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# The poset of posets

Serge Bouc

**Abstract:** Let  $X$  be a finite set. This paper describes some topological and combinatorial properties of the poset  $\Omega_X$  of order relations on  $X$ . In particular, the homotopy type of all the intervals in  $\Omega_X$  is precisely determined, and the Möbius function of  $\Omega_X$  is computed.

**AMS subject classification:** 06A11, 05E45, 55U10

**Keywords:** poset, order, Möbius function, homotopy type

## 1. Introduction

Let  $X$  be a finite set. In this paper, we consider some topological and combinatorial properties of the *poset of posets* on  $X$ : it is the set  $\Omega_X$  of order relations on  $X$ , ordered by inclusion of subsets of the cartesian product  $X \times X$ .

In particular we show that the intervals  $]R, S[_{\Omega_X}$  in this poset, for  $R \subseteq S$ , are either contractible, or have the homotopy type of a sphere of dimension  $|S - R - 2|$ . As a consequence, we determine the Möbius function of the poset  $\Omega_X$ . We also show that the upper intervals  $]R, \cdot[_{\Omega_X}$  are either contractible, or have the homotopy type of a sphere  $S^{d_R}$ . Each case can be precisely determined, and in particular the dimension  $d_R$  of the sphere involved in the second case can be computed explicitly from the relation  $R$ .

The initial motivation for considering the poset  $\Omega_X$  is a joint work with Jacques Thévenaz ([3]), in which the Möbius function of  $\Omega_X$  appears (even though the exact value of this function was not needed for our purpose in that paper). Apart from methods introduced by Quillen in his seminal paper [4], the present paper is self contained.

## 2. Relations, posets

Let  $X$  denote a finite set, of cardinality  $n$ .

- A *relation* on  $X$  is by definition a subset of the cartesian product  $X \times X$ .
- A *subrelation* of a relation  $S$  is just a subset  $R \subseteq S$  of  $S$ . A *proper subrelation* of  $S$  is a proper subset  $R \subset S$ .
- If  $R$  and  $S$  are relations on  $X$ , their *composition*  $R \circ S$ , also called the

product  $RS$  of  $R$  and  $S$ , is defined by

$$R \circ S = \{(x, y) \in X \times X \mid \exists z \in X, (x, z) \in R \text{ and } (z, y) \in S\} .$$

- The relation  $\Delta (= \Delta_X) = \{(x, x) \mid x \in X\}$  is called the *equality relation* on  $X$ .
- If  $R$  is a relation on  $X$ , then the *opposite* relation  $R^{op}$  is defined by

$$R^{op} = \{(x, y) \in X \times X \mid (y, x) \in R\} .$$

- A relation  $R \subseteq X \times X$  is called:
  - *reflexive* if  $(x, x) \in R, \forall x \in X$ . Equivalently  $\Delta \subseteq R$ .
  - *transitive* if  $\forall x, y, z \in X, (x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R$ . Equivalently  $R^2 \subseteq R$ .
  - *a preorder* if  $R$  is reflexive and transitive. Equivalently  $\Delta \subseteq R = R^2$ .
  - *symmetric* if  $(x, y) \in R \Rightarrow (y, x) \in R$ . Equivalently  $R = R^{op}$ .
  - *an equivalence relation* if it is a preorder, and  $R$  is symmetric. Equivalently  $\Delta \subseteq R = R^{op} = R^2$ .
  - *antisymmetric* if  $\forall x, y \in X, (x, y) \in R \text{ and } (y, x) \in R \Rightarrow x = y$ . Equivalently  $R \cap R^{op} \subseteq \Delta$ . A subrelation of an antisymmetric relation is antisymmetric.
  - *an order* if  $R$  is a preorder, and  $R$  is antisymmetric. Equivalently  $R = R^2$  and  $R \cap R^{op} = \Delta$ .
- A set endowed with an order relation is called a partially ordered set, or *poset* for short.

Let  $\Omega_X$  denote the set of order relations on  $X$ . The inclusion of subsets of  $X \times X$  induces an order relation on  $\Omega_X$ . The poset  $(\Omega_X, \subseteq)$  is called the *poset of posets* on  $X$ .

### 3. The Frattini subrelation

From now on, the set  $X$  will be fixed, and often understood. In particular, the poset  $\Omega_X$  will be denoted by  $\Omega$ .

**3.1. Notation :** Let  $R$  be an order on  $X$ . For  $x, y \in X$ , write  $x \leq_R y$  if  $(x, y) \in R$ , and  $x <_R y$  if  $(x, y) \in R$  and  $x \neq y$ . Set moreover

$$\begin{aligned} [x, y]_R &= \{z \in X \mid x \leq_R z \leq_R y\} \text{ ,} \\ ]x, y[_R &= \{z \in X \mid x <_R z <_R y\} \text{ ,} \\ ]\cdot, x]_R &= \{z \in X \mid z \leq_R x\} \\ ]\cdot, x[_R &= \{z \in X \mid z <_R x\} \\ [x, \cdot]_R &= \{z \in X \mid x \leq_R z\} \\ [x, \cdot[_R &= \{z \in X \mid x <_R z\} \text{ .} \end{aligned}$$

If  $R \in \Omega$ , a pair  $(x, y) \in X \times X$  is called adjacent for  $R$  if  $x <_R y$  and  $]x, y[_R = \emptyset$ . The set of adjacent pairs for the relation  $R$  is denoted by  $\mathcal{M}_R$ .

