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Asymptotic behaviour of the cross-variation of some integral long memory processes

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Abstract. We study the asymptotic behaviour of the cross-variation of two-dimensional processes having the form of a Young integral with respect to a fractional Brownian motion of index $H > \frac{1}{2}$. When H is smaller than or equal to $\frac{3}{4}$, we show asymptotic mixed normality. When H is strictly bigger than $\frac{3}{4}$, we obtain a limit that is expressed in terms of the difference of two independent Rosenblatt processes.

1 Introduction

1.1 Foreword and main results

In the near past, there have been many applications of stochastic differential equations (SDE) driven by fractional Brownian motion (fBm) in different areas of mathematical modeling. To name but a few, we mention the use of such equations as a model for meteorological phenomena [1, 10], protein dynamics [5], or noise in electrical networks [6].

Here, we consider more generally a two-dimensional stochastic process $\{X_t\}_{t \in [0, T]} = \{(X_t^{(1)}, X_t^{(2)})\}_{t \in [0, T]}$ of the form

$$X_t^{(i)} = x^{(i)} + \int_0^t \sigma_s^{i,1} dB_s^{(i)} + \int_0^t \sigma_s^{i,2} dB_s^{(2)}, \quad t \in [0, T], \quad i = 1, 2. \quad (1.1)$$

In (1.1), $B = (B^{(1)}, B^{(2)})$ is a two-dimensional fractional Brownian motion of Hurst index $H > \frac{1}{2}$, whereas σ is a 2×2 matrix-valued process. The

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case where X solves a fractional SDE corresponds to $\sigma_t = \sigma(X_t)$, with $\sigma : \mathbb{R}^2 \rightarrow \mathcal{M}_2(\mathbb{R})$ deterministic. Since we are assuming that $H > \frac{1}{2}$, by imposing appropriate conditions on σ (see Section 2 for the details) one may and will assume throughout the text that $\int_0^t \sigma_s^{i,j} dB_s^{(j)}$ is understood in the Young [14] sense (see again Section 2 for the details).

In this paper, we are concerned with the asymptotic behaviour of the cross-variation associated to X on $[0, T]$, which is the sequence of stochastic processes defined as:

$$J_n(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \Delta X_{k2^{-n}}^{(1)} \Delta X_{k2^{-n}}^{(2)}, \quad n \geq 1, t \in [0, T]. \quad (1.2)$$

Here, $\Delta X_{k2^{-n}}^{(i)}$ is shorthand for the increment $X_{(k+1)2^{-n}}^{(i)} - X_{k2^{-n}}^{(i)}$. We then have the following two theorems. They may be of interest to solve problems arising from statistics, as for instance the problem of testing the hypothesis (H_0) : “ $\sigma^{1,2} = \sigma^{2,1} = 0$ ” in (1.1).

Theorem 1.1. *For any $t \in [0, T]$,*

$$2^{n(2H-1)} J_n(t) \xrightarrow{\text{prob}} \int_0^t (\sigma_s^{1,1} \sigma_s^{1,2} + \sigma_s^{2,1} \sigma_s^{2,2}) ds \quad \text{as } n \rightarrow \infty. \quad (1.3)$$

Theorem 1.2. *Assume $\sigma^{1,2} = \sigma^{2,1} = 0$ and let*

$$a_n := \begin{cases} 2^{n(2H-\frac{1}{2})} & \text{if } \frac{1}{2} < H < \frac{3}{4} \\ \frac{2^n}{\sqrt{n}} & \text{if } H = \frac{3}{4} \\ 2^n & \text{if } \frac{3}{4} < H < 1 \end{cases}. \quad (1.4)$$

Then, for some explicit constant C_H depending only on H ,

$$a_n J_n \xrightarrow{\text{fdd}} \int_0^\cdot \sigma_s^{1,1} \sigma_s^{2,2} dZ_s \quad \text{as } n \rightarrow \infty.$$

Here, ‘fdd’ stands for the ‘convergence of finite-dimensional distributions’. In addition, the definition of Z is according to the value of H . More precisely, Z equals C_H times W when $H \in (\frac{1}{2}, \frac{3}{4}]$, with C_H given by (3.8)-(3.9) and with W a Brownian motion independent of \mathcal{F} ; and $Z = R^{(1)} - R^{(2)}$ when $H \in (\frac{3}{4}, 1)$, with $R^{(k)}$ the Rosenblatt process constructed from the fractional Brownian motion

$$\beta^{(k)} = \frac{1}{\sqrt{2}} (B^{(1)} + (-1)^{k+1} B^{(2)}), \quad k = 1, 2,$$

see Definition 3.4.

