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# On the path cover number of $k$-assignable arbitrarily partitionable graphs

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**Abstract**

A graph $G$ is arbitrarily partitionable if for every partition $\pi = (n_1, n_2, ..., n_p)$ of $|V(G)|$ there is a partition $(V_1, V_2, ..., V_p)$ of $V(G)$ such that $G[V_i]$ is a connected graph on $n_i$ vertices for every $i \in \{1, 2, ..., p\}$. If additionally any $k$ arbitrary vertices of $G$ can each be assigned to one part of the resulting vertex partition, then $G$ is a $k$-assignable arbitrarily partitionable graph. All $k$-assignable arbitrarily partitionable graphs exhibited so far have an Hamiltonian path. Using the notion of path cover, we show that this Hamiltonian condition is not a necessary one, in the sense that $k$-assignable arbitrarily partitionable graphs can have arbitrarily small longest paths (compared to their orders).

## 1 Introduction

An *$n$-sequence* $\pi = (n_1, n_2, ..., n_p)$ is a sequence of positive integers summing up to some integer $n \geq 1$. By the *size* of $\pi$, we refer to its number of elements, that is $p$. We say that $\pi$ is *realizable* in some graph $G$ with order $n$ if there is a *realization* of $\pi$ in $G$, i.e. a partition $(V_1, V_2, ..., V_p)$ of $V(G)$ such that $G[V_i]$ is a connected subgraph on $n_i$ vertices for every $i \in \{1, 2, ..., p\}$. In case all $n$-sequences are realizable in $G$, we say that $G$ is *arbitrarily partitionable* (AP for short).

A $k$-tuple $P = (v_1, v_2, ..., v_k)$ of $k \leq p$ distinct vertices of $G$ is called a *$k$-prescription* of $G$. We say that $\pi$ is *$P$-realizable* in $G$ if there is a *$P$-realization* of $\pi$ in $G$, i.e. a realization $(V_1, V_2, ..., V_p)$ such that $v_1 \in V_1, v_2 \in V_2, ..., v_k \in V_k$. In other words, not only we want the realization to satisfy the sizes and connectivity constraints, but also $k$ prescribed vertices must each belong to one of the resulting subgraphs (each $i^{th}$ prescribed vertex must belong to the subgraph whose order is the $i^{th}$ element of $\pi$ by convention). We say that $G$ is $k$-*assignable* arbitrarily partitionable (AP+$k$ for short) if every $n$-sequence with size at least $k$ is $P$-realizable in $G$ for every $k$-prescription $P$ of $G$.

Perhaps one of the most influential result regarding the partition of graphs into connected subgraphs is the following result proved independently by Györi and Lovász in the 1970's.

**Theorem 1** ([7] and [5], independently)**.** *Every $|V(G)|$-sequence with size $k$ is $P$-realizable in any $k$-connected graph $G$ for every $k$-prescription $P$ of $G$.*

The introduction of AP graphs is much more recent and may be attributed to [1], wherein AP graphs are introduced as graphs modelling networks with convenient sharing properties. AP+$k$ graphs were then introduced in [2] notably to bind AP graphs and the vertex prescription notion from Theorem 1. Only a few things are known about AP+$k$ graphs though. It was shown in [2] that $k^{th}$ powers of *traceable* graphs, i.e. graphs with an Hamiltonian path, and *Hamiltonian* graphs are AP+($k-1$) and AP+($2k-1$), respectively. The minimum size of an AP+$k$ graph on $n$ vertices was investigated in [3], wherein it is shown that such graphs have at least $\lceil \frac{n(k+1)}{2} \rceil$ edges. This lower bound cannot be improved, due to the following observation resulting from the fact that prescribing a vertex to a part with size 1 is like removing it from the graph.

**Observation 2.** *Every AP+$k$ graph is $(k+1)$-connected.*

Of course checking whether a graph is AP+$k$ is laborious due to the number of sequences and prescriptions to consider. Regarding the AP+$k$ graphs exhibited in the mentioned above works, the checking process was actually made easier because these graphs have good Hamiltonian properties, which turned out to be convenient in our context. To be convinced of that statement, just note that every traceable graph is AP, every Hamiltonian graph is AP+1, and every *Hamiltonian-connected* graph, i.e. which has an Hamiltonian path joining every two of its vertices, is AP+2 (see [3]).