**3.2.** Let  $R$  be reflexive relation on  $X$ . Then  $\Delta \subseteq R \subseteq R^2 \subseteq \dots \subseteq R^m$ , and there is an integer  $m$  such that  $R^m = R^{m+1}$ . This limit  $\bar{R} = R^m$  is a preorder, called the *transitive closure* of  $R$ .

Let  $R, S \in \Omega$ . Notation 3.1 becomes

$$\begin{aligned} [R, S]_\Omega &= \{T \in \Omega \mid R \subseteq T \subseteq S\} \\ ]R, S[_\Omega &= \{T \in \Omega \mid R \subset T \subset S\} \\ ]R, \cdot[_\Omega &= \{T \in \Omega \mid R \subset T\} \text{ .} \end{aligned}$$

An order  $R$  is said to be *maximal* in an order  $S$  if the pair  $(R, S)$  is adjacent in the poset  $\Omega$ , i.e. if  $R \subset S$  and  $]R, S[_\Omega = \emptyset$ .

The subposet  $[R, S]_\Omega$  of  $\Omega$  is a *lattice*: the meet  $T \wedge T'$  of  $T, T' \in [R, S]_\Omega$  is the intersection  $T \cap T'$ , and their join is  $T \vee T' = \overline{T \cup T'}$  (which is indeed an order, as it is a preorder contained in  $S$ ).

**3.3. Notation :** When  $S$  is a relation on  $X$ , denote by  $S^{(2)} = S \times S^{op}$  the relation on  $X \times X$  defined by

$$((x, x'), (y, y')) \in S^{(2)} \Leftrightarrow (x, y) \in S \text{ and } (y', x') \in S \text{ .}$$

If  $S$  is an order on  $X$ , then  $S^{(2)}$  is an order on  $X \times X$ . In this case

$$(x, y) \leq_{S^{(2)}} (x', y') \Leftrightarrow x \leq_S x' \text{ and } y' \leq_S y \text{ .}$$

### 3.4. Lemma :

1. Let  $S$  be an order on  $X$ , and  $(x, y)$  be an adjacent pair for  $S$ . Then  $R = S - \{(x, y)\}$  is an order on  $X$ , and  $R$  is maximal in  $S$ .
2. Let  $R, S \in \Omega$  with  $R \subset S$ . If  $R$  is maximal in  $S$ , then the set  $S - R$  consists of a single pair  $(x, y)$ , which is adjacent for  $S$ , and a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ .
3. Let  $R \in \Omega$ , and let  $(x, y)$  be a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ . Then  $S = R \sqcup \{(x, y)\}$  is an order on  $X$ , and  $R$  is maximal in  $S$ .

**Proof :** For Assertion 1, the relation  $R = S - \{(x, y)\}$  is reflexive, as  $x \neq y$ . It is also antisymmetric, since it is contained in  $S$ . Showing that it is transitive amounts to showing that if  $a, b, c \in X$  with  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R$ . The assumptions imply  $(a, c) \in S$ , as  $S$  is transitive. If  $(a, c) \notin R$ , then  $(a, c) = (x, y)$ . Then  $a = x$  and  $c = y$ , and moreover  $(x, b) \in S$  and  $(b, y) \in S$ . Since  $(x, y)$  is adjacent for  $S$ , it follows that  $x = b$  or  $b = y$ . If  $x = b$ , then  $(b, c) = (x, y) \notin R$ , and if  $b = y$ , then  $(a, b) = (x, y) \notin R$ . This contradiction shows that  $(a, c) \in R$ , and Assertion 1 follows.

For Assertion 2, let  $R$  be maximal in  $S$ , and let  $(x, y) \in S - R$  (so that in particular  $x \neq y$ ). Then  $(x, y) \in (X \times X) - (R \cup R^{op})$ , for otherwise  $(x, y)$  and  $(y, x)$  are both in  $S$ . Moreover, the transitive closure  $\overline{R \sqcup \{(x, y)\}}$  is equal to  $S$ . Thus for any  $(a, b) \in S - R$ , there exists a chain of pairs

$$(3.5) \quad (a_0, a_1), (a_1, a_2), \dots, (a_{k-1}, a_k)$$

with  $a_0 = a$  and  $a_k = b$ , and  $(a_i, a_{i+1}) \in R \sqcup \{(x, y)\}$ , for  $i \in \{0, \dots, k-1\}$ . Since  $R$  is transitive, we can assume that there are no two consecutive  $i$  in  $\{0, \dots, k-1\}$  such that  $(a_i, a_{i+1}) \in R$ . If  $(a_i, a_{i+1}) = (x, y)$  for two distinct values  $l$  and  $l'$  of  $i$ , then we can assume  $l' = l + 2$ , and  $(a_{l+1}, a_{l+2}) \in R$ . Then  $a_{l+1} = y$  and  $a_{l+2} = x$ , hence  $(y, x) \in R \subseteq S$ . It follows that  $x = y$ , as  $S$  is antisymmetric.