Our results are close in spirit to those contained in [3], where central limit theorems for power variations of integral fractional processes are investigated. Actually, we must acknowledge that [3] has been a strong source of inspiration to us. Nevertheless, our analysis of J_n required extra efforts compared to [3], as we are here dealing with a *two-dimensional* fractional Brownian motion on one hand and we are able to cover the case where $H > \frac{3}{4}$ on the other hand (the authors of [3] did not consider this latter case since, according to them, “the problem is more involved because non-central limit theorems are required”).

1.2 Plan of the paper

The rest of the paper is as follows. Section 2 contains a thorough description of the framework in which our study takes place (in particular, we recall the definition of the Young integral and we provide its main properties). Section 3 gathers several preliminary results that will be essential for proving our main results. Finally, proofs of Theorems 1.1 and 1.2 are given in Section 4.

2 Our framework

In this section, we describe the framework used throughout the paper and we fix a parameter $\alpha \in (0, 1)$.

We let C^α denote the set of Hölder continuous functions of index $\alpha \in (0, 1)$, that is, the set of those functions $f : [0, T] \rightarrow \mathbb{R}$ satisfying

$$|f|_\alpha := \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\alpha} < \infty. \quad (2.5)$$

Also, we set $\|f\|_\alpha := |f|_\alpha + |f|_\infty$, with $|f|_\infty = \sup_{0 \leq t \leq T} |f(t)|$.

For a fixed $f \in C^\alpha$, we consider the operator $T_f : C^1 \rightarrow C^1$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0, T].$$

Let $\beta \in (0, 1)$ be such that $\alpha + \beta > 1$. Then T_f extends, in a unique way, to an operator $T_f : C^\beta \rightarrow C^\beta$, which further satisfies

$$\|T_f(g)\|_\beta \leq (1 + C_{\alpha,\beta})(1 + T^\beta)\|f\|_\alpha\|g\|_\beta,$$

with $C_{\alpha,\beta} = \frac{1}{2} \sum_{n=1}^{\infty} 2^{-n(\alpha+\beta-1)} < \infty$. See, e.g., [7, Theorem 3.1] for a proof.

Definition 2.1. Let $\alpha, \beta \in (0, 1)$ be such that $\alpha + \beta > 1$. Let $f \in C^\alpha$ and $g \in C^\beta$. The Young integral $\int_0^\cdot f(u)dg(u)$ is then defined as being $T_f(g)$.

The Young integral satisfies (see, e.g., [7, inequality (3.3)]) that, for any $a, b \in [0, T]$ with $a < b$,

$$\left| \int_a^b (f(u) - f(a)) dg(u) \right| \leq C_{\alpha, \beta} |f|_{\alpha} |g|_{\beta} (b - a)^{\alpha + \beta}. \quad (2.6)$$

As we said in the Introduction, we let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional fractional Brownian motion defined on a probability space (Ω, \mathcal{F}, P) . We assume further that \mathcal{F} is the σ -field generated by B . We also suppose that the Hurst parameter H of B is the same for the two components and that it is strictly bigger than $\frac{1}{2}$.

Let $\alpha \in (0, 1)$ and let $\sigma^{i,j} : \Omega \times [0, T] \rightarrow \mathbb{R}$, $i, j = 1, 2$, be four given stochastic processes that are measurable with respect to \mathcal{F} . We will assume throughout the text that the following two additional assumptions on α and $\sigma^{i,j}$ take place:

(A) $\alpha \in (\frac{1}{4} + \frac{H}{2}, H)$,

(B) For each pair $(i, j) \in \{1, 2\}^2$, the random variable $\|\sigma^{i,j}\|_{\alpha}$ has moments of all orders.