We herein consider the opposite question, namely does an AP+$k$ graph necessarily have to be (kind of) Hamiltonian? We answer this question in the negative throughout. Namely, we show that AP+$k$ graphs can have arbitrarily small longest paths (compared to their orders). For this purpose, we consider the notion of *path cover*, where a path cover of some graph $G$ is a collection of vertex-disjoint paths covering all vertices of $G$. It is worth mentioning that the notion of path cover and Theorem 1 were already bind in e.g. [6], where a path version of Theorem 1 is investigated.

This paper is organized as follows. First, we introduce some terminology and preliminary results in Section 2. We then show, in Section 3, that, given some non-traceable AP+$k$ graph, we can construct AP+$k$ graphs with arbitrarily large path cover numbers. In Section 4, we prove that such non-traceable AP+$k$ graphs can be constructed inductively starting from a small non-traceable AP+1 graph. Combining these results, we are able to construct AP+$k$ graphs with arbitrarily large path cover numbers, and hence longest paths arbitrarily smaller than their orders.

## 2 Preliminaries

Let $G$ be a graph. We denote $\varsigma(G)$ the order of a longest path of $G$. In case a path has order 1 (and hence length 0), we call this path *trivial*. The *path cover number* of $G$ is defined as

$$\mu(G) := \inf\{|C| : C \text{ is a path cover of } G\}.$$

Now consider two positive integers $\nu \geq 1$ and $\ell \geq 1$, and $\ell$ graphs $G_1, G_2, ..., G_\ell$. By $K_\nu(G_1, G_2, ..., G_\ell)$, we refer to the graph obtained as follows:

1. take the disjoint union of $G_1, G_2, ..., G_\ell$,
2. add $\nu$ new vertices $u_1, u_2, ..., u_\nu$ to the graph,
3. turn $u_1, u_2, ..., u_\nu$ into *universal* vertices, i.e. add an edge between every $u_i$ and every other vertex of $K_\nu(G_1, G_2, ..., G_\ell)$.

As a first result, observe that the path cover number of any graph $K_\nu(G_1, G_2, ..., G_\ell)$ is quite related to the path cover numbers of $G_1, G_2, ..., G_\ell$.

**Observation 3.** *We have* $\mu(K_\nu(G_1, G_2, ..., G_\ell)) = (\sum_{i=1}^{\ell} \mu(G_i)) - \nu$.

*Proof.* Typically a minimum path cover of $K_\nu(G_1, G_2, ..., G_\ell)$ is obtained by considering minimum path covers of $G_1, G_2, ..., G_\ell$ and then "glueing" the ends of some of the resulting paths thanks to the universal vertices. In particular, since $K_\nu(G_1, G_2, ..., G_\ell)$ has $\nu$ universal vertices, we can "replace" $\nu + 1$ vertex-disjoint paths $P_1, P_2, ..., P_{\nu+1}$ from minimum path covers of $G_1, G_2, ..., G_\ell$ with the long path $P_1 u_1 P_2 u_2 P_3 ... P_\nu u_\nu P_{\nu+1}$, where $u_1, u_2, ..., u_\nu$ are the universal vertices of $K_\nu(G_1, G_2, ..., G_\ell)$. □

Previous Observation 3 implies the following.

**Observation 4.** *Let* $G_1, G_2, ..., G_\ell$ *be* $\ell$ *graphs with* $\mu(G_1) \geq \mu(G_2) \geq ... \geq \mu(G_\ell)$. *Then we have* $\mu(K_\nu(G_1, G_2, ..., G_\ell)) > \mu(G_1)$ *whenever* $\nu < \sum_{i=2}^{\ell} \mu(G_i)$.

*Proof.* The result follows by just replacing $\mu(K_\nu(G_1, G_2, ..., G_\ell))$ in the inequality $\mu(K_\nu(G_1, G_2, ..., G_\ell)) > \mu(G_1)$ with its explicit value given in Observation 3. $\square$

We now point out the following relationship between $\mu(G)$ and $\varsigma(G)$ for every graph $G$ with order $n$.

