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The poset of posets

Serge Bouc

Abstract: Let X be a finite set. This paper describes some topological and combinatorial properties of the poset Ω_X of order relations on X . In particular, the homotopy type of all the intervals in Ω_X is precisely determined, and the Möbius function of Ω_X is computed.

AMS subject classification: 06A11, 05E45, 55U10

Keywords: poset, order, Möbius function, homotopy type

1. Introduction

Let X be a finite set. In this paper, we consider some topological and combinatorial properties of the *poset of posets* on X : it is the set Ω_X of order relations on X , ordered by inclusion of subsets of the cartesian product $X \times X$.

In particular we show that the intervals $]R, S[_{\Omega_X}$ in this poset, for $R \subseteq S$, are either contractible, or have the homotopy type of a sphere of dimension $|S - R - 2|$. As a consequence, we determine the Möbius function of the poset Ω_X . We also show that the upper intervals $]R, \cdot[_{\Omega_X}$ are either contractible, or have the homotopy type of a sphere S^{d_R} . Each case can be precisely determined, and in particular the dimension d_R of the sphere involved in the second case can be computed explicitly from the relation R .

The initial motivation for considering the poset Ω_X is a joint work with Jacques Thévenaz ([3]), in which the Möbius function of Ω_X appears (even though the exact value of this function was not needed for our purpose in that paper). Apart from methods introduced by Quillen in his seminal paper [4], the present paper is self contained.

2. Relations, posets

Let X denote a finite set, of cardinality n .

- A *relation* on X is by definition a subset of the cartesian product $X \times X$.
- A *subrelation* of a relation S is just a subset $R \subseteq S$ of S . A *proper subrelation* of S is a proper subset $R \subset S$.
- If R and S are relations on X , their *composition* $R \circ S$, also called the

product RS of R and S , is defined by

$$R \circ S = \{(x, y) \in X \times X \mid \exists z \in X, (x, z) \in R \text{ and } (z, y) \in S\} .$$

- The relation $\Delta (= \Delta_X) = \{(x, x) \mid x \in X\}$ is called the *equality relation* on X .
- If R is a relation on X , then the *opposite* relation R^{op} is defined by

$$R^{op} = \{(x, y) \in X \times X \mid (y, x) \in R\} .$$

- A relation $R \subseteq X \times X$ is called:
 - *reflexive* if $(x, x) \in R, \forall x \in X$. Equivalently $\Delta \subseteq R$.
 - *transitive* if $\forall x, y, z \in X, (x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R$. Equivalently $R^2 \subseteq R$.
 - *a preorder* if R is reflexive and transitive. Equivalently $\Delta \subseteq R = R^2$.
 - *symmetric* if $(x, y) \in R \Rightarrow (y, x) \in R$. Equivalently $R = R^{op}$.
 - *an equivalence relation* if it is a preorder, and R is symmetric. Equivalently $\Delta \subseteq R = R^{op} = R^2$.
 - *antisymmetric* if $\forall x, y \in X, (x, y) \in R \text{ and } (y, x) \in R \Rightarrow x = y$. Equivalently $R \cap R^{op} \subseteq \Delta$. A subrelation of an antisymmetric relation is antisymmetric.
 - *an order* if R is a preorder, and R is antisymmetric. Equivalently $R = R^2$ and $R \cap R^{op} = \Delta$.
- A set endowed with an order relation is called a partially ordered set, or *poset* for short.

Let Ω_X denote the set of order relations on X . The inclusion of subsets of $X \times X$ induces an order relation on Ω_X . The poset (Ω_X, \subseteq) is called the *poset of posets* on X .

3. The Frattini subrelation

From now on, the set X will be fixed, and often understood. In particular, the poset Ω_X will be denoted by Ω .

3.1. Notation : Let R be an order on X . For $x, y \in X$, write $x \leq_R y$ if $(x, y) \in R$, and $x <_R y$ if $(x, y) \in R$ and $x \neq y$. Set moreover

$$\begin{aligned} [x, y]_R &= \{z \in X \mid x \leq_R z \leq_R y\} \ , \\]x, y[_R &= \{z \in X \mid x <_R z <_R y\} \ , \\]\cdot, x]_R &= \{z \in X \mid z \leq_R x\} \\]\cdot, x[_R &= \{z \in X \mid z <_R x\} \\ [x, \cdot]_R &= \{z \in X \mid x \leq_R z\} \\ [x, \cdot[_R &= \{z \in X \mid x <_R z\} \ . \end{aligned}$$

If $R \in \Omega$, a pair $(x, y) \in X \times X$ is called adjacent for R if $x <_R y$ and $]x, y[_R = \emptyset$. The set of adjacent pairs for the relation R is denoted by \mathcal{M}_R .