Observe that $\alpha + H > 1$ due to both (A) and $H > \frac{1}{2}$, so that the integrals in (1.1) are well-defined in the Young sense. Also, recall the following variant of the Garcia-Rodemich-Rumsey Lemma [4]: for any $q > 1$, there exists a constant $c_{\alpha, q} > 0$ (depending only on α and q) such that

$$|B^{(i)}|_{\alpha}^q \leq c_{\alpha, q} \iint_{[0, T]^2} \frac{|B_u^{(i)} - B_v^{(i)}|^q}{|u - v|^{2 + q\alpha}} dudv. \quad (2.7)$$

Using (2.7), one deduces that $|B^{(i)}|_{\alpha}$ has moments of all orders.

3 Preliminaries

3.1 Breuer-Major theorem

The next statement is a direct consequence of the celebrated Breuer-Major [2] theorem (see [7, Section 7.2] for a modern proof).

Theorem 3.1 (Breuer-Major). *Let β be a (one-dimensional) fractional Brownian motion of index $H \in (0, \frac{3}{4}]$. Then, as $n \rightarrow \infty$ and with W a standard Brownian motion,*

(i) if $H < \frac{3}{4}$ then

$$\left\{ 2^{-n/2} \sum_{k=0}^{\lfloor 2^{nt} \rfloor - 1} [(\beta_{k+1} - \beta_k)^2 - 1] \right\}_{t \in [0, T]} \xrightarrow{\text{fdd}} \frac{1}{2} \sum_{k \in \mathbb{Z}} (|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H})^2 \{W_t\}_{t \in [0, T]};$$

(ii) if $H = \frac{3}{4}$ then

$$\left\{ \frac{2^{-n/2}}{\sqrt{n}} \sum_{k=0}^{\lfloor 2^{nt} \rfloor - 1} [(\beta_{k+1} - \beta_k)^2 - 1] \right\}_{t \in [0, T]} \xrightarrow{\text{fdd}} \frac{3}{4} \log 2 \{W_t\}_{t \in [0, T]}.$$

By a scaling argument (to pass from k to $k2^{-n}$) and by using the seminal result of Peccati and Tudor [9] (to allow an extra F), one immediately deduces from Theorem 3.1 the following corollary.

Corollary 3.2. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in (0, \frac{3}{4}]$. Then, as $n \rightarrow \infty$ and with W a (one-dimensional) standard Brownian motion independent of β , we have, for any random vector $F = (F_1, \dots, F_d)$ measurable with respect to β ,*

(i) if $H < \frac{3}{4}$ then

$$\left\{ F, 2^{n(2H-\frac{1}{2})} \sum_{k=0}^{\lfloor 2^{nt} \rfloor - 1} [(\beta_{(k+1)2^{-n}}^{(1)} - \beta_{k2^{-n}}^{(1)})^2 - (\beta_{(k+1)2^{-n}}^{(2)} - \beta_{k2^{-n}}^{(2)})^2] \right\}_{t \in [0, T]} \xrightarrow{\text{fdd}} \{F, C_H W_t\}_{t \in [0, T]},$$

where

$$C_H = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} (|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H})^2 \quad (3.8)$$

(ii) if $H = \frac{3}{4}$ then

$$\left\{ F, \frac{2^n}{\sqrt{n}} \sum_{k=0}^{\lfloor 2^{nt} \rfloor - 1} [(\beta_{(k+1)2^{-n}}^{(1)} - \beta_{k2^{-n}}^{(1)})^2 - (\beta_{(k+1)2^{-n}}^{(2)} - \beta_{k2^{-n}}^{(2)})^2] \right\}_{t \in [0, T]} \xrightarrow{\text{fdd}} \{F, C_{3/4} W_t\}_{t \in [0, T]},$$

where

$$C_{3/4} = \frac{3\sqrt{2}}{4} \log 2. \quad (3.9)$$

3.2 Taquu's theorem and the Rosenblatt process

Taquu's theorem [11] aims to describe the fluctuations of the quadratic variation of the fractional Brownian motion when the Hurst index H is strictly bigger than $\frac{3}{4}$, that is, for the range of values which are not covered by the Breuer-Major Theorem 3.1. We state here a version that fits into our framework. With respect to the original statement, it is worthwhile noting that, in Theorem 3.3 (whose proof may be found in [8]), the convergence is in $L^2(\Omega)$ (and not only in law). As anticipated, this fact will play a crucial role in our proof of Theorem 1.2.