It follows that the pair  $(x, y)$  occurs only once in the sequence 3.5. Hence  $(a, x) \in R$  and  $(y, b) \in R$ , i.e.  $(a, b) \leq_{R^{(2)}} (x, y)$ . By symmetry of the roles of  $(x, y)$  and  $(a, b)$ , it follows that  $(a, b) = (x, y)$ . Thus  $S - R$  consists of a single pair  $(x, y)$ . Moreover if  $z \in X$  is such that  $(x, z)$  and  $(z, y)$  are both in  $S$ , then either  $z = y$  or  $(x, z) \neq (x, y)$ , thus  $(x, z) \in R$ . Similarly, either  $z = x$  or  $(z, y) \in R$ . Since  $(x, y) \notin R$ , one of the pairs  $(x, z)$  or  $(z, y)$  is not in  $R$ , hence  $x = z$  or  $z = y$ . Hence  $(x, y)$  is adjacent for  $S$ .

Moreover if  $(x', y') \in (X \times X) - (R \cup R^{op})$  and  $(x', y') \leq_{R^{(2)}} (x, y)$ , then  $(x', x) \in R$  and  $(y, y') \in R$ . Since  $R \subset S$  and  $(x, y) \in S$ , it follows that

$(x', y') \in S$ . If  $(x', y') \neq (x, y)$ , it follows that  $(x', y') \in R$ . Hence  $(x, y)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ , and this completes the proof of Assertion 2.

For Assertion 3, let  $(x, y)$  be a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ . Then as in the proof of Assertion 2, the transitive closure  $S = \overline{R \sqcup \{(x, y)\}}$  consists of the union of  $R$  with the set of pairs  $(a, b) \in X \times X$  such that  $(a, x) \in R$  and  $(y, b) \in R$ , i.e.  $(a, b) \leq_{R^{(2)}} (x, y)$ . Since  $(x, y)$  is minimal for  $R^{(2)}$ , it follows that  $S = R \sqcup \{(x, y)\}$ . This relation  $S$  is clearly antisymmetric if  $(x, y) \notin R \cup R^{op}$ , hence  $S$  is an order and Assertion 3 follows.  $\square$

**3.6.** By analogy with the case of subgroups of a group, we set:

**3.7. Definition :** *Let  $S$  be an order on  $X$ . The Frattini subrelation  $\Phi(S)$  of  $S$  is defined as the intersection of all the maximal order subrelations of  $S$ .*

**3.8. Proposition :** *Let  $S$  be an order on  $X$ . Then*

$$\Phi(S) = \Delta \cup (S - \Delta)^2 .$$

*In other words, a pair  $(x, y)$  of  $X \times X$  is in  $\Phi(S)$  if and only either  $x = y$ , or  $x <_S y$  and  $]x, y[_S \neq \emptyset$ .*

**Proof :** By Lemma 3.4, the maximal order subrelations of  $S$  are the relations  $S - \{(x, y)\}$ , where  $(x, y)$  is adjacent for  $S$ . It follows that  $\Phi(S)$  consists of the difference  $S - \mathcal{M}_S$ . The proposition follows.  $\square$

## 4. Intervals

**4.1. Theorem :** *Let  $R \subseteq S$  be two orders on  $X$ .*

1. *If  $\Phi(S) \not\subseteq R$ , i.e. if there exist  $x, y, z \in X$  such that  $x <_S y <_S z$  and  $x \not\leq_R z$ , then the poset  $]R, S[_\Omega$  is contractible.*
2. *If  $\Phi(S) \subseteq R$ , then any subset of  $S$  containing  $R$  is an order. In particular, the poset  $]R, S[_\Omega$  is isomorphic to the poset of proper non empty subsets of  $S - R$ , and it has the homotopy type of a sphere of dimension  $|S - R - 2|$ .*

**Proof :** If  $T \in ]R, S[_\Omega$ , then there is a maximal element  $U$  of  $]R, S[_\Omega$  which contains  $T$ , hence  $T \cup \Phi(S)$ . Thus  $\overline{T \cup \Phi(S)} \in ]R, S[_\Omega$ . Now if  $\Phi(S) \not\subseteq R$ , the maps of posets  $T \mapsto \overline{T \cup \Phi(S)} \mapsto \overline{R \cup \Phi(S)}$  show that  $]R, S[_\Omega$  is *conically contractible*, in the sense of [4] 1.5.

If  $\Phi(S) \subseteq R$ , then  $R = S - A$ , where  $A$  is a subset of  $\mathcal{M}_S$ . The map  $\delta : T \mapsto T - R$  is a map of posets from  $]R, S[_\Omega$  to the set  $]\emptyset, A[$  of proper non empty subsets of  $A$ , ordered by inclusion.

