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How Vertex reinforced jump process arises naturally

Xiaolin ZENG

Abstract

We prove that the only nearest neighbor jump process with local dependence on the occupation times satisfying the partial exchangeability property is the vertex reinforced jump process, under some technical conditions. This result gives a counterpart to the characterization of edge reinforced random walk given by Rolles [9].

1 Introduction

One of the most remarkable results in probabilistic symmetries is the de Finetti's theorem [3], which states that the law of any exchangeable sequence valued in a finite state space is in fact a mixture of i.i.d. sequences. This theorem has a geometrical interpretation via Choquet's theorem. More precisely, the subspace of exchangeable probabilities forms a convex, and those probabilities given by i.i.d. sequences are exactly the extreme points of the convex.

In the 1920s, W.E. Johnson conjectured that, under some technical conditions, if a process X_n is exchangeable and $\mathbb{P}(X_{n+1} = i | X_0, \dots, X_n)$ depends only on the number of times i occurs and the total steps n , then X_n is nothing but the famous Polya urn: drawing balls uniformly from an urn and put back one additional ball with same color as the drawn one. This is a process with linear reinforcement. In term of random walk the natural counterpart of Polya urn is the edge reinforced random walk (ERRW): Diaconis conjectured that this process have the same characterization as Polya urn. In [9] S.W.W.Rolles have shown that both conjectures are true under technical conditions.

The vertex reinforced jump process (VRJP) is a linearly reinforced process in continuous time. In a recent paper, Sabot and Tarres [10] have shown that ERRW is a mixture of VRJP, which indicates that the VRJP are building blocks of ERRW, thus should share a similar characterization. We prove that this is true in the sense that we give a counterpart of the characterization given by Rolles.

2 Definitions and results

Let G be a connected graph such that each vertex have finite degree, define its vertex set V and edge set E . Denote $i \sim j$ if $\{i, j\} \in E$, assume that G contains no loops (edges with one endpoint).

Definition 1. We call $(X_t)_{t \geq 0}$ a nearest neighbor jump processes on G , if it is a random process which is right continuous without explosion, and each jump is from some vertex i to one of its neighbors j (i.e. $i \sim j$).

Definition 2. A nearest neighbor jump process X_t is a unique mixture of Markov jump process if there exists a unique probability measure μ on Markov jump processes such that $\mathcal{L}(X_t) = \int \mathcal{L}(Y_t) \mu(dY)$, where \mathcal{L} denotes the law of respective processes. If for μ a.e. the Markov process is reversible, then the process is a unique mixture of reversible Markov process.

Freedman introduced the notion of partial exchangeability in continuous time in [7]. Define the transition count from i to j of a finite string of states of length n : $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ to be

$$N_{i,j}(\xi) = \#\{k, 0 \leq k \leq n-1, \xi_k = i, \xi_{k+1} = j\}.$$

Define two finite strings of states ξ, η to be equivalent and denoted $\xi \sim \eta$, if ξ and η start at the same state and the transition count from i to j of any pair (i, j) are equal for ξ and η , i.e. $N_{i,j}(\xi) = N_{i,j}(\eta)$ for all (i, j) .

Definition 3 (Freedman). *A continuous process X_t is partially exchangeable if for each $h > 0$, the law of $\{X_{nh} : n = 1, 2, \dots\}$ satisfies the following property: for any $\xi \sim \eta$ of length l ,*

$$\mathbb{P}(X_0 = \xi_0, \dots, X_{lh} = \xi_l) = \mathbb{P}(X_0 = \eta_0, \dots, X_{lh} = \eta_l).$$

We recall the de Finetti's theorem for continuous time Markov chain introduced by Freedman [7] here,

Theorem 1. *Let X_t be a continued time process starting at $i_0 \in G$, X_t is mixture of Markov jump process starting at i_0 if*

1. X_t has no fixed points of discontinuity, more precisely, for every t , if $t_n \rightarrow t$, then $\mathbb{P}(X_{t_n} \rightarrow X_t) = 1$;
2. X_t is recurrent;
3. X_t is partially exchangeable.

Next, we define the vertex reinforced jump process X_t . Let $(W_e)_{e \in E}$ be weights on edges, the process X_t starts at time 0 at some vertex i_0 , and if X is at vertex $i \in V$ at time t , then, conditioned on the past, the process jumps to a neighbor j of i with rate $W_{i,j}(1 + l_j(t))$, where for $e = \{i, j\}$, $W_{i,j} = W_e$ and $l_j(t)$ is the local time of vertex j at time t :

$$l_j(t) := \int_0^t \mathbf{1}_{X_s=j} ds.$$

This process turns out to be partially exchangeable within a time scale: let

$$D(s) = \sum_{i \in V} (l_i(s)^2 + 2l_i(s)),$$

then the process $Y_t = X_{D^{-1}(t)}$ is a mixture of Markov chains, c.f. [10] Theorem 2.

Now we can state our main theorem.

Theorem 2. *Let X_t be a nearest neighbor jump process on G satisfying the following assumptions:*

1. For all $i \in V$, there exists \mathbb{R}^+ homeomorphisms h_i such that X is partially exchangeable within the time scale $D(s) = \sum_{i \in V} h_i(l_i(s))$;
2. G is strongly connected (i.e. any two adjacent vertices are in a cycle);
3. The process, at vertex i at time t , jumps to a neighbor j of i with rate $f_{i,j}(l_j(t))$ for some continuous functions $f_{i,j}$

Then X is a vertex reinforced jump process within time scale, i.e. there exists another time scale \tilde{D} such that $X_{\tilde{D}^{-1}(t)}$ is a vertex reinforced jump process.

Remarks 1. Note that we do not a priori require $f_{i,j} = f_{j,i}$, i.e. there is no assumption of reversibility for X_t ; however the VRJP is a mixture of reversible Markov jump process within time change.