3.2. Let R be reflexive relation on X . Then $\Delta \subseteq R \subseteq R^2 \subseteq \dots \subseteq R^m$, and there is an integer m such that $R^m = R^{m+1}$. This limit $\bar{R} = R^m$ is a preorder, called the *transitive closure* of R .

Let $R, S \in \Omega$. Notation 3.1 becomes

$$\begin{aligned} [R, S]_\Omega &= \{T \in \Omega \mid R \subseteq T \subseteq S\} \\]R, S[_\Omega &= \{T \in \Omega \mid R \subset T \subset S\} \\]R, \cdot[_\Omega &= \{T \in \Omega \mid R \subset T\} \ . \end{aligned}$$

An order R is said to be *maximal* in an order S if the pair (R, S) is adjacent in the poset Ω , i.e. if $R \subset S$ and $]R, S[_\Omega = \emptyset$.

The subposet $[R, S]_\Omega$ of Ω is a *lattice*: the meet $T \wedge T'$ of $T, T' \in [R, S]_\Omega$ is the intersection $T \cap T'$, and their join is $T \vee T' = \overline{T \cup T'}$ (which is indeed an order, as it is a preorder contained in S).

3.3. Notation : When S is a relation on X , denote by $S^{(2)} = S \times S^{op}$ the relation on $X \times X$ defined by

$$((x, x'), (y, y')) \in S^{(2)} \Leftrightarrow (x, y) \in S \text{ and } (y', x') \in S \ .$$

If S is an order on X , then $S^{(2)}$ is an order on $X \times X$. In this case

$$(x, y) \leq_{S^{(2)}} (x', y') \Leftrightarrow x \leq_S x' \text{ and } y' \leq_S y \ .$$

3.4. Lemma :

1. Let S be an order on X , and (x, y) be an adjacent pair for S . Then $R = S - \{(x, y)\}$ is an order on X , and R is maximal in S .
2. Let $R, S \in \Omega$ with $R \subset S$. If R is maximal in S , then the set $S - R$ consists of a single pair (x, y) , which is adjacent for S , and a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$.
3. Let $R \in \Omega$, and let (x, y) be a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$. Then $S = R \sqcup \{(x, y)\}$ is an order on X , and R is maximal in S .

Proof : For Assertion 1, the relation $R = S - \{(x, y)\}$ is reflexive, as $x \neq y$. It is also antisymmetric, since it is contained in S . Showing that it is transitive amounts to showing that if $a, b, c \in X$ with $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. The assumptions imply $(a, c) \in S$, as S is transitive. If $(a, c) \notin R$, then $(a, c) = (x, y)$. Then $a = x$ and $c = y$, and moreover $(x, b) \in S$ and $(b, y) \in S$. Since (x, y) is adjacent for S , it follows that $x = b$ or $b = y$. If $x = b$, then $(b, c) = (x, y) \notin R$, and if $b = y$, then $(a, b) = (x, y) \notin R$. This contradiction shows that $(a, c) \in R$, and Assertion 1 follows.

For Assertion 2, let R be maximal in S , and let $(x, y) \in S - R$ (so that in particular $x \neq y$). Then $(x, y) \in (X \times X) - (R \cup R^{op})$, for otherwise (x, y) and (y, x) are both in S . Moreover, the transitive closure $\overline{R \sqcup \{(x, y)\}}$ is equal to S . Thus for any $(a, b) \in S - R$, there exists a chain of pairs

$$(3.5) \quad (a_0, a_1), (a_1, a_2), \dots, (a_{k-1}, a_k)$$

with $a_0 = a$ and $a_k = b$, and $(a_i, a_{i+1}) \in R \sqcup \{(x, y)\}$, for $i \in \{0, \dots, k-1\}$. Since R is transitive, we can assume that there are no two consecutive i in $\{0, \dots, k-1\}$ such that $(a_i, a_{i+1}) \in R$. If $(a_i, a_{i+1}) = (x, y)$ for two distinct values l and l' of i , then we can assume $l' = l + 2$, and $(a_{l+1}, a_{l+2}) \in R$. Then $a_{l+1} = y$ and $a_{l+2} = x$, hence $(y, x) \in R \subseteq S$. It follows that $x = y$, as S is antisymmetric.