**Claim:** for any subset  $B \in ]\emptyset, A[$ , the relation  $R \cup B$  is an order on  $X$ : indeed  $R \cup B = S - C$ , where  $C = A - B$ . The case  $|C| = 1$  is Assertion 1 of Lemma 3.4. The general case of the claim follows by induction on  $|C|$ , from the fact that if  $T \subseteq S$ , then  $\mathcal{M}_S \cap T \subseteq \mathcal{M}_T$ .

Now the map  $v : B \mapsto R \cup B$  is a map of posets from  $]\emptyset, A[$  to  $]R, S[_\Omega$ . Clearly the maps  $\delta$  and  $v$  are inverse isomorphisms of posets, and the poset of proper non empty subsets of a finite set of cardinality  $c$  has the homotopy type of the sphere  $S^{c-2}$ : indeed, its geometric realization is the boundary of the standard  $(c - 1)$ -simplex in  $\mathbb{R}^c$ , consisting of the points  $(x_1, \dots, x_c) \in \mathbb{R}^c$  such that  $x_i \geq 0$  and  $\sum_{i=1}^c x_i = 1$ . □

**4.2. Corollary :** *Let  $R \subseteq S$  be orders on  $X$ . Then the Möbius function  $\mu_\Omega(R, S)$  of the poset  $\Omega$  is equal to 0 if  $\Phi(S) \not\subseteq R$ , and to  $(-1)^{|S-R|}$  otherwise.* □

**Proof :** Indeed, the Möbius function  $\mu_\Omega(R, S)$  is equal to the reduced Euler-Poincaré characteristic of the poset  $]R, S[_\Omega$ . □

## 5. Upper intervals

Let  $R$  be an order on  $X$ . This section deals with the homotopy type of the poset  $]R, \cdot[_\Omega$ .

**5.1. Lemma :** *Let  $R$  be an order on  $X$ , and  $(a, b) \in (X \times X)$ . Then  $(a, b)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$  if and only if  $(a, b) \notin R \cup R^{op}$  and*

$$(5.2) \quad ]\cdot, a[_R \subseteq ]\cdot, b[_R \text{ and } ]b, \cdot[_R \subseteq ]a, \cdot[_R \cdot$$

**Proof :** If  $(a, b)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ , then  $(a, b) \notin R \cup R^{op}$ . If  $x \in X$  and  $x <_R a$ , then  $(x, b) <_{R^{(2)}} (a, b)$ .

Hence  $(x, b) \in R \cup R^{op}$ . If  $(x, b) \in R^{op}$ , then  $b \leq_R x <_R a$ , hence  $(b, a) \in R$ , a contradiction. Hence  $(x, b) \in R$ , i.e.  $x \leq_R b$ . Moreover  $x \neq b$  since  $(b, a) \notin R$ . Hence  $x <_R b$ , i.e.  $x \in ]\cdot, b[_R$ . This shows that  $]\cdot, a[_R \subseteq ]\cdot, b[_R$ . Similarly, if  $y \in X$  and  $b <_R y$ , then  $(a, y) <_{R^{(2)}} (a, b)$ , so  $(a, y) \in R \cup R^{op}$ . If  $(a, y) \in R^{op}$ , then  $b <_R y \leq_R a$ , and  $(b, a) \in R$ , a contradiction. Hence  $(a, y) \in R$ , i.e.  $a \leq_R y$ . Moreover  $a \neq y$  since  $b \not\leq_R a$ . Hence  $y \in ]a, \cdot[_R$ , thus  $]b, \cdot[_R \subseteq ]a, \cdot[_R$ , and Condition 5.2 holds.

Conversely, if  $(a, b) \notin R \cup R^{op}$ , and if Condition 5.2 holds, suppose that  $(x, y) \in X \times X$  and  $(x, y) \leq_{R^{(2)}} (a, b)$ , i.e.  $x \leq_R a$  and  $b \leq_R y$ . If  $x \neq a$ , then  $x \in ]\cdot, a[_R \subseteq ]\cdot, b[_R$ , hence  $x <_R b \leq_R y$ , thus  $(x, y) \in R$ . Similarly if  $b \neq y$ , then  $y \in ]b, \cdot[_R \subseteq ]a, \cdot[_R$ , hence  $x \leq_R a <_R y$ , and  $(x, y) \in R$ . Hence if  $(x, y) <_{R^{(2)}} (a, b)$ , then  $(x, y) \in R \subseteq R \cup R^{op}$ , so  $(a, b)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ .  $\square$

This motivates the following notation:

**5.3. Notation :** Let  $R \in \Omega$ . Set

$$\mathcal{E}_R = \{(a, b) \in (X \times X) - (R \cup R^{op}) \mid ]\cdot, a[_R \subseteq ]\cdot, b[_R \text{ and } ]b, \cdot[_R \subseteq ]a, \cdot[_R\} .$$

