By the proof above, the inequalities in (1), the inequalities in (3) when  $q > 0$ , and the inequalities in (6) and (7) are equalities. The equality in (6) gives that  $S = S_k \cup S_{r+1}$  when  $p + q < q + 1$  and  $S = S_k \cup S_{r+1} \cup S_r$  when  $p + q \geq q + 1$ , while the second equality in (7) gives that  $S' = S'_r$  which is stable. The first equality in (1) gives that every vertex in  $S_{r+1}$  has exactly one neighbor in  $S_k$ , while the second equality gives that every vertex in  $S_k$  has exactly  $k - 1$  neighbors in  $S_{r+1}$  and so exactly  $r + 1 - k$  neighbors in  $S'$ . We claim that  $p + q \geq q + 1$  is impossible, for otherwise  $p \geq 1$  implying  $\frac{r+1-k}{k} \geq 1$  and so  $r > k$ . Then by Lemma 1, every vertex in  $S_r$  is adjacent to some vertex in  $S'_k = \emptyset$  as  $S' = S'_r$ , a contradiction. Hence,  $p + q < q + 1$  and so  $S = S_k \cup S_{r+1}$ . Letting  $F_1 = S_{r+1}$ ,  $F_2 = S_k$  and  $F_3 = S'$ , we have that  $G$  is in  $\mathcal{H}_{r,k}$ . On the other hand, any graph  $G$  in  $\mathcal{H}_{r,k}$  satisfies  $\Gamma_{\times k}(G) \geq \frac{krn}{kr+r+1-k}$  and so  $\Gamma_{\times k}(G) = \frac{krn}{kr+r+1-k}$ .  $\square$

The extremal graphs in  $\mathcal{H}_{r,k}$  for Theorem 2 contain claws. For claw-free  $r$ -regular graphs, the upper bound in Theorem 2 for upper 2-tuple domination can be improved.

**Theorem 3** *If  $G$  is a claw-free  $r$ -regular graph of order  $n$  with  $r \geq 3$ , then  $\Gamma_{\times 2}(G) \leq \frac{2n}{3}$ .*

**Proof:** Let  $S, S', S_i$  and  $S'_i$  be defined as in the proof of Theorem 2 with  $k = 2$ . For  $2 \leq i \leq r + 1$ , we further write

$$S_{2,i} = \{u \in S_2: |N(u) \cap S_i| = 1\} \quad \text{and} \quad S_{i,2} = \{u \in S_i: |N(u) \cap S_2| \geq 1\}.$$

Then  $|S_2| = \sum_{i=2}^{r+1} |S_{2,i}|$ . By the definition of  $S_{2,i}$  and  $S_{i,2}$ , for  $3 \leq i \leq r + 1$ ,

$$|S_{i,2}| \leq e[S_{i,2}, S_{2,i}] = |S_{2,i}|. \tag{8}$$

By Lemma 1,  $S_{r+1,2} = S_{r+1}$  and so

$$|S_{r+1}| \leq |S_{2,r+1}| \leq |S_2|. \tag{9}$$

We then consider  $e[S'_r, S]$ . Since  $S'_r$  is stable and  $G$  is claw-free, every vertex in  $S$  has at most 2 neighbors in  $S'_r$ . First, every vertex  $u \in S_{2,r+1}$  has no neighbor in  $S'_r$ . Otherwise, if such a neighbor  $v$  exists, then  $u$  has another neighbor  $w$  in  $S'$  and a neighbor  $x$  in  $S_{r+1}$ . Since  $x$  has only neighbors in  $S$ , it is not adjacent to  $v$  nor to  $w$ . Also,  $v$  is not adjacent to  $w$  since it has only neighbors in  $S$ . Hence a claw occurs at  $u$ , a contradiction. Second, for  $3 \leq i \leq r$ , every vertex  $u \in S_i \setminus S_{i,2}$  has no neighbor in  $S_2$  and hence, by Lemma 1, it has a neighbor in  $S'_2$ . Thus, if  $u$  is in  $S_r$  ( $i = r$ ), then this neighbor in  $S'_2$  is the only neighbor of  $u$  in  $S'$ , and so  $u$  has no neighbor in  $S'_r$ . Else, if  $3 \leq i \leq r - 1$ , then  $u$  has at most one neighbor in  $S'_r$ , or it would be the center of a claw. Finally, by definition, every vertex in  $S_{r+1}$  has no neighbor in  $S'_r$ . These give the first inequality in (10), while (8) and (9) give the second inequality in (10).