It follows that the pair (x, y) occurs only once in the sequence 3.5. Hence $(a, x) \in R$ and $(y, b) \in R$, i.e. $(a, b) \leq_{R^{(2)}} (x, y)$. By symmetry of the roles of (x, y) and (a, b) , it follows that $(a, b) = (x, y)$. Thus $S - R$ consists of a single pair (x, y) . Moreover if $z \in X$ is such that (x, z) and (z, y) are both in S , then either $z = y$ or $(x, z) \neq (x, y)$, thus $(x, z) \in R$. Similarly, either $z = x$ or $(z, y) \in R$. Since $(x, y) \notin R$, one of the pairs (x, z) or (z, y) is not in R , hence $x = z$ or $z = y$. Hence (x, y) is adjacent for S .

Moreover if $(x', y') \in (X \times X) - (R \cup R^{op})$ and $(x', y') \leq_{R^{(2)}} (x, y)$, then $(x', x) \in R$ and $(y, y') \in R$. Since $R \subset S$ and $(x, y) \in S$, it follows that

$(x', y') \in S$. If $(x', y') \neq (x, y)$, it follows that $(x', y') \in R$. Hence (x, y) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$, and this completes the proof of Assertion 2.

For Assertion 3, let (x, y) be a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$. Then as in the proof of Assertion 2, the transitive closure $S = \overline{R \sqcup \{(x, y)\}}$ consists of the union of R with the set of pairs $(a, b) \in X \times X$ such that $(a, x) \in R$ and $(y, b) \in R$, i.e. $(a, b) \leq_{R^{(2)}} (x, y)$. Since (x, y) is minimal for $R^{(2)}$, it follows that $S = R \sqcup \{(x, y)\}$. This relation S is clearly antisymmetric if $(x, y) \notin R \cup R^{op}$, hence S is an order and Assertion 3 follows. \square

3.6. By analogy with the case of subgroups of a group, we set:

3.7. Definition : *Let S be an order on X . The Frattini subrelation $\Phi(S)$ of S is defined as the intersection of all the maximal order subrelations of S .*

3.8. Proposition : *Let S be an order on X . Then*

$$\Phi(S) = \Delta \cup (S - \Delta)^2 .$$

In other words, a pair (x, y) of $X \times X$ is in $\Phi(S)$ if and only either $x = y$, or $x <_S y$ and $]x, y[_S \neq \emptyset$.

Proof : By Lemma 3.4, the maximal order subrelations of S are the relations $S - \{(x, y)\}$, where (x, y) is adjacent for S . It follows that $\Phi(S)$ consists of the difference $S - \mathcal{M}_S$. The proposition follows. \square

4. Intervals

4.1. Theorem : *Let $R \subseteq S$ be two orders on X .*

1. *If $\Phi(S) \not\subseteq R$, i.e. if there exist $x, y, z \in X$ such that $x <_S y <_S z$ and $x \not\leq_R z$, then the poset $]R, S[_\Omega$ is contractible.*
2. *If $\Phi(S) \subseteq R$, then any subset of S containing R is an order. In particular, the poset $]R, S[_\Omega$ is isomorphic to the poset of proper non empty subsets of $S - R$, and it has the homotopy type of a sphere of dimension $|S - R - 2|$.*

Proof : If $T \in]R, S[_\Omega$, then there is a maximal element U of $]R, S[_\Omega$ which contains T , hence $T \cup \Phi(S)$. Thus $\overline{T \cup \Phi(S)} \in]R, S[_\Omega$. Now if $\Phi(S) \not\subseteq R$, the maps of posets $T \mapsto \overline{T \cup \Phi(S)} \mapsto \overline{R \cup \Phi(S)}$ show that $]R, S[_\Omega$ is *conically contractible*, in the sense of [4] 1.5.

If $\Phi(S) \subseteq R$, then $R = S - A$, where A is a subset of \mathcal{M}_S . The map $\delta : T \mapsto T - R$ is a map of posets from $]R, S[_\Omega$ to the set $]\emptyset, A[$ of proper non empty subsets of A , ordered by inclusion.

Claim: for any subset $B \in]\emptyset, A[$, the relation $R \cup B$ is an order on X : indeed $R \cup B = S - C$, where $C = A - B$. The case $|C| = 1$ is Assertion 1 of Lemma 3.4. The general case of the claim follows by induction on $|C|$, from the fact that if $T \subseteq S$, then $\mathcal{M}_S \cap T \subseteq \mathcal{M}_T$.