Theorem 3.3 (Taquu). *Let β be a (one-dimensional) fractional Brownian motion of index $H \in (\frac{3}{4}, 1)$. Then, for any $t \in [0, T]$, the sequence*

$$2^{n(1-2H)} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} [2^{2nH} (\beta_{(k+1)2^{-n}} - \beta_{k2^{-n}})^2 - 1] \quad (3.10)$$

converges in $L^2(\Omega)$ as $n \rightarrow \infty$.

Definition 3.4. *Let the assumption of Theorem 3.3 prevail and denote by R_t the limit of (3.10). The process $R = \{R_t\}_{t \in [0, T]}$ is called the Rosenblatt process constructed from β .*

For the main properties of the Rosenblatt process R , we refer the reader to Taquu [12] or Tudor [13]. See also [7, Section 7.3]. A direct corollary of Theorem 3.3 is as follows.

Corollary 3.5. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in (\frac{3}{4}, 1)$. Then, for any $t \in [0, T]$,*

$$2^n \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} [(\beta_{(k+1)2^{-n}}^{(1)} - \beta_{k2^{-n}}^{(1)})^2 - (\beta_{(k+1)2^{-n}}^{(2)} - \beta_{k2^{-n}}^{(2)})^2] \xrightarrow{L^2(\Omega)} R_t^{(1)} - R_t^{(2)}$$

as $n \rightarrow \infty$, where $R^{(i)}$ is the Rosenblatt process constructed from the fractional Brownian motion $\beta^{(i)}$, $i = 1, 2$.

3.3 Two simple lemmas

To complete the proofs of Theorems 1.1 and 1.2, we will also need the following two simple lemmas.

Lemma 3.6. *Let B and σ be as in Section 2. Then there exists a constant $C = C(\alpha, H, T, \{\sigma^{i,j}\}_{i,j=1,2}) > 0$ such that, for any $i, j = 1, 2$, any $n \geq 1$ and any $k \in \{0, \dots, \lfloor 2^n T \rfloor - 1\}$,*

$$\left\| \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{i,j} - \sigma_{k2^{-n}}^{i,j}) dB_s^j \right\|_{L^2(\Omega)} \leq C 2^{-2n\alpha}, \quad (3.11)$$

$$\left\| \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{i,j} dB_s^j \right\|_{L^2(\Omega)} \leq C 2^{-nH}. \quad (3.12)$$

Proof. Without loss of generality, one may and will assume that $i = j = 1$. Using (2.6) with $\beta = \alpha$, we have, almost surely,

$$\left| \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{1,1} - \sigma_{k2^{-n}}^{1,1}) dB_s^1 \right| \leq C_{\alpha,\alpha} |\sigma^{1,1}|_{\alpha} |B^1|_{\alpha} 2^{-2n\alpha}.$$

Using Cauchy-Schwarz inequality, one deduces

$$\begin{aligned} & E \left[\left(\int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{1,1} - \sigma_{k2^{-n}}^{1,1}) dB_s^1 \right)^2 \right] \\ & \leq C_{\alpha,\alpha}^2 \sqrt{E[|\sigma^{1,1}|_{\alpha}^4]} \sqrt{E[|B^1|_{\alpha}^4]} 2^{-4n\alpha} = C 2^{-4n\alpha}, \end{aligned}$$

thus yielding (3.11). On the other hand, one has

$$\begin{aligned} & \left\| \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{i,j} dB_s^j \right\|_{L^2(\Omega)} \\ & \leq \left\| \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{i,j} - \sigma_{k2^{-n}}^{i,j}) dB_s^j \right\|_{L^2(\Omega)} + \left\| \sigma_{k2^{-n}}^{i,j} \Delta B_{k2^{-n}}^j \right\|_{L^2(\Omega)} \\ & \leq C 2^{-2n\alpha} + C 2^{-2nH}, \quad \text{by (3.11) and because of (B)} \\ & \leq C 2^{-nH}, \quad \text{using (A),} \end{aligned}$$