Remarks 2. Concerning the third assumption, we cannot prove the result with rate $f_{i,j}(l_i, l_j)$, but the case where $f_{i,j}(l_i, l_j) = f_i(l_i)f_j(l_j)$ can be treated. In fact, by applying a time change, the process with rate function of the form $f_i(l_i)f_j(l_j)$ can be reduced to our theorem.

In section 3, we introduce an equivalent notion of partial exchangeability and, as an example, we give a different proof of partial exchangeability of VRJP within a time scale. Section 4 contains the proof of Theorem 2.

3 The two notions of partial exchangeability

3.1 Partial exchangeability, infinitesimal point of view

Consider a nearest neighbor jump process on G satisfying the third assumption of Theorem 2. As we have assumed regularity on the path of the process (c.f. Definition 1), to describe the law of our process, it is enough to describe the probability of the following events:

$$\sigma = \{X_{[0,t_1[} = i_0, X_{[t_1,t_2[} = i_1, X_{[t_2,t_3[} = i_2, \dots, X_{[t_{n-1},t_n[} = i_{n-1}, X_{[t_n,t]} = i_n\},$$

which is also denoted

$$\sigma : i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n}$$

in the sequel and we call such an event *trajectory*.

It turns out that when the jump rate is a continuous function of local times, the law of our process can be characterized by some function, which will be called *density* in the sequel. More precisely:

Definition 4. We say that X_t admits a density if, for all t and for all trajectories

$$\sigma = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n},$$

there exists an explicit function d_σ , which is a function of t_k and i_k , such that for all bounded measurable test function Ψ defined on the trajectories up to time t ,

$$\mathbb{E}(\Psi(X_u, u \leq t)) = \sum_{n \geq 1} \sum_{i_0, \dots, i_n} \int d_\sigma \Psi(\sigma) dt_1 \cdots dt_n + d_{i_0 \xrightarrow{t}} \Psi(i_0 \xrightarrow{t})$$

and obviously we have $d_{i_0 \xrightarrow{t}} = \mathbb{P}(X_s = i_0, 0 \leq s \leq t)$ for the trajectory with no jump. We will call such a function d_σ the density of σ .

Let us now give a notion of partial exchangeability for continuous time process in terms of density. Define two trajectories σ and τ to be equivalent and denoted $\sigma \sim \tau$, if their discrete chain strings are equivalent and the local times are equal at each vertex. Formally,

Definition 5. Let

$$\sigma = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n},$$

$$\tau = j_0 \xrightarrow{s_1} j_1 \xrightarrow{s_2 - s_1} j_2 \cdots j_{n-1} \xrightarrow{s_n - s_{n-1}} j_n \xrightarrow{t - s_n} .$$

Then σ and τ are equivalent if and only if

$$\begin{cases} \forall i \in V, l_i^\sigma(t) = l_i^\tau(t) \\ \forall i, j, N_{i,j}(\sigma) = N_{i,j}(\tau). \end{cases}$$

where $N_{i,j}(\sigma)$ denotes the number of jumps from i to j in σ , i.e. $N_{i,j}(\sigma) = N_{i,j}((i_0, \dots, i_n))$, and $l_i^\sigma(t) = \int_0^t \mathbb{1}_{\sigma_s=i} ds$ denotes the local time.

Definition 6. A continuous time nearest neighbor jump process is said to be partially exchangeable in density if the densities are equal for any two equivalent trajectories. More precisely, the density depends only on the local times and the transition counts.

3.2 Equivalence of the two notions

It turns out that in the case of nearest neighbor jump process with continuous rate functions, the notion of partial exchangeability in Definition 3 and in Definition 6 are equivalent.

Proposition 1. If a continuous time nearest neighbor jump process is partially exchangeable in the sense of Definition 6, then it is partially exchangeable in Freedman's definition.

Proof. Suppose that the process X_t is partially exchangeable in density, let $h > 0$, consider the event $I = \{X_0 = i_0, X_h = i_1, \dots, X_{nh} = i_n\}$, let $(j_0 = i_0, j_1, \dots, j_n)$ be an equivalent string of (i_0, \dots, i_n) , and $J = \{X_0 = j_0, X_h = j_1, \dots, X_{nh} = j_n\}$.

We can construct an application T which maps one continuous trajectory to another in such a way that T maps bijectively from I to J .

More precisely, as these two trajectories are equivalent, for any pair of neighbors (i, j) , there are exactly the same number of transition counts from i to j . Let us define T to be the transformation which is a permutation of the time segmentations $[lh, (l+1)h)$ of size h ; which, for any k , moves the k th transition $i \xrightarrow{k\text{th}} j$ of I to the k th transition $i \xrightarrow{k\text{th}} j$ of J , and leaving the last time segmentation $[nh, \infty)$ invariant. Figure 1 illustrates an example of such application.

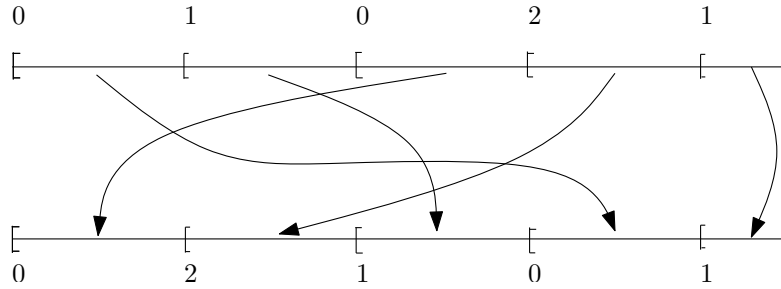


Figure 1: The transformation T for $I = \{X_0 = 0, X_h = 1, X_{2h} = 0, X_{3h} = 2, X_{4h} = 1\}$ and $J = \{X_0 = 0, X_h = 2, X_{2h} = 1, X_{3h} = 0, X_{4h} = 1\}$.