**5.4. Theorem :** Let  $R$  be an order on  $X$ .

1. If there exists  $(a, b) \in \mathcal{E}_R$  such that  $(b, a) \notin \mathcal{E}_R$ , then  $]R, \cdot[_\Omega$  is contractible.
2. Otherwise  $\mathcal{E}_R = (\mathcal{E}_R)^{op}$ , and  $E_R = \Delta \sqcup \mathcal{E}_R$  is an equivalence relation on  $X$ , which can be defined by

$$(a, b) \in E_R \Leftrightarrow \begin{cases} ]\cdot, a[_R = ]\cdot, b[_R \\ ]a, \cdot[_R = ]b, \cdot[_R \end{cases} .$$

If  $X_1, \dots, X_r$  are the equivalence classes for this relation, let  $\Omega_i = \Omega_{X_i}$  and  $\Delta_i = \Delta_{X_i}$ , for  $i \in \{1, \dots, r\}$ . Then there is a homotopy equivalence

$$]R, \cdot[_\Omega \cong ]\Delta_1, \cdot[_{\Omega_1} * ]\Delta_2, \cdot[_{\Omega_2} * \dots * ]\Delta_r, \cdot[_{\Omega_r} ,$$

where  $P * Q$  in the right hand side denotes the join of two posets  $P$  and  $Q$  (cf. [4] Proposition 1.9).

**Proof :** If  $(a, b) \in \mathcal{E}_R$ , then  $(a, b)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ , by Lemma 5.1. It follows from Lemma 3.4 that the

relation  $S = R \sqcup \{(a, b)\}$  is an order on  $X$ . The poset  $[S, \cdot]_{\Omega}$  has a smallest element  $S$ , hence it is a contractible subposet of  $]R, \cdot]_{\Omega}$ .

Let  $i : [S, \cdot]_{\Omega} \hookrightarrow ]R, \cdot]_{\Omega}$  denote the inclusion map. For  $T \in ]R, \cdot]_{\Omega}$ , the poset

$$i_T = \{U \in [S, \cdot]_{\Omega} \mid U \supseteq T\}$$

is equal to the set of order relations containing  $S \cup T$ . If  $\overline{S \cup T}$  is an order, i.e. if it is antisymmetric, then  $i_T$  has a smallest element  $\overline{S \cup T}$ . Otherwise  $i_T$  is empty.

Let

$$A_S = \{T \in ]R, \cdot]_{\Omega} \mid \overline{S \cup T} \text{ is antisymmetric}\} .$$

In particular  $A_S \supseteq [S, \cdot]_{\Omega}$ , and the inclusion  $i : [S, \cdot]_{\Omega} \hookrightarrow A_S$  is a homotopy equivalence ([4], Proposition 1.6). Hence  $A_S$  is contractible.

Now it is easy to check that  $\overline{S \cup T} = \{(a, b)\} \cup T$  is antisymmetric if and only if  $(b, a) \notin T$ .

Let

$$(A_S)_* = \{T \in A_S \mid \cdot, T]_{A_S} \text{ is not contractible}\} .$$

Then by standard results (e.g. [2] or [1], Proposition 6.6.5), if  $B$  is any poset with  $(A_S)_* \subseteq B \subseteq A_S$ , the inclusion  $B \hookrightarrow A_S$  is a homotopy equivalence. Now  $\cdot, T]_{A_S} = ]R, T]_{\Omega}$ , for any  $T \in A_S$ . Let

$$B = \{T \in ]R, \cdot]_{\Omega} \mid (b, a) \notin T \text{ and } \Phi(T) \subseteq R\} .$$

Then  $(A_S)_* \subseteq B \subseteq A_S$  by Theorem 4.1, thus  $B$  is contractible.

Let  $T \in ]R, \cdot]_{\Omega}$  such that  $\Phi(T) \subseteq R$ . If  $(b, a) \in T$ , then  $R \sqcup \{(b, a)\} \subseteq T$  is an order, by Theorem 4.1. By Lemma 3.4, the pair  $(b, a)$  is a minimal element of  $(X \times X) - (R \cup R^{op})$  for the relation  $R^{(2)}$ . Hence  $(b, a) \in \mathcal{E}_R$ , by Lemma 5.1.

Hence if there exist  $(a, b) \in \mathcal{E}_R$  such that  $(b, a) \notin \mathcal{E}_R$ , then  $(]R, \cdot]_{\Omega})_* = B$  is contractible, which proves Assertion 1.