**Observation 5.** *We have* $\varsigma(G) \leq n - \mu(G) + 1$.

*Proof.* Assume the claim is not true. Then $\mu(G) > n - \varsigma(G) + 1$. Now consider any (not necessarily minimum) path cover $C$ of $G$ including one path with order $\varsigma(G)$. Clearly we have $|C| \leq n - \varsigma(G) + 1$ since, in the worst case, all paths of $C$ which do not have order $\varsigma(G)$ are trivial. We hence have

$$n - \varsigma(G) + 1 < \mu(G) \leq |C| \leq n - \varsigma(G) + 1,$$

a contradiction. $\square$

## 3 Path cover number and AP+$k$ graphs

In this section, we show that, given some non-traceable AP+$k$ graph, we can construct AP+$k$ graphs with arbitrarily large path cover numbers (and hence arbitrarily small longest paths according to Observation 5). This relies on the following lemma.

**Lemma 6.** *Let $G_1, G_2, ..., G_\ell$ be $\ell$ AP+$k$ graphs. Then $K_\nu(G_1, G_2, ..., G_\ell)$ is AP+$k$ whenever $\nu \geq k + \ell - 1$.*

*Proof.* Let $G = K_\nu(G_1, G_2, ..., G_\ell)$ be a graph with order $n$ for given AP+$k$ graphs $G_1, G_2, ..., G_\ell$, and denote $u_1, u_2, ..., u_\nu$ the universal vertices of $G$. Consider further any $n$-sequence $\pi = (n_1, n_2, ..., n_p)$ and $k$-prescription $P = (v_1, v_2, ..., v_k)$ of $G$.

Our strategy for deducing a $P$-realization of $\pi$ in $G$ relies on the fact that if any partial part of the realization is missing some vertices but contains one of the universal vertices, say $u_1$, of $G$, then we can easily complete this part with arbitrary unused vertices since $u_1$ neighbours any other vertex of $G$. This property allows us to proceed as follows. Consider one of the $G_i$'s, say $G_1$, and pick as many parts of the realization as possible in $G_1$ using the fact that $G_1$ is AP. In case some parts fit exactly in $G_1$, i.e. there is some subsequence of $\pi$ which sums up to $|V(G_1)|$, everything is fine. Otherwise, we can "fill" $G_1$ with some parts, but one part exceeds from $G_1$. In this situation, we have to add a universal vertex to this part so that it can be completed later (if necessary). The only things we have to make sure of are that we respect the prescription $P$, and that not too many parts have to be completed once we filled all of the $G_i$'s with as many parts as possible (i.e. at most $\nu$, the number of universal vertices in $G$).

We now explicit the two steps of the described above strategy, which we refer to as the Filling and Completing Steps.

**Filling Step.**

The Filling step is achieved within two steps: we first deal with the prescribed universal vertices before considering the $G_i$'s and filling them as described above.

**Step 1.** For every vertex $v_i \in P \cap U$, start with the partial part $\{v_i\}$ which will be completed during the Completing Step (this will be possible since this part already contains a universal vertex).

**Step 2.** Let $U := \{u_1, u_2, ..., u_\nu\}$. At any moment of the procedure, we denote by $\pi_r := (r_1,\ r_2,\ ...,\ r_q)$ the sequence made up of the *remaining* non-prescribed part sizes of $\pi$, i.e. which have not been considered yet. In particular, we start with $\pi_r = (n_{k+1}, n_{k+2}, ..., n_p)$ (but we always refer to the elements of $\pi_r$ as $r_1, r_2, ..., r_q$ for the sake of simplicity).

Now consider the $G_i$'s in order. We assume throughout that we deal with $G_1$ for the sake of clarity. We distinguish several cases.

**Case 1:** $V(G_1) \cap P = \emptyset$.

We are in the situation where $G_1$ contains no prescribed vertices. If there is an $i \in \{1, 2, ..., q\}$ such that $r_1 + r_2 + ... + r_i = |V(G_1)|$, then the parts with size $r_1, r_2, ..., r_i$ of the realization can be deduced by considering a realization of $(r_1, r_2, ..., r_i)$ in $G_1$, which exists since $G_1$ is AP. In this situation, no part has to be completed.

If $r_1 + r_2 + ... + r_q < |V(G_1)|$, then we can deduce a realization of $(r_1, r_2, ..., r_q, |V(G_1)| - (r_1 + r_2 + ... + r_q))$ in $G_1$, which again exists. Again, no part has to be completed (actually, the vertices from the part with size $|V(G_1)| - (r_1 + r_2 + ... + r_q)$ will be available during the Completing Step to fill some partial parts).