Let

$$\sigma = k_0 \xrightarrow{s_1} k_1 \xrightarrow{s_2} k_2 \cdots k_{N-1} \xrightarrow{s_N} k_N \xrightarrow{s_{N+1}}$$

be one trajectory of the event I , it is not hard to check that

$$T(\sigma) = k'_0 \xrightarrow{s'_1} k'_1 \xrightarrow{s'_2} k'_2 \cdots k'_{N-1} \xrightarrow{s'_N} k'_N \xrightarrow{s'_{N+1}}$$

is a trajectory of the event J , and that T is one-one and on-to (c.f. Figure 2). If we fix the total number of jumps N and the discrete trajectory (k_0, k_1, \dots, k_N) , then T can be thought as a substitution of integration. Thus

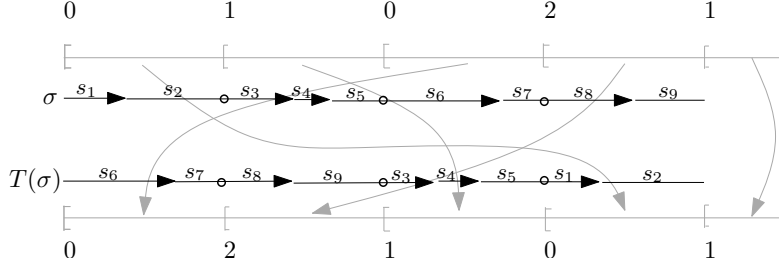


Figure 2: An example of σ and $T(\sigma)$.

$$\begin{aligned} \mathbb{P}(I) &= \sum_N \sum_{k_0, k_1, \dots, k_N} \int \mathbf{1}_{s_1, \dots, s_{N+1} \in I(N, k_0, \dots, k_N)} d_\sigma ds_1 \cdots ds_{N+1} \\ &= \sum_N \sum_{k'_0, k'_1, \dots, k'_N} \int \mathbf{1}_{s_1, \dots, s_N \in I'(N, k'_0, \dots, k'_N)} d_{T(\sigma)} ds_1 \cdots ds_{N+1} = \mathbb{P}(J), \end{aligned}$$

where $I(N, k_0, \dots, k_N)$ is the subset of \mathbb{R}^{N+1} defined as the set of (s_1, \dots, s_{N+1}) such that the event $k_0 \xrightarrow{s_1} k_1 \xrightarrow{s_2} \dots k_N \xrightarrow{s_{N+1}}$ is in I ; and $I'(N, k_0, \dots, k_N)$ is its image by applying T ; see Figure 2 for a concrete example. As T preserves local times and the numbers of transition counts, these two integrals are whence equal. \square

Proposition 2. *If a jump process is partially exchangeable in Freedman's sense, and its jump rate is a continuous function of local times (in which case the density can be written down explicitly), then it is also partially exchangeable in density.*

Proof. Let X_t denote this process, for $h > 0$, consider the σ -algebra $\mathcal{F}_h = \sigma(X_{nh}, n \geq 0)$, let

$$\mathcal{F}_0 = \sigma(\cup_{h>0} \mathcal{F}_h)$$

and

$$\mathcal{F} = \sigma(X_t, t \geq 0).$$

As in [7], we only consider h running through the binary rationals. Note that $\mathcal{F}_0 = \mathcal{F}$ thanks to the right continuity of the trajectories.

Let $\sigma = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_n \xrightarrow{t-t_n}$ be a trajectory with n jumps (say $n \geq 1$ to avoid triviality). Let $\{X^{(h)} \sim \sigma/h\}$ denotes the event

$$\{X_0 = \sigma_0, X_h = \sigma_h, \dots, X_{Nh} = \sigma_{Nh}, \text{ with } N = \lfloor t/h \rfloor\}.$$

It turns out that

$$d_\sigma = \lim_{h \rightarrow 0} \mathbb{P}(X^{(h)} \sim \sigma/h) h^{-n}.$$

In fact, let $\Psi = \mathbf{1}_{X^{(h)} \sim \sigma/h}$, by definition of d_σ ,

$$\mathbb{E}(\Psi(X_u, u \leq t)) = \mathbb{P}(X^{(h)} \sim \sigma/h) = \sum_{k \geq 1} \sum_{i_1, \dots, i_k} \int d_\tau \Psi(\tau) dt_1 \cdots dt_k \quad (1)$$

where

$$\tau = i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{k-1} \xrightarrow{t_k-t_{k-1}} i_n \xrightarrow{t-t_n}.$$

When h is small enough, the sum in (1) must be over $k \geq n$, and we have

$$\mathbb{P}(X^{(h)} \sim \sigma/h) = \mathbb{P}_1 + \mathbb{P}_2.$$

where for some $p_k, k = 1, \dots, n$ depending on h

$$\mathbb{P}_1 = \mathbb{P}((X_u)_{0 \leq u \leq t} \text{ makes } n \text{ jumps at times } s_1, \dots, s_n \text{ with } s_k \in (p_k h, (p_k + 1)h])$$

$$\mathbb{P}_2 = \mathbb{P}((X_u)_{0 \leq u \leq t} \text{ makes more than } n + 1 \text{ jumps and } X^{(h)} \sim \sigma/h)$$

Note that the jump rates are bounded from both below and above, by considering Poisson processes of jump rates C , where C denotes the upper bound of the rate, we have

$$\mathbb{P}_2 \leq \sum_{l \geq 1} \frac{(C(n + [l/2])h)^{n+l}}{(n+l)! e^{C(n+[l/2])h}} [t/h]^l.$$