Otherwise  $\mathcal{E}_R \subseteq \mathcal{E}_R^{op}$ , hence  $\mathcal{E}_R = \mathcal{E}_R^{op}$ , and

$$\mathcal{E}_R = \{(a, b) \in (X \times X) - (R \cup R^{op}) \mid \cdot, a]_{[R=]} \cdot, b]_{[R]} \text{ and } ]a, \cdot]_{[R=]} b, \cdot]_{[R]}\} .$$

Then  $E_R = \Delta \sqcup \mathcal{E}_R$  is an equivalence relation: it is obviously reflexive and symmetric. To show that it is transitive, it suffices to show that if  $(a, b), (b, c) \in \mathcal{E}_R$ , then  $(a, c) \in E_R$ . The assumption implies that  $\cdot, a]_{[R=]} \cdot, b]_{[R]} \cdot, c]_{[R]}$  and  $]a, \cdot]_{[R=]} b, \cdot]_{[R=]} c, \cdot]_{[R]}$ . If  $(a, c) \in R$ , then either  $a = c$  or  $a \in \cdot, c]_{[R]}$ . In the latter case  $a \in \cdot, a]_{[R]}$ , a contradiction. Thus  $a = c$ . Now if  $(a, c) \in R^{op}$ , then  $c \leq_R a$ . Thus again if  $a \neq c$ , then  $c \in \cdot, a]_{[R=]} \cdot, c]_{[R]}$ , a contradiction. In both cases  $a = c$ . Thus either  $a = c$ , or  $(a, c) \notin R \cup R^{op}$ , and then  $(a, c) \in \mathcal{E}_R$ . Hence  $E_R$  is transitive, so it is an equivalence relation.

Let  $X_1, \dots, X_r$  be the equivalence classes of  $E_R$ . If  $T$  is an order containing  $R$ , for each  $i \in \{1, \dots, r\}$ , denote by  $T|_i$  the restriction of  $T$  to  $X_i$ , i.e.  $T|_i = T \cap (X_i \times X_i)$ . Then  $T|_i$  is an order on  $X_i$ , in other words  $T|_i \in \Omega_i = \Omega_{X_i}$ . Let

$$P = [\Delta_1, \cdot]_{\Omega_1} \times [\Delta_2, \cdot]_{\Omega_2} \times \dots \times [\Delta_r, \cdot]_{\Omega_r}$$

denote the product poset. The map

$$e : [R, \cdot]_{\Omega} \rightarrow P$$

defined by  $e(T) = (T|_1, \dots, T|_r)$  is a map of posets.

**Claim:** Let  $(T_1, \dots, T_r) \in P$ . Then  $S = R \cup T_1 \cup \dots \cup T_r$  is an order on  $X$ .

Indeed  $S$  is clearly reflexive. It is also antisymmetric: if  $(a, b) \in S \cap S^{op}$ , then we can assume that:

- either  $(a, b) \in R \cap R^{op}$ , and then  $a = b$ , since  $R$  is antisymmetric.
- or  $(a, b) \in R$  and  $(b, a) \in T_i - R$ , for some  $i \in \{1, \dots, r\}$ . Then  $a \neq b$ , thus  $(b, a) \in \mathcal{E}_R$ , hence  $] \cdot, a[_R = ] \cdot, b[_R$ , and  $a \in ] \cdot, b[_R = ] \cdot, a[_R$ , a contradiction.
- or  $(a, b) \in T_i - R$  (thus  $a \neq b$ ) and  $(b, a) \in T_j$ , for some  $i, j \in \{1, \dots, r\}$ . In particular  $(a, b) \in E_R$ , so  $a$  and  $b$  are in the same equivalence class for  $E_R$ , hence  $i = j$ . But then  $(a, b) \in T_i \cap T_i^{op}$ , thus  $a = b$  since  $T_i$  is antisymmetric. This is again a contradiction.

Hence  $S$  is antisymmetric. It is also transitive, for if  $(a, b) \in S - \Delta$  and  $(b, c) \in S - \Delta$ , then:

- either  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R \subseteq S$ , since  $R$  is transitive.
- or  $(a, b) \in R - \Delta$  and  $(b, c) \in T_i - R$ , for some  $i \in \{1, \dots, r\}$ . Then  $a <_R b$ , thus  $a \in ] \cdot, b[_R = ] \cdot, c[_R$ , and then  $(a, c) \in R \subseteq S$ .
- or  $(a, b) \in T_i - R$  and  $(b, c) \in R - \Delta$ , for some  $i \in \{1, \dots, r\}$ . Then  $b <_R c$ , i.e.  $c \in ]b, \cdot[_R = ]a, \cdot[_R$ , hence  $(a, c) \in R \subseteq S$ .
- or  $(a, b) \in T_i - R$  and  $(b, c) \in T_j - R$ , for some  $i, j \in \{1, \dots, r\}$ . Then  $(a, b) \in \mathcal{E}_R$  and  $(b, c) \in \mathcal{E}_R$ , and  $a, b, c$  are in the same equivalence class for  $E_R$ . Hence  $i = j$ , thus  $a <_{T_i} b <_{T_i} c$ , thus  $a <_{T_i} c$ , and  $(a, c) \in T_i \subseteq S$ .

This completes the proof of the above claim.

Now let

$$f : P \rightarrow [R, \cdot]_{\Omega}$$

be the map defined by  $f(T_1, \dots, T_r) = R \cup T_1 \cup \dots \cup T_r$ . It is clearly a map of posets.

