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MIMO Conditional Integrator Control for Unmanned Airlaunch

Van Cuong Nguyen and Gilney Damm

Abstract

A Nonlinear Multiple Input Multiple Output (MIMO) controller based on the conditional integrator technique is designed for the robust stabilization of a new satellite launching strategy called (unmanned) airlaunch. This control technique performs as a nonlinear robust controller when the system is outside a boundary layer defined by the controller (large errors), and as a nonlinear controller that provides an integral term when the system enters into the boundary layer (small errors). While the airlaunch strategy consists in using a two-stages launching system. The first stage is composed of an airplane (manned or unmanned) that carries a rocket launcher which constitute the subsequent stages. The control objective is to stabilize the aircraft in the launch phase. It is developed separately for two nonlinear MIMO motion modes of the model, the longitudinal mode and lateral mode, and is then applied to the full model of the aircraft. The considered model is highly nonlinear, mostly as a consequence of possible large angle of attack, sideslip and roll angle. Finally, the present work illustrates through simulations the good performance of the proposed control algorithm.

Keywords: Conditional Integrator; Airlaunch; F-16 Aircraft; Non-linear System

1 Introduction

Satellites launching is a strategical activity today. Launchers are able to carry from micro-satellites of some tens of kilograms up to 10 tons in the case of French Ariane 5 launcher. Recently, new applications have called upon very small satellites mostly used in groups (see [1]). These small satellites need a new class of launchers since launching implies in many fixed costs that are independent of the size and weight of the launched device. For this reason, the ratio price-per-kilogram launched in space becomes too high. A quite logical solution in this case would be to pack many small devices to be launched together. Unfortunately this implies many additional risks in the split phase and is not envisaged.

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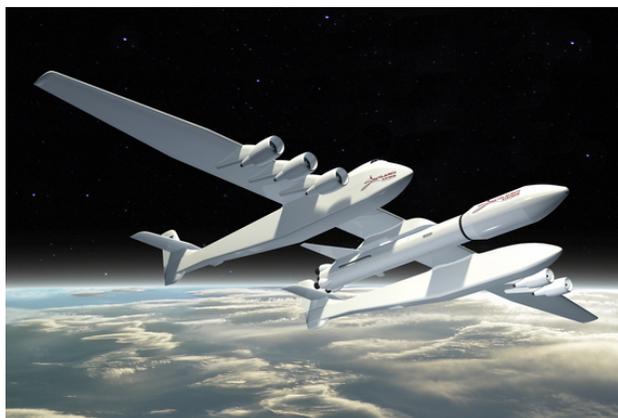


Figure 1: Stratolaunch system credited © Stratolaunch Systems, Inc.

A more efficient solution in this case is to use the procedure of airlaunch (see [12], [4]). It consists of using a two stages launching system (see Fig. 1). The first stage is composed of an air vehicle (manned or unmanned) that carries (inside, beneath or above) a launcher which constitutes the second stage. There are many advantages in airlaunch, mainly because there is no need for specific large non populated launching areas. The aerial vehicle takes-off from a standard runway and fly to open ocean, avoiding populated areas or ship and airplane paths. For this reason there is also a minimization of weather constraints, since the vehicle can fly to open sky, and as consequence the launch delay can be significantly shortened. Similarly, instead of waiting for specific launch windows (to attain desired orbits), the vehicle may be flown to a better suited launch point, with a better alignment with the desired orbit. The fact that the first stage is a reusable aerial vehicle allows a much smaller launching delay. In the same way, launching reuse time may be very short (one or two days). These characteristics provide great flexibility, and allow to deploy small satellites designed for specific tasks of communication or data gathering in real time for urgent situations.

Airlaunch provides the advantages of two stage launchers. The second (dropped) stage may use specific nozzles and propellants for the low outside atmospheric pressure at altitude (20 000 meters or 60000 feet). This is obtained without the complex, expensive and relatively dangerous high pressure ground-launched first stage that is replaced by the aerial vehicle. Most current airlaunch projects use standard or lightly modified airplanes as first stage. For example, there has been tests using F15, C17, B52, L-1011 in Rascal, Quick-Reach, Proteus and Pegasus projects.

It is important to remark that airplanes use the wing's lift force to fly. For this reason, higher (low altitude) air density benefit the flight while the aircraft uses standard fuel to keep flying. A first stage rocket would use a much more complex, dangerous and expensive fuel while in this higher air density. From

a certain altitude, air density is too low to be useful for an airplane, while not representing anymore a drawback for rockets.

Unlike the before-mentioned projects, other ones aim in developing an air-launch system that uses an Unmanned Aerial Vehicle (UAV) instead of a standard aircraft with a human pilot inboard. The current paper addresses their launching phase, it intends to introduce modeling and a robust nonlinear controller for this delicate procedure. In fact, airlaunch may be very challenging since the rocket may be almost as heavy as the UAV. This means that the aircraft will instantaneously lose almost half of its mass. As consequence, the stabilization task is much more complex during and after the launching phase with a much more adverse mass ratio. To our best knowledge, it does not exist an equivalent research line, and then there is no results in the literature considering this problem.

The paper is organized as follows: in section 2, we describe the nonlinear mathematical system model. A conditional integrator control design and its application to the full system model is discussed in section 3. The paper is completed by computer simulations and conclusions.

2 Modeling

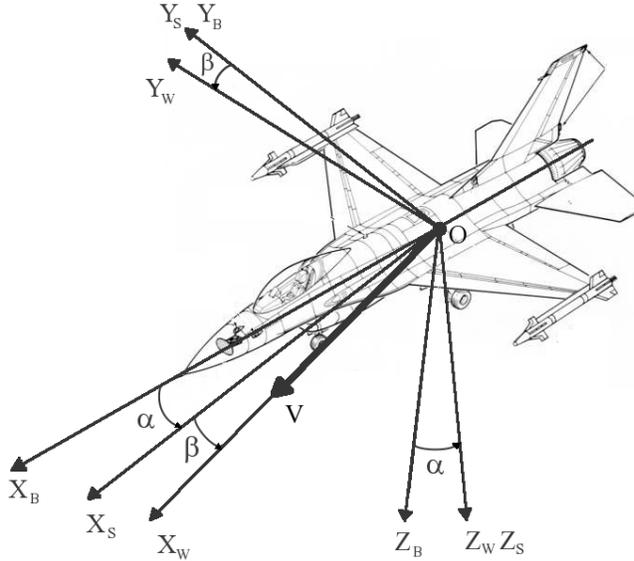


Figure 2: Frames: Body fixed axes $OX_B Y_B Z_B$, Stability axes $OX_S Y_S Z_S$, Aerodynamic axes $OX_W Y_W Z_W$

Modeling an air launch system and its separation phase is a difficult task which requires many data and informations about the real system. We have based our work on the model of an F-16 aircraft (that already has been tested for manned airlaunch). We have also used it to verify in simulations the reactions of the system when modeling the air launch phase.

We suppose that the studied system is a set of a reusable launch vehicle and a down stage, whose mass is equivalent to the reusable launch vehicle's own mass. For the sake of simplicity, this set is considered as a complete aircraft before air launching. In this paper the airlaunch system is considered as a hybrid system composed by two continuous models that are switched in three phases, representing the system before, during and after the separation phase (see [11]).

1. before the separation \Rightarrow the first aircraft model (representing the UAV and the rocket) is in a stable operating condition
2. during the separation \Rightarrow a second aircraft model representing only the UAV, starting on the previous operating condition, is disturbed by impulses on forces and moments representing a not perfect separation inside

a time interval T_{int} . Furthermore the initial conditions, inherited from the first phase, are not an equilibrium point for the second aircraft model.

3. after the separation \Rightarrow the disturbances stop (UAV and rocket are not in physical contact anymore).

It can be shown that the effect of launching the rocket from the UAV impacts most the lift force, and the roll and pitch moments. We suppose that these perturbing force and moments are constant during interval T_{int} , and we represent then F_{z_p} , L_p and M_p for the perturbations on the lift force, on the roll moment and pitch moment respectively.

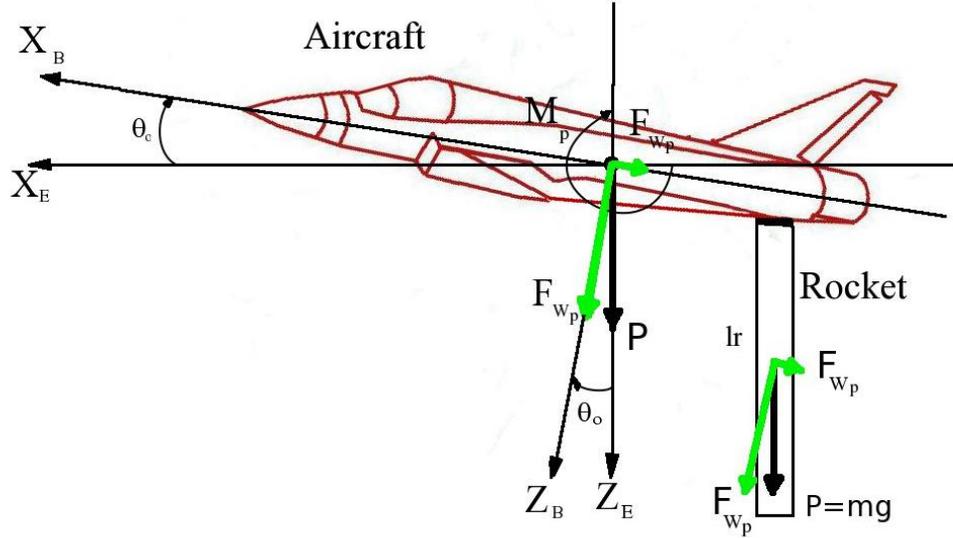


Figure 3: The launcher attached to the aircraft carrier in the worst case

In the present work we will study a worst case of disturbance. We consider that the separation phase is not simultaneous in all links that attach the rocket and the UAV. For this reason, the rocket remains attached to one end of the UAV during T_{int} . We have then studied how long the disturbance could last and that the control algorithm could still stabilize the aircraft back.

We suppose that:

- the perturbation on lift force during T_{int} is equal to the air launch vehicle's mass, that means $F_{w_p} = m_r g \cos \theta_0$.
- the perturbation on drag force is $F_{u_p} = -F_{w_p} \sin \theta_0 = -m_r g \sin \theta_0$ where θ_0 is the initial pitch angle of the first model at the launching phase and m_r is the rocket's weight.

- the perturbation on pitch moment during T_{int} is an worst case that is represented by the rocket that remains attached to the aircraft by only one end during T_{int} , applying a rotational movement to the aircraft, so a moment with value $M_p = m_r g l_r \cos \theta_0 / 2$ where l_r is the rocket length.
- the perturbation on roll moment during T_{int} is small because of the rocket shape (long and thin).
- the model following the launch phase is the F-16 model. Its initial condition is the state at an equilibrium point of the model previous the launch phase that is the F-16 model but with twice its standard mass. That means the rocket's mass is equal to the aircraft carrier. It is a hypothesis in the worst case because in practice the aircraft carrier's weight is greater than the rocket's weight.

The model of the dynamic airlaunch after the split phase is described by the Newton-Euler's law in the aerodynamic axes ($OX_W Y_W Z_W$ in Fig. 2) i.e. the reference frame attached to the airspeed vector V (see [6], [13] and [14]), instead of using variables in the body fixed axes because of the measurability of these state variables:

$$\left\{ \begin{array}{l} \dot{\alpha} = -\cos \alpha \tan \beta p + q - \sin \alpha \tan \beta r - \frac{\sin \alpha}{mV \cos \beta} (T + F_x) + \frac{\cos \alpha}{mV \cos \beta} F_z \\ \quad + \frac{g}{V \cos \beta} [\sin \alpha \cos \theta + \cos \alpha \cos \phi \cos \theta] \\ \dot{\beta} = \sin \alpha p - \cos \alpha r - \frac{\cos \alpha \sin \beta}{mV} [T + F_x] + \frac{\cos \beta}{mV} F_y - \frac{\sin \alpha \sin \beta}{mV} F_z + \frac{g}{V} [\cos \alpha \sin \beta \sin \theta \\ \quad + \cos \beta \cos \theta \sin \phi - \sin \alpha \sin \beta \cos \phi \cos \theta] \\ \dot{V} = \frac{\cos \alpha \cos \beta}{m} [T + F_x] + \frac{\sin \beta}{m} F_y + \frac{\sin \alpha \cos \beta}{m} F_z + g [\cos \alpha \cos \beta \sin \theta \\ \quad + \sin \beta \sin \phi \cos \theta + \sin \alpha \cos \beta \cos \phi \cos \theta] \\ \dot{p} = \frac{1}{I_{xx} I_{zz} - I_{xz}^2} [(I_{yy} I_{zz} - I_{zz}^2 - I_{xz}^2) r q - I_{xz} (I_{xx} + I_{zz} - I_{yy}) p q + I_{zz} L - I_{xz} N] \\ \dot{q} = \frac{1}{I_{yy}} [(I_{zz} - I_{xx}) p r + I_{xz} (p^2 - r^2) + M] \\ \dot{r} = \frac{1}{I_{xx} I_{zz} - I_{xz}^2} [(-I_{xx} I_{yy} + I_{zz}^2 + I_{xz}^2) p q + I_{xz} (I_{xx} + I_{zz} - I_{yy}) r q + I_{xx} N - I_{xz} L] \\ \dot{\phi} = p + \tan \theta (q \sin \phi + r \cos \phi) \\ \dot{\theta} = q \cos \phi - r \sin \phi \\ \dot{\psi} = \frac{q \sin \phi + r \cos \phi}{\cos \theta} \end{array} \right. \quad (1)$$

in which $I_{xx}, I_{yy}, I_{zz}, I_{xz}$ are the moments of inertia, m is the mass of the system (kg) and g the gravity constant. $\alpha, \beta, V, p, q, r, \phi, \theta, \psi$ are the state variables of the airlaunch aircraft model, they are the angle of attack, sideslip, airspeed, roll rate, pitch rate, yaw rate, roll angle, pitch angle and yaw angle respectively. $\alpha, \beta, \phi, \theta, \psi$ are expressed in rad , p, q, r in rad/s and V in m/s . T is the thrust force, F_x, F_y, F_z and L, M, N are aerodynamic forces and moments respectively. All forces and moments are expressed in N and Nm.