Applying Stirling's formula gives $P_2 \leq O(h^{n+1})$, thus P_2 can be dropped when taking the limit. In addition,

$$\mathbb{P}_1 = \int_{p_n h}^{(p_n+1)h} \cdots \int_{p_1 h}^{(p_1+1)h} d_\sigma dt_1 \cdots dt_n,$$

note that here d_σ depends only on t_1, \dots, t_n and it is an absolutely integrable function, by Lebesgue differentiation theorem (Theorem 1.6.19 [13]) $\lim_{h \rightarrow 0} \mathbb{P}_1/h^n = d_\sigma$. Now let

$$\begin{array}{ccc} d_\sigma & \longleftarrow & \mathbb{P}(X_{ih} \sim \sigma/h)h^{-n} \\ & & | \quad | \\ d_\tau & \longleftarrow & \mathbb{P}(X_{ih} \sim \tau/h)h^{-n} \end{array}$$

$\sigma \sim \tau$, when h is sufficiently small, proceeding as in the diagram shows that $d_\sigma = d_\tau$. \square

3.3 Example: VRJP is partially exchangeable within a time change

Recall that $Y_s = X_{D^{-1}(s)}$, we can write down the density of the trajectory σ of the (time changed) VRJP process Y (For convenience, write s_{n+1} for s in the sequel), where

$$\sigma := i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2 - s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n - s_{n-1}} i_n \xrightarrow{s - s_n}$$

its density is (c.f. [11])

$$\begin{aligned} d_\sigma = & \left(\frac{1}{2}\right)^n \prod_{k=1}^n W_{i_{k-1}, i_k} \prod_{i \in V, i \neq i_n} \frac{1}{\sqrt{1 + S_i(s)}} \prod_{k=1}^{n+1} ds_k \\ & \cdot \exp\left(-\sum_{i \sim j} \frac{W_{i,j}}{2} (\sqrt{(S_i(s) + 1)(S_j(s) + 1)} - 1)\right), \end{aligned} \quad (2)$$

which depends only on final local times and transition counts, thus by Proposition 1, Y is partially exchangeable. On finite graph it is rather easy to prove that the VRJP is recurrent (for example, using a representation of VRJP by time changed Poisson point process as in [10], and then use an argument as in [2] or [12]). Therefore, Y is a mixture of Markov jump process; in addition, Sabot and Tarres have computed the mixing measure in [10].

For convenient, we include a proof of this in the sequel, as a corollary of the Proposition 3, since the mechanisms of this proof enlightens the proof of the main theorem.

4 Proof of theorem 1

4.1 Computation of densities

Let X be a nearest neighbor jump process on G satisfying the assumptions of Theorem 2, in particular, recall the time scale

$$D(s) = \sum_{i \in V} h_i(l_i(s)). \quad (3)$$

Let $l_i(t)$ be the local time of the process X at vertex i at time t . Let us denote the process after time change to be

$$Y_t = X_{D^{-1}(t)}, \quad (4)$$

let

$$S_i(s) = \int_0^s \mathbb{1}_{Y_u=i} du \quad (5)$$

denote the local time of Y . Consider the trajectory

$$\sigma : i_0 \xrightarrow{t_1} i_1 \xrightarrow{t_2-t_1} i_2 \cdots i_{n-1} \xrightarrow{t_n-t_{n-1}} i_n \xrightarrow{t-t_n} \quad (6)$$

where $0 < t_1 < \cdots < t_n < t$, by applying the time change, the corresponding trajectory for Y is

$$\sigma_Y : i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2-s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n-s_{n-1}} i_n \xrightarrow{s-s_n}$$

where $s_k = D(t_k)$.

Proposition 3. *With the same settings as in equations (3) (4) (5) (6), the density of the trajectory σ_Y for Y is*

$$d_\sigma^Y = \exp \left(- \int_0^s \sum_{j \sim Y_v} \frac{f_{Y_v,j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv \right) \prod_{k=1}^n \frac{f_{i_{k-1},i_k}(h_{i_k}^{-1}(S_{i_k}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{i_{k-1}}(s_k)))}.$$

Proof. Remark that if at time t_k the process just jumps to i_k , then during the time interval $[t_k, t_{k+1})$, the jump rate to any adjacent vertex j is $f_{i_k,j}(l_j(t))$, which is constant equal to $f_{i_k,j}(l_j(t_k))$. Therefore, the holding time $\tau_{k+1} := t_{k+1} - t_k$ is exponentially distributed of rate $\sum_{j \sim i_k} f_{i_k,j}(l_j(t_k))$.

On the other hand, using an elementary property of exponential variable, the probability that at time t_{k+1} the process jumps to i_{k+1} is

$$\frac{f_{i_k,i_{k+1}}(l_{i_{k+1}}(t_k))}{\sum_{j \sim i_k} f_{i_k,j}(l_j(t_k))}.$$

Combining these and using substitution $t_k = \tau_1 + \cdots + \tau_k$, the density d_σ for the process X admits the following explicit form:

$$d_\sigma = \exp \left(- \int_0^t \sum_{j \sim X_u} f_{X_u,j}(l_j(u)) du \right) \prod_{k=1}^n f_{i_{k-1},i_k}(l_{i_k}(t_{k-1})).$$

Recall that in (3) we assumed that $h_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are diffeomorphisms satisfying $h_i(0) = 0$.

Next we compute the same density but for the process $Y_s = X_{D^{-1}(s)}$, as we have $S_i(D(s)) = h_i(l_i(s))$, derivation leads to

$$S_i(D(s))' = D'(s) \mathbb{1}_{Y_{D(s)}=i} = h'_i(l_i(s)) \mathbb{1}_{X_s=i}.$$

Hence

$$(D^{-1}(t))' = \frac{1}{D'(D^{-1}(t))} = \frac{1}{h'_{Y_{t^-}} \circ h_{Y_{t^-}}^{-1}(S_{Y_{t^-}}(t))}$$

$$l_{i_k}(t_{k-1}) = h_{i_k}^{-1}(S_{i_k}(D(t_{k-1}))) = h_{Y_{S_k}}^{-1}(S_{Y_{S_k}}(s_{k-1})).$$

By substitution $s = D^{-1}(t)$, we have

$$d_\sigma^Y = \exp\left(-\int_0^s \sum_{j \sim Y_v} \frac{f_{Y_v, j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv\right) \prod_{k=1}^n \frac{f_{i_{k-1}, i_k}(h_{i_k}^{-1}(S_{i_k}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{i_{k-1}}(s_k)))}.$$

□

Back to the partial exchangeability of VRJP

Proof. Apply the previous proposition to VRJP, where $f_{i,j}(l_j) = W_{i,j}(1+l_j)$ and $h_i(l_i) = l_i^2 + 2l_i$.