$$r|S'_r| = e[S'_r, S] \leq \sum_{i=2}^r 2|S_{2,i}| + \sum_{i=3}^r |S_{i,2}| + \sum_{i=3}^{r-1} |S_i| \leq 3|S_2| - 3|S_{r+1}| + \sum_{i=3}^{r-1} |S_i|. \tag{10}$$

Formula (5) with  $k = 2$  gives

$$\sum_{i=2}^r (r+1-i)|S_i| = \sum_{i=2}^r i|S'_i|. \quad (11)$$

Formula (3) with  $k = 2$  gives

$$\sum_{i=3}^{r+1} |S_i| \leq |S_2| + 2|S'_2|. \quad (12)$$

Adding  $\frac{r-3}{r} \cdot (9)$ ,  $\frac{1}{r} \cdot (10)$ , (11) and  $\frac{r-3}{2} \cdot (12)$  gives

$$\frac{r-1}{2}(|S_2| + |S_r| + |S_{r+1}|) + \sum_{i=3}^{r-1} \left(\frac{3r-1}{2} - i - \frac{1}{r}\right)|S_i| \leq (r-1)(|S'_2| + |S'_r|) + \sum_{i=3}^{r-1} i|S'_i| \quad (13)$$

The left side is bounded below by  $\frac{r-1}{2}|S|$ , the right above by  $(r-1)|S'|$ . Thus we get  $|S| \leq 2|S'|$  and finally,  $\Gamma_{\times 2}(G) \leq \frac{2n}{3}$ .  $\square$

### 3 Complexity results

The upper domination problem was shown to be NP-complete by Cheston, Fricke, Hedetniemi and Jacobs [6]. However, Cockayne, Favaron, Payan and Thomason [7] proved that  $\Gamma(G) = \beta_0(G)$  for any bipartite graph  $G$ , and so  $\Gamma(G)$  can be computed for bipartite graphs in polynomial time. It was also shown by Jacobson and Peters [11] that  $\Gamma(G) = \beta_0(G)$  for any chordal graph  $G$ , and so  $\Gamma(G)$  can be computed for chordal graphs in polynomial time. Besides, Hare, Hedetniemi, Laskar, Peters and Wimer [10] also established a polynomial algorithm for determining  $\Gamma(G)$  on generalized series-parallel graphs.

On the other hand, we shall prove that the  $k$ -tuple domination problem, with  $k \geq 2$  fixed, is NP-complete for bipartite graphs and for chordal graphs. The proofs are separated into the cases of  $k = 2$  and of  $k \geq 3$ . We consider the decision problem version as follows.

#### Upper $k$ -tuple domination problem (U $k$ TD)

**Instance:** A graph  $G = (V, E)$  and a positive integer  $s \leq |V|$ .

**Question:** Does  $G$  have a minimal upper  $k$ -dominating set of cardinality at least  $s$ ?

To show that U2TD is NP-complete, we will make use of the well-known NP-complete problem 3-SAT [8].

#### One-in-three 3SAT (OneIn3SAT)

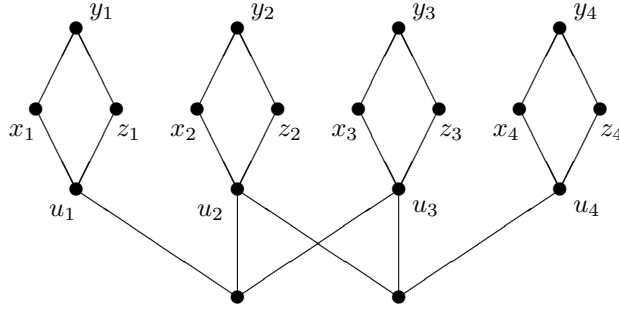
**Instance:** A set  $U = \{u_1, \dots, u_n\}$  of  $n$  variables and a collection  $\mathcal{C} = \{c_1, \dots, c_m\}$  of  $m$  clauses over  $U$  such that each clause  $c \in \mathcal{C}$  has  $|c| = 3$  and no clause contains a negated variable.