which is the desired claim (3.12). \square

Lemma 3.7. *Let $g, h : [0, T] \rightarrow \mathbb{R}$ be two continuous functions, let $\gamma \in \mathbb{R}$, and let us write $\Delta h_{k2^{-n}}$ to denote the increment $h((k+1)2^{-n}) - h(k2^{-n})$. If*

$$\forall t \in [0, T] \cap \mathbb{Q} : \lim_{n \rightarrow \infty} 2^{n\gamma} \sum_{k=0}^{\lfloor 2^n T \rfloor - 1} \mathbf{1}_{[0,t]}(k2^{-n}) (\Delta h_{k2^{-n}})^2 = t, \quad (3.13)$$

then, for all $t \in [0, T]$,

$$\lim_{n \rightarrow \infty} 2^{n\gamma} \sum_{k=0}^{\lfloor 2^n T \rfloor - 1} g(k2^{-n}) \mathbf{1}_{[0,t]}(k2^{-n}) (\Delta h_{k2^{-n}})^2 = \int_0^t g(s) ds.$$

Proof. Since $t \mapsto 2^{n\gamma} \sum_{k=0}^{2^n-1} \mathbf{1}_{[0,t]}(k2^{-n}) (\Delta h_{k2^{-n}})^2$ is non-decreasing, it is straightforward to deduce from (3.13) that, for all $t \in [0, T]$,

$$\lim_{n \rightarrow \infty} 2^{n\gamma} \sum_{k=0}^{\lfloor 2^n T \rfloor - 1} \mathbf{1}_{[0,t]}(k2^{-n}) (\Delta h_{k2^{-n}})^2 = t.$$

Otherwise stated, the cumulative distribution function (cdf) of the compactly supported measure

$$\nu_n(dx) = 2^{n\gamma} \sum_{k=0}^{\lfloor 2^n T \rfloor - 1} (\Delta h_{k2^{-n}})^2 \delta_{k2^{-n}}(dx),$$

where δ_a stands for the Dirac mass at a , converges pointwise to the cdf of the Lebesgue measure on $[0, T]$. Since g is continuous, it is then a routine exercise to deduce that our desired claim holds true. \square

3.4 An auxiliary result

As anticipated, the following result will represent a central ingredient in the proof of both Theorems 1.1 and 1.2.

Proposition 3.8. *Let*

$$K_n(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} u_{k2^{-n}} \Delta B_{k2^{-n}}^{(1)} \Delta B_{k2^{-n}}^{(2)}, \quad t \in [0, T],$$

and let a_n be given by (1.4). Then

$$a_n K_n \xrightarrow{\text{fdd}} \int_0^\cdot u_s dZ_s \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

Here, Z is as in the statement of Theorem 1.2.

Proof. The proof is divided into several steps.

Step 1: Rotation trick. Let $\beta^{(1)} = \frac{1}{\sqrt{2}}(B^{(1)} + B^{(2)})$ and $\beta^{(2)} = \frac{1}{\sqrt{2}}(B^{(1)} - B^{(2)})$. It is easy to check that $\beta^{(1)}$ and $\beta^{(2)}$ are two independent fractional

Brownian motions of index H . For $l \in \{0, \dots, 2^m - 1\}$ where m is an integer less than n , set

$$Z_{l,n}^m = a_n \sum_{k=0}^{2^{n-m}-1} (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}).$$

Assume that $H \leq \frac{3}{4}$. Then, according to Corollary 3.2 and for any fixed \mathcal{F} -measurable random vector F ,

$$(F, Z_{l,n}^m, l = 0, \dots, 2^m - 1) \xrightarrow{\text{law}} (F, Z_{(l+1)2^{-m}} - Z_{l2^{-m}}, l = 0, \dots, 2^m - 1)$$

as $n \rightarrow \infty$, where Z is as in the statement of Theorem 1.2. Assume conversely that $H > \frac{3}{4}$. This time, according to Corollary 3.5,

$$(Z_{i,l,n}^m, l = 0, \dots, 2^m - 1) \xrightarrow{L^2(\Omega)} (Z_{(l+1)2^{-m}} - Z_{l2^{-m}}, l = 0, \dots, 2^m - 1),$$

as $n \rightarrow \infty$, where Z is again as in the statement of Theorem 1.2.