If we are not in one of the two previous cases, then there is an $i \in \{1, 2, ..., q\}$ such that $r_1 + r_2 + ... + r_{i-1} < |V(G_1)|$ and $r_1 + r_2 + ... + r_i > |V(G_1)|$. Let $r_i'$ and $r_i''$ be two positive integers such that $r_i = r_i' + r_i''$, and $r_1 + r_2 + ... + r_{i-1} + r_i' = |V(G_1)|$. Since $G_1$ is AP, we can deduce a realization of $(r_1, r_2, ..., r_{i-1}, r_i')$ in $G_1$. In this situation, the part $V_i$ with size $r_i'$ is incomplete and has to be completed with $r_i'' \geq 1$ additional vertices. Then pick an unused vertex from $U$, and add it to $V_i$ so that $V_i$ can be completed during the Completing Step (if necessary).

In any of these three situations, remove the elements off $\pi_r$ which have already been treated.

**Case 2:** $G_1$ *includes* $t$ *of the prescribed vertices, where* $1 \leq t \leq k$.

Denote $v_1, v_2, ..., v_t$ the vertices of $V(G_1) \cap P$. We consider three main cases. At first, if $n_1+n_2+...+n_t = |V(G_1)|$, then just consider a $(v_1, v_2, ..., v_t)$-realization of $(n_1, n_2, ..., n_t)$ in $G_1$, which exists since $G_1$ is AP+$k$ with $k \geq t$. Note that in doing so, no partial part will have to be completed during the Completing Step.

Now, if on the one hand we have $n_1 + n_2 + ... + n_t > |V(G_1)|$, then let $i \in \{1, 2, ..., t\}$ be the value for which $n_1+n_2+...+n_{i-1}+(t-(i-1)) < |V(G_1)|$ and $n_1 + n_2 + ... + n_i + (t-i) > |V(G_1)|$. Set $n'_i := |V(G_1)| - (n_1 + n_2 + ... + n_{i-1}) - (t-i)$. Since $G_1$ is AP+$k$ with $k \geq t$, we can deduce a $(v_1, v_2, ..., v_t)$-realization of $(n_1, n_2, ..., n_{i-1}, n'_i, 1, 1, ..., 1)$, where the value 1 is repeated $t-i$ times, in $G_1$. In doing so, note that all of the prescribed vertices $v_1, v_2, ..., v_t$ belong to distinct parts. Besides, the parts which are intended to have size $n_1, n_2, ..., n_{i-1}$ are complete, while the $i^{th}$ part is missing $n_i - n'_i$ vertices, and the other parts are missing all but one vertex (except if these parts are intended to have size 1). Then add an unused universal vertex in each of the parts which have to be completed (there are at most $t - i + 1$ of them).

If, on the other hand, we have $n_1+n_2+...+n_t < |V(G_1)|$, then proceed as follows. First, if $\pi_r$ is empty, then just consider a $(v_1, v_2, ..., v_t)$-realization of $(n_1, n_2, ..., n_t, |V(G_1)| - (n_1 + n_2 + ... + n_t))$ in $G_1$, which exists since $G_1$ is AP+$k$ with $k \geq t$. The vertices from the resulting part with size $|V(G_1)| - (n_1 + n_2 + ... + n_t)$ will actually be used during the Completing Step to complete some partial parts. Second, i.e. $\pi_r$ has elements, let $i \in \{1, 2, ..., q+1\}$ be the maximum index for which $n_1 + n_2 + ... + n_t + r_1 + r_2 + ... + r_{i-1} < |V(G_1)|$. In case $i = q + 1$, i.e. all of the parts with size $n_1, n_2, ..., n_t$ as well as those whose sizes are not prescribed can be picked in $G_1$, just consider a $(v_1, v_2, ..., v_t)$-realization of

$$(n_1, n_2, ..., n_t, r_1, r_2, ..., r_q, |V(G_1)| - (n_1 + n_2 + ... + n_t + r_1 + r_2 + ... + r_q))$$

in $G_1$, which exists since $G_1$ is AP+$k$ with $t \leq k$. Again, the vertices from the extra part will be available for the Completing Step. If we are not in this case, i.e. $i < q + 1$, then again split $r_i$ into two integers $r'_i$ and $r''_i$ (with $r_i = r'_i + r''_i$) such that $n_1 + n_2 + ... + n_t + r_1 + r_2 + ... + r_{i-1} + r'_i = |V(G_1)|$, and consider a $(v_1, v_2, ..., v_t)$-realization of $(n_1, n_2, ..., n_t, r_1, r_2, ..., r_{i-1}, r'_i)$ in $G_1$. Again, add an unused universal vertex to the part with size $r'_i$ so that it can eventually be completed with $r''_i - 1$ additional vertices during the Completing Step.