Now the map $v : B \mapsto R \cup B$ is a map of posets from $]\emptyset, A[$ to $]R, S[_\Omega$. Clearly the maps δ and v are inverse isomorphisms of posets, and the poset of proper non empty subsets of a finite set of cardinality c has the homotopy type of the sphere S^{c-2} : indeed, its geometric realization is the boundary of the standard $(c - 1)$ -simplex in \mathbb{R}^c , consisting of the points $(x_1, \dots, x_c) \in \mathbb{R}^c$ such that $x_i \geq 0$ and $\sum_{i=1}^c x_i = 1$. \square

4.2. Corollary : *Let $R \subseteq S$ be orders on X . Then the Möbius function $\mu_\Omega(R, S)$ of the poset Ω is equal to 0 if $\Phi(S) \not\subseteq R$, and to $(-1)^{|S-R|}$ otherwise.* \square

Proof : Indeed, the Möbius function $\mu_\Omega(R, S)$ is equal to the reduced Euler-Poincaré characteristic of the poset $]R, S[_\Omega$. \square

5. Upper intervals

Let R be an order on X . This section deals with the homotopy type of the poset $]R, \cdot[_\Omega$.

5.1. Lemma : *Let R be an order on X , and $(a, b) \in (X \times X)$. Then (a, b) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$ if and only if $(a, b) \notin R \cup R^{op}$ and*

$$(5.2) \quad]\cdot, a[_R \subseteq]\cdot, b[_R \text{ and }]b, \cdot[_R \subseteq]a, \cdot[_R \cdot$$

Proof : If (a, b) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$, then $(a, b) \notin R \cup R^{op}$. If $x \in X$ and $x <_R a$, then $(x, b) <_{R^{(2)}} (a, b)$.

Hence $(x, b) \in R \cup R^{op}$. If $(x, b) \in R^{op}$, then $b \leq_R x <_R a$, hence $(b, a) \in R$, a contradiction. Hence $(x, b) \in R$, i.e. $x \leq_R b$. Moreover $x \neq b$ since $(b, a) \notin R$. Hence $x <_R b$, i.e. $x \in]\cdot, b[_R$. This shows that $]\cdot, a[_R \subseteq]\cdot, b[_R$. Similarly, if $y \in X$ and $b <_R y$, then $(a, y) <_{R^{(2)}} (a, b)$, so $(a, y) \in R \cup R^{op}$. If $(a, y) \in R^{op}$, then $b <_R y \leq_R a$, and $(b, a) \in R$, a contradiction. Hence $(a, y) \in R$, i.e. $a \leq_R y$. Moreover $a \neq y$ since $b \not\leq_R a$. Hence $y \in]a, \cdot[_R$, thus $]b, \cdot[_R \subseteq]a, \cdot[_R$, and Condition 5.2 holds.

Conversely, if $(a, b) \notin R \cup R^{op}$, and if Condition 5.2 holds, suppose that $(x, y) \in X \times X$ and $(x, y) \leq_{R^{(2)}} (a, b)$, i.e. $x \leq_R a$ and $b \leq_R y$. If $x \neq a$, then $x \in]\cdot, a[_R \subseteq]\cdot, b[_R$, hence $x <_R b \leq_R y$, thus $(x, y) \in R$. Similarly if $b \neq y$, then $y \in]b, \cdot[_R \subseteq]a, \cdot[_R$, hence $x \leq_R a <_R y$, and $(x, y) \in R$. Hence if $(x, y) <_{R^{(2)}} (a, b)$, then $(x, y) \in R \subseteq R \cup R^{op}$, so (a, b) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$. \square

This motivates the following notation:

5.3. Notation : Let $R \in \Omega$. Set

$$\mathcal{E}_R = \{(a, b) \in (X \times X) - (R \cup R^{op}) \mid]\cdot, a[_R \subseteq]\cdot, b[_R \text{ and }]b, \cdot[_R \subseteq]a, \cdot[_R\} .$$

5.4. Theorem : Let R be an order on X .

1. If there exists $(a, b) \in \mathcal{E}_R$ such that $(b, a) \notin \mathcal{E}_R$, then $]R, \cdot[_\Omega$ is contractible.
2. Otherwise $\mathcal{E}_R = (\mathcal{E}_R)^{op}$, and $E_R = \Delta \sqcup \mathcal{E}_R$ is an equivalence relation on X , which can be defined by

$$(a, b) \in E_R \Leftrightarrow \begin{cases}]\cdot, a[_R =]\cdot, b[_R \\]a, \cdot[_R =]b, \cdot[_R \end{cases} .$$

If X_1, \dots, X_r are the equivalence classes for this relation, let $\Omega_i = \Omega_{X_i}$ and $\Delta_i = \Delta_{X_i}$, for $i \in \{1, \dots, r\}$. Then there is a homotopy equivalence

$$]R, \cdot[_\Omega \cong]\Delta_1, \cdot[_{\Omega_1} *]\Delta_2, \cdot[_{\Omega_2} * \dots *]\Delta_r, \cdot[_{\Omega_r} ,$$

where $P * Q$ in the right hand side denotes the join of two posets P and Q (cf. [4] Proposition 1.9).