These aerodynamic forces and moments are function of all considered states. In this model, these aerodynamic forces and moments are under look-up table from wind tunnel data measurements as may be found in [9]. Finally, the control inputs are respectively the aileron (δ_a), rudder (δ_r) and elevator (δ_e) angles.

This model is based on wind tunnel data from NASA, considering the following conditions:

- angle of attack is in the range of $[-10^\circ, 45^\circ]$ and sideslip of $[-30^\circ, 30^\circ]$
- flap deflection is ignored
- physical constraints for aileron ($|\delta_a| \leq 21.5^\circ$), rudders ($|\delta_e| \leq 25^\circ$) and elevator ($|\delta_r| \leq 30^\circ$)
- all actuators are modeled as a first order model ($\tau = 0.0495s$) with limit rates $60^\circ/s$ for aileron and elevator, and $120^\circ/s$ for rudder.

In particular, we use the low quality mode of the F-16 model, and the aerodynamic data is interpolated and extrapolated linearly in simulation from tables found in [9].

3 Control design

3.1 Conditional integrator control design

The MIMO conditional integrator (CI) controller design for the output regulation of a class of minimum-phase nonlinear systems in case of asymptotically constant references is studied in [7], [5] and extended in [3]. Our present work is dedicated to use a nonlinear extension of these results developed in [2] and [10], for stabilizing an unmanned aircraft during and immediately after the airlaunch phase.

This modified CI is composed of two main nonlinear terms, one saturated and one that may grow unbounded. The condition is responsible to choose where each term dominates the controller. In this way it behaves differently following some conditions (how big the errors are) that are tuned by the design parameters.

We remind the theory of conditional integrator control design for a nonlinear system.

Consider the system:

$$\begin{cases} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= f(e_1, e_2) + g(e_1, e_2)u \end{cases} \quad (2)$$

where $e_1(t) \in \mathbb{R}^n$ is an output error vector, $e_2 = \dot{e}_1$, $u \in \mathbb{R}^n$ control input and $f(e_1, e_2) \in \mathbb{R}^n$, $g(e_1, e_2) \in \mathbb{R}^{n \times n}$ are continuous functions.

Let us define an intermediate variable:

$$s = k_0\sigma + K_1e_1 + e_2 \quad (3)$$

where $\sigma \in \mathbb{R}^n$ is the output of the conditional integrator

$$\dot{\sigma} = -k_0\sigma + \mu \text{sat}(s/\mu) \quad (4)$$

in which μ is the boundary layer, k_0 is a positive parameter, $K_1 \in \mathbb{R}^{n \times n}$ is a positive definite matrix.

The saturation function is determined as:

$$\text{sat}(s/\mu) = \begin{cases} s/\|s\| & \text{if } \|s\| \geq \mu \\ s/\mu & \text{if } \|s\| < \mu \end{cases} \quad (5)$$

We denote \mathcal{O}_μ as the region neighborhood of $(e_1, e_2) = (0, 0)$ with a radius R_μ for $\|s\| < \mu$

$$\mathcal{O}_\mu = \{e = (e_1, e_2) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|e\| \leq R_\mu\} \quad (6)$$

We state the following assumptions on the forcing terms $f(e_1, e_2)$ and $g(e_1, e_2)$ to design the control algorithm.

Assumption 3.1 $f(e_1, e_2)$ is bounded by a function $\gamma(\|e_1\| + \|e_2\|)$ of class \mathcal{K} and a positive constant $\Delta_0 = \|f(0, 0)\|$

$$\|f(e_1, e_2)\| \leq \gamma(\|e_1\| + \|e_2\|) + \Delta_0 \quad (7)$$

for $(e_1, e_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and while the sliding surface does not enter the boundary layer, i.e. $\|s\| \geq \mu$.

Inside the boundary layer, the function $f(e_1, e_2)$ is required to be Lipschitz for $(e_1, e_2) \in \mathcal{O}_\mu$, as a consequence

$$\|f(e_1, e_2) - f(0, 0)\| \leq L_1 \|K_1 e_1\| + L_2 \|e_2\| \quad (8)$$

$\gamma(\|e_1\| + \|e_2\|)$ is also required to be Lipschitz for $(e_1, e_2) \in \mathcal{O}_\mu$:

$$\gamma(\|e_1\| + \|e_2\|) \leq \gamma_1 \|K_1 e_1\| + \gamma_2 \|e_2\| \quad (9)$$

where L_1 and L_2 , γ_1 and $\gamma_2 \in \mathbb{R}^+$.

□

Assumption 3.2 $g(e_1, e_2)$ is invertible for $(e_1, e_2) \in \mathbb{R}^n \times \mathbb{R}^n$.

□

Following these assumptions, the controller u defined below in (10) can be applied to (2) to stabilize the system.

$$u = -\pi(e_1, e_2) \text{sat}(s/\mu) \quad (10)$$

in which,

$$\pi(\cdot) = (\pi_0 + \gamma(\cdot) + k_0 \mu + \Delta_0) g^{-1}(\cdot) \quad (11)$$

π_0 and k_0 are positive constants, μ is the boundary layer as defined above.

Remark 1 *It is important to remark that the control can even grow unbounded since the term (11) is not necessarily bounded. Functions $\gamma(\cdot)$ and $g^{-1}(\cdot)$ can grow continuously. For this reason we call this controller a modified Conditional Integrator, composed of two terms (see (10-11)) one saturated and one not. The later will dominate for small errors, and as a consequence the controller will behave as an integrator. In the case of large errors, it is the first that dominates, and the controller acts as a robust controller.*

The stability in the general case of the control law (10) for system (2) is demonstrated in [2].

3.2 Lateral control design

In the lateral control design, we assume that all longitudinal state variables are null or constant, only lateral states are time varying. Moreover it is assumed that the airspeed's response is much slower than other states, and that the control surface deflection has no effects on the aerodynamic force components (lift and drag) but only on moments. Aerodynamic force F_y and moments L , N are calculated by their aerodynamic coefficients (see more in [6]).

$$F_y = (C_y(\beta) + (C_{y_p}(\alpha)p + C_{y_r}(\alpha)r)\bar{b}/(2V))\bar{q}S$$

$$L = (C_l(\beta) + C_{l_p}(\alpha, \beta)p\bar{b}/(2V) + C_{l_r}(\alpha, \beta)r\bar{b}/(2V) + C_{l_{\delta_a}}(\alpha)\delta_a + C_{l_{\delta_r}}(\alpha)\delta_r)\bar{q}S\bar{b}$$

$$N = (C_n(\beta) + C_{n_p}(\alpha, \beta)p\bar{b}/(2V) + C_{n_r}(\alpha, \beta)r\bar{b}/(2V) + C_{n_{\delta_a}}(\alpha)\delta_a + C_{n_{\delta_r}}(\alpha)\delta_r)\bar{q}S\bar{b}$$

By replacing F_y , moments L , N and $\alpha = \alpha_0$, $\theta = \theta_0$ in (1), the lateral nonlinear dynamic model used for the control design procedure is consequently reduced as:

$$\begin{cases} \dot{\beta} = \frac{1}{mV}(-\cos(\alpha_0)\sin(\beta)(T + C_x(\alpha_0)\bar{q}S) + \cos(\beta)C_y(\beta)\bar{q}S - \sin(\alpha_0)\sin(\beta)C_z(\alpha_0, \beta)\bar{q}S) \\ \quad + \sin(\alpha_0)p - \cos(\alpha_0)r + \frac{\rho S}{4m}(\cos(\beta)C_{y_p}(\alpha_0)\bar{b}p + \cos(\beta)C_{y_r}(\alpha_0)\bar{b}r) \\ \quad + \frac{g}{V}(\cos(\alpha_0)\sin(\beta)\sin(\theta_0) + \cos(\beta)\cos(\theta_0)\sin(\phi) - \sin(\alpha_0)\sin(\beta)\cos(\phi)) \\ \dot{\phi} = p + \cos(\phi)\tan(\theta_0)r \\ \dot{p} = I_3C_l(\alpha_0, \beta)\bar{q}S\bar{b} + I_4C_n(\alpha_0, \beta)\bar{q}S\bar{b} + \frac{\rho V S \bar{b}}{4}[(I_3C_{l_p}(\alpha_0) \\ \quad + I_4C_{n_p}(\alpha_0))p + (I_3C_{l_r}(\alpha_0) + I_4C_{n_r}(\alpha_0))r] \\ \quad + \bar{q}S[(I_3C_{l_{\delta_a}}(\alpha_0) + I_4C_{n_{\delta_a}}(\alpha_0))\delta_a + (I_3C_{l_{\delta_r}}(\alpha_0) + I_4C_{n_{\delta_r}}(\alpha_0))\delta_r] \\ \dot{r} = I_4C_l(\alpha_0, \beta)\bar{q}S\bar{b} + I_9C_n(\alpha_0, \beta)\bar{q}S\bar{b} + \frac{\rho V S \bar{b}}{4}[(I_4C_{l_p}(\alpha_0) \\ \quad + I_9C_{n_p}(\alpha_0))p + (I_4C_{l_r}(\alpha_0) + I_9C_{n_r}(\alpha_0))r] \\ \quad + \bar{q}S[(I_4C_{l_{\delta_a}}(\alpha_0) + I_9C_{n_{\delta_a}}(\alpha_0))\delta_a + (I_4C_{l_{\delta_r}}(\alpha_0) + I_9C_{n_{\delta_r}}(\alpha_0))\delta_r] \end{cases} \quad (12)$$

in which S is the wing area, \bar{q} dynamic pressure, \bar{b} is reference wing span, $I_3 = \frac{I_{zz}}{(I_{xx}I_{zz} - I_{xz}^2)}$, $I_4 = \frac{I_{xz}}{(I_{xx}I_{zz} - I_{xz}^2)}$, $I_9 = \frac{I_{xx}}{(I_{xx}I_{zz} - I_{xz}^2)}$. $C_y(\alpha, \delta_e)$, $C_{y_p}(\alpha_0)$, $C_{y_r}(\alpha_0)$, $C_l(\alpha_0, \beta)$, $C_n(\alpha_0, \beta)$, $C_{l_p}(\alpha_0)$, $C_{n_p}(\alpha_0)$, $C_{l_r}(\alpha_0)$, $C_{n_r}(\alpha_0)$, $C_{l_{\delta_a}}(\alpha_0)$, $C_{n_{\delta_a}}(\alpha_0)$, $C_{l_{\delta_r}}(\alpha_0)$, $C_{n_{\delta_r}}(\alpha_0)$ are lateral aerodynamic coefficients taken from [8].

The previous equation can be rearranged as:

$$\begin{cases} \begin{bmatrix} \dot{\beta} \\ \dot{\phi} \end{bmatrix} = f_{11}^\beta(\beta, \phi) + f_{12}^\beta(\beta, \phi) \begin{bmatrix} p \\ r \end{bmatrix} \\ \begin{bmatrix} \dot{p} \\ \dot{r} \end{bmatrix} = f_{21}^\beta(\beta, \phi) + f_{22}^\beta(\beta, \phi) \begin{bmatrix} p \\ r \end{bmatrix} + g_2^\beta(\beta, \phi) \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \end{cases} \quad (13)$$

where $f_{11}^\beta(\cdot)$, $f_{12}^\beta(\cdot)$, $f_{13}^\beta(\cdot)$, $f_{21}^\beta(\cdot)$, $f_{22}^\beta(\cdot)$, and $g_2^\beta(\cdot)$ represent the terms of (12) respectively (see Appendix A).