Also, a straightforward calculation leads to

$$\sup_{n \geq 1} \sup_{t \in [0, T]} E \left[\left(a_n \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (B_{(k+1)2^{-n}}^{(1)} - B_{k2^{-n}}^{(1)}) (B_{(k+1)2^{-n}}^{(2)} - B_{k2^{-n}}^{(2)}) \right)^2 \right] < \infty.$$

We then deduce, using an hypercontractivity argument, that, for any real number $p \geq 1$,

$$\sup_{n \geq 1} \sup_{t \in [0, T]} E \left[\left| a_n \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (B_{(k+1)2^{-n}}^{(1)} - B_{k2^{-n}}^{(1)}) (B_{(k+1)2^{-n}}^{(2)} - B_{k2^{-n}}^{(2)}) \right|^p \right] < \infty. \quad (3.15)$$

Step 2: Coarse graining on K_n . We shall apply a coarse graining transformation to K_n . More precisely, observe that, for any integer m less than n ,

$$\begin{aligned} K_n(t) &= \sum_{l=0}^{\lfloor 2^m t \rfloor - 1} \sum_{k=0}^{2^{n-m}-1} u_{l2^{-m}+k2^{-n}} (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) \\ &\quad \times (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}) \\ &= K_{1,m,n}(t) + K_{2,m,n}(t), \end{aligned}$$

with

$$\begin{aligned}
K_{1,m,n}(t) &= \sum_{l=0}^{\lfloor 2^m t \rfloor - 1} u_{l2^{-m}} \sum_{k=0}^{2^{n-m}-1} (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) \\
&\quad \times (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}) \\
K_{2,m,n}(t) &= \sum_{l=0}^{\lfloor 2^m t \rfloor - 1} \sum_{k=0}^{2^{n-m}-1} (u_{l2^{-m}+k2^{-n}} - u_{l2^{-m}}) \\
&\quad \times (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}).
\end{aligned}$$

Step 3: Study of $K_{2,m,n}$. Fix $m \geq 1$ and $t \in [0, T]$, and let us introduce, for any $n \geq m$ and any $x \in [l2^{-m}, (l+1)2^{-m}]$,

$$\begin{aligned}
X_n^{m,t}(x) &= a_n \sum_{l=0}^{\lfloor 2^m t \rfloor - 1} (u_x - u_{l2^{-m}}) \\
&\quad \times \sum_{k=0}^{2^{n-m}-1} (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}).
\end{aligned}$$

The sequence $\{X_n^{m,t}\}_{n \geq m}$ is tight in $\mathcal{C}([l2^{-m}, (l+1)2^{-m}])$ endowed with the supremum norm. Indeed,

$$\begin{aligned}
&X_n^{m,t}(y) - X_n^{m,t}(x) \\
&= (u_y - u_x) \times a_n \sum_{l=0}^{\lfloor 2^m t \rfloor - 1} \sum_{k=0}^{2^{n-m}-1} (B_{l2^{-m}+(k+1)2^{-n}}^{(1)} - B_{l2^{-m}+k2^{-n}}^{(1)}) (B_{l2^{-m}+(k+1)2^{-n}}^{(2)} - B_{l2^{-m}+k2^{-n}}^{(2)}) \\
&= (u_y - u_x) \times a_n \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (B_{(k+1)2^{-n}}^{(1)} - B_{k2^{-n}}^{(1)}) (B_{(k+1)2^{-n}}^{(2)} - B_{k2^{-n}}^{(2)}),
\end{aligned}$$

so that, using (3.15) for the first inequality and **(B)** for the second one,

$$E \left[|X_n^{m,t}(y) - X_n^{m,t}(x)|^2 \right] \leq C_H \sqrt{E[|u_y - u_x|^4]} \leq C_H |y - x|^{2\alpha}.$$