Proof : If $(a, b) \in \mathcal{E}_R$, then (a, b) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$, by Lemma 5.1. It follows from Lemma 3.4 that the

relation $S = R \sqcup \{(a, b)\}$ is an order on X . The poset $[S, \cdot]_{\Omega}$ has a smallest element S , hence it is a contractible subposet of $]R, \cdot]_{\Omega}$.

Let $i : [S, \cdot]_{\Omega} \hookrightarrow]R, \cdot]_{\Omega}$ denote the inclusion map. For $T \in]R, \cdot]_{\Omega}$, the poset

$$i_T = \{U \in [S, \cdot]_{\Omega} \mid U \supseteq T\}$$

is equal to the set of order relations containing $S \cup T$. If $\overline{S \cup T}$ is an order, i.e. if it is antisymmetric, then i_T has a smallest element $\overline{S \cup T}$. Otherwise i_T is empty.

Let

$$A_S = \{T \in]R, \cdot]_{\Omega} \mid \overline{S \cup T} \text{ is antisymmetric}\} .$$

In particular $A_S \supseteq [S, \cdot]_{\Omega}$, and the inclusion $i : [S, \cdot]_{\Omega} \hookrightarrow A_S$ is a homotopy equivalence ([4], Proposition 1.6). Hence A_S is contractible.

Now it is easy to check that $\overline{S \cup T} = \overline{\{(a, b)\} \cup T}$ is antisymmetric if and only if $(b, a) \notin T$.

Let

$$(A_S)_* = \{T \in A_S \mid \cdot, T|_{A_S} \text{ is not contractible}\} .$$

Then by standard results (e.g. [2] or [1], Proposition 6.6.5), if B is any poset with $(A_S)_* \subseteq B \subseteq A_S$, the inclusion $B \hookrightarrow A_S$ is a homotopy equivalence. Now $\cdot, T|_{A_S} =]R, \cdot]_{\Omega}$, for any $T \in A_S$. Let

$$B = \{T \in]R, \cdot]_{\Omega} \mid (b, a) \notin T \text{ and } \Phi(T) \subseteq R\} .$$

Then $(A_S)_* \subseteq B \subseteq A_S$ by Theorem 4.1, thus B is contractible.

Let $T \in]R, \cdot]_{\Omega}$ such that $\Phi(T) \subseteq R$. If $(b, a) \in T$, then $R \sqcup \{(b, a)\} \subseteq T$ is an order, by Theorem 4.1. By Lemma 3.4, the pair (b, a) is a minimal element of $(X \times X) - (R \cup R^{op})$ for the relation $R^{(2)}$. Hence $(b, a) \in \mathcal{E}_R$, by Lemma 5.1.

Hence if there exist $(a, b) \in \mathcal{E}_R$ such that $(b, a) \notin \mathcal{E}_R$, then $(]R, \cdot]_{\Omega})_* = B$ is contractible, which proves Assertion 1.

Otherwise $\mathcal{E}_R \subseteq \mathcal{E}_R^{op}$, hence $\mathcal{E}_R = \mathcal{E}_R^{op}$, and

$$\mathcal{E}_R = \{(a, b) \in (X \times X) - (R \cup R^{op}) \mid \cdot, a|_{[R=]} \cdot, b|_{[R=]} \text{ and }]a, \cdot|_{[R=]} b, \cdot|_{[R=]}\} .$$

