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► **To cite this version:**

Nicolas Hussenot. The leaves of the Fatou set accumulate on the leaves of the Julia set. 2013. hal-00881844v1

HAL Id: hal-00881844

<https://hal.science/hal-00881844v1>

Preprint submitted on 9 Nov 2013 (v1), last revised 20 Aug 2015 (v2)

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The leaves of the Fatou set accumulate on the leaves of the Julia set

Nicolas Hussenot

November 9, 2013

Abstract

In 2001, E. Ghys, X. Gomez-Mont and J. Saludes defined in [GGS] the Fatou and Julia components of transversely holomorphic foliations on compact manifolds. It is a partition of the manifold in two saturated sets: the Fatou set which is open and represents the non-chaotic part of the foliation and its complementary set, the Julia set. Using the brownian motion transverse to the leaves, it is proved that, if the foliation is taut and if F is a wandering component of the Fatou set, then almost every point of the topological boundary ∂F (almost for any harmonic measure on ∂F) is a limit point of each leaf of F .

1 Introduction

Let M be a compact, connected manifold of real dimension $d + 2$ endowed with a transversely holomorphic foliation \mathcal{F} . In 2001, Etienne Ghys, Xavier Gomez-Mont and Jordi Saludes defined in [GGS], the Fatou and Julia sets of such a foliation. It is a partition of the manifold in two saturated sets: the Fatou set is defined as the set of points x in M such that there exists a basic normal vector field (i.e a section of the normal bundle to the foliation constant along the leaves) which does not vanish at x (we will come back later to the demanded regularity for X). The Fatou set is then an open set. Its complementary set is a closed set called Julia set. Given a connected component F of the Fatou set, it is easy to prove (integrating the basic normal vector fields), that the group of homeomorphisms of M which preserve the foliation acts transitively on F . Using this homogeneity property and by

analogy with Molino's theory ([Mo]), Ghys, Gomez-Mont and Saludes prove that there are only three exclusive cases for a connected component F_k of the Fatou set (cf theorem 2.4). If \mathcal{F}_k denotes the restriction of \mathcal{F} to F_k , then the three exclusive cases are:

1. F_k is a wandering component: the leaves of \mathcal{F}_k are closed in F_k
2. F_k is a semi-wandering component: the closure of the leaves of \mathcal{F}_k form a real codimension 1 foliation of F_k which has a structure of a fibre bundle over a 1-dimensional manifold.
3. F_k is a dense component: all the leaves of \mathcal{F}_k are dense in F_k

In [GGS], the authors give a complete description of the Fatou set studying in details each of the three previous cases. In this article, we are interested in the following question: does the Julia set play the role of an attractive set for the leaves of the Fatou set? Using probabilistic tools, we prove the following theorem which is a beginning of positive answer:

Theorem 1.1. *Let M be a compact, connected manifold endowed with a transversely holomorphic foliation \mathcal{F} . Suppose that \mathcal{F} is taut. Let F be a wandering component of the Fatou set. Let x_0 be a point of F and μ be the exit measure of F of a brownian motion starting from x_0 . Then, μ -almost every point of ∂F is a limit point of each leaf of F .*

Remarks 1.2. 1. *The hypothesis that the foliation is taut is essential in our proof but we think that the theorem is also true without this hypothesis. Nevertheless, this hypothesis is not so strong: for example a foliation without invariant transverse measure is taut. (see [S2] or [CC1] for some more details about taut foliations).*

2. *Let us give the main ideas of the proof of theorem (1.1): if F is a wandering component, the leaf space $\Sigma := F/\mathcal{F}$ is a finite Riemann surface. Using the tautness of the foliation, we prove that one can find a complete metric g in M and a complete metric h in Σ such that the projection $p : (F, g) \rightarrow (\Sigma, h)$ preserves the brownian motion. We conclude using the fact that a brownian motion is recurrent in a finite Riemann surface.*

2 Definitions

In this first section, we are going to define the Fatou and Julia components of a transversely holomorphic foliation following the presentation of [GGs].