Let us define $x_1^\beta = [\beta, \phi]^T$, $x_2^\beta = \dot{x}_1^\beta = [\dot{\beta}, \dot{\phi}]^T$ and $u^\beta = (\delta_a, \delta_r)^T$, that allow us to rewrite equation (13) to:

$$\begin{cases} \dot{x}_1^\beta &= x_2^\beta \\ \dot{x}_2^\beta &= F^{\beta'}(x_1^\beta, x_2^\beta) + G^{\beta'}(x_1^\beta, x_2^\beta)u^\beta \end{cases} \quad (14)$$

where

$$\begin{cases} F^{\beta'}(\cdot) = \left(\frac{\partial f_{11}^\beta(\cdot)}{\partial x_1^\beta} + (f^\beta(\cdot) + f_{12}^\beta(\cdot)f_{22}^\beta(\cdot))(f_{12}^\beta(\cdot))^{-1} \right) x_2^\beta \\ \quad - (f^\beta(\cdot) + f_{12}^\beta(\cdot)f_{22}^\beta(\cdot))(f_{12}^\beta(\cdot))^{-1} f_{11}^\beta(\cdot) + f_{12}^\beta(\cdot)f_{21}^\beta(\cdot) \\ G^{\beta'}(\cdot) = f_{12}^\beta(\cdot)g_2^\beta(\cdot) \\ f^\beta(\cdot)[p, r]^T = \frac{\partial(f_{12}^\beta(\cdot)[p, r]^T)}{\partial x_1^\beta} \end{cases} \quad (15)$$

We define an output error vector $e_1^\beta = x_1^\beta - x_{1ref}^\beta$ and $e_2^\beta = \dot{e}_1^\beta$ where $x_{1ref}^\beta = (\beta_{ref}, \phi_{ref})^T$ is the output reference considered as constant. Equation (15) can be transformed into (16) with two new state variables e_1^β and e_2^β .

$$\begin{cases} \dot{e}_1^\beta = e_2^\beta \\ \dot{e}_2^\beta = F^\beta(e_1^\beta, e_2^\beta) + G^\beta(e_1^\beta, e_2^\beta)u^\beta \end{cases} \quad (16a)$$

$$\quad (16b)$$

$G^\beta(x_1^\beta, x_2^\beta)$ is invertible in the considered domain of $x_1^\beta = [\beta, \phi]^T$, $\dot{x}_1^\beta = x_2^\beta$ with $\beta \in (-30^\circ, 30^\circ)$ and $\phi \in (-180^\circ, 180^\circ)$.

Application of control law (10) for system (16) leads to the controller:

$$\begin{cases} u^\beta &= -\pi^\beta(e_1^\beta, e_2^\beta) \text{sat}(s^\beta/\mu^\beta) \\ \pi^\beta(\cdot) &= (\pi_0^\beta + \gamma^\beta(\cdot) + k_0^\beta \mu^\beta + \Delta_0^\beta)(G^\beta(\cdot))^{-1} \end{cases} \quad (17)$$

with

$$\begin{cases} s^\beta = k_0^\beta \sigma^\beta + K_1^\beta e_1^\beta + e_2^\beta \\ \dot{\sigma}^\beta = -k_0^\beta \sigma^\beta + \mu^\beta \text{sat}(s^\beta/\mu^\beta) \end{cases} \quad (18)$$

where π_0^β is a constant large enough, k_0^β is a positive parameter, μ^β is the boundary layer and K_1^β is a positive definite matrix chosen such a way that $K_1^\beta + sI_2$ is Hurwitz.

◇

Theorem 3.1 System (16) with $F^\beta(\cdot)$ which satisfies assumption 3.1, $G^\beta(\cdot)$ satisfying assumption 3.2, and applying the control law (17- 18), will globally reach an arbitrary error region in finite time, and there on will be exponentially stabilized towards its equilibrium point.

□

Proof: In order to demonstrate the exponential stability of designed controller (17) and (18) for the lateral mode in (16) which is a nonlinear MIMO system where the sideslip and roll angle are the outputs and aileron and rudder are the inputs, we will consider two regions: outside the boundary layer ($\|s^\beta\| \geq \mu^\beta$) and inside the boundary layer ($\|s^\beta\| \leq \mu^\beta$).

3.2.1 In the region $\|s^\beta\| \geq \mu^\beta$, $\text{sat}(s^\beta/\mu^\beta) = s^\beta/\|s^\beta\|$

The derivative of variable s can be expressed as:

$$\begin{aligned}\dot{s}^\beta &= k_0^\beta \dot{\sigma}^\beta + K_1^\beta \dot{e}_1^\beta + \dot{e}_2^\beta \\ &= -k_0^\beta s^\beta + k_0^\beta \mu^\beta \text{sat}(s^\beta/\mu^\beta) + k_0^\beta (K_1^\beta e_1^\beta + e_2^\beta) + K_1^\beta e_2^\beta + F^\beta(\cdot) + G^\beta(\cdot)u^\beta\end{aligned}$$

Now by letting

$$\Delta^\beta(\cdot) = k_0^\beta (K_1^\beta e_1^\beta + e_2^\beta) + K_1^\beta e_2^\beta + F^\beta(\cdot) \quad (19)$$

Its derivative becomes:

$$\dot{s}^\beta = -k_0^\beta s^\beta + k_0^\beta \mu^\beta \text{sat}(s^\beta/\mu^\beta) + \Delta^\beta(\cdot) + G^\beta(\cdot)u^\beta \quad (20)$$

Because of boundedness of $F^\beta(x_1^\beta, x_2^\beta, \theta)$, $\Delta^\beta(\cdot)$ is bounded by a function of $\gamma^\beta(\|e_1^\beta\| + \|e_2^\beta\|)$ (where $\gamma^\beta(\cdot)$ is a class \mathcal{K} function) and a positive constant Δ_0^β (assumption 3.1):

$$\|\Delta^\beta(e_1^\beta, e_2^\beta)\| \leq \gamma^\beta(\|e_1^\beta\| + \|e_2^\beta\|) + \Delta_0^\beta \quad (21)$$

and as a consequence,

$$\|\Delta^\beta(e_1^\beta = 0, e_2^\beta = 0)\| = \|F^\beta(0, 0)\| \leq \Delta_0^\beta \quad (22)$$

for $(e_1^\beta, e_2^\beta) \in \mathbb{R}^n \times \mathbb{R}^n$.

Let's consider the product $(s^\beta)^T \dot{s}^\beta$

$$(s^\beta)^T \dot{s}^\beta = -(s^\beta)^T k_0^\beta s^\beta + k_0^\beta \mu^\beta (s^\beta)^T \text{sat}(s^\beta/\mu^\beta) + (s^\beta)^T \Delta^\beta(e_1^\beta, e_2^\beta) + (s^\beta)^T G^\beta(e_1^\beta, e_2^\beta)u^\beta \quad (23)$$

This product $(s^\beta)^T \dot{s}^\beta$ can be developed with the definition of saturation function (5):

$$\begin{aligned}(s^\beta)^T \dot{s}^\beta &= -(s^\beta)^T k_0^\beta s^\beta + \mu^\beta (s^\beta)^T k_0^\beta s^\beta / \|s^\beta\| + (s^\beta)^T \Delta^\beta(\cdot) - (s^\beta)^T G^\beta(\cdot) \pi^\beta(\cdot) s^\beta / \|s^\beta\| \\ &\leq -(s^\beta)^T k_0^\beta s^\beta + \mu^\beta (s^\beta)^T k_0^\beta s^\beta / \|s^\beta\| + \|\Delta^\beta(\cdot)\| \|s^\beta\| - (s^\beta)^T G^\beta(\cdot) \pi^\beta(\cdot) s^\beta / \|s^\beta\| \\ &\leq -(s^\beta)^T k_0^\beta s^\beta + \mu^\beta (s^\beta)^T k_0^\beta s^\beta / \|s^\beta\| + (\gamma^\beta(\cdot) + \Delta_0^\beta) \|s^\beta\| - (s^\beta)^T G^\beta(\cdot) \pi^\beta(\cdot) s^\beta / \|s^\beta\| \\ &\leq -(s^\beta)^T k_0^\beta s^\beta - (s^\beta)^T (G^\beta(\cdot) \pi^\beta(\cdot) - (\mu^\beta k_0^\beta + \gamma^\beta(\cdot) + \Delta_0^\beta) I_n) s^\beta / \|s^\beta\|\end{aligned} \quad (24)$$

Replacing the control law in (17) and (18), the term $(s^\beta)^T \dot{s}^\beta$ can be expressed as:

$$\begin{aligned}(s^\beta)^T \dot{s}^\beta &\leq -(s^\beta)^T k_0^\beta s^\beta - (s^\beta)^T (G^\beta(\cdot) \pi^\beta(\cdot) - (\mu^\beta k_0^\beta + \gamma^\beta(\cdot) + \Delta_0^\beta) I_n) s^\beta / \|s^\beta\| \\ &\leq -(s^\beta)^T k_0^\beta s^\beta - (s^\beta)^T \pi_0^\beta s^\beta / \|s^\beta\| \\ &\leq -k_0^\beta \|s^\beta\|^2 - \pi_0^\beta \|s^\beta\|\end{aligned} \quad (25)$$

The product $(s^\beta)^T \dot{s}^\beta$ is then not positive and we have also

$$\begin{aligned} \frac{d(\|s^\beta\|^2)}{dt} &= 2\|s^\beta\| \frac{d(\|s^\beta\|)}{dt} = 2 \frac{(s^\beta)^T \dot{s}^\beta}{dt} \leq 2(-\pi_0^\beta \|s^\beta\| - k_0^\beta \|s^\beta\|^2) \\ \therefore \frac{d(\|s^\beta\|)}{dt} &\leq -\pi_0^\beta - k_0^\beta \|s^\beta\| \\ \therefore \|s^\beta(t)\| &\leq \|s^\beta(0)\| - \pi_0^\beta t - \|s^\beta(0)\|(e^\beta)^{-k_0^\beta t} - 1 \end{aligned} \quad (26)$$

Then $s^\beta(t)$ reaches the boundary layer μ^β in finite time.

3.2.2 In the region $\|s\| \leq \mu^\beta$, $\text{sat}(s/\mu^\beta) = s/\mu^\beta$

Consider again (18) and (20), which inside the boundary layer may be rewritten as:

$$\begin{cases} \dot{\sigma}^\beta = -k_0^\beta \sigma^\beta + s^\beta & (27a) \\ \dot{e}_1^\beta = -K_1^\beta e_1^\beta + s^\beta - k_0^\beta \sigma^\beta & (27b) \\ \dot{s}^\beta = \Delta^\beta(\cdot) - G^\beta(\cdot) \pi^\beta(\cdot) s^\beta / \mu^\beta & (27c) \end{cases}$$

It can be shown that this system has an equilibrium point: $\bar{e}_1^\beta = \bar{e}_2^\beta = 0$, $s^\beta = \bar{s}^\beta$, $\sigma^\beta = \bar{\sigma}^\beta$ with $\bar{s}^\beta = k_0^\beta \bar{\sigma}^\beta = \mu^\beta (\pi^\beta(0,0))^{-1} (G^\beta(0,0))^{-1} F^\beta(0,0) = \mu^\beta F^\beta(0,0) / (\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta)$.

System (27) may be rewritten with respect to \bar{s}^β and $\bar{\sigma}^\beta$:

$$\begin{cases} \dot{\tilde{\sigma}}^\beta = -k_0^\beta \tilde{\sigma}^\beta + \tilde{s}^\beta & (28a) \\ \dot{e}_1^\beta = -K_1^\beta e_1^\beta + \tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta & (28b) \\ \dot{\tilde{s}}^\beta = \Delta^\beta(\cdot) - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s}^\beta / \mu^\beta - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{\sigma}^\beta / \mu^\beta & (28c) \end{cases}$$

where $\tilde{\sigma}^\beta = \sigma^\beta - \bar{\sigma}^\beta$, $\tilde{s}^\beta = s^\beta - \bar{s}^\beta$.

$F^\beta(x_1^\beta, x_2^\beta)$ is a Lipschitz function inside the boundary region e.g. $\|s^\beta\| \leq \mu^\beta$, such that:

$$\|F^\beta(e_1^\beta, e_2^\beta) - F^\beta(0,0)\| \leq l_1^\beta \|e_1^\beta\| + l_2^\beta \|e_2^\beta\| \quad (29)$$

where l_1^β and $l_2^\beta \in \mathbb{R}^+$.

Following assumption 3.1, $\gamma(\cdot)$ is also a Lipschitz function:, such that:

$$\gamma^\beta(\cdot) \|F^\beta(0,0)\| \leq (\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta) (\gamma_1^\beta \|e_1^\beta\| + \gamma_2^\beta \|e_2^\beta\|) \quad (30)$$

where γ_1^β and $\gamma_2^\beta \in \mathbb{R}^+$.

We would like to demonstrate that every trajectory of system (28) starting inside the boundary layer, will approach the equilibrium point as time tends to infinity when the control law (10) is applied. Toward that end, we take:

$$W^\beta = \frac{\lambda_1^\beta}{2} (\tilde{\sigma}^\beta)^T \tilde{\sigma}^\beta + \frac{\lambda_2^\beta}{2} (e_1^\beta)^T e_1^\beta + \frac{(\tilde{s}^\beta)^T \tilde{s}^\beta}{2} \quad (31)$$

as a Lyapunov candidate, where λ_1^β and λ_2^β are positive constants.