Then $E_R = \Delta \sqcup \mathcal{E}_R$ is an equivalence relation: it is obviously reflexive and symmetric. To show that it is transitive, it suffices to show that if $(a, b), (b, c) \in \mathcal{E}_R$, then $(a, c) \in E_R$. The assumption implies that $\cdot, a|_{[R=]} \cdot, b|_{[R=]} \cdot, c|_{[R=]}$ and $]a, \cdot|_{[R=]} b, \cdot|_{[R=]} c, \cdot|_{[R=]}$. If $(a, c) \in R$, then either $a = c$ or $a \in \cdot, c|_{[R]}$. In the latter case $a \in \cdot, a|_{[R]}$, a contradiction. Thus $a = c$. Now if $(a, c) \in R^{op}$, then $c \leq_R a$. Thus again if $a \neq c$, then $c \in \cdot, a|_{[R=]} \cdot, c|_{[R]}$, a contradiction. In both cases $a = c$. Thus either $a = c$, or $(a, c) \notin R \cup R^{op}$, and then $(a, c) \in \mathcal{E}_R$. Hence E_R is transitive, so it is an equivalence relation.

Let X_1, \dots, X_r be the equivalence classes of E_R . If T is an order containing R , for each $i \in \{1, \dots, r\}$, denote by $T|_i$ the restriction of T to X_i , i.e. $T|_i = T \cap (X_i \times X_i)$. Then $T|_i$ is an order on X_i , in other words $T|_i \in \Omega_i = \Omega_{X_i}$. Let

$$P = [\Delta_1, \cdot]_{\Omega_1} \times [\Delta_2, \cdot]_{\Omega_2} \times \dots \times [\Delta_r, \cdot]_{\Omega_r}$$

denote the product poset. The map

$$e : [R, \cdot]_{\Omega} \rightarrow P$$

defined by $e(T) = (T|_1, \dots, T|_r)$ is a map of posets.

Claim: Let $(T_1, \dots, T_r) \in P$. Then $S = R \cup T_1 \cup \dots \cup T_r$ is an order on X .

Indeed S is clearly reflexive. It is also antisymmetric: if $(a, b) \in S \cap S^{op}$, then we can assume that:

- either $(a, b) \in R \cap R^{op}$, and then $a = b$, since R is antisymmetric.
- or $(a, b) \in R$ and $(b, a) \in T_i - R$, for some $i \in \{1, \dots, r\}$. Then $a \neq b$, thus $(b, a) \in \mathcal{E}_R$, hence $] \cdot, a[_R =] \cdot, b[_R$, and $a \in] \cdot, b[_R =] \cdot, a[_R$, a contradiction.
- or $(a, b) \in T_i - R$ (thus $a \neq b$) and $(b, a) \in T_j$, for some $i, j \in \{1, \dots, r\}$. In particular $(a, b) \in E_R$, so a and b are in the same equivalence class for E_R , hence $i = j$. But then $(a, b) \in T_i \cap T_i^{op}$, thus $a = b$ since T_i is antisymmetric. This is again a contradiction.

Hence S is antisymmetric. It is also transitive, for if $(a, b) \in S - \Delta$ and $(b, c) \in S - \Delta$, then:

- either $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R \subseteq S$, since R is transitive.
- or $(a, b) \in R - \Delta$ and $(b, c) \in T_i - R$, for some $i \in \{1, \dots, r\}$. Then $a <_R b$, thus $a \in] \cdot, b[_R =] \cdot, c[_R$, and then $(a, c) \in R \subseteq S$.
- or $(a, b) \in T_i - R$ and $(b, c) \in R - \Delta$, for some $i \in \{1, \dots, r\}$. Then $b <_R c$, i.e. $c \in]b, \cdot[_R =]a, \cdot[_R$, hence $(a, c) \in R \subseteq S$.
- or $(a, b) \in T_i - R$ and $(b, c) \in T_j - R$, for some $i, j \in \{1, \dots, r\}$. Then $(a, b) \in \mathcal{E}_R$ and $(b, c) \in \mathcal{E}_R$, and a, b, c are in the same equivalence class for E_R . Hence $i = j$, thus $a <_{T_i} b <_{T_i} c$, thus $a <_{T_i} c$, and $(a, c) \in T_i \subseteq S$.

This completes the proof of the above claim.

Now let

$$f : P \rightarrow [R, \cdot]_{\Omega}$$

be the map defined by $f(T_1, \dots, T_r) = R \cup T_1 \cup \dots \cup T_r$. It is clearly a map of posets.