Moreover for any  $T \in [R, \cdot]_{\Omega}$

$$fe(T) = R \cup T_{|1} \cup \dots \cup T_{|r} \subseteq T \quad .$$

Conversely, for any  $(T_1, \dots, T_r) \in P$

$$ef(T_1, \dots, T_r) = e(R \cup T_1 \cup \dots \cup T_r) = (T'_1, \dots, T'_r) \quad ,$$

where  $T'_i = (R \cup T_1 \cup \dots \cup T_r) \cap (X_i \times X_i) \supseteq T_i$ , for any  $i \in \{1, \dots, k\}$ . Actually  $T'_i = T_i$ , since  $R \cap (X_i \times X_i) = \Delta_i$  : indeed  $R \cap (X_i \times X_i) \subseteq R \cap E_R = \Delta$ . Hence  $ef$  is the identity map of the poset  $P$ . This also shows that  $e(R) = (\Delta_1, \dots, \Delta_r)$ .

Conversely, suppose that  $T \in [R, \cdot]_{\Omega}$  is such that  $e(T) = (\Delta_1, \dots, \Delta_r)$ . Then  $T_{|i} = \Delta_i$  for each  $i \in \{1, \dots, r\}$ . If  $T \neq R$ , let  $(a, b) \in T - R$ , minimal for the relation  $R^{(2)}$ . Then  $a \neq b$ , and  $(b, a) \notin R$ , since  $R \subseteq T$  and  $T$  is antisymmetric. Hence  $(a, b)$  is also a minimal element of  $(X \times X) - (R \cup R^{op})$ , thus  $(a, b) \in \mathcal{E}_R$ . It follows that  $(a, b) \in T_i$  for some  $i \in \{1, \dots, r\}$ . As  $a \neq b$ , this contradicts the assumption  $T_{|i} = \Delta_i$ . Hence  $T = R$ .

Finally, let  $(T_1, \dots, T_r) \in P$  such that  $f(T_1, \dots, T_r) = R$ . It follows that  $(T_1, \dots, T_r) = ef(T_1, \dots, T_r) = e(R) = (\Delta_1, \dots, \Delta_r)$ .

Now the maps  $e$  and  $f$  restrict to maps of posets

$$]R, \cdot[_{\Omega} \xrightleftharpoons[f]{e} ([\Delta_1, \cdot[_{\Omega_1} \times [\Delta_2, \cdot[_{\Omega_2} \times \dots \times [\Delta_r, \cdot[_{\Omega_r}) - \{(\Delta_1, \dots, \Delta_r)\}$$

such that  $fe \subseteq \text{Id}$  and  $ef = \text{Id}$ . Hence  $e$  and  $f$  are inverse homotopy equivalences ([4] 1.3). This completes the proof of the Theorem, since the poset

$$([\Delta_1, \cdot[_{\Omega_1} \times [\Delta_2, \cdot[_{\Omega_2} \times \dots \times [\Delta_r, \cdot[_{\Omega_r}) - \{(\Delta_1, \dots, \Delta_r)\}$$

is homotopic to the join  $] \Delta_1, \cdot[_{\Omega_1} * ] \Delta_2, \cdot[_{\Omega_2} * \dots * ] \Delta_r, \cdot[_{\Omega_r}$ , by [4] Proposition 1.9.  $\square$

**5.5. Theorem :** *Let  $X$  be a set of cardinality  $n$ . Then the poset  $\Omega_X - \{\Delta_X\}$  of non trivial orders on  $X$  has the homotopy type of a sphere of dimension  $n - 2$ .*

**Proof :** Set  $\Omega^{\sharp} = \Omega - \{\Delta\}$  (where as above  $\Omega = \Omega_X$  and  $\Delta = \Delta_X$ ). If  $R \in \Omega^{\sharp}$ , then  $] \cdot, R[_{\Omega^{\sharp}} = ] \Delta, R[_{\Omega}$ . If  $\Phi(R) \neq \Delta$ , then  $] \Delta, R[_{\Omega}$  is contractible by Theorem 4.1. It follows that the inclusion

$$B = \{R \in \Omega^{\sharp} \mid \Phi(R) = \Delta\} \hookrightarrow \Omega^{\sharp}$$

is a homotopy equivalence.

Now saying that  $\Phi(R) = \Delta$  is equivalent to saying that there is no chain  $x <_R y <_R z$  in  $X$ . If  $R \in B$ , let

$$\begin{aligned} R_- &= \{x \in X \mid \exists y \in X, x <_R y\} \\ R_+ &= \{y \in X \mid \exists x \in X, x <_R y\} \\ R_0 &= X - (R_- \cup R_+) . \end{aligned}$$

Define  $h(R) = R \cup (R_- \times R_+)$ . Then  $h(R)$  is an order on  $X$ : it is obviously reflexive, since if  $(a, b) \in h(R) \cap h(R)^{op}$ , then:

- either  $(a, b) \in R$  and  $(b, a) \in R$ , hence  $a = b$ .
- or  $(a, b) \in R$  and  $(b, a) \in R_- \times R_+$ . Then  $b \in R_-$  and  $a \in R_+$ . Moreover  $a \neq b$ , and  $a <_R b$ . Thus  $a \in R_-$  and  $b \in R_+$ . This is a contradiction since  $R_- \cap R_+ = \emptyset$ .
- or  $(b, a) \in R$  and  $(a, b) \in R_- \times R_+$ . The argument of the previous case applies to the pair  $(b, a)$ , and this gives a contradiction again.
- or  $(a, b) \in R_- \times R_+$  and  $(b, a) \in R_- \times R_+$ . Again, since  $R_- \cap R_+ = \emptyset$ , this is a contradiction.