The density d_σ^Y is

$$\frac{1}{2^n} \exp\left(-\int_0^s \sum_{j \sim Y_u} \frac{W_{Y_u, j} \sqrt{S_j(u) + 1}}{2\sqrt{S_{Y_u}(u) + 1}} du\right) \prod_{k=1}^n \left(W_{i_{k-1}, i_k} \frac{\sqrt{S_{i_k}(s_{k-1}) + 1}}{\sqrt{S_{i_{k-1}}(s_k) + 1}}\right).$$

As our path is left continuous without explosion, starting at i_0 , if we calculate the product through the path, by telescopic simplification, it results that the product reduces to

$$\prod_{i \in V, i \neq i_n} \frac{1}{\sqrt{S_i(s) + 1}} \prod_{k=1}^n W_{i_{k-1}, i_k}.$$

While the integral inside the exponential becomes

$$\begin{aligned} & \int_0^s \sum_{j \sim Y_u} \frac{W_{Y_u, j} \sqrt{S_j(u) + 1}}{2\sqrt{S_{Y_u}(u) + 1}} du \\ &= \sum_{k=1}^{n+1} \int_{s_{k-1}}^{s_k} \sum_{j \sim Y_u} \frac{W_{Y_u, j} \sqrt{S_j(u) + 1}}{2\sqrt{S_{Y_u}(u) + 1}} du \\ &= \sum_{k=1}^{n+1} \sum_{j \sim i_{k-1}} \frac{W_{i_{k-1}, j}}{2} \sqrt{S_j(s_{k-1}) + 1} \left(\sqrt{S_{i_{k-1}}(s_k) + 1} - \sqrt{S_{i_{k-1}}(s_{k-1}) + 1}\right) \\ &= \sum_{i \sim j} \frac{W_{i, j}}{2} \sum_{k=1}^{n+1} \mathbb{1}_{i_{k-1}=i} \left(\sqrt{(S_j(s_{k-1}) + 1)(S_i(s_k) + 1)} - \sqrt{(S_j(s_{k-1}) + 1)(S_i(s_{k-1}) + 1)}\right). \end{aligned}$$

summing through the path, for every pair i, j , there goes another telescopic simplification, which gives (2), and expression (2) depend only on final local times and transition counts, the result hence follows. □

4.2 Determination of time change h

In the sequel we work with the time changed process Y , to simplify notations, we will write d_σ for d_σ^Y when it does not lead to any confusion. By Proposition 3, the density of certain trajectory contains an exponential term and a product term, let us denote

$$d_\sigma = \exp\left(-\int \sigma\right) \cdot \prod \sigma,$$

with

$$\begin{cases} \int \sigma = \int_0^s \sum_{j \sim Y_v} \frac{f_{Y_v, j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv \\ \prod \sigma = \prod_{k=1}^n \frac{f_{i_{k-1}, i_k}(h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{Y_{s_{k-1}}}(s_k)))} \end{cases}$$

Where the exponential term stems from those exponential waiting times, and the product term corresponds to the probability of the discrete chain.

The heuristics of this subsection is the following: as we assumed partial exchangeability, if we consider two equivalent trajectories, then their densities share the same expression, by comparing them we can hence deduce certain equalities involving $f_{i,j}$ and h_i etc. It turns out that these equalities determine h_i s then $f_{i,j}$ s.

The following fact is simple but important, suppose that at time s , the random walker arrives at i_0 , each vertex i has accumulated local time $l_i := S_i(s)$; then it jumps to i_1 after an amount of time t , by Proposition 3, the density has acquired a multiplicative factor

$$\exp \left(- \int_s^{s+t} \sum_{j \sim i_0} \frac{f_{i_0, j} \circ h_j^{-1}(l_j)}{h'_{i_0} \circ h_{i_0}^{-1}(l_{i_0} + v)} dv \right) \cdot \frac{f_{i_0, i_1} \circ h_{i_1}^{-1}(l_{i_1})}{h'_{i_0} \circ h_{i_0}^{-1}(l_{i_0} + t)}. \quad (7)$$

This fact is in constant use in the sequel, when we explicit the density of certain trajectory.

Lemma 1. *Let $\sigma = i_0 \xrightarrow{s_1} i_1 \xrightarrow{s_2-s_1} i_2 \cdots i_{n-1} \xrightarrow{s_n-s_{n-1}} i_n \xrightarrow{s-s_n}$ be a trajectory, then $\int \sigma = \int \tilde{\sigma} + \int \hat{\sigma}$ where*

$$\int \tilde{\sigma} = \int_0^s \sum_{j \in \sigma, j \sim Y_v} \frac{f_{Y_v, j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv, \quad \int \hat{\sigma} = \int_0^s \sum_{j \notin \sigma, j \sim Y_v} \frac{f_{Y_v, j}(h_j^{-1}(S_j(v)))}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv$$

and if τ is such that $\tau \sim \sigma$, then $\int \hat{\sigma} = \int \hat{\tau}$.