Moreover for any $T \in [R, \cdot]_{\Omega}$

$$fe(T) = R \cup T_{|1} \cup \dots \cup T_{|r} \subseteq T \quad .$$

Conversely, for any $(T_1, \dots, T_r) \in P$

$$ef(T_1, \dots, T_r) = e(R \cup T_1 \cup \dots \cup T_r) = (T'_1, \dots, T'_r) \quad ,$$

where $T'_i = (R \cup T_1 \cup \dots \cup T_r) \cap (X_i \times X_i) \supseteq T_i$, for any $i \in \{1, \dots, k\}$. Actually $T'_i = T_i$, since $R \cap (X_i \times X_i) = \Delta_i$: indeed $R \cap (X_i \times X_i) \subseteq R \cap E_R = \Delta$. Hence ef is the identity map of the poset P . This also shows that $e(R) = (\Delta_1, \dots, \Delta_r)$.

Conversely, suppose that $T \in [R, \cdot]_{\Omega}$ is such that $e(T) = (\Delta_1, \dots, \Delta_r)$. Then $T_{|i} = \Delta_i$ for each $i \in \{1, \dots, r\}$. If $T \neq R$, let $(a, b) \in T - R$, minimal for the relation $R^{(2)}$. Then $a \neq b$, and $(b, a) \notin R$, since $R \subseteq T$ and T is antisymmetric. Hence (a, b) is also a minimal element of $(X \times X) - (R \cup R^{op})$, thus $(a, b) \in \mathcal{E}_R$. It follows that $(a, b) \in T_i$ for some $i \in \{1, \dots, r\}$. As $a \neq b$, this contradicts the assumption $T_{|i} = \Delta_i$. Hence $T = R$.

Finally, let $(T_1, \dots, T_r) \in P$ such that $f(T_1, \dots, T_r) = R$. It follows that $(T_1, \dots, T_r) = ef(T_1, \dots, T_r) = e(R) = (\Delta_1, \dots, \Delta_r)$.

Now the maps e and f restrict to maps of posets

$$]R, \cdot[_{\Omega} \xrightleftharpoons[f]{e} ([\Delta_1, \cdot[_{\Omega_1} \times [\Delta_2, \cdot[_{\Omega_2} \times \dots \times [\Delta_r, \cdot[_{\Omega_r}] - \{(\Delta_1, \dots, \Delta_r)\})$$

such that $fe \subseteq \text{Id}$ and $ef = \text{Id}$. Hence e and f are inverse homotopy equivalences ([4] 1.3). This completes the proof of the Theorem, since the poset

$$([\Delta_1, \cdot[_{\Omega_1} \times [\Delta_2, \cdot[_{\Omega_2} \times \dots \times [\Delta_r, \cdot[_{\Omega_r}] - \{(\Delta_1, \dots, \Delta_r)\})$$

is homotopic to the join $] \Delta_1, \cdot[_{\Omega_1} *] \Delta_2, \cdot[_{\Omega_2} * \dots *] \Delta_r, \cdot[_{\Omega_r}$, by [4] Proposition 1.9. \square

5.5. Theorem : *Let X be a set of cardinality n . Then the poset $\Omega_X - \{\Delta_X\}$ of non trivial orders on X has the homotopy type of a sphere of dimension $n - 2$.*

Proof : Set $\Omega^{\sharp} = \Omega - \{\Delta\}$ (where as above $\Omega = \Omega_X$ and $\Delta = \Delta_X$). If $R \in \Omega^{\sharp}$, then $] \cdot, R[_{\Omega^{\sharp}} =] \Delta, R[_{\Omega}$. If $\Phi(R) \neq \Delta$, then $] \Delta, R[_{\Omega}$ is contractible by Theorem 4.1. It follows that the inclusion

$$B = \{R \in \Omega^{\sharp} \mid \Phi(R) = \Delta\} \hookrightarrow \Omega^{\sharp}$$

is a homotopy equivalence.

Now saying that $\Phi(R) = \Delta$ is equivalent to saying that there is no chain $x <_R y <_R z$ in X . If $R \in B$, let

$$\begin{aligned} R_- &= \{x \in X \mid \exists y \in X, x <_R y\} \\ R_+ &= \{y \in X \mid \exists x \in X, x <_R y\} \\ R_0 &= X - (R_- \cup R_+) . \end{aligned}$$

Define $h(R) = R \cup (R_- \times R_+)$. Then $h(R)$ is an order on X : it is obviously reflexive, since if $(a, b) \in h(R) \cap h(R)^{op}$, then:

- either $(a, b) \in R$ and $(b, a) \in R$, hence $a = b$.
- or $(a, b) \in R$ and $(b, a) \in R_- \times R_+$. Then $b \in R_-$ and $a \in R_+$. Moreover $a \neq b$, and $a <_R b$. Thus $a \in R_-$ and $b \in R_+$. This is a contradiction since $R_- \cap R_+ = \emptyset$.
- or $(b, a) \in R$ and $(a, b) \in R_- \times R_+$. The argument of the previous case applies to the pair (b, a) , and this gives a contradiction again.
- or $(a, b) \in R_- \times R_+$ and $(b, a) \in R_- \times R_+$. Again, since $R_- \cap R_+ = \emptyset$, this is a contradiction.