Let M be a compact, connected manifold of real dimension $d+2$ endowed with a transversely holomorphic foliation \mathcal{F} . Such a foliation may be defined by an atlas $(U_i, \varphi_i, \gamma_{ij})$. The U_i are open sets in M covering M . The maps $\varphi_i : U_i \rightarrow \mathbb{C}$ are submersions with connected fibres and the maps $\gamma_{ij} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$ are biholomorphisms and satisfy $\varphi_i = \gamma_{ij} \circ \varphi_j$.

Look at (M, \mathcal{F}) as a foliation of complex codimension 1. Denote TM the tangent bundle, $T\mathcal{F}$ the subbundle of TM consisting of those vectors which are tangent to the leaves and $\nu^{1,0}$ the quotient bundle $TM/T\mathcal{F}$. One has the following exact sequence:

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \longrightarrow \nu^{1,0} \longrightarrow 0 \quad (1)$$

Let E be a vector bundle over M . We say that the germ of a section X of E at a point x of M has modulus of continuity $\epsilon \log \epsilon$ if there is a positive constant C , a coordinate chart U containing x , and a trivialization of the bundle over U such that for x_1, x_2 in U , we have:

$$|X(x_2) - X(x_1)| < -C |x_1 - x_2| \log(|x_1 - x_2|)$$

Denote $\mathcal{C}^{\epsilon \log \epsilon}(E)$ the sheaf of sections of the vector bundle E which have a modulus of continuity $\epsilon \log \epsilon$. The sheaves of sections of (1) of modulus of continuity $\epsilon \log \epsilon$ give rise to the following exact sequence of fine sheaves:

$$0 \longrightarrow \mathcal{C}^{\epsilon \log \epsilon}(T\mathcal{F}) \longrightarrow \mathcal{C}^{\epsilon \log \epsilon}(TM) \longrightarrow \mathcal{C}^{\epsilon \log \epsilon}(\nu^{1,0}) \longrightarrow 0$$

The normal bundle $\nu^{1,0}$ is flat along the leaves. Indeed, it can be defined by the cocycle: $U_i \cap U_j \rightarrow \mathbb{C}^*$, $p \mapsto \gamma'_{ji}(\varphi_i(p))$. These maps are constant along the fibres of φ_i so they are constant along the leaves. Any time that we have a bundle which is flat along the leaves, we can define the sections of this bundle which are constant along the leaves. We call these sections basic sections.

As $\nu^{1,0}$, the bundle $\nu^{1,0} \otimes \nu^{0,1*}$ is flat along the leaves. Denote $L_{\mathcal{F}}^{\infty}(\nu^{1,0} \otimes \nu^{0,1*})$ the sheaf of basic sections of this bundle which are essentially bounded. Denote also $\mathcal{C}_{\mathcal{F}}(\nu^{1,0})$ the sheaf of continuous basic sections of the bundle $\nu^{1,0}$ satisfying: $\forall \sigma \in \mathcal{C}_{\mathcal{F}}(\nu^{1,0}), \bar{\partial} \sigma \in L_{\mathcal{F}}^{\infty}(\nu^{1,0} \otimes \nu^{0,1*})$. Denote $\mathcal{C}_{\mathcal{F}}^{\epsilon \log \epsilon}(TM) :=$

$\pi^{-1}(\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ (where π is the projection $\pi : \mathcal{C}^{\epsilon \log \epsilon}(TM) \longrightarrow \mathcal{C}^{\epsilon \log \epsilon}(\nu^{1,0})$).