Its derivative can be easily developed as:

$$\begin{aligned}\dot{W} &= \lambda_1 k_0^\beta (\tilde{\sigma}^\beta)^T \dot{\tilde{\sigma}}^\beta + \lambda_2 (e_1^\beta)^T K_1^\beta \dot{e}_1^\beta + (\tilde{s}^\beta)^T \dot{\tilde{s}}^\beta \\ &= \lambda_1 k_0^\beta (\tilde{\sigma}^\beta)^T (-k_0^\beta \tilde{\sigma}^\beta + \tilde{s}^\beta) + \lambda_2 (e_1^\beta)^T K_1^\beta (-K_1^\beta e_1^\beta + \tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta) \\ &\quad + (\tilde{s}^\beta)^T (\Delta^\beta(\cdot) - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s}^\beta / \mu^\beta - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s} / \mu^\beta)\end{aligned}\quad (32)$$

Since $\|s^\beta\| \leq \mu^\beta$, $\Delta^\beta(\cdot)$ can be expressed as:

$$\Delta^\beta(\cdot) = k_0^\beta \tilde{s}^\beta - (k_0^\beta)^2 \tilde{\sigma}^\beta - (K_1^\beta)^2 e_1^\beta + K_1^\beta \tilde{s}^\beta - k_0^\beta K_1^\beta \tilde{\sigma}^\beta + F^\beta(\cdot) \quad (33)$$

Replacing system (28) and $\Delta^\beta(\cdot)$ into the derivative of the Lyapunov function, we have then (reminding that $\tilde{s}^\beta = \mu^\beta (\pi^\beta(0, 0))^{-1} (G^\beta(0, 0))^{-1} F^\beta(0, 0)$ and $\pi^\beta(0, 0) = (\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta) (G^\beta(0, 0))^{-1}$):

$$\begin{aligned}\dot{W} &= \lambda_1 k_0^\beta (\tilde{\sigma}^\beta)^T (-k_0^\beta \tilde{\sigma}^\beta + \tilde{s}^\beta) + \lambda_2 (e_1^\beta)^T K_1^\beta (-K_1^\beta e_1^\beta + \tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta) \\ &\quad + (\tilde{s}^\beta)^T (k_0^\beta \tilde{s}^\beta - (k_0^\beta)^2 \tilde{\sigma}^\beta - (K_1^\beta)^2 e_1^\beta + K_1^\beta \tilde{s}^\beta - k_0^\beta K_1^\beta \tilde{\sigma}^\beta - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s}^\beta / \mu^\beta) \\ &\quad + (\tilde{s}^\beta)^T (F^\beta(\cdot) - G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s}^\beta / \mu^\beta) \\ &= -\lambda_1 (k_0^\beta)^2 (\tilde{\sigma}^\beta)^T \tilde{\sigma}^\beta + \lambda_1 k_0^\beta (\tilde{\sigma}^\beta)^T \tilde{s}^\beta - \lambda_2 (e_1^\beta)^T (K_1^\beta)^2 e_1^\beta + \lambda_2 (e_1^\beta)^T K_1^\beta (\tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta) \\ &\quad + ((\tilde{s}^\beta)^T (k_0^\beta I_n + K_1^\beta) \tilde{s}^\beta + (\tilde{s}^\beta)^T (k_0^\beta + K_1^\beta) k_0^\beta \tilde{\sigma}^\beta - (\tilde{s}^\beta)^T (K_1^\beta)^2 e_1^\beta - (\tilde{s}^\beta)^T G^\beta(\cdot) \pi^\beta(\cdot) \tilde{s}^\beta / \mu^\beta) \\ &\quad + (\tilde{s}^\beta)^T (F^\beta(\cdot) - F^\beta(0, 0) - \frac{\gamma^\beta(\cdot)}{\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta} F^\beta(0, 0))\end{aligned}$$

Using equations (16a), (28b) and Assumption 29, we can have:

$$\begin{aligned}& (\tilde{s}^\beta)^T (F^\beta(\cdot) - F^\beta(0, 0) - \frac{\gamma^\beta(\cdot)}{\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta} F^\beta(0, 0)) \\ & \leq \|\tilde{s}\| (l_1^\beta \|K_1^\beta e_1^\beta\| + l_2^\beta \|e_2\|) + \frac{\gamma^\beta(\cdot)}{\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta} \|\tilde{s}\| \|F^\beta(0, 0)\| \\ & \leq \|\tilde{s}\| (l_1^\beta \|K_1^\beta e_1^\beta\| + l_2^\beta \|e_2\|) + \frac{\Delta_0^\beta}{\pi_0^\beta + k_0^\beta \mu^\beta + \Delta_0^\beta} \|\tilde{s}\| (\gamma_1^\beta \|K_1^\beta e_1^\beta\| + \gamma_2^\beta \|e_2\|) \\ & \leq \|\tilde{s}\| (l_1^\beta \|K_1^\beta e_1^\beta\| + l_2^\beta \|e_2\|) + \|\tilde{s}\| (\gamma_1^\beta \|K_1^\beta e_1^\beta\| + \gamma_2^\beta \|e_2\|) \\ & \leq (l_1^\beta + \gamma_1^\beta) \|\tilde{s}\| \|K_1^\beta e_1^\beta\| + (l_2^\beta + \gamma_2^\beta) \|\tilde{s}\| \|e_2\| \\ & \leq \frac{(l_1^\beta + \gamma_1^\beta)}{2} ((\tilde{s}^\beta)^T \tilde{s} + (e_1^\beta)^T (K_1^\beta)^2 e_1^\beta) + \frac{(l_2^\beta + \gamma_2^\beta)}{2} ((\tilde{s}^\beta)^T \tilde{s} + e_2^T e_2) \\ & \leq \frac{(l_1^\beta + \gamma_1^\beta)}{2} ((\tilde{s}^\beta)^T \tilde{s} + (e_1^\beta)^T (K_1^\beta)^2 e_1^\beta) + \frac{(l_2^\beta + \gamma_2^\beta)}{2} ((\tilde{s}^\beta)^T \tilde{s} \\ & \quad + 3((\tilde{s}^\beta)^T \tilde{s} + (k_0^\beta)^2 (\tilde{\sigma}^\beta)^T \tilde{\sigma}^\beta + e_1^\beta (K_1^\beta)^2 e_1^\beta)) \\ & \leq \frac{3(l_2^\beta + \gamma_2^\beta)}{2} (k_0^\beta)^2 (\tilde{\sigma}^\beta)^T \tilde{\sigma}^\beta + \frac{(l_1^\beta + \gamma_1^\beta) + 3(l_2^\beta + \gamma_2^\beta)}{2} ((e_1^\beta)^T (K_1^\beta)^2 e_1^\beta) \\ & \quad + \frac{(l_1^\beta + \gamma_1^\beta) + 4(l_2^\beta + \gamma_2^\beta)}{2} ((\tilde{s}^\beta)^T \tilde{s}) \\ & \leq c_1 (k_0^\beta)^2 (\tilde{\sigma}^\beta)^T \tilde{\sigma}^\beta + c_2 (\tilde{s}^\beta)^T \tilde{s} + c_3 (e_1^\beta)^T (K_1^\beta)^2 e_1^\beta\end{aligned}\quad (34)$$

where $c_1 = \frac{3(l_2^\beta + \gamma_2^\beta)}{2}$ and $c_2 = \frac{(l_1^\beta + \gamma_1^\beta) + 3(l_2^\beta + \gamma_2^\beta)}{2}$ and $c_3 = \frac{(l_1^\beta + \gamma_1^\beta) + 4(l_2^\beta + \gamma_2^\beta)}{2}$.

Using (28) and (34), the derivative of the Lyapunov function is developed:

$$\begin{aligned}
\dot{W} &= -\lambda_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta + \lambda_1(\tilde{\sigma}^\beta)^T k_0^\beta \tilde{s} - \lambda_2(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta + \lambda_2(e_1^\beta)^T K_1^\beta (\tilde{s} - k_0^\beta \tilde{\sigma}^\beta) \\
&\quad + ((\tilde{s}^\beta)^T(k_0^\beta I_n + K_1^\beta)\tilde{s} - (\tilde{s}^\beta)^T(k_0^\beta + K_1^\beta)k_0^\beta \tilde{\sigma}^\beta - (\tilde{s}^\beta)^T(K_1^\beta)^2 e_1^\beta - (\tilde{s}^\beta)^T G^\beta(\cdot)\pi^\beta(\cdot)\tilde{s}^\beta/\mu^\beta) \\
&\quad + (\tilde{s}^\beta)^T(F^\beta(\cdot) - G^\beta(\cdot)\pi^\beta(\cdot)\tilde{s}/\mu^\beta) \\
&\leq -\lambda_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta + \lambda_1/2((\tilde{s}^\beta)^T\tilde{s}^\beta + (k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta) - \lambda_2(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta \\
&\quad + \lambda_2/2((e_1^\beta)^T(K_1^\beta)^2 e_1^\beta + (\tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta)^T(\tilde{s}^\beta - k_0^\beta \tilde{\sigma}^\beta)) + ((\tilde{s}^\beta)^T(k_0^\beta I_n + K_1^\beta)\tilde{s}^\beta \\
&\quad + 1/2((\tilde{s}^\beta)^T(k_0^\beta I_n + K_1^\beta)^2 \tilde{s}^\beta + \lambda_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta) + 1/2((\tilde{s}^\beta)^T(K_1^\beta)^2 \tilde{s}^\beta + (e_1^\beta)^T(K_1^\beta)^2 e_1^\beta) \\
&\quad - (\tilde{s}^\beta)^T G^\beta(\cdot)\pi^\beta(\cdot)\tilde{s}^\beta/\mu^\beta + c_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta + c_2(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta + c_3(\tilde{s}^\beta)^T\tilde{s}^\beta) \\
&\leq -\lambda_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta + \lambda_1/2((\tilde{s}^\beta)^T\tilde{s}^\beta + (k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta) - \lambda_2(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta \\
&\quad + \lambda_2/2((e_1^\beta)^T(K_1^\beta)^2 e_1^\beta + 2((\tilde{s}^\beta)^T\tilde{s}^\beta + (k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta)) + ((\tilde{s}^\beta)^T(k_0^\beta I_n + K_1^\beta)\tilde{s}^\beta \\
&\quad + 1/2((\tilde{s}^\beta)^T(k_0^\beta I_n + K_1^\beta)^2 \tilde{s}^\beta + (k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta) + 1/2((\tilde{s}^\beta)^T(K_1^\beta)^2 \tilde{s}^\beta + (e_1^\beta)^T(K_1^\beta)^2 e_1^\beta) \\
&\quad - (\tilde{s}^\beta)^T G^\beta(\cdot)\pi^\beta(\cdot)\tilde{s}^\beta/\mu^\beta + c_1(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta + c_2(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta + c_3(\tilde{s}^\beta)^T\tilde{s}^\beta) \\
&\leq -(\lambda_1(k_0^\beta)^2 - \lambda_1/2(k_0^\beta)^2 - \lambda_2(k_0^\beta)^2 - 1/2(k_0^\beta)^2 - c_1(k_0^\beta)^2)(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta \\
&\quad - (e_1^\beta)^T(\lambda_2(K_1^\beta)^2 - \lambda_2/2(K_1^\beta)^2 - 1/2(K_1^\beta)^2 - c_2(K_1^\beta)^2)e_1^\beta \\
&\quad - (\tilde{s}^\beta)^T((G^\beta(\cdot)\pi^\beta(\cdot)/\mu^\beta - (k_0^\beta I_n + K_1^\beta) - \lambda_1/2I_n - \lambda_2 I_n \\
&\quad - 1/2(k_0^\beta I_n + K_1^\beta)^2 - 1/2(K_1^\beta)^2 - c_3 I_n)\tilde{s}^\beta) \\
&\leq -(\lambda_1/2 - \lambda_2 - 1/2 - c_1)(k_0^\beta)^2(\tilde{\sigma}^\beta)^T\tilde{\sigma}^\beta - (\lambda_2/2 - 1/2 - c_2)(e_1^\beta)^T(K_1^\beta)^2 e_1^\beta \\
&\quad - (\tilde{s}^\beta)^T((\pi_0^\beta + k_0^\beta \mu^\beta + \gamma^\beta(\cdot) + \Delta_0^\beta)/\mu^\beta - (k_0^\beta I_n + K_1^\beta) - 1/2(k_0^\beta I_n + K_1^\beta)^2 \\
&\quad - 1/2(K_1^\beta)^2 - (\lambda_1/2 + \lambda_2 + c_3)I_n)\tilde{s}^\beta
\end{aligned} \tag{35}$$

It can be verified that by taking $\lambda_1^\beta, \lambda_2^\beta$ and $\pi^\beta(\cdot)$ large enough and μ^β small enough, the derivative of Lyapunov function is negative. We establish the additional design parameter's condition:

$$\begin{cases} \lambda_1/2 - \lambda_2 - 1/2 & > c_1 \\ \lambda_2/2 - 1/2 & > c_2 \\ (\frac{\pi_0^\beta + k_0^\beta \mu^\beta + \gamma^\beta(\cdot) + \Delta_0^\beta}{\mu^\beta}) I_n > ((k_0^\beta I_n + K_1^\beta) - 1/2(k_0^\beta I_n + K_1^\beta)^2 - 1/2(K_1^\beta)^2) \\ & + (\lambda_1/2 + \lambda_2 + c_3)I_n \end{cases} \tag{36}$$

Inequality (36) implies that the design condition of parameter k_0^β and matrix K_1^β must satisfy:

$$\begin{cases} \lambda_1 - 2\lambda_2 & > 1 + 2c_1 \\ \lambda_2 & > 1 + 2c_2 \\ (\frac{\pi_0^\beta + \Delta_0^\beta}{\mu^\beta}) I_n - K_1^\beta - 1/2(k_0^\beta I_n + K_1^\beta)^2 - 1/2(K_1^\beta)^2 > (\frac{\lambda_1}{2} + \lambda_2 + c_3)I_n \end{cases} \tag{37}$$

In this way, $W^\beta(t)$ satisfies $W^\beta(t) > 0$ and $\dot{W}^\beta < -w_0 W^\beta$ (where w_0 is a positive constant) for all $\sigma^\beta \neq \bar{\sigma}^\beta$, $e_1^\beta \neq 0$ and $s^\beta \neq \bar{s}^\beta$. Then $W^\beta(t)$ reaches exponentially zero when time tends to infinite. As consequence, the output error $e_1^\beta(t)$ tends to zero, σ^β and s^β tend to their equilibrium values as time tends to infinite. We may assure the exponential stability of the system in the region of $\|s^\beta\| \leq \mu^\beta$.