Proof. Note that for $j \notin \sigma$, $S_j(u) = 0$ for all $u \leq s$. Let \hat{H}_i be a primitive of $\frac{1}{h'_i \circ h_i^{-1}}$,

$$\begin{aligned} \int \hat{\sigma} &= \sum_{j \notin \sigma} \int_0^s \mathbb{1}_{Y_v \sim j} \frac{f_{Y_v, j}(0)}{h'_{Y_v}(h_{Y_v}^{-1}(S_{Y_v}(v)))} dv \\ &= \sum_{j \notin \sigma, i \in \sigma, j \sim i} f_{i, j}(0) \int_0^s \frac{\mathbb{1}_{Y_v = i}}{h'_i(h_i^{-1}(S_i(v)))} dv \\ &= \sum_{j \notin \sigma, i \in \sigma, j \sim i} f_{i, j}(0) (\hat{H}(S_i(s)) - \hat{H}(0)) \end{aligned}$$

which depends only on final local times, thus if $\tau \sim \sigma$, then $\int \hat{\tau} = \int \hat{\sigma}$. \square

In the sequel cst denotes some constant, which can vary from line to line.

Lemma 2. *If the process X admits such a time change D which makes it partially exchangeable in density, then for any $i \sim j$, there exists some constants $\lambda_{i,j}$ such that*

$$f_{i,j}(x) = \lambda_{i,j} h'_j(x), \quad \forall x \geq 0. \quad (8)$$

Proof. Let $\epsilon > 0$, consider the following two trajectories for the process Y :

$$\begin{aligned}\sigma &= i \xrightarrow{\epsilon} j \xrightarrow{\epsilon} i \xrightarrow{t} j \xrightarrow{s} i \xrightarrow{\cdot} \\ \tau &= i \xrightarrow{t} j \xrightarrow{s} i \xrightarrow{\epsilon} j \xrightarrow{\epsilon} i \xrightarrow{\cdot}\end{aligned}$$

Note that σ and τ have the same transition counts and the final local times on vertex i, j are respectively equal. Thus the densities of these trajectories are a.s. equal by partial exchangeability. By Lemma 1,

$$d_\sigma = \prod \sigma \cdot \exp\left(\int \tilde{\sigma} + \int \hat{\sigma}\right),$$

with

$$\begin{cases} \prod \sigma = \frac{f_{i,j} \circ h_j^{-1}(0)}{h'_i \circ h_i^{-1}(\epsilon)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon)}{h'_j \circ h_j^{-1}(\epsilon)} \cdot \frac{f_{i,j} \circ h_j^{-1}(\epsilon)}{h'_i \circ h_i^{-1}(\epsilon+t)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h'_j \circ h_j^{-1}(\epsilon+s)} \\ \int \tilde{\sigma} = \int_0^\epsilon \frac{f_{i,j} \circ h_j^{-1}(0)}{h'_i \circ h_i^{-1}(v)} dv + \int_0^\epsilon \frac{f_{j,i} \circ h_i^{-1}(\epsilon)}{h'_j \circ h_j^{-1}(v)} dv + \int_0^t \frac{f_{i,j} \circ h_j^{-1}(\epsilon)}{h'_i \circ h_i^{-1}(\epsilon+v)} dv + \int_0^s \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h'_j \circ h_j^{-1}(\epsilon+v)} dv. \end{cases}$$

$$d_\tau = \prod \tau \cdot \exp\left(\int \tilde{\tau} + \int \hat{\tau}\right),$$

with

$$\begin{cases} \prod \tau = \frac{f_{i,j} \circ h_j^{-1}(0)}{h'_i \circ h_i^{-1}(t)} \cdot \frac{f_{j,i} \circ h_i^{-1}(t)}{h'_j \circ h_j^{-1}(s)} \cdot \frac{f_{i,j} \circ h_j^{-1}(s)}{h'_i \circ h_i^{-1}(t+\epsilon)} \cdot \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h'_j \circ h_j^{-1}(\epsilon+s)} \\ \int \tilde{\tau} = \int_0^t \frac{f_{i,j} \circ h_j^{-1}(0)}{h'_i \circ h_i^{-1}(v)} dv + \int_0^s \frac{f_{j,i} \circ h_i^{-1}(t)}{h'_j \circ h_j^{-1}(v)} dv + \int_0^\epsilon \frac{f_{i,j} \circ h_j^{-1}(s)}{h'_i \circ h_i^{-1}(t+v)} dv + \int_0^\epsilon \frac{f_{j,i} \circ h_i^{-1}(\epsilon+t)}{h'_j \circ h_j^{-1}(s+v)} dv; \end{cases}$$

where we do not explicit $\int \hat{\sigma}$ and $\int \hat{\tau}$ as they cancel when we compare these expressions (c.f. Lemma 1).

Letting $\epsilon \rightarrow 0$ yields that $\exp(\int \tilde{\sigma}) = \exp(\int \tilde{\tau})$; therefore $\prod \sigma = \prod \tau$, i.e.

$$\forall s, t, \quad \frac{f_{i,j} \circ h_j^{-1}(s)}{h'_i \circ h_i^{-1}(s)} \cdot \frac{f_{j,i} \circ h_i^{-1}(t)}{h'_j \circ h_j^{-1}(t)} = cst.$$

Now fix t , let s vary, whence

$$\forall s, \quad f_{i,j} \circ h_j^{-1}(s) = cst \cdot h'_i \circ h_i^{-1}(s),$$

and let $\lambda_{i,j}$ denotes this constant, as h_j^{-1} is a diffeomorphism, its range is \mathbb{R}^+ , which allows us to conclude. \square

The next lemma states in some sense that the exponential part and the product part appearing in the density of a trajectory can be treated separately.

Lemma 3. *Let σ, τ be two trajectories, and denote*

$$d_\sigma = \exp\left(\int \sigma\right) \cdot \prod \sigma, \quad d_\tau = \exp\left(\int \tau\right) \cdot \prod \tau,$$

if $\sigma \sim \tau$, then $\prod \sigma = \prod \tau$.