Moreover, see the conclusion of Step 1, $\{X_n^{m,t}\}_{n \geq m}$ converges fdd to $x \mapsto$

$$\begin{aligned}
& \sum_{l=0}^{2^m-1} (u_x - u_{l2^{-m}}) \Delta Z_{l2^{-m}}. \text{ As a result, for any } \delta > 0, \\
& \limsup_{n \rightarrow \infty} P(|a_n K_{2,m,n}(t)| \geq \delta) \\
& \leq \limsup_{n \rightarrow \infty} P \left(\sup_{x \in [l2^{-m}, (l+1)2^{-m}]} |X_n^{m,t}(x)| \geq \delta \right) \\
& \leq P \left(\sup_{x \in [l2^{-m}, (l+1)2^{-m}]} \left| \sum_{l=0}^{\lfloor 2^{mt} \rfloor - 1} (u_x - u_{l2^{-m}}) \Delta Z_{l2^{-m}} \right| \geq \frac{\delta}{2} \right) \\
& \leq \frac{2}{\delta} E \left[\sup_{x \in [l2^{-m}, (l+1)2^{-m}]} \left| \sum_{l=0}^{\lfloor 2^{mt} \rfloor - 1} (u_x - u_{l2^{-m}}) \Delta Z_{l2^{-m}} \right| \right] \\
& \leq \frac{C}{\delta} 2^{m(1-\alpha-\beta)},
\end{aligned}$$

where β is chosen so that $\alpha + \beta > 1$ (which is always possible since, whenever $H \leq \frac{3}{4}$ (resp. $H > \frac{3}{4}$), β must be indeed smaller than $\frac{1}{2}$ (resp. $2H - 1$) but may be chosen as closely to $\frac{1}{2}$ as wanted).

Step 4: Study of $K_{1,m,n}$. As $n \rightarrow \infty$ and then $m \rightarrow \infty$, $a_n K_{1,m,n}$ converges fdd to $\sum_{l=0}^{\lfloor 2^{mt} \rfloor - 1} u_{l2^{-m}} \Delta Z_{l2^{-m}}$ which, itself, converges in $L^2(\Omega)$ to $\int_0^t u_s dZ_s$.

Step 5: Conclusion. The desired conclusion (3.14) follows by plugging together the results of Step 2, Step 3 and Step 4. □

4 Proof of our main results

4.1 Proof of Theorem 1.1

We divide it into several steps.

Step 1. Recall J_n from (1.2). One can write

$$\begin{aligned}
J_n(t) &= \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \left(\int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,1} dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,2} dB_s^2 \right) \\
&\quad \times \left(\int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{2,1} dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{2,2} dB_s^2 \right) \\
&=: A_n(t) + R_{1,n}(t) + R_{2,n}(t),
\end{aligned}$$

with

$$A_n(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (\sigma_{k2^{-n}}^{1,1} \Delta B_{k2^{-n}}^1 + \sigma_{k2^{-n}}^{1,2} \Delta B_{k2^{-n}}^2) (\sigma_{k2^{-n}}^{2,1} \Delta B_{k2^{-n}}^1 + \sigma_{k2^{-n}}^{2,2} \Delta B_{k2^{-n}}^2), \quad (4.16)$$

$$\begin{aligned}
R_{1,n}(t) &= \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \left(\int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,1} dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,2} dB_s^2 \right) \\
&\quad \times \left(\int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{2,1} - \sigma_{k2^{-n}}^{2,1}) dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{2,2} - \sigma_{k2^{-n}}^{2,2}) dB_s^2 \right), \quad (4.17)
\end{aligned}$$

$$\begin{aligned}
R_{2,n}(t) &= \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (\sigma_{k2^{-n}}^{2,1} \Delta B_{k2^{-n}}^1 + \sigma_{k2^{-n}}^{2,2} \Delta B_{k2^{-n}}^2) \\
&\quad \times \left(\int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{1,1} - \sigma_{k2^{-n}}^{1,1}) dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{1,2} - \sigma_{k2^{-n}}^{1,2}) dB_s^2 \right). \quad (4.18)
\end{aligned}$$