□

3.3 Longitudinal control design

As in the case of lateral control design, in the longitudinal case it is considered that only longitudinal state variables are time varying. It is a single input single output system where angle of attack is the output and elevator is the input. Aerodynamic forces F_x , F_z and moment M can be calculated by its aerodynamic coefficients (see more in [6]).

$$F_x = (C_x(\alpha) + \bar{c}C_{x_q}(\alpha)q/(2V))\bar{q}S$$

$$F_z = (C_z(\alpha, \beta) + \bar{c}C_{z_q}(\alpha)q/(2V))\bar{q}S$$

$$M = (C_m(\alpha) + C_{m_q}(\alpha)q\bar{c}/(2V) + C_{m_{\delta_e}}(\alpha)\delta_e)\bar{q}S\bar{c}$$

By replacing F_x , F_z , moment M and $\beta = 0$, $\phi = 0$, $p = 0$, $r = 0$ in (1), the model for longitudinal dynamic can be written as:

$$\begin{cases} \dot{\alpha} = \frac{1}{mV} [-\sin \alpha(T + C_x(\alpha)\bar{q}S) + \cos \alpha C_z(\alpha)\bar{q}S] + q + \frac{\rho S}{4m} (-\sin \alpha C_{x_q}(\alpha)\bar{c} \\ \quad + \cos \alpha C_{z_q}(\alpha)\bar{c})q + \frac{g}{V} \cos(\theta - \alpha) \\ \dot{q} = I_7 \bar{q} S (C_m(\alpha)\bar{c} + C_{m_q}(\alpha)\bar{c}q + C_{m_{\delta_e}}(\alpha)\bar{c}\delta_e) \\ \dot{\theta} = q \end{cases} \quad (38)$$

in which \bar{c} mean aerodynamic chord, $I_7 = 1/I_{yy}$, $C_x(\alpha)$, $C_{x_q}(\alpha)$, $C_z(\alpha)$, $C_{z_q}(\alpha)$, $C_m(\alpha)$, $C_{m_q}(\alpha)$, $C_{m_{\delta_e}}(\alpha)$ are aerodynamic coefficients taken from [8].

Equation (38) can be rearranged as:

$$\begin{cases} \dot{\theta} &= q \\ \dot{\alpha} &= f_{11}^\alpha(\alpha) + (1 + f_{12}^\alpha(\alpha))q + f_{13}^\alpha(\alpha, \theta) \\ \dot{q} &= f_{21}^\alpha(\alpha) + f_{22}^\alpha(\alpha)q + g_2^\alpha(\alpha)\delta_e \end{cases} \quad (39)$$

where $f_{11}^\alpha(\alpha)$, $f_{12}^\alpha(\alpha)$, $f_{13}^\alpha(\alpha, \theta)$, $f_{21}^\alpha(\alpha)$, $f_{22}^\alpha(\alpha)$ and $g_2^\alpha(\alpha)$ represent the terms of (38) respectively (see Appendix B).

Let us define $x_1^\alpha = \alpha$, $x_2^\alpha = \dot{x}_1^\alpha = \dot{\alpha}$ and $u^\alpha = \delta_e$, which allow us to rewrite (39) into:

$$\dot{\theta} = \eta^\alpha(x_1^\alpha, x_2^\alpha, \theta) \quad (40a)$$

$$\dot{x}_1^\alpha = x_2^\alpha$$

$$\dot{x}_2^\alpha = F^{\alpha'}(x_1^\alpha, x_2^\alpha) + h'(x_1^\alpha, x_2^\alpha, \theta) + G^{\alpha'}(x_1^\alpha, x_2^\alpha)u^\alpha \quad (40b)$$

where

$$\begin{cases} \eta^\alpha(\cdot) = (x_2^\alpha - f_{11}^\alpha(x_1^\alpha) - f_{13}^\alpha(x_1^\alpha, \theta))/(1 + f_{12}^\alpha(x_1^\alpha)) \\ F^{\alpha'}(\cdot) = \frac{\partial f_{11}^\alpha(x_1^\alpha)}{\partial x_1^\alpha} x_2^\alpha + (1 + f_{12}^\alpha(x_1^\alpha))f_{21}^\alpha(x_1^\alpha) + \frac{\partial(1+f_{12}^\alpha(x_1^\alpha))}{\partial x_1^\alpha} x_2^\alpha \\ \quad + (1 + f_{12}^\alpha(x_1^\alpha))f_{22}^\alpha(x_1^\alpha) \frac{(x_2^\alpha - f_{11}^\alpha(x_1^\alpha))}{(1 + f_{12}^\alpha(x_1^\alpha))} \\ h'(\cdot) = \frac{\partial f_{12}^\alpha(x_1^\alpha)}{\partial x_1^\alpha} x_2^\alpha \frac{(-f_{13}^\alpha(x_1^\alpha, \theta))}{(1 + f_{12}^\alpha(x_1^\alpha))} - f_{13}^\alpha(x_1^\alpha, \theta)f_{22}^\alpha(x_1^\alpha) + \frac{\partial f_{13}^\alpha(x_1^\alpha, \theta)}{\partial x_1^\alpha} x_2^\alpha \\ \quad + \frac{\partial f_{13}^\alpha(x_1^\alpha, \theta)}{\partial \theta} \frac{(x_2^\alpha - f_{11}^\alpha(x_1^\alpha) - f_{13}^\alpha(x_1^\alpha, \theta))}{(1 + f_{12}^\alpha(x_1^\alpha))} \\ G^{\alpha'}(\cdot) = (1 + f_{12}^\alpha(x_1^\alpha))g_2^\alpha(x_1^\alpha) \end{cases} \quad (41)$$

In (40b), $F^\alpha(\cdot)$ is function of x_1^α and x_2^α . Function $h'(\cdot)$ is function of x_1^α , x_2^α and θ , it can be expressed as:

$$h'(\cdot) = -\frac{g}{V} \frac{\cos(\theta-\alpha)}{(1+f_{12}^\alpha(x_1^\alpha))} \frac{\partial f_{12}^\alpha(x_1^\alpha)}{\partial x_1^\alpha} x_2^\alpha - \frac{g}{V} \cos(\theta-\alpha) f_{22}^\alpha(x_1^\alpha) + \frac{g}{V} \sin(\theta-\alpha) x_2^\alpha \frac{f_{12}^\alpha(x_1^\alpha)}{(1+f_{12}^\alpha(x_1^\alpha))} + \left(\frac{g}{V}\right)^2 \frac{\sin(\theta-\alpha) \cos(\theta-\alpha)}{(1+f_{12}^\alpha(x_1^\alpha))} + \frac{g}{V} \frac{\sin(\theta-\alpha) f_{11}^\alpha(x_1^\alpha)}{(1+f_{12}^\alpha(x_1^\alpha))} \quad (42)$$

Remark 2 We have some remarks:

- f_{11}^α , f_{12}^α , f_{13}^α , f_{21}^α , f_{22}^α and g_2^α are function of aerodynamic coefficients under analytical forms by interpolation from wind tunnel test data. F^α and h' , formed from these functions, can be then bounded by a class \mathcal{K} function and be a Lipschitz function in a flying envelop.
- $G^{\alpha'}$ is invertible in the flight domain, that means $\alpha \in (-10^\circ, 45^\circ)$ and $\theta \in (-90^\circ, 90^\circ)$.

As a consequence, they fulfill Assumptions 3.1 and 3.2.

We define now the reference for the angle of attack α_{ref} considered as constant in this study, and the error vector of angle of attack $e_1^\alpha = x_1^\alpha - x_{1ref}^\alpha = \alpha - \alpha_{ref}$ and the variable $e_2^\alpha = \dot{e}_1^\alpha$. Equation (40b) can be transformed into:

$$\begin{cases} \dot{e}_1^\alpha = e_2^\alpha & (43a) \\ \dot{e}_2^\alpha = F^\alpha(e_1^\alpha, e_2^\alpha) + h(e_1^\alpha, e_2^\alpha, \theta) + G^\alpha(e_1^\alpha, e_2^\alpha)u^\alpha & (43b) \end{cases}$$

Since function $h(\cdot)$ depends on θ under cosines and sinus functions, it is easy to show that $h(\cdot)$ is bounded by a function of class \mathcal{K} $\gamma_1^\alpha(\cdot)$ and a constant H_0 .

$$|h(e_1^\alpha, e_2^\alpha, \theta)| \leq \gamma_1^\alpha(|e_1^\alpha| + |e_2^\alpha|) + H_0 \quad (44)$$

Here we remark that $G^\alpha(x_1^\alpha, x_2^\alpha)$ is invertible for the entire domain of attraction of the system. It is important to remark that pitch angle θ is not included in the considered system, and is left free in the analysis that follows. The reason for this is to acknowledge the possibility of airlaunching under different possible angles θ , and even under a looping like trajectory that naturally brings the aircraft away of the rocket.

Applying the control algorithm presented in (10) for system (43) which in this case is a nonlinear single input single output system, gives the controller:

$$\begin{cases} u^\alpha & = -\pi^\alpha(e_1^\alpha, e_2^\alpha) \text{sat}(s^\alpha/\mu^\alpha) \\ \pi^\alpha(\cdot) & = (\pi_0^\alpha + \gamma^\alpha(\cdot) + k_0^\alpha \mu^\alpha + \Delta_0^\alpha)(G^\alpha(\cdot))^{-1} \end{cases} \quad (45)$$

with

$$\begin{cases} s^\alpha = k_0^\alpha \sigma^\alpha + K_1^\alpha e_1^\alpha + e_2^\alpha \\ \dot{\sigma}^\alpha = -k_0^\alpha \sigma^\alpha + \mu^\alpha \text{sat}(s^\alpha/\mu^\alpha) \end{cases} \quad (46)$$

where π_0^α , μ^α , K_1^α and k_0^α are positive constants.

◇

Theorem 3.2 System (43) with $F^\alpha(\cdot)$ satisfying assumption 3.1, $G^\alpha(\cdot)$ which satisfies assumption 3.2, and the control law (45-46), will globally reach an arbitrary error region in finite time, and there on will be exponentially stabilized towards its equilibrium point.

□

Proof: In order to demonstrate the exponential stability of designed controller (45) and (46) for the longitudinal mode in (43), we will consider in two regions outside the boundary layer ($|s^\alpha| \geq \mu^\alpha$) and inside the boundary layer ($|s^\alpha| \leq \mu^\alpha$). The longitudinal mode in this study is a single input single output system.

3.3.1 In the region $|s^\alpha| \geq \mu^\alpha$, $\text{sat}(s^\alpha/\mu^\alpha) = s^\alpha/|s^\alpha|$

The derivative of the integral error measurement surface can be then expressed as:

$$\dot{s}^\alpha = k_0^\alpha \dot{\sigma}^\alpha + k_1^\alpha \dot{e}_1^\alpha + \dot{e}_2^\alpha \quad (47)$$

From (43) and (46), the previous equation may be written again :

$$\begin{aligned} \dot{s}^\alpha &= k_0^\alpha (-k_0^\alpha \sigma^\alpha + \mu^\alpha \text{sat}(s^\alpha/\mu^\alpha)) + k_1^\alpha \dot{e}_2^\alpha + \dot{e}_2^\alpha \\ &= k_0^\alpha (-(s^\alpha - (k_1^\alpha e_1^\alpha + e_2^\alpha)) + \mu^\alpha \text{sat}(s^\alpha/\mu^\alpha)) + k_1^\alpha \dot{e}_2^\alpha + \dot{e}_2^\alpha \\ &= -k_0^\alpha s^\alpha + k_0^\alpha \mu^\alpha \text{sat}(s^\alpha/\mu^\alpha) + k_0^\alpha (k_1^\alpha e_1^\alpha + e_2^\alpha) + k_1^\alpha \dot{e}_2^\alpha + f^\alpha(\cdot) + g^\alpha(\cdot) u^\alpha \end{aligned}$$

Now by letting

$$\delta^\alpha(\cdot) = k_0^\alpha (k_1^\alpha e_1^\alpha + e_2^\alpha) + k_1^\alpha \dot{e}_2^\alpha + f^\alpha(\cdot) + h(\cdot) \quad (48)$$

The derivative of the integral error measurement surface becomes:

$$\dot{s}^\alpha = -k_0^\alpha s^\alpha + k_0^\alpha \mu^\alpha \text{sat}(s^\alpha/\mu^\alpha) + \delta^\alpha(\cdot) + g^\alpha(\cdot) u^\alpha \quad (49)$$

The term $f^\alpha(x_1^\alpha, x_2^\alpha)$ is bounded by a function $\gamma_2^\alpha(|e_1^\alpha| + |e_2^\alpha|)$ outside the boundary region e.g. $|s^\alpha| \geq \mu^\alpha$ and a positive constant f_0^α , where $\gamma_2^\alpha(\cdot)$ is a class \mathcal{K} function.

$$|f^\alpha(e_1^\alpha, e_2^\alpha)| \leq \gamma_2^\alpha(|e_1^\alpha| + |e_2^\alpha|) + f_0^\alpha \quad (50)$$

Function $h(\cdot)$ is also bounded in the same way (see (44)). Then $\delta^\alpha(\cdot)$ is bounded by a $\gamma^\alpha(\cdot)$ class \mathcal{K} function and a positive constant δ_0^α :

$$\begin{aligned} |\delta^\alpha(e_1^\alpha, e_2^\alpha)| &\leq \gamma_1^\alpha(|e_1^\alpha| + |e_2^\alpha|) + h_0 + \gamma_2^\alpha(|e_1^\alpha| + |e_2^\alpha|) + f_0^\alpha \\ |\delta^\alpha(e_1^\alpha, e_2^\alpha)| &\leq \gamma^\alpha(|e_1^\alpha| + |e_2^\alpha|) + \delta_0^\alpha \end{aligned} \quad (51)$$

where $\gamma^\alpha(\cdot) = \gamma_1^\alpha(\cdot) + \gamma_2^\alpha(\cdot)$ and $\delta_0^\alpha = h_0 + f_0^\alpha$
and as a consequence,

$$|\delta^\alpha(e_1^\alpha = 0, e_2^\alpha = 0)| = |f^\alpha(0, 0)| \leq \delta_0^\alpha \quad (52)$$

for $(e_1^\alpha, e_2^\alpha) \in \mathbb{R}^n \times \mathbb{R}^n$.