Again, after any of these cases, remove the first elements from $\pi_r$ whose associated connected subgraphs have already been (possibly partially) picked.

**Completing Step.**

After the Filling Step, any part $V_i$ of the realization is either complete, i.e. it already has size $n_i$, or it is missing some vertices but $V_i$ contains a universal vertex by construction. Then just add $n_i - |V_i|$ unused vertices to $V_i$, i.e. vertices which belong to some extra parts. This part still induces a connected subgraph since it contains a universal vertex.

Regarding the correctness of the process, note that all of the parts induce connected subgraphs, have the correct sizes, and include a prescribed vertex (if required). Furthermore note that the number of partial parts we have to deal with during the Completing Step is at most $k + \ell - 1$. Indeed, by Step 1 at most $k$ such partial parts may result (this upper bound is typically reached in the extremal case where all prescribed vertices are also universal), while at most $\ell - 1$ such partial parts may arise during Step 2 (this is reached when the picked parts exceed from the $\ell - 1$ first $G_i$'s considered). The $P$-realization of $\pi$ in $G$ is then always eventually obtained under the assumption $\nu \geq k + \ell - 1$, while one can imagine situations in which the Filling and Completing Steps cannot provide a solution when $\nu < k + \ell - 1$. □

Using Lemma 6, we now prove the main result of this section.

**Theorem 7.** *Let $G_1, G_2, ..., G_{k+2}$ be $k+2$ copies of a non-traceable AP+$k$ graph $G$. Then $\mu(K_{2k+1}(G_1, G_2, ..., G_{k+2})) > \mu(G)$ and $K_{2k+1}(G_1, G_2, ..., G_{k+2})$ is AP+$k$.*

*Proof.* Since $2k + 1 \geq k + (k + 2) - 1$, the graph $K_{2k+1}(G_1, G_2, ..., G_{k+2})$ is AP+$k$ according to Lemma 6. Besides, since we have

$$\mu(G_1) = \mu(G_2) = ... = \mu(G_{k+2}) = \mu(G) \geq 2$$

by the non-traceability of $G$, we get

$$\sum_{i=2}^{k+2} \mu(G_i) \geq 2(k + 1) > 2k + 1.$$

We then also have $\mu(K_{2k+1}(G_1, G_2, ..., G_{k+2})) > \mu(G)$ according to Observation 4, as claimed. □

**Corollary 8.** *Provided a non-traceable AP+$k$ graph, we can construct AP+$k$ graphs with arbitrarily large path cover numbers and arbitrarily small longest paths (compared to their orders).*

*Proof.* Let $G$ be such a non-traceable AP+$k$ graph, and let $(G_0, G_1, G_2, ..., G_q)$ be the sequence of $q + 1$ graphs defined inductively as follows.

- $G_0 := G$.

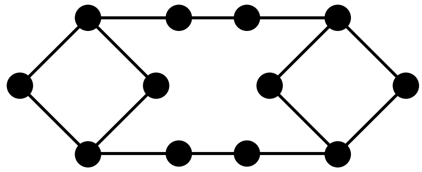

Figure 1: A non-traceable AP+1 graph.

- For every $i \in \{1, 2, ..., q\}$, set $G_i := K_{2k+1}(G_{i-1}, G_{i-1}, ..., G_{i-1})$ where $G_i$ is made up of $k + 2$ components isomorphic to $G_{i-1}$.

According to Theorem 7, all of the graphs from $(G_0, G_1, G_2, ..., G_q)$ are AP+$k$ since $G_0 = G$ is AP+$k$ by assumption. Besides, we have

$$\mu(G_q) > \mu(G_{q-1}) > ... > \mu(G_0).$$

For this reason, if we write $\varsigma(G_i) = |V(G_i)| - c_i$ for every $i \in \{0, 1, ..., q\}$, where $c_i \geq 1$ is some integer, then we get

$$c_q > c_{q-1} > ... > c_0$$

according to Observation 5. This completes the proof. □

# 4 Non-traceable AP+$k$ graphs

Recall that Corollary 8 relies on the assumption that there are non-traceable AP+$k$ graphs for some $k \geq 1$. We describe below an inductive construction providing a non-traceable AP+$k$ graph from AP+$(k-1)$ graphs with large path cover numbers. As a basis, this construction then requires a non-traceable AP+1 graph. Consider the graph depicted in Figure 1. This graph is obviously non-traceable, and it was shown in [4] that this graph is also AP+1 (checking this property is actually easy due to the symmetric structure and the small order of this graph). Using Corollary 8, we hence directly get the following.