We have the following exact sequence:

$$0 \longrightarrow \mathcal{C}^{\epsilon \log \epsilon}(T\mathcal{F}) \longrightarrow \mathcal{C}_{\mathcal{F}}^{\epsilon \log \epsilon}(TM) \longrightarrow \mathcal{C}_{\mathcal{F}}(\nu^{1,0}) \longrightarrow 0$$

The first sheaf is a fine one. So, $H^1(M, C^{\epsilon \log \epsilon}(T\mathcal{F})) = 0$, which gives rise to the following exact sequence of global sections:

$$0 \rightarrow H^0(M, C^{\epsilon \log \epsilon}(T\mathcal{F})) \rightarrow H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM)) \rightarrow H^0(M, C_{\mathcal{F}}(\nu^{1,0})) \rightarrow 0$$

This implies that we can lift any basic normal vector field $X \in H^0(M, C_{\mathcal{F}}(\nu^{1,0}))$ to a vector field in M with modulus of continuity $\epsilon \log \epsilon$. Such vector fields have the property to be uniquely integrable.

Summarizing, we have the following:

Lemma 2.1. [GGS]

1. Any basic normal vector field $X \in H^0(M, C_{\mathcal{F}}(\nu^{1,0}))$ can be lifted to a vector field in $H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM))$.
2. Any vector field $X \in H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM))$ gives rise to a global 1-parameter flow $\phi : M \times \mathbb{R} \rightarrow M$ preserving the foliation.

We can now define the Fatou and Julia sets of a transversely holomorphic foliated compact manifold (M, \mathcal{F}) :

Definition 2.2. • The Julia set of (M, \mathcal{F}) is the closed saturated set where all the elements of $H^0(M, C_{\mathcal{F}}(\nu^{1,0}))$ vanish :

$$Julia(\mathcal{F}) = \{x \in M \text{ tel que } X(x) = 0 \quad \forall X \in H^0(M, C_{\mathcal{F}}(\nu^{1,0}))\}$$

- The Fatou set of (M, \mathcal{F}) is the open and saturated set defined as the complement of the Julia set:

$$Fatou(\mathcal{F}) = M - Julia(\mathcal{F})$$

Using the existence of basic normal vector fields in every point of the Fatou set, the authors prove the following homogeneity property:

Proposition 2.3. [GGS] Let F_k be a connected component of the Fatou set. Given x_1 and x_2 two points in F_k , there is a \mathcal{F} -preserving homeomorphism of M sending x_1 to x_2 .

Proof. First, for the tangent direction, given a point x in F_k , it is easy to find \mathcal{F} preserving homeomorphisms sending x to any point located in the same leaf as x (we just have to integrate vector fields tangent to the leaves).

For the transversal direction, we use the fact that if x is in the Fatou set, then there exists $X \in H^0(M, C_{\mathcal{F}}(\nu^{1,0}))$ with $X(x) \neq 0$. Let \tilde{X} and \tilde{Y} in $H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM))$ lifting X and iX and consider the map:

$$\begin{aligned} \phi : \mathbb{C} \times M &\rightarrow M \\ (te^{i\theta}, x) &\mapsto \phi(te^{i\theta}, x) \end{aligned}$$

which associates to $(te^{i\theta}, x)$ the point $\phi(te^{i\theta}, x)$ of M obtained by flowing with the vector field $\tilde{X}_\theta := \tilde{X} \cos \theta + \tilde{Y} \sin \theta$. As \tilde{X}_θ is a basic vector field, if we fix $te^{i\theta}$, the map $\phi(\cdot, te^{i\theta})$ preserve \mathcal{F} . With this method, we get \mathcal{F} preserving homeomorphisms sending x to any point located in a transverse topological disc centered in x .