Proof. We have $S_{Y_{s_k}}(s_k) = S_{Y_{s_k}}(s_{k-1})$, thus Lemma 2 yields that $f_{i_{k-1}, i_k} \circ h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})) = \lambda_{i_{k-1}, i_k} h'_{i_k} \circ h_{i_k}^{-1}(S_{Y_{s_k}}(s_k))$. Whence the product part is

$$\prod \sigma = \prod_{k=1}^n \frac{f_{i_{k-1}, i_k}(h_{i_k}^{-1}(S_{Y_{s_k}}(s_{k-1})))}{h'_{i_{k-1}}(h_{i_{k-1}}^{-1}(S_{Y_{s_{k-1}}}(s_k)))} ds_k = \prod_{k=1}^n \lambda_{i_{k-1}, i_k} \prod_{i \in V} \frac{1}{h'_i \circ h_i^{-1}(S_i(t_n))},$$

and the last term depends only on the transition counts and final local times. \square

Lemma 4. Let $H_i = h'_i \circ h_i^{-1}$, then for some constant A_i ,

$$(H_i^2)' = A_i \text{ and if } i \sim j, \text{ then } \lambda_{i,j} A_j = \lambda_{j,i} A_i.$$

Remarks 3. The latest equality tells that the process is reversible. However, we did not assume the reversibility of the process, but vertex reinforced jump processes are reversible (as a mixture of reversible Markov jump process), so are the edge reinforced random walks. In contrast, directed edge reinforced random walks are mixtures of non reversible Markov chains, with independent Dirichlet environments. We can hence expect that the reversibility is a consequence of a non oriented linear reinforcement (where linearity corresponds to partial exchangeability).

Proof. Recall that we have assumed that the graph is strongly connected, i.e. if i, j are two adjacent vertices, there exists a shortest cycle $i_1 \sim i_2 \sim i_3 \cdots \sim i_n \sim i_1$ with $i_1 = i, i_n = j$ and the i_k s are distinct.

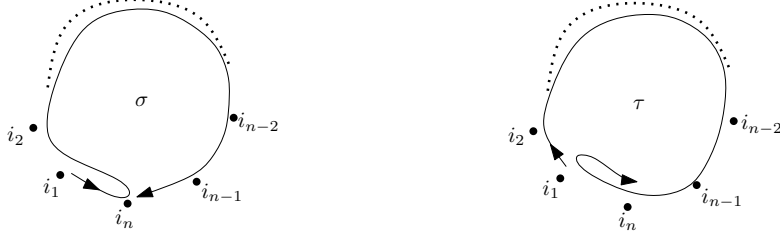


Figure 3: the trajectories σ and τ in Lemma 4.

Let $(i_1 = i, i_2, i_3, \dots, i_n = j)$ be a cycle as described, consider the trajectories (c.f. Figure 3)

$$\begin{aligned} \sigma &= i_1 \xrightarrow{r_1} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_2 \xrightarrow{s_2} i_3 \cdots i_{n-2} \xrightarrow{s_{n-2}} i_{n-1} \xrightarrow{s_{n-1}} i_n \\ \tau &= i_1 \xrightarrow{r_1} i_2 \xrightarrow{s_2} i_3 \cdots i_{n-2} \xrightarrow{s_{n-2}} i_{n-1} \xrightarrow{s_{n-1}} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_n. \end{aligned}$$

As $\sigma \sim \tau$, by Lemma 3 and Lemma 1, $\int \tilde{\sigma} = \int \tilde{\tau}$. Also let

$$\begin{aligned} \sigma' &= i_1 \xrightarrow{r_1} i_n \xrightarrow{r_2} i_1 \xrightarrow{s_1} i_2 \xrightarrow{s_2} i_1 \\ \tau' &= i_1 \xrightarrow{r_1} i_2 \xrightarrow{s_2} i_1 \xrightarrow{s_1} i_n \xrightarrow{r_2} i_1, \end{aligned}$$

thus $\int \tilde{\sigma}' = \int \tilde{\tau}'$. We are going to compute explicitly $\int \tilde{\sigma}$, $\int \tilde{\tau}$ etc, using (7), let $s = r_1 + r_2 + s_1 + \cdots + s_{n-1}$ and recall that \hat{H}_i is a primitive of $\frac{1}{h'_i \circ h_i^{-1}}$.

$$\begin{aligned} \int \tilde{\sigma} &= \sum_{(i,j) \in \sigma^2, i \sim j} \lambda_{i,j} \int_0^s \mathbb{1}_{Y_v=i} \frac{h'_j \circ h_j^{-1}(v)}{h'_i \circ h_i^{-1}(v)} dv \\ &= \lambda_{i_1, i_2} H_{i_2}(0) (\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(0)) + \lambda_{i_2, i_1} H_{i_1}(r_1 + s_1) (\hat{H}_{i_2}(s_2) - \hat{H}_{i_2}(0)) \\ &\quad + \lambda_{i_1, i_n} \left(H_{i_n}(0) (\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_n}(r_2) (\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_n, i_1} H_{i_1}(r_1) (\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) + \lambda_{i_n, i_{n-1}} H_{i_{n-1}}(0) (\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) \\ &\quad + \lambda_{i_{n-1}, i_n} H_{i_n}(r_2) (\hat{H}_{i_{n-1}}(s_{n-1}) - \hat{H}_{i_{n-1}}(0)) + \Delta \end{aligned}$$

where Δ is defined as follows: let $Q_k := H_{i_k}(0)(\hat{H}_{i_{k-1}}(s_{i_{k-1}}) - \hat{H}_{i_{k-1}}(0))$ and $Q'_k := H_{i_k}(s_k)(\hat{H}_{i_{k+1}}(s_{i_{k+1}}) - \hat{H}_{i_{k+1}}(0))$,

$$\Delta = \sum_{k=3}^{n-1} \lambda_{i_{k-1}, i_k} Q_k + \lambda_{i_k, i_{k-1}} Q'_{k-1}.$$