Let's consider the product $s^\alpha \dot{s}^\alpha$ (since s is a scalar)

$$s^\alpha \dot{s}^\alpha = -k_0^\alpha (s^\alpha)^2 + k_0^\alpha \mu^\alpha s^\alpha \text{sat}(s^\alpha / \mu^\alpha) + s^\alpha \delta^\alpha(e_1^\alpha, e_2^\alpha) + s^\alpha g^\alpha(e_1^\alpha, e_2^\alpha) u^\alpha \quad (53)$$

This product $(s^\alpha)^T \dot{s}^\alpha$ can be developed with the definition of saturation function (5):

$$\begin{aligned} s^\alpha \dot{s}^\alpha &= -k_0^\alpha (s^\alpha)^2 + \mu^\alpha k_0^\alpha (s^\alpha)^2 / |s^\alpha| + s^\alpha \delta^\alpha(\cdot) - s^\alpha g^\alpha(\cdot) \pi^\alpha(\cdot) s^\alpha / |s^\alpha| \\ &\leq -k_0^\alpha (s^\alpha)^2 + \mu^\alpha k_0^\alpha (s^\alpha)^2 / |s^\alpha| + |\delta^\alpha(\cdot)| |s^\alpha| - s^\alpha g^\alpha(\cdot) \pi^\alpha(\cdot) s^\alpha / |s^\alpha| \\ &\leq -k_0^\alpha (s^\alpha)^2 + \mu^\alpha k_0^\alpha (s^\alpha)^2 / |s^\alpha| + (\gamma^\alpha(\cdot) + \delta_0^\alpha) |s^\alpha| - s^\alpha g^\alpha(\cdot) \pi^\alpha(\cdot) s^\alpha / |s^\alpha| \\ &\leq -k_0^\alpha (s^\alpha)^2 - s^\alpha (g^\alpha(\cdot) \pi^\alpha(\cdot) - (\mu^\alpha k_0^\alpha + \gamma^\alpha(\cdot) + \delta_0^\alpha)) s^\alpha / |s^\alpha| \end{aligned} \quad (54)$$

Replacing the control law in (45) and (46), the term $s^\alpha \dot{s}^\alpha$ can be expressed as:

$$\begin{aligned} s^\alpha \dot{s}^\alpha &\leq -k_0^\alpha (s^\alpha)^2 - s^\alpha (g^\alpha(\cdot) \pi^\alpha(\cdot) - (\mu^\alpha k_0^\alpha + \gamma^\alpha(\cdot) + \delta_0^\alpha)) s^\alpha / |s^\alpha| \\ &\leq -k_0^\alpha (s^\alpha)^2 - \pi_0^\alpha (s^\alpha)^2 / |s^\alpha| \\ &\leq -k_0^\alpha (s^\alpha)^2 - \pi_0^\alpha |s^\alpha| \end{aligned} \quad (55)$$

The product $s^\alpha \dot{s}^\alpha$ is then not positive and we have also

$$\begin{aligned} \frac{d((s^\alpha)^2)}{dt} &= 2|s^\alpha| \frac{d(|s^\alpha|)}{dt} = 2 \frac{s^\alpha \dot{s}^\alpha}{dt} \leq 2(-\pi_0^\alpha |s^\alpha| - k_0^\alpha (s^\alpha)^2) \\ \therefore \frac{d(|s^\alpha|)}{dt} &\leq -\pi_0^\alpha - k_0^\alpha |s^\alpha| \\ \therefore |s^\alpha(t)| &\leq |s^\alpha(0)| - \pi_0^\alpha t - |s^\alpha(0)| (e^\alpha)^{-k_0^\alpha t} - 1 \end{aligned} \quad (56)$$

Then the integral error measurement surface $s^\alpha(t)$ reaches the boundary layer μ^α in finite time.

3.3.2 In the region $|s| \leq \mu^\alpha$, $\text{sat}(s/\mu^\alpha) = s/\mu^\alpha$.

Consider again (46) and (49), which inside the boundary layer may be rewritten as:

$$\dot{\sigma}^\alpha = -k_0^\alpha \sigma^\alpha + s^\alpha \quad (57a)$$

$$\dot{e}_1^\alpha = -k_1^\alpha e_1^\alpha + s^\alpha - k_0^\alpha \sigma^\alpha \quad (57b)$$

$$\dot{s}^\alpha = \delta^\alpha(\cdot) - g^\alpha(\cdot) \pi^\alpha(\cdot) s^\alpha / \mu^\alpha \quad (57c)$$

It can be shown that this system has an equilibrium point: $\bar{e}_1^\alpha = \bar{e}_2^\alpha = 0$, $s^\alpha = \bar{s}^\alpha$, $\sigma^\alpha = \bar{\sigma}^\alpha$ with $\bar{s} = k_0^\alpha \bar{\sigma}^\alpha = \mu^\alpha (\pi^\alpha(0,0))^{-1} (g^\alpha(0,0))^{-1} (f^\alpha(0,0) + h(0,0,\theta)) = (f^\alpha(0,0) + h(0,0,\theta)) / (\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)$.

System (57) may be rewritten with respect to \bar{s}^α and $\bar{\sigma}^\alpha$:

$$\begin{cases} \dot{\tilde{\sigma}}^\alpha = -k_0^\alpha \tilde{\sigma}^\alpha + \tilde{s}^\alpha & (58a) \\ \dot{e}_1^\alpha = -k_1^\alpha e_1^\alpha + \tilde{s}^\alpha - k_0^\alpha \tilde{\sigma}^\alpha & (58b) \\ \dot{\tilde{s}}^\alpha = \delta^\alpha(\cdot) - g^\alpha(\cdot) \pi^\alpha(\cdot) \tilde{s}^\alpha / \mu^\alpha - g^\alpha(\cdot) \pi^\alpha(\cdot) \bar{s}^\alpha / \mu^\alpha & (58c) \end{cases}$$

where $\tilde{\sigma}^\alpha = \sigma^\alpha - \bar{\sigma}^\alpha$, $\tilde{s}^\alpha = s^\alpha - \bar{s}^\alpha$.

Function $f^\alpha(x_1^\alpha, x_2^\alpha)$ is Lipschitz inside the boundary region e.g. $|s^\alpha| \leq \mu^\alpha$ (see remark 2), such that:

$$|f^\alpha(e_1^\alpha, e_2^\alpha) - f^\alpha(0,0)| \leq l_1^\alpha |k_1^\alpha e_1^\alpha| + l_2^\alpha |e_2^\alpha| \quad (59)$$

where l_1^α and $l_2^\alpha \in \mathbb{R}^+$.

$\gamma_2^\alpha(\cdot)$ and $\gamma_1^\alpha(\cdot)$ is chosen so that are differentiable, then Lipschitz. As a consequence, function $\gamma^\alpha(\cdot)$ is then Lipschitz.

$$\gamma^\alpha(\cdot) \leq d_1^\alpha |k_1^\alpha e_1^\alpha| + d_2^\alpha |e_2^\alpha| \quad (60)$$

where d_1^α and $d_2^\alpha \in \mathbb{R}^+$.

We would like to demonstrate that every trajectory of system (58) starting inside the boundary layer, will approach the equilibrium point as time tends to infinity when the control law (45) is applied. Toward that end, we take:

$$W^\alpha = \frac{\lambda_1^\alpha}{2} k_0^\alpha (\tilde{\sigma}^\alpha)^2 + \frac{\lambda_2^\alpha}{2} k_1^\alpha (e_1^\alpha)^2 + \frac{(\tilde{s}^\alpha)^2}{2} \quad (61)$$

as a Lyapunov candidate, where λ_1^α and λ_2^α are positive constants.

Since $|s^\alpha| \leq \mu^\alpha$, $\delta^\alpha(\cdot)$ can be expressed as:

$$\delta^\alpha(\cdot) = (k_0^\alpha + k_1^\alpha) \tilde{s}^\alpha - (k_0^\alpha + k_1^\alpha) k_0^\alpha \tilde{\sigma}^\alpha - (k_1^\alpha)^2 e_1^\alpha + f^\alpha(\cdot) + h(\cdot) \quad (62)$$

From (58) and reminding that $\bar{s} = (f^\alpha(0,0) + h(0,0,\theta)) / (\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)$ and $g^\alpha(\cdot) \pi^\alpha(\cdot) = (\pi_0^\alpha + k_0^\alpha \mu^\alpha + \gamma^\alpha(\cdot) + \delta_0^\alpha)$, the derivative of the Lyapunov

function is developed:

$$\begin{aligned}
\dot{W}^\alpha &= \lambda_1^\alpha k_0^\alpha \tilde{\sigma}^\alpha \dot{\tilde{\sigma}}^\alpha + \lambda_2^\alpha k_1^\alpha e_1^\alpha \dot{e}_1^\alpha + \tilde{s}^\alpha \dot{\tilde{s}}^\alpha \\
&= \lambda_1^\alpha \tilde{\sigma}^\alpha k_0^\alpha (-k_0^\alpha \tilde{\sigma}^\alpha + \tilde{s}^\alpha) + \lambda_2^\alpha k_1^\alpha e_1^\alpha (-k_1^\alpha e_1^\alpha + \tilde{s}^\alpha - k_0^\alpha \tilde{\sigma}^\alpha) \\
&\quad + \tilde{s}^\alpha (\delta^\alpha(\cdot) - g^\alpha(\cdot)) \pi^\alpha(\cdot) \tilde{s}^\alpha / \mu^\alpha - g^\alpha(\cdot) \pi^\alpha(\cdot) \tilde{s}^\alpha / \mu^\alpha \\
&= -\lambda_1^\alpha (k_0^\alpha \tilde{\sigma}^\alpha)^2 + \lambda_1^\alpha k_0^\alpha \tilde{\sigma}^\alpha \tilde{s}^\alpha - \lambda_2^\alpha (k_1^\alpha e_1^\alpha)^2 + \lambda_2^\alpha k_1^\alpha e_1^\alpha (\tilde{s}^\alpha - k_0^\alpha \tilde{\sigma}^\alpha) \\
&\quad + (k_0^\alpha + k_1^\alpha) (\tilde{s}^\alpha)^2 - (k_0^\alpha + k_1^\alpha) k_0^\alpha \tilde{s}^\alpha \tilde{\sigma}^\alpha - (k_1^\alpha)^2 e_1^\alpha \tilde{s}^\alpha - g^\alpha(\cdot) \pi^\alpha(\cdot) / \mu^\alpha (\tilde{s}^\alpha)^2 \\
&\quad + \tilde{s}^\alpha (f^\alpha(\cdot) + h(\cdot) - g^\alpha(\cdot)) \pi^\alpha(\cdot) \tilde{s}^\alpha / \mu^\alpha \\
&= -\lambda_1^\alpha (k_0^\alpha \tilde{\sigma}^\alpha)^2 + \lambda_1^\alpha k_0^\alpha \tilde{\sigma}^\alpha \tilde{s}^\alpha - \lambda_2^\alpha (k_1^\alpha e_1^\alpha)^2 + \lambda_2^\alpha k_1^\alpha e_1^\alpha (\tilde{s}^\alpha - k_0^\alpha \tilde{\sigma}^\alpha) \\
&\quad + (k_0^\alpha + k_1^\alpha) (\tilde{s}^\alpha)^2 - (k_0^\alpha + k_1^\alpha) k_0^\alpha \tilde{s}^\alpha \tilde{\sigma}^\alpha - (k_1^\alpha)^2 e_1^\alpha \tilde{s}^\alpha - g^\alpha(\cdot) \pi^\alpha(\cdot) / \mu^\alpha (\tilde{s}^\alpha)^2 \\
&\quad + \tilde{s}^\alpha (f^\alpha(\cdot) + h(\cdot) - \frac{\pi_0^\alpha + k_0^\alpha \mu^\alpha + \gamma(\cdot) + \delta_0^\alpha}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} (f^\alpha(0, 0) + h(0, 0, \theta))) \\
&= -\lambda_1^\alpha (k_0^\alpha \tilde{\sigma}^\alpha)^2 + \lambda_1^\alpha k_0^\alpha \tilde{\sigma}^\alpha \tilde{s}^\alpha - \lambda_2^\alpha (k_1^\alpha e_1^\alpha)^2 + \lambda_2^\alpha k_1^\alpha e_1^\alpha (\tilde{s}^\alpha - k_0^\alpha \tilde{\sigma}^\alpha) \\
&\quad + (k_0^\alpha + k_1^\alpha) (\tilde{s}^\alpha)^2 - (k_0^\alpha + k_1^\alpha) k_0^\alpha \tilde{s}^\alpha \tilde{\sigma}^\alpha - (k_1^\alpha)^2 e_1^\alpha \tilde{s}^\alpha - g^\alpha(\cdot) \pi^\alpha(\cdot) / \mu^\alpha (\tilde{s}^\alpha)^2 \\
&\quad + \tilde{s}^\alpha (f^\alpha(\cdot) - f^\alpha(0, 0) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} f^\alpha(0, 0)) \\
&\quad + \tilde{s}^\alpha (h^\alpha(\cdot) - h^\alpha(0, 0, \theta) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} h^\alpha(0, 0, \theta))
\end{aligned} \tag{63}$$