**Question:** Is there a truth assignment  $\mathcal{A}: U \rightarrow \{\text{true}, \text{false}\}$  for  $U$  such that each clause in  $\mathcal{C}$  has exactly one true literal?

**Theorem 4** *The upper 2-tuple domination problem is NP-complete even restricted on bipartite graphs or on split graphs, and hence also on chordal graphs.*

**Proof.** Obviously, U2TD is in NP. We shall show the NP-completeness of U2TD for bipartite graphs by reducing OneIn3SAT to it in polynomial time. Let  $U = \{u_1, u_2, \dots, u_n\}$  and  $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$  be an instance  $I$  of OneIn3SAT. We transform  $I$  to the instance  $(G_I, s)$  of U2TD in which  $s = 3n + m$  and  $G_I$  is the bipartite graph formed as follows.

Corresponding to each variable  $u_i$  we associate a cycle  $C_i = u_i x_i y_i z_i u_i$ . Corresponding to each 3-element clause  $c_j$  we associate a vertex named  $w_j$ . Joining the vertex  $u_i$  to the vertex  $w_j$  if and only if the literal  $u_i$  belongs to the clause  $c_j$ . According to the above construction, it is easy to see that  $G_I$  is a bipartite graph and the construction is accomplished in polynomial time. The graph  $G_I$  associate with  $(u_1 \vee u_2 \vee u_3) \wedge (u_2 \vee u_3 \vee u_4)$  is shown in Figure 2.



**Fig. 2:** The graph  $G_I$  for  $(u_1 \vee u_2 \vee u_3) \wedge (u_2 \vee u_3 \vee u_4)$ .

We next show that  $I$  has a satisfying truth assignment if and only if  $G_I$  has a minimal upper 2-tuple dominating set of cardinality at least  $s = 3n + m$ .

Suppose first  $I$  has a satisfying truth assignment  $\mathcal{A}$ . A minimal upper 2-tuple dominating set  $S$  of  $G_I$  of cardinality  $s$  is constructed as follows. Let  $w_j$  belong to  $S$  for all  $1 \leq j \leq m$ . For each  $1 \leq i \leq n$ , if  $\mathcal{A}(u_i) = \{\text{true}\}$ , then let  $u_i, x_i$  and  $z_i$  be in  $S$ ; otherwise, let  $x_i, y_i$  and  $z_i$  be in  $S$ . Clearly,  $|N[v] \cap S| \geq 2$  for each vertex  $v \in V(G_I)$  and  $S$  satisfies the conditions of Lemma 1, and so  $S$  is a minimal upper 2-tuple dominating set of  $G_I$  with cardinality  $s = 3n + m$ .

On the other hand, assume that  $S$  is a minimal upper 2-tuple dominating set of  $G_I$  with cardinality at least  $s = 3n + m$ . It follows from Lemma 1 that  $|S \cap V(C_i)| \leq 3$  for  $1 \leq i \leq n$ . Further,  $|S \cap \{w_1, w_2, \dots, w_m\}| \leq m$  and so  $|S| \leq 3n + m$ . Notice that  $|S| \geq 3n + m$  by the assumption. Hence,  $|S \cap V(C_i)| = 3$  for  $1 \leq i \leq n$  and  $\{w_1, w_2, \dots, w_m\} \subseteq S$ . Let  $\{u_{j_1}, u_{j_2}, u_{j_3}\}$  be the open neighborhood of  $w_j$  in  $G_I$  for  $1 \leq j \leq m$ . Since  $S$  is a minimal upper 2-tuple dominating set,  $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \geq 1$ . We claim that  $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| = 1$ . Otherwise,  $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \geq 2$ . Then the degree of  $w_j$  in  $G_I[S]$  is more than 1. However,  $w_j$  does not satisfy the conditions of Lemma 1 because  $|S \cap V(C_i)| = 3$  for  $1 \leq i \leq n$ , a contradiction. Let  $\mathcal{A}: U \rightarrow \{\text{true}, \text{false}\}$  be defined by  $\mathcal{A}(u_i) = \{\text{true}\}$  if  $u_i \in S$  and  $\mathcal{A}(u_i) = \{\text{false}\}$  if  $u_i \notin S$ . By the construction of  $G_I$ , we have each clause  $c_j$  of  $I$  contains only one variable  $u_i$  belonging to  $S$ . So  $\mathcal{A}$  is a satisfying truth assignment for  $I$ . Consequently,  $I$  has a satisfying truth assignment if and only if  $G_I$  has a minimal upper 2-tuple dominating set of cardinality at least  $s = 3n + m$ . This completes the proof for bipartite graphs.