For $\tilde{\tau}$ we have:

$$\begin{aligned} \int \tilde{\tau} &= \sum_{(i,j) \in \tau^2, i \sim j} \lambda_{i,j} \int_0^s \mathbf{1}_{Y_v=i} \frac{h'_j \circ h_j^{-1}(v)}{h'_i \circ h_i^{-1}(v)} dv \\ &= \lambda_{i_1, i_2} H_{i_2}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_2}(s_2)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \\ &\quad + \lambda_{i_2, i_1} H_{i_1}(r_1)(\hat{H}_{i_2}(s_2) - \hat{H}_{i_2}(0)) \\ &\quad + \lambda_{i_1, i_n} \left(H_{i_n}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_n}(r_2)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_n, i_1} H_{i_1}(r_1)(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) + \lambda_{i_n, i_{n-1}} H_{i_{n-1}}(s_{n-1})(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) \\ &\quad + \lambda_{i_{n-1}, i_n} H_{i_n}(0)(\hat{H}_{i_{n-1}}(s_{n-1}) - \hat{H}_{i_{n-1}}(0)) + \Delta \end{aligned}$$

with the same Δ . Also

$$\begin{aligned} \int \tilde{\sigma}' &= \lambda_{i_1, i_2} \left(H_{i_2}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_2}(0)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_2, i_1} H_{i_1}(r_1 + s_1)(\hat{H}_{i_2}(s_2) - \hat{H}_{i_2}(0)) \\ &\quad + \lambda_{i_1, i_n} \left(H_{i_n}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_n}(r_2)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_n, i_1} H_{i_1}(r_1)(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) \\ \\ \int \tilde{\tau}' &= \lambda_{i_1, i_2} \left(H_{i_2}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_2}(s_2)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_2, i_1} H_{i_1}(r_1)(\hat{H}_{i_2}(s_2) - \hat{H}_{i_2}(0)) \\ &\quad + \lambda_{i_1, i_n} \left(H_{i_n}(0)(\hat{H}_{i_1}(r_1) - \hat{H}_{i_1}(0)) + H_{i_n}(0)(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \right) \\ &\quad + \lambda_{i_n, i_1} H_{i_1}(r_1 + s_1)(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)). \end{aligned}$$

Recall that $\int \sigma - \int \sigma' = \int \tau - \int \tau'$, which leads to

$$\begin{aligned} &\lambda_{i_n, i_{n-1}} H_{i_{n-1}}(0)(\tilde{H}_{i_n}(r_2) - \tilde{H}_{i_n}(0)) + \lambda_{i_{n-1}, i_n} H_{i_n}(r_2)(\hat{H}_{i_{n-1}}(s_{n-1}) - \hat{H}_{i_{n-1}}(0)) \\ &= \lambda_{i_1, i_n} (H_{i_n}(r_2) - H_{i_n}(0))(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) \\ &\quad + \lambda_{i_n, i_1} (H_{i_1}(r_1) - H_{i_1}(r_1 + s_1))(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) \\ &\quad + \lambda_{i_n, i_{n-1}} H_{i_{n-1}}(s_{n-1})(\tilde{H}_{i_n}(r_2) - \tilde{H}_{i_n}(0)) + \lambda_{i_{n-1}, i_n} H_{i_n}(0)(\hat{H}_{i_{n-1}}(s_{n-1}) - \hat{H}_{i_{n-1}}(0)) \end{aligned}$$

letting $s_{n-1} \rightarrow 0$ leads to

$$\begin{aligned} &\lambda_{i_1, i_n} (H_{i_n}(r_2) - H_{i_n}(0))(\hat{H}_{i_1}(r_1 + s_1) - \hat{H}_{i_1}(r_1)) = \\ &\lambda_{i_n, i_1} (H_{i_1}(r_1 + s_1) - H_{i_1}(r_1))(\hat{H}_{i_n}(r_2) - \hat{H}_{i_n}(0)) \end{aligned}$$

as i_1, i_n, r_2, s_1, r_1 are arbitrary, divide the formula by $r_2 s_1$ and let r_2, s_1 go to zero leads to

$$\lambda_{i_1, i_n} H'_{i_n}(0) \hat{H}'_{i_1}(r_1) = \lambda_{i_n, i_1} H'_{i_1}(r_1) \hat{H}'_{i_n}(0),$$

finally note that $\hat{H}'_i = 1/H_i$, thus $\lambda_{i_1, i_n} (H_{i_n}^2)'(0) = \lambda_{i_n, i_1} (H_{i_1}^2)'(r_1)$. \square

Lemma 5. For all $i \sim j$, let $W_{i,j} = \lambda_{i,j}A_j/2 = \lambda_{j,i}A_i/2$, there exists constant D_j depends only on j , such that $f_{i,j}(x) = W_{i,j}x + D_j$.

Proof. As $(H_j^2(s))' = A_j$, there exists B_j such that $H_j^2(s) = A_j s + B_j$, therefore

$$f_{i,j} \circ h_j^{-1}(s) = \lambda_{i,j}H_j(s) = \lambda_{i,j}\sqrt{A_j s + B_j}.$$

On the other hand, $(h_j^{-1})'(s) = \frac{1}{\sqrt{A_j s + B_j}}$, thus for some C_j ,

$$h_j^{-1}(s) = \frac{2}{A_j}\sqrt{A_j s + B_j} + C_j.$$

$f_{i,j}(h_j^{-1}(s)) = f_{i,j}(\frac{2}{A_j}\sqrt{A_j s + B_j} + C_j) = \lambda_{i,j}\sqrt{A_j s + B_j}$, which leads to

$$f_{i,j}(x) = W_{i,j}x + D_j,$$

where D_j is some constant depends only on j . Applying the time change

$$D(s) = \sum_i \frac{l_i(s) - D_i}{D_i},$$

the resulting process will be of jump rate

$$W_{i,j}D_i D_j(1 + T_j(t))$$

where $T_j(t)$ is the local time for the time changed process $Z_t = X_{D^{-1}(t)}$. \square

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