Using equations (43a) and (58b), the relation in (59) and in (60) can be expressed as:

$$\begin{aligned}
&\tilde{s}^\alpha (f^\alpha(e_1^\alpha, e_2^\alpha) - f^\alpha(0, 0) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} f^\alpha(0, 0)) \\
&\leq |\tilde{s}^\alpha| (|f^\alpha(e_1^\alpha, e_2^\alpha) - f^\alpha(0, 0)| + |\frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} f^\alpha(0, 0)|) \\
&\leq |\tilde{s}^\alpha| (l_1^\alpha |k_1^\alpha e_1^\alpha| + l_2^\alpha |e_2^\alpha| + d_1^\alpha |k_1^\alpha e_1^\alpha| + d_2^\alpha |e_2^\alpha|) \\
&\leq (d_1^\alpha + l_1^\alpha) |e_1^\alpha| |\tilde{s}^\alpha| + (d_2^\alpha + l_2^\alpha) |e_2^\alpha| |\tilde{s}^\alpha| \\
&\leq 1/2(d_1^\alpha + l_1^\alpha) ((k_1^\alpha e_1^\alpha)^2 + (\tilde{s}^\alpha)^2) + 1/2(d_2^\alpha + l_2^\alpha) ((\tilde{s}^\alpha)^2 \\
&\quad + 3((\tilde{s}^\alpha)^2 + ((k_1^\alpha e_1^\alpha)^2 + (k_0^\alpha \tilde{\sigma}^\alpha)^2)) \\
&\leq c_1 (k_0^\alpha \tilde{\sigma}^\alpha)^2 + c_2 (k_1^\alpha e_1^\alpha)^2 + c_3 (\tilde{s}^\alpha)^2
\end{aligned} \tag{64}$$

where $c_1 = 3/2(d_2^\alpha + l_2^\alpha)$, $c_2 = 1/2(d_1^\alpha + l_1^\alpha) + 3/2(d_2^\alpha + l_2^\alpha)$ and $c_3 = 1/2(d_1^\alpha + l_1^\alpha) + 2(d_2^\alpha + l_2^\alpha)$. and then,

$$\begin{aligned}
\dot{W}^\alpha &\leq -\lambda_1^\alpha (k_0^\alpha \tilde{\sigma}^\alpha)^2 + 1/2 \lambda_1^\alpha ((k_0^\alpha \tilde{\sigma}^\alpha)^2 + (\tilde{s}^\alpha)^2) - \lambda_2^\alpha (k_1^\alpha e_1^\alpha)^2 \\
&\quad + 1/2 \lambda_2^\alpha ((k_1^\alpha e_1^\alpha)^2 + 2((\tilde{s}^\alpha)^2 + (k_0^\alpha \tilde{\sigma}^\alpha)^2)) + (k_0^\alpha + k_1^\alpha) (\tilde{s}^\alpha)^2 \\
&\quad + 1/2 ((k_0^\alpha \tilde{\sigma}^\alpha)^2 + ((k_0^\alpha + k_1^\alpha) \tilde{s}^\alpha)^2) + 1/2 ((k_1^\alpha e_1^\alpha)^2 + (k_1^\alpha \tilde{s}^\alpha)^2) - g^\alpha(\cdot) \pi^\alpha(\cdot) / \mu^\alpha (\tilde{s}^\alpha)^2 \\
&\quad + c_1 (k_0^\alpha \tilde{\sigma}^\alpha)^2 + c_2 (k_1^\alpha e_1^\alpha)^2 + c_3 (\tilde{s}^\alpha)^2 + |h^\alpha(\cdot) - h^\alpha(0, 0, \theta) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} h^\alpha(0, 0, \theta)| |\tilde{s}^\alpha| \\
&\leq -(1/2 \lambda_1^\alpha - \lambda_2^\alpha - 1/2 - c_1) (k_0^\alpha \tilde{\sigma}^\alpha)^2 - (1/2 \lambda_2^\alpha - 1/2 - c_2) (k_1^\alpha e_1^\alpha)^2 \\
&\quad - (g^\alpha(\cdot) \pi^\alpha(\cdot) / \mu^\alpha - 1/2 \lambda_1^\alpha - \lambda_2^\alpha - (k_0^\alpha + k_1^\alpha) - 1/2 (k_0^\alpha + k_1^\alpha)^2 - 1/2 (k_1^\alpha)^2 - c_3) (\tilde{s}^\alpha)^2 \\
&\quad + |h^\alpha(\cdot) - h^\alpha(0, 0, \theta) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} h^\alpha(0, 0, \theta)| |\tilde{s}^\alpha|
\end{aligned} \tag{65}$$

Because inside the boundary region e.g. $|s^\alpha| \leq \mu^\alpha$, functions $h(\cdot)$ and $\gamma^\alpha(\cdot)$ are bounded, the following term is bounded by a positive constant $C\mu$:

$$|h^\alpha(\cdot) - h^\alpha(0, 0, \theta) - \frac{\gamma^\alpha(\cdot)}{(\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)} h^\alpha(0, 0, \theta)| |\tilde{s}^\alpha| \leq C\mu \tag{66}$$

For any value of θ , it is very important to remark that this variable was not included in this analysis, even if its derivative is. We will discuss this point

again later, but the main reason is that the airlaunch may be performed under any pitch angle θ , and even under a looping-like trajectory. From (65), the derivative of the Lyapunov function can be further developed:

$$\begin{aligned}
\dot{W}^\alpha &\leq -(1/2\lambda_1^\alpha - \lambda_2^\alpha - 1/2 - c_1)(k_0^\alpha \tilde{\sigma}^\alpha)^2 - (1/2\lambda_2^\alpha - 1/2 - c_2)(k_1^\alpha e_1^\alpha)^2 \\
&\quad - (g^\alpha(\cdot)\pi^\alpha(\cdot)/\mu^\alpha - 1/2\lambda_1^\alpha - \lambda_2^\alpha - (k_0^\alpha + k_1^\alpha) - 1/2(k_0^\alpha + k_1^\alpha)^2 - 1/2(k_1^\alpha)^2 - c_3)(\tilde{s}^\alpha)^2 \\
&\quad + C\mu \\
&\leq -\frac{\lambda_1^\alpha w_1}{2} k_0^\alpha (\tilde{\sigma}^\alpha)^2 - \frac{\lambda_2^\alpha w_2}{2} k_1^\alpha (e_1^\alpha)^2 - \frac{w_3}{2} (\tilde{s}^\alpha)^2 + C\mu \\
&\leq w_0 \left(-\frac{\lambda_1^\alpha w_1}{2w_0} (\tilde{\sigma}^\alpha)^2 - \frac{\lambda_2^\alpha w_2}{2w_0} (e_1^\alpha)^2 - \frac{w_3}{2w_0} (\tilde{s}^\alpha)^2 \right) + C\mu \\
&\leq w_0 \left(-\frac{\lambda_1^\alpha}{2} (\tilde{\sigma}^\alpha)^2 - \frac{\lambda_2^\alpha}{2} (e_1^\alpha)^2 - \frac{1}{2} (\tilde{s}^\alpha)^2 \right) + C\mu \\
&\leq -w_0 W^\alpha + C\mu
\end{aligned} \tag{67}$$

where

$$\begin{cases} w_1 = (1 - \frac{2\lambda_2^\alpha + 1/2 + 2c_1}{\lambda_1^\alpha}) k_0^\alpha \\ w_2 = (1 - \frac{1+2c_2}{\lambda_2^\alpha}) k_1^\alpha \\ w_3 = 2((\pi_0^\alpha + k_0^\alpha \mu^\alpha + \delta_0^\alpha)/\mu^\alpha - 1/2\lambda_1^\alpha - \lambda_2^\alpha - (k_0^\alpha + k_1^\alpha) - 1/2(k_0^\alpha + k_1^\alpha)^2 - 1/2(k_1^\alpha)^2 - c_3) \\ w_0 = \min(w_1, w_2, w_3) \end{cases} \tag{68}$$

The derivative of Lyapunov function negative implies that $\lambda_1^\alpha, \lambda_2^\alpha$ and π_0^α large enough and μ^α small enough, to satisfy:

$$\begin{cases} 1 > \frac{2\lambda_2^\alpha + 1/2 + 2c_1}{\lambda_1^\alpha} \\ 1 > \frac{1+2c_2}{\lambda_2^\alpha} \\ (\pi_0^\alpha + \delta_0^\alpha)/\mu^\alpha - k_1^\alpha - 1/2(k_0^\alpha + k_1^\alpha)^2 - 1/2(k_1^\alpha)^2 > 1/2\lambda_1^\alpha + \lambda_2^\alpha + c_3 \end{cases} \tag{69}$$

In this way, $W^\alpha(t)$ satisfies $W^\alpha(t) > 0$ and $\dot{W}^\alpha < -w_0 W^\alpha + C\mu$ (where w_0 is a positive constant) for all $\sigma^\alpha \neq \bar{\sigma}^\alpha$, $e_1^\alpha \neq 0$ and $s^\alpha \neq \bar{s}^\alpha$. Then $W^\alpha(t)$ is ultimately bounded towards a neighborhood of zero when time tends to infinite, independent of the pitch angle θ . As consequence, $e_1^\alpha(t)$, σ^α and s^α tend to a region around their equilibrium point as time tends to infinite and for every pitch angle θ . The system is then said to be ultimately bounded to a small region which is function of pitch angle θ . Finally, it is interesting to remark that, as said before, variable θ is left free in order to allow situations as a looping, where θ is continuously varying. Its derivative on the other hand is bounded, and also goes to a residual set. In fact, the best trajectory for airlaunch is still an open problem. There are proposals of launching in horizontal, constant angle (climbing), concave or convex (zero gravity instant) trajectories.

□

Since the airspeed control is only a secondary objective, we design a simple PI controller for the thrust to regulate airspeed. Its form is:

$$T = -k_P(V - V_{ref}) - k_I(\dot{V} - \dot{V}_{ref})$$

where V_{ref} is the airspeed reference, $k_P = 1242$ and $k_I = 955$.

4 Simulation Results

In section 3, the design methodology of the conditional integrator controller to stabilize the angle of attack, sideslip and roll angle is proposed when full knowledge of the aerodynamic characteristics is available. This section presents numerical simulation results for the controller to demonstrate the performance of the proposed conditional integrator control laws in the drop phase.

As mentioned in section 2, we have considered the launch phase as impulses on aerodynamic force and moments during a time interval T_{int} , and that the model following the launch phase is that of an F-16. This model is used since it has already been applied for (manned) airlaunch, and because its nonlinear model, wind tunnel informations and data are widely known and used for control design. It is important to remark that the model used in the following simulations is even more complete than that used in the control design, for example it includes actuator dynamics and their limitations. As a consequence, simulations also illustrate some properties of robustness to unmodeled dynamics.

In the following simulations, we have simultaneously applied the SISO longitudinal controller for angle of attack, and the MIMO lateral one for the sideslip and roll angles in the full nonlinear F-16 aircraft model. We may note that the control inputs are limited by their physical bounds introduced in section 2.

The control law in (11) whose $\pi(\cdot)$ can be written simpler as:

$$\pi(\cdot) = (\pi_0 + \gamma(\cdot))G(\cdot)^{-1} \quad (70)$$

in which, $\gamma(\cdot) = \gamma_1 \|e_1\|^2 + \gamma_2 \|e_2\|^2$, γ_1 and γ_2 are positive constant. $G(\cdot)^{-1}$ is defined in (43) or in (16) depending on longitudinal mode or lateral mode.

Application of this control law to two motion modes presented in subsections (3.3) and (3.2) is done by determining the set of parameters π_0^i , γ_1^i , γ_2^i , μ^i , K_1^i and k_0^i with $i = \alpha, \beta$ corresponding to longitudinal mode and lateral mode respectively. We use the design parameters in the Table 1 for the longitudinal controller, and the design parameters in Table 2 for the lateral controller.

π_0^α	μ^α	γ_1^α and γ_2^α	k_0^α	K_1^α
25.0	1.0	0.001 and 0.001	2.0	2.0

Table 1: Parameters for the longitudinal mode

π_0^β	μ^β	γ_1^β and γ_2^β	k_0^β	K_1^β
10 0.0 0.0 10	1.0	0.01 and 0.01	0.8 0.0 0.0 0.8	1.2 0.0 0.0 1.2

Table 2: Parameters for the lateral mode

We will stabilize the second model following the launch phase to its equilibrium point $(V, h) = (154\text{m/s}, 6500\text{m})$ corresponding to angle of attack α_r to 2.7° , sideslip β_r to 0° , and roll angle ϕ_r to 0° .