A composition of maps of both types prove the assertion. ■

The last proposition is the key to prove the following theorem:

Theorem 2.4. [GGS] *Let (M, \mathcal{F}) be a transversely holomorphic foliated compact manifold. Let \mathcal{F}_k be the restriction of \mathcal{F} to a connected component F_k of Fatou(\mathcal{F}). Then, there are 3 exclusive cases:*

1. *Wandering component: the leaves of \mathcal{F}_k are closed in F_k*
2. *semi-wandering component: the closure of the leaves of \mathcal{F}_k form a real codimension 1 foliation of F_k which has a structure of a fibre bundle over a 1-dimensional manifold.*
3. *Dense component: all the leaves of \mathcal{F}_k are dense in F_k*

For the wandering case, one can prove the following:

Theorem 2.5. [GGS] *Let F_k be a wandering component of the Fatou set. Then, the leaf space $\Sigma_k := F_k/\mathcal{F}_k$ is a finite Riemann surface. In other words, Σ_k is a compact Riemann surface minus a finite number of points.*

Remark 2.6. *Note the analogy with the finiteness theorem of Ahlfors which asserts that if Γ is a finitely generated kleinian group, then the quotient of the discontinuity set $\Omega(\Gamma)$ by the action of Γ is a finite Riemann surface.*

3 Harmonic morphisms are brownian path preserving

We start with some basic facts about harmonic morphisms. The reader who wants to know more about this theory could refer, for example, to the survey of John C.Wood ([W]) or to the book of H.Urakawa ([U]).

Let (M, g) and (N, h) be two C^∞ Riemannian manifolds with dimensions are respectively m and n . Denote $\Delta_M = \text{div}(\text{grad})$ the Laplace-Beltrami operator in M . A C^∞ map $f : M \rightarrow \mathbb{R}$ satisfying $\Delta_M f = 0$ is called a harmonic map.

Definition 3.1. A C^∞ map $\Phi : (M, g) \rightarrow (N, h)$ is a harmonic morphism if for any harmonic map $f : V \rightarrow \mathbb{R}$ defined in an open subset V of N with $\Phi^{-1}(V)$ non empty, $f \circ \Phi$ is a harmonic map in $\Phi^{-1}(V)$.

Definition 3.2. A C^∞ map $\Phi : (M, g) \rightarrow (N, h)$ is said to be horizontally conformal if for any point p in M , the differential map $D\Phi_p$ sends conformally the horizontal space $\ker(D\Phi_p)^\perp$ in $T_{\Phi(p)}N$, in other words $D\Phi_p$ is onto and there exists a real $\lambda(p) \neq 0$ such that for all $X, Y \in \ker(D\Phi_p)^\perp$:

$$h_{\phi(p)}(D\Phi_p(X), D\Phi_p(Y)) = \lambda(p)^2 g_p(X, Y)$$

The following characterisation of harmonic morphisms will be useful later:

Theorem 3.3. [BE] Suppose that the dimension of N is $n = 2$. Let $\Phi : (M, g) \rightarrow (N, h)$ be a horizontally conformal map. Then Φ is a harmonic morphism if and only if the fibres of Φ are minimal sub-manifolds of M for the metric g .

Let (M, g) be a connected Riemannian manifold with bounded geometry. The brownian motion on (M, g) is the diffusion process associated to the Laplace-Beltrami operator Δ . It is defined on a probability space (Ω, \mathbb{P}) and denoted $(B_t)_{t \geq 0}$. In 1940, Paul Lévy prove that a conformal map is brownian path preserving (see [Le]).

The following result is a generalisation of Paul Lévy's result. It asserts that the maps between Riemannian manifolds which are brownian path preserving are the harmonic morphisms. This property has been proved in the case of harmonic morphisms between euclidean spaces in [BCD]. The general case of harmonic morphisms between Riemannian manifolds has been proved later in [D].

Theorem 3.4. [D]

Let $(M, g), (N, h)$ be two C^∞ Riemannian manifolds and $\Phi : M \rightarrow N$ be a C^∞ map. Φ is a harmonic morphism if and only if the following is satisfied: let $(B_t)_{t \geq 0}$ be a Brownian motion in M starting from a point a and stopped in a stopping-time T . There is a strictly increasing process $\sigma : \Omega \times [0, T[\rightarrow [0, +\infty[$ and a brownian motion $(B'_s)_{s \geq 0}$ starting from $\phi(a)$ such that

$$\Phi \circ B = B' \circ \sigma$$

Remarks 3.5. 1. $\Phi \circ B = B' \circ \sigma$ means: for all $\omega \in \Omega$ and for all $t \in [0, T(\omega)[$, we have: $\Phi \circ B_t(\omega) = B'_{\sigma_t(\omega)}(\omega)$.