To deal with split graphs, we add edges to make  $\{u_1, y_1, u_2, y_2, \dots, u_n, y_n\}$  a clique. The same arguments hold, and show the NP-completeness of the upper 2-tuple domination problem for split graphs.  $\square$

**Theorem 5** For any fixed integer  $k \geq 3$ , the  $k$ -tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.

**Proof:** For any bipartite graph  $G$  on  $n$  vertices, consider the bipartite graph  $G'$  obtained by the following process. For each vertex  $v$  of  $G$ , we add a copy of  $K_{k-1, k-1}$ , denoted  $G_v$  with bipartition denoted  $(X_v, Y_v)$ . Then we link by an edge the vertex  $v$  to  $k - 2$  vertices in  $X_v$ . The widget added to each vertex  $v$  is drawn in Figure 3.

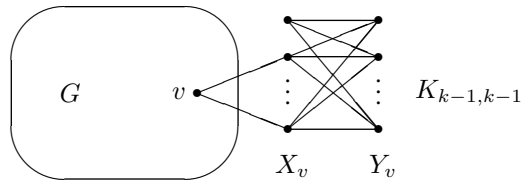


Fig. 3: The widget  $G_v$  added to each vertex  $v$  in  $G$ .

We claim that  $G$  has a minimal upper 2-tuple dominating set of size at least  $s$  if and only if  $G'$  has a minimal upper  $k$ -tuple dominating set of size at least  $s + 2(k - 1)n$ .

Clearly, if  $S$  is a minimal upper 2-tuple dominating set of  $G$  with size at least  $s$ , then  $S \cup (\bigcup_{v \in V(G)} (X_v \cup Y_v))$  is a minimal upper  $k$ -tuple dominating set of  $G'$  with size at least  $s + 2(k - 1)n$ .

On the other hand, suppose  $S'$  is a minimal upper  $k$ -tuple dominating set of  $G'$  with size at least  $s + 2(k - 1)n$ . Since every vertex in  $Y_v$  is of degree  $k - 1$ ,  $S'$  necessarily includes  $\bigcup_{v \in V(G)} (X_v \cup Y_v)$ . Let  $S = S' \setminus (\bigcup_{v \in V(G)} (X_v \cup Y_v))$ . Every vertex  $v$  in  $V(G)$  is dominated precisely  $k - 2$  times by vertices from  $S' \setminus S$ . Therefore,  $v$  is dominated by at least two vertices in  $S$  and so  $S$  is a 2-tuple dominating set of  $G$ . By Lemma 1,  $v$  is dominated by some  $u \in V(G')$  with  $N_{G'}[u] = k$ . This vertex  $u$  must be in  $V(G)$  and  $N_G[u] = 2$ . By Lemma 1 again,  $S$  is a minimal upper 2-tuple dominating set of  $G$  with size at least  $s$ .

The NP-completeness of the upper  $k$ -tuple domination problem for bipartite graphs then follows from the NP-completeness of the upper 2-tuple domination problem for bipartite graphs.

To deal with the chordal case, we start with a chordal graph  $G$  and for each  $v \in V(G)$  we add to  $G_v$  edges joining any pair of vertices in  $X_v$ , forming a clique. The same arguments hold, and show the NP-completeness of the upper  $k$ -tuple domination problem for chordal graphs as a consequence of the NP-completeness of the upper 2-tuple domination problem for chordal graphs.  $\square$



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