Step 2. Let us prove the convergence of $2^{n(2H-1)} R_{i,n}(t)$, $i = 1, 2$, $t \in [0, T]$, in $L^1(\Omega)$ towards zero. Using Cauchy-Schwarz and Lemma 3.6, we see that

$$\begin{aligned}
\|R_{1,n}(t)\|_{L^1(\Omega)} &\leq \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \left\| \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,1} dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} \sigma_s^{1,2} dB_s^2 \right\|_{L^2(\Omega)} \\
&\quad \times \left\| \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{2,1} - \sigma_{k2^{-n}}^{2,1}) dB_s^1 + \int_{k2^{-n}}^{(k+1)2^{-n}} (\sigma_s^{2,2} - \sigma_{k2^{-n}}^{2,2}) dB_s^2 \right\|_{L^2(\Omega)} \\
&\leq C 2^{-n(H+2\alpha-1)}.
\end{aligned}$$

Thanks to our assumption **(A)**, one deduces that $2^{n(2H-1)} \|R_{1,n}(t)\|_{L^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Similarly, one may prove that $2^{n(2H-1)} \|R_{2,n}(t)\|_{L^1(\Omega)} \rightarrow 0$.

Step 3. Let us now consider A_n . One has

$$\begin{aligned} A_n(t) &= \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (\sigma_{k2^{-n}}^{1,1} \Delta B_{k2^{-n}}^1 + \sigma_{k2^{-n}}^{1,2} \Delta B_{k2^{-n}}^2) (\sigma_{k2^{-n}}^{2,1} \Delta B_{k2^{-n}}^1 + \sigma_{k2^{-n}}^{2,2} \Delta B_{k2^{-n}}^2) \\ &=: A_{1,n}(t) + A_{2,n}(t) + S_n(t), \end{aligned}$$

with

$$A_{i,n}(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \sigma_{k2^{-n}}^{1,i} \sigma_{k2^{-n}}^{2,i} (\Delta B_{k2^{-n}}^i)^2, \quad i = 1, 2, \quad (4.19)$$

$$S_n(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (\sigma_{k2^{-n}}^{1,1} \sigma_{k2^{-n}}^{2,2} + \sigma_{k2^{-n}}^{1,2} \sigma_{k2^{-n}}^{2,1}) \Delta B_{k2^{-n}}^1 \Delta B_{k2^{-n}}^2. \quad (4.20)$$

Using Proposition 3.8 and whatever the value of H compared to $\frac{3}{4}$, one immediately checks that $2^{n(2H-1)} S_n(t)$ converges in law to zero, thus in probability. On the other hand, fix $i \in \{1, 2\}$ and recall that, for any $t \in [0, T]$, almost surely,

$$\lim_{n \rightarrow \infty} 2^{n(2H-1)} \sum_{k=0}^{\lfloor 2^n T \rfloor - 1} \mathbf{1}_{[0,t]}(k2^{-n}) (\Delta B_{k2^{-n}}^i)^2 = t.$$

We then deduce that, with probability 1, assumption (3.13) holds true with $h = B^i$ and $\gamma = 2H - 1$. Lemma 3.7 applies and yields that, almost surely,

$$2^{n(2H-1)} A_{i,n}(t) \rightarrow \int_0^t \sigma_s^{1,i} \sigma_s^{2,i} ds.$$

Step 4. Plugging together the conclusions of Steps 1 to 3 completes the proof of Theorem 1.1. \square

4.2 Proof of Theorem 1.2

Recall from the previous section that $J_n = A_{1,n} + A_{2,n} + S_n + R_{1,n} + R_{2,n}$, with $A_{i,n}$, S_n , $R_{1,n}$ and $R_{2,n}$ given by (4.19), (4.20), (4.17) and (4.18) respectively. Using the estimates of Step 2 in the previous section, we easily obtain that, under **(A)**, $a_n R_{i,n}(t)$ tends to zero in $L^1(\Omega)$ as $n \rightarrow \infty$, $i = 1, 2$, $t \in [0, T]$. Moreover, the quantities $A_{1,n}$ and $A_{2,n}$ given by (4.19) equal zero when $\sigma^{1,2} = \sigma^{2,1} = 0$. As a result, the asymptotic behavior of $a_n J_n$ is the same as that of $a_n S_n$, and the desired conclusion follows directly from Proposition 3.8. \square

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