**Corollary 9.** *AP+1 graphs can have arbitrarily large path cover numbers and arbitrarily small longest paths (compared to their orders).*

Assuming that AP+$(k-1)$ graphs can have arbitrarily large path cover numbers, we show below that we can deduce a non-traceable AP+$k$ graph.

**Lemma 10.** *Let $G_1, G_2, ..., G_\ell$ be $\ell$ AP+$(k-1)$ graphs. Then $K_\nu(G_1, G_2, ..., G_\ell)$ is AP+$k$ whenever $\nu \geq \max\{k + \ell - 1, \ell + 2\}$.*

*Proof.* Let $G = K_\nu(G_1, G_2, ..., G_\ell)$ be such a graph with order $n$ for some graphs $G_1, G_2, ..., G_\ell$. Let further $\pi = (n_1, n_2, ..., n_p)$ be an $n$-sequence, and $P = (v_1, v_2, ..., v_k)$ be a $k$-prescription of $G$. We exhibit a $P$-realization of $\pi$ in $G$ in a very same manner as in the proof of Lemma 6 (we hence use the same terminology throughout this proof).

Any situation described in the proof of Lemma 6 can actually be handled similarly since $\nu \geq \max\{k + \ell - 1, \ell + 2\} \geq k + \ell - 1$. The only new case we have to consider is when the $k$ prescribed vertices are all located in $G_1$ while $G_1$ is "only" AP+$(k-1)$. In such a situation the Filling Step has to be handled as follows. If $n_1 + n_2 + ... + n_k = |V(G_1)|$, then we can deduce a $(v_1, v_2, ..., v_k)$-realization of $(n_1, n_2, ..., n_k)$ in $G_1$ using Theorem 1 since $G_1$ is AP+$(k-1)$ and hence $k$-connected (see Observation 2). No part has to be completed via the Completing Step in this situation.

In case $n_1 + n_2 + ... + n_k > |V(G_1)|$, we can proceed as in Step 2 from the proof of Theorem 6. Let $i \in \{1, 2, ..., k\}$ be the index for which $n_1 + n_2 + ... + n_{i-1} + (k - (i-1)) < |V(G_1)|$ and $n_1 + n_2 + ... + n_i + (k - i) > |V(G_1)|$. Again, set $n'_i := |V(G_1)| - (n_1 + n_2 + ... + n_{i-1}) - (k - i)$. By the $k$-connectivity of $G_1$, using Theorem 1 we can deduce a $(v_1, v_2, ..., v_k)$-realization of $(n_1, n_2, ..., n_{i-1}, n'_i, 1, 1, ..., 1)$ in $G_1$, with the value 1 being repeated $k - i$ times. Again, the $i'^{th}$ parts, with $i' \geq i$, are (possibly) incomplete, so just add a universal vertex to the corresponding partial parts so that these parts can be completed during the Completing Step.

Finally consider the case where $n_1 + n_2 + ... + n_k < |V(G_1)|$. Then let $i \in \{1, 2, ..., q\}$ be the index for which $n_1 + n_2 + ... + n_k + r_1 + r_2 + ... + r_{i-1} < |V(G_1)|$ and $n_1 + n_2 + ... + n_k + r_1 + r_2 + ... + r_i > |V(G_1)|$. We can suppose that such an index exists since $\pi_r$ cannot be empty at the beginning of the Filling Step (otherwise by the $k$-connectivity of $G$ we could directly deduce a $P$-realization of $\pi$ using Theorem 1). In particular, if a component contains the $k$ prescribed vertices, then consider this component as $G_1$ first so that $\pi_r$ is not empty. Again, split $r_i$ into two non-null elements $r'_i$ and $r''_i$ such that $r_i = r'_i + r''_i$ and $n_1 + n_2 + ... + n_k + r_1 + r_2 + ... + r_{i-1} + r'_i = |V(G_1)|$. Consider a $(v_1, v_2, ..., v_{k-1})$-realization $(V_1, V_2, ..., V_k, U_1, U_2, ..., U_i)$ of $(n_1, n_2, ..., n_k, r_1, r_2, ..., r_{i-1}, r'_i)$ in $G_1$, which exists since $G_1$ is AP+$(k-1)$. The only requirement which may not be fulfilled is the membership of $v_k$ to $V_k$. If this is already met, then we are done. Otherwise, we modify the parts as follows.