Finally if $(a, b) \in h(R)$ and $(b, c) \in h(R)$, then either $a = b$ or $b = c$, so $h(R)$ is trivially transitive. It follows that $h(R) \in \Omega^\#$. Clearly $R \subseteq h(R)$.

If $R \subseteq S \in B$, then clearly $R_- \subseteq S_-$ and $R_+ \subseteq S_+$, thus $h(R) \subseteq h(S)$. Moreover $h(R)_- = R_-$ and $h(R)_+ = R_+$, thus $h^2 = h$. It follows that h induces a homotopy equivalence

$$B \rightarrow h(B) = \{R \in B \mid h(R) = R\} ,$$

inverse to the inclusion $h(B) \hookrightarrow B$. Here $h(B)$ is viewed as a subposet of B , i.e. $h(B)$ is ordered by inclusion.

Now if $R \in B$ and $h(R) = R$, then the pair $\pi(R) = (R_-, R_+)$ is a pair (V, U) of disjoint non empty subsets of X . Conversely, such a pair (V, U) yields an element $\omega(V, U)$ of $h(B)$ defined by $\omega(V, U) = \Delta \cup (V \times U)$. The maps π and ω induce inverse isomorphisms of posets between $h(B)$ and the poset

$$D_2(X) = \{(V, U) \mid \emptyset \neq V \subset X, \emptyset \neq U \subset X, V \cap U = \emptyset\} ,$$

ordered by the product order $(V', U') \subseteq (V, U) \Leftrightarrow V' \subseteq V$ and $U' \subseteq U$ (note that if U and V are disjoint non empty subsets of X , then U and V are *proper* subsets of X).

Let $D_1(X)$ denote the set of non empty proper subsets of X , ordered by inclusion. The projection map $g : D_2(X) \rightarrow D_1(X)$ defined by $g(V, U) = U$ is a map of posets. Moreover for $W \in D_1(X)$

$$g_W = \{(V, U) \in D_2(X) \mid g(V, U) \supseteq W\} = \{(V, U) \in D_2(X) \mid U \supseteq W\}$$

is contractible via the maps $(V, U) \mapsto (V, W) \subseteq (X - W, W)$. It follows that g is a homotopy equivalence ([4], Proposition 1.6). Hence the posets Ω^\sharp and $D_1(X)$ are homotopy equivalent.

This completes the proof, since if X has cardinality n , then $D_1(X)$ has the homotopy type of a sphere of dimension $n - 2$. \square

5.6. Corollary : *Let R be an order on X . Then:*

1. *if $\mathcal{E}_R \neq \mathcal{E}_R^{op}$, the poset $]R, \cdot[_\Omega$ is contractible.*
2. *if $\mathcal{E}_R = \mathcal{E}_R^{op}$, then $E_R = \Delta \sqcup \mathcal{E}_R$ is an equivalence relation on X . The poset $]R, \cdot[_\Omega$ is homotopy equivalent to a sphere of dimension $n - r - 1$, where r is the number of equivalence classes of E_R .*

Proof : Assertion 1 follows from Theorem 5.4. For Assertion 2, by Theorem 5.4, there is a homotopy equivalence

$$]R, \cdot[_\Omega \cong]\Delta_1, \cdot[_{\Omega_1} *]\Delta_2, \cdot[_{\Omega_2} * \dots *]\Delta_r, \cdot[_{\Omega_r} ,$$

where $\Omega_i = \Omega_{X_i}$ is the poset of posets on the equivalence class X_i of X for the relation E_R .

By Theorem 5.5, each poset $]\Delta_i, \cdot[_{\Omega_i}$ has the homotopy type of a sphere of dimension $|X_i| - 2$. Since for $d, e \in \mathbb{N}$, the join $S^d * S^e$ is homotopy equivalent to S^{d+e+1} (where S^d denotes a sphere of dimension d), it follows that $]R, \cdot[_\Omega$ has the homotopy type of a sphere of dimension

$$(|X_1| - 2) + (|X_2| - 2) + \dots + (|X_r| - 2) + r - 1 = n - 2r + r - 1 = n - r - 1 ,$$

as was to be shown. \square

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