Finally if  $(a, b) \in h(R)$  and  $(b, c) \in h(R)$ , then either  $a = b$  or  $b = c$ , so  $h(R)$  is trivially transitive. It follows that  $h(R) \in \Omega^\#$ . Clearly  $R \subseteq h(R)$ .

If  $R \subseteq S \in B$ , then clearly  $R_- \subseteq S_-$  and  $R_+ \subseteq S_+$ , thus  $h(R) \subseteq h(S)$ . Moreover  $h(R)_- = R_-$  and  $h(R)_+ = R_+$ , thus  $h^2 = h$ . It follows that  $h$  induces a homotopy equivalence

$$B \rightarrow h(B) = \{R \in B \mid h(R) = R\} ,$$

inverse to the inclusion  $h(B) \hookrightarrow B$ . Here  $h(B)$  is viewed as a subposet of  $B$ , i.e.  $h(B)$  is ordered by inclusion.

Now if  $R \in B$  and  $h(R) = R$ , then the pair  $\pi(R) = (R_-, R_+)$  is a pair  $(V, U)$  of disjoint non empty subsets of  $X$ . Conversely, such a pair  $(V, U)$  yields an element  $\omega(V, U)$  of  $h(B)$  defined by  $\omega(V, U) = \Delta \cup (V \times U)$ . The maps  $\pi$  and  $\omega$  induce inverse isomorphisms of posets between  $h(B)$  and the poset

$$D_2(X) = \{(V, U) \mid \emptyset \neq V \subset X, \emptyset \neq U \subset X, V \cap U = \emptyset\} ,$$

ordered by the product order  $(V', U') \subseteq (V, U) \Leftrightarrow V' \subseteq V$  and  $U' \subseteq U$  (note that if  $U$  and  $V$  are disjoint non empty subsets of  $X$ , then  $U$  and  $V$  are *proper* subsets of  $X$ ).

Let  $D_1(X)$  denote the set of non empty proper subsets of  $X$ , ordered by inclusion. The projection map  $g : D_2(X) \rightarrow D_1(X)$  defined by  $g(V, U) = U$  is a map of posets. Moreover for  $W \in D_1(X)$

$$g_W = \{(V, U) \in D_2(X) \mid g(V, U) \supseteq W\} = \{(V, U) \in D_2(X) \mid U \supseteq W\}$$

is contractible via the maps  $(V, U) \mapsto (V, W) \subseteq (X - W, W)$ . It follows that  $g$  is a homotopy equivalence ([4], Proposition 1.6). Hence the posets  $\Omega^\sharp$  and  $D_1(X)$  are homotopy equivalent.

This completes the proof, since if  $X$  has cardinality  $n$ , then  $D_1(X)$  has the homotopy type of a sphere of dimension  $n - 2$ .  $\square$

**5.6. Corollary :** *Let  $R$  be an order on  $X$ . Then:*

1. *if  $\mathcal{E}_R \neq \mathcal{E}_R^{op}$ , the poset  $]R, \cdot[_\Omega$  is contractible.*
2. *if  $\mathcal{E}_R = \mathcal{E}_R^{op}$ , then  $E_R = \Delta \sqcup \mathcal{E}_R$  is an equivalence relation on  $X$ . The poset  $]R, \cdot[_\Omega$  is homotopy equivalent to a sphere of dimension  $n - r - 1$ , where  $r$  is the number of equivalence classes of  $E_R$ .*

**Proof :** Assertion 1 follows from Theorem 5.4. For Assertion 2, by Theorem 5.4, there is a homotopy equivalence

$$]R, \cdot[_\Omega \cong ]\Delta_1, \cdot[_{\Omega_1} * ]\Delta_2, \cdot[_{\Omega_2} * \dots * ]\Delta_r, \cdot[_{\Omega_r} ,$$

where  $\Omega_i = \Omega_{X_i}$  is the poset of posets on the equivalence class  $X_i$  of  $X$  for the relation  $E_R$ .

By Theorem 5.5, each poset  $]\Delta_i, \cdot[_{\Omega_i}$  has the homotopy type of a sphere of dimension  $|X_i| - 2$ . Since for  $d, e \in \mathbb{N}$ , the join  $S^d * S^e$  is homotopy equivalent to  $S^{d+e+1}$  (where  $S^d$  denotes a sphere of dimension  $d$ ), it follows that  $]R, \cdot[_\Omega$  has the homotopy type of a sphere of dimension

$$(|X_1| - 2) + (|X_2| - 2) + \dots + (|X_r| - 2) + r - 1 = n - 2r + r - 1 = n - r - 1 ,$$

as was to be shown.  $\square$

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