Assume $v_k \in U_1$ for the sake of simplicity. Let further $u_1$ and $u_2$ be two unused universal vertices of $G$. Now let

$$U'_1 := U_1 - \{v_k\} \cup \{u_1\}$$

and

$$V'_k := V_k - \{w_1, w_2\} \cup \{u_2, v_k\},$$

where $w_1$ and $w_2$ are any two arbitrary adjacent vertices of $V_k$. Clearly $G[U_1']$ is connected due to the presence of $u_1$ in this subgraph (which still has order $r_1$). Similarly, the part $V_k'$ still induces a connected subgraph (due to the presence of $u_2$) on $n_k$ vertices, and now contains $v_k$. Now $w_1$ and $w_2$ belong to no part, but then we can try to complete the part $U_i$ with $w_1$ and $w_2$ (recall that $U_i$ is missing $r_i''$ vertices). More precisely, if $r_i'' = 1$, then the part is only missing one vertex, so add a universal vertex to it, and create (possibly) partial parts of the realization $U_{i+1}$ and possibly $U_{i+2}$ (intended to have size $r_{i+1}$ and $r_{i+2}$, respectively) containing $w_1$, $w_2$, and at most one universal vertex, and inducing connected subgraphs (if $r_{i+1} \geq 2$, then $U_{i+1}$ is sufficient). If $r_i'' = 2$, then add a universal vertex to $U_i$, as well as, say, $w_1$, so that $U_i$ has the required size $r_i$. Now create the (possibly partial) part $U_{i+1}$ by adding $w_2$ to $U_{i+1}$, as well as a universal vertex if $r_{i+1} \geq 2$. In any case, $w_2$ belongs to some part, which can be completed during the Completing Step if necessary. Now if $r_i'' \geq 3$, then add $w_1$, $w_2$, and a universal vertex to the part $U_i$.

Again, in any of these cases, remove the part sizes of $\pi_r$ whose associated parts have already been picked.

The only thing to be aware of, is that the number of needed universal vertices is $\max\{k + \ell - 1, \ell + 2\}$. All arguments from the proof of Lemma 6 still work since $\max\{k + \ell - 1, \ell + 2\} \geq k + \ell - 1$. For the new situations, note that, in the worst case, we may need two universal vertices to “move” the prescribed vertex $v_k$ from a part to another one, plus another universal vertex to complete a part which is missing only one vertex (i.e. when $r_i'' = 1$), and a last universal vertex to complete the part which is intended to contain $w_1$ and $w_2$. Then what remain is $\ell - 1$ components which do not include prescribed vertices, and $\nu - 4$ universal vertices. As seen in the proof of Lemma 6, we then need at most $\ell - 2$ universal vertices to pick the remaining parts in the remaining components. For this additional case, we thus need at most $4 + \ell - 2 = \ell + 2$ universal vertices. Again, for bad values of $\nu$, we can deduce situations where a $P$-realization of $\pi$ in $G$ cannot be deduced. □

We are now ready to express the main result of this paper.

**Theorem 11.** *AP+k graphs can have arbitrarily large path cover numbers and arbitrarily small longest paths (compared to their orders).*

*Proof.* Recall that the claim is already true for $k = 1$, see Corollary 9. Now assume it is true for every value of $k$ up to some $i - 1 \geq 1$, and put $k = i$. The only thing we need to show is that there exists a non-traceable AP+$k$ graph $G$ so that Corollary 8 directly implies the claim. Said differently, we want $G$ to have $\mu(G) \geq 2$.

Choose $\nu \geq \max\{k+1, 4\}$, and let $G_1$ and $G_2$ be two AP+$(k-1)$ graphs satisfying $\mu(G_1) + \mu(G_2) \geq \nu + 2$. Such graphs exist according to the induction hypothesis. Now set $G := K_\nu(G_1, G_2)$. Then $G$ is an AP+$k$ graph according to Lemma 10, and is not traceable since we have

$$\mu(G) = \mu(G_1) + \mu(G_2) - \nu$$

according to Observation 3, implying $\mu(G) \geq 2$. This completes the proof. □

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