Its initial condition is the final state of the first model ($\alpha = 12.5^\circ$, $\beta = 0^\circ$ and $\phi = 0^\circ$) as in Section 2. Moreover, we add on its initial condition a small disturbance on system output. That means the initial condition of second model is ($\alpha_0 = 17.5^\circ$, $\beta_0 = 4^\circ$ and $\phi_0 = 10^\circ$) for all numerical simulations.

The second model is disturbed on its aerodynamic force and moments during an interval

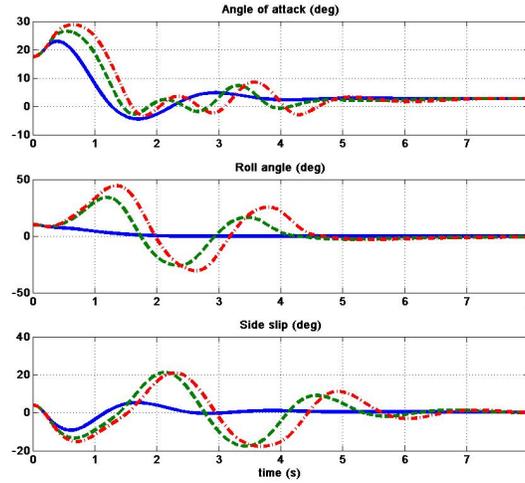


Figure 4: System Output: Angle of attack, Sideslip Angle and Roll angle stabilized

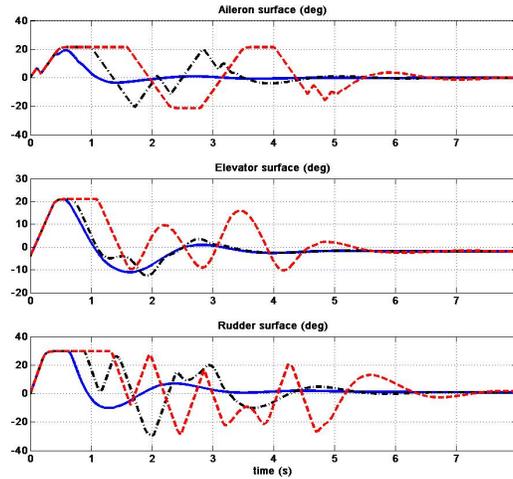


Figure 5: Control surface: Aileron, Elevator and Rudder

We simulate three sets of time lengths:

1. $T_{int} = 0.2s$ (corresponding to solid lines in Fig. 4 to Fig. 6) produces damped oscillations.
2. $T_{int} = 0.227s$, the Conditional Integrator is able to stabilize the system as can be seen in the dashed lines in Fig. 4 to Fig. 6, but the bounds on actuator dynamics are reached.
3. $T_{int} = 0.3s$ (corresponding to dash dotted lines in Fig. 4 to Fig. 6) one can see that the controller is still able to stabilize the system.

Fig. 4 illustrates the convergence of the system output to the operating point of the aircraft at the end of 5s without static steady error for the three cases of $T_{int} = (0.2s, 0.227s, 0.3s)$. All system outputs are still under their physical limitation.

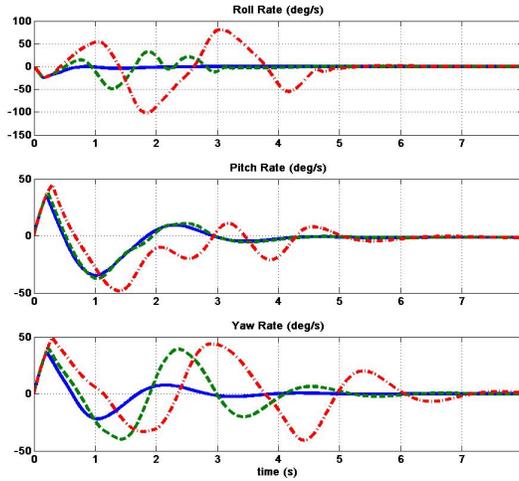


Figure 6: State variables: Angular rates of system

Figs. 6 shows that angular rates converge to zero in all cases. In Fig. 5, it can be seen that the control variables are saturated by their physical limitations due to a high perturbation on aerodynamic forces and moments.

Finally we show in Fig. 7 and Fig. 8 that the system will be unstable for an interval $T_{int} = 0.39s$.

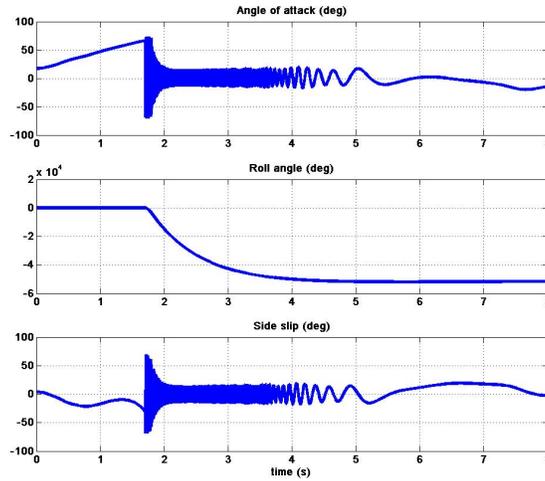


Figure 7: Angle of attack, Sideslip Angle and Roll angle unstable

5 Conclusion

We have developed a MIMO nonlinear Conditional Integrator Controller for the staging phase of an airlaunch system. Because our airlaunch system has a down stage mass close to the launch vehicle, the separation phase produces large changes in aerodynamic force and moment, as shown in section 2, which may turn the system unstable. These changes may, specially in the case of a non perfect separation phase, produce large impulses on forces and moments, making the system unstable. These impulses are considered to last a time interval, that is then evaluated in simulations.

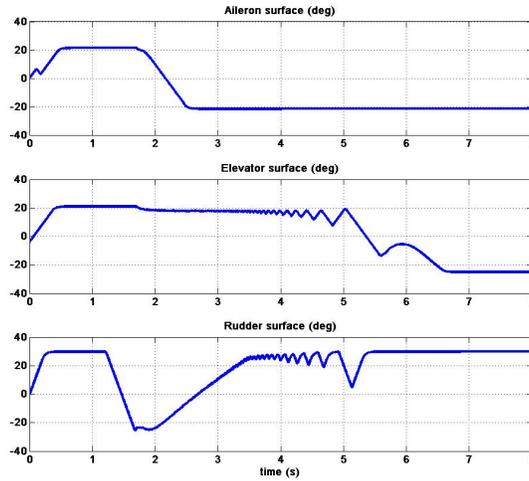


Figure 8: Saturation of Aileron, Elevator and Rudder

To stabilize the airlaunch system after this stage separation phase, a conditional integrator control is considered. This controller is designed using an F-16 MIMO nonlinear model representing the aircraft during and after dropping the second stage, and as a consequence disturbed by large impulses.

For a perturbation on aerodynamic forces and moments during an interval T_{int} , the stability of the system after the drop stage may be assured for small time intervals. When T_{int} becomes large, the system becomes unstable even with the proposed controller, that is bounded by actuator's limitations.

References

- [1] M.M. Burlacu and J. Kohlenberg. An analysis of the nanosatellites launches between 2004 and 2007. 3rd International Conference on Systems and Networks Communications, pages 292–297, 2008.
- [2] Gilney Damm and Van Cuong Nguyen. Mimo conditional servo-compensator control for a class of nonlinear systems (submitted). 2012.
- [3] Gilney Damm and Van Cuong Nguyen. Mimo conditional integrator control for a class of nonlinear systems. In In proceedings of 2011 15th International Conference, Sinaia, Romania, October 14-16, 2011.
- [4] Gary C. Hudson. Quickreach responsive launch system. In 4th Responsive Space Conference, Los Angeles, CA, USA, 2006.
- [5] Hassan K. Khalil. On the design of robust servomechanisms for minimum phase nonlinear systems. Robust Nonlinear Control 2000, John Wiley & Sons, 10:339–361, 2000.

- [6] Taeyoung Lee and Youdan Kim. *Nonlinear adaptive flight control using backstepping and neural networks controller*. Journal of guidance, control and dynamics, 24:675–682, 2001.
- [7] Nazmi A. Mahmoud and Hassan K. Khalil. *Robust control for a nonlinear servomechanism problem*. International Journal of Control, 66:779–802, April 1, 1997.
- [8] Eugene Morelli. *Global nonlinear parametric modeling with application to f-16 aerodynamics*. In Proceedings of the 1998 American Control Conference, Philadelphia, PA , USA, 2:997–1001, 1998.
- [9] L.T. Nguyen, M.E. Ogburn, and P. Deal. *Simulator study of fighter airplane with relaxed longitudinal static stability*. Technical report NASA, page 1538, 1979.
- [10] Van Cuong Nguyen and Gilney Damm. *Modeling and conditional integrator control of an unmanned aerial vehicle for airlaunch*. In Proceedings of the 2012 American Control Conference (ACC), Montreal, Canada, June 27-29, 2012.
- [11] Van Cuong Nguyen, Gilney Damm, Naoufel Azouz, Claire Vasiljevic, and Etienne COLLE. *Modeling and lqr robust control of an unmanned aerial vehicle for airlaunch*. In Proceedings of 2012 International Conference on Unmanned Aircraft Systems, Philadelphia, PA USA, June 2012.
- [12] Marti Sarigul-Klijn and Nesrin Sarigul-Klijn. *Selection of a carrier aircraft and a launch method for air launching space vehicles*. In AIAA SPACE Conference and Exposition, San Diego, California, USA, 2008.
- [13] Lars Sonneveldt. *Nonlinear F-16 Model Description*. Delft University of Technology, Netherlands, 2006.
- [14] Peter H. Zipfel. *Modeling and Simulation of Aerospace Vehicle Dynamics*, 2nd edition. AIAA, January, 2001.

A Functions in the lateral mode

$$f_{11}^\beta(\cdot) = \begin{bmatrix} \frac{1}{mV}(-\cos(\alpha_0)\sin(\beta)(T + C_x(\alpha_0)\bar{q}S) + \cos(\beta)C_y(\beta)\bar{q}S - \sin(\alpha_0)\sin(\beta)C_z(\alpha_0, \beta)\bar{q}S) \\ 0 \\ + \frac{g}{V}(\cos(\alpha_0)\sin(\beta)\sin(\theta_0) + \cos(\beta)\cos(\theta_0)\sin(\phi) - \sin(\alpha_0)\sin(\beta)\cos(\phi)) \end{bmatrix}$$

$$f_{12}^\beta(\cdot) = \begin{bmatrix} \sin(\alpha_0) + \frac{\rho S}{4m}\cos(\beta)C_{y_p}(\alpha_0)\bar{b} & -\cos(\alpha_0) + \frac{\rho S}{4m}\cos(\beta)C_{y_r}(\alpha_0)\bar{b} \\ 1 & \cos(\phi)\tan(\theta_0) \end{bmatrix}$$

$$f_{21}^\beta(\cdot) = \begin{bmatrix} I_3C_l(\alpha_0, \beta)\bar{q}S\bar{b} + I_4C_n(\alpha_0, \beta)\bar{q}S\bar{b} \\ I_4C_l(\alpha_0, \beta)\bar{q}S\bar{b} + I_9C_n(\alpha_0, \beta)\bar{q}S\bar{b} \end{bmatrix}$$

$$f_{22}^\beta(\cdot) = \frac{\rho V S \bar{b}}{4} \begin{bmatrix} (I_3 C_{l_p}(\alpha_0) + I_4 C_{n_p}(\alpha_0)) & (I_3 C_{l_r}(\alpha_0) + I_4 C_{n_r}(\alpha_0)) \\ (I_4 C_{l_p}(\alpha_0) + I_9 C_{n_p}(\alpha_0)) & (I_4 C_{l_r}(\alpha_0) + I_9 C_{n_r}(\alpha_0)) \end{bmatrix}$$

$$g_2^\beta(\cdot) = \bar{q} S \begin{bmatrix} (I_3 C_{l_{\delta_a}}(\alpha_0) + I_4 C_{n_{\delta_a}}(\alpha_0)) & (I_3 C_{l_{\delta_r}}(\alpha_0) + I_4 C_{n_{\delta_r}}(\alpha_0)) \\ (I_4 C_{l_{\delta_a}}(\alpha_0) + I_9 C_{n_{\delta_a}}(\alpha_0)) & (I_4 C_{l_{\delta_r}}(\alpha_0) + I_9 C_{n_{\delta_r}}(\alpha_0)) \end{bmatrix}$$

B Functions in the longitudinal mode

$$f_{11}^\alpha(\alpha) = \frac{1}{mV} [-\sin \alpha (T + C_x(\alpha) \bar{q} S) + \cos \alpha C_z(\alpha) \bar{q} S]$$

$$f_{12}^\alpha(\alpha) = \frac{\rho S}{4m} (-\sin \alpha C_{x_q}(\alpha) \bar{c} + \cos \alpha C_{z_q}(\alpha) \bar{c})$$

$$f_{13}^\alpha(\alpha, \theta) = \frac{g}{V} \cos(\theta_0 - \alpha)$$

$$f_{21}^\alpha(\alpha) = I_7 \bar{q} S (C_m(\alpha) \bar{c})$$

$$f_{22}^\alpha(\alpha) = C_{m_q}(\alpha) \bar{c}$$

$$g_2^\alpha(\alpha) = C_{m_{\delta_e}}(\alpha) \bar{c}$$