2. The time change scale is explicitly given by the following:

$$\sigma_\omega(t) = \int_0^t \lambda^2(B_u(\omega)) du$$

where λ is the dilatation coefficient of Φ (defined in (3.2))

4 Proof of theorem (1.1)

Let (M, \mathcal{F}) be a transversely holomorphic foliated compact manifold. Suppose that the foliation is taut. Let F be a wandering component of the Fatou set and denote $\Sigma = F/\mathcal{F}$ the leaf space, which is a finite Riemann surface. Let x_0 be a point in F and μ be the exit measure of F for a brownian motion starting from x_0 . The goal of this part is to prove that, for μ -almost every $a \in \partial F$ and for all $x \in F$, the leaf containing x accumulates on a . Let us start with an idea of the proof: the first part of the proof consists in showing that, with the hypothesis that the foliation is taut, we can put a metric g in M and a metric h in Σ such that the projection $p : (F, g|_F) \rightarrow (\Sigma, h)$ is a harmonic morphism. Now, let $(B_t)_{t \geq 0}$ be a brownian motion starting from x_0 and stopped at the exit time T of F . Let a be a point in the support of μ . If V_a is a neighborhood of a , then with a strictly positive probability, $B_t \in V_a$ for t close enough to T . As p is a harmonic morphism, it is brownian path preserving. So, $p(B_{\sigma^{-1}(s)})$ is a brownian path in Σ stopped at the time $\lim_{t \rightarrow T} \sigma_t$. One can prove that $\lim_{t \rightarrow T} \sigma_t = \infty$ and so $p(B_{\sigma^{-1}(s)})$ is a brownian path defined for all the times $s \in [0, \infty[$. As Σ is a Riemann surface of finite type, the brownian motion is recurrent in Σ . So, if U is any open set in F , there is a

sequence $(s_n)_{n \in \mathbb{N}}$ converging to infinity such that $p(B_{\sigma^{-1}(s_n)})$ belong to $p(U)$. This implies that there exists a sequence $(t_n)_{n \in \mathbb{N}}$ converging to T such that B_{t_n} belong to $p^{-1}(p(U)) = \text{sat}(U)$ ($\text{sat}(U)$ is the saturated set of U for the foliation). This proves that for any open set U in F and every neighborhood V_a of a , $\text{sat}(U) \cap V_a \neq \emptyset$. We will see that this property implies the theorem.

Now, let us precise the previous ideas. Put on the Riemann surface Σ a metric h with constant curvature $+1$, 0 or -1 in its conformal class. We have the following:

Lemma 4.1. *There exists a metric g in M satisfying:*

1. *the fibres of $p : (F, g|_F) \longrightarrow (\Sigma, h)$ are minimal.*
2. *$p : (F, g|_F) \longrightarrow (\Sigma, h)$ is horizontally conform. In other words, there exists a continuous map $\lambda : F \rightarrow \mathbb{R}^*$ such that if X and Y are two vector fields orthogonal to the fibres of p , for all x in F :*

$$h_{p(x)}(p'_x(X(x)), p'_x(Y(x))) = \lambda^2(x) \cdot g_x(X(x), Y(x))$$

Proof. Let g_0 be a metric in M such that all the leaves of \mathcal{F} are minimal. Let \mathcal{A} be an atlas with a finite number of foliated charts: $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^2$ and λ_α a partition of unity associated to this atlas. Let e_1 and e_2 be two vector fields in U_α satisfying:

- $(\phi_\alpha)_*(e_i) = \frac{\partial}{\partial x_i}$, for $i = 1, 2$
- $e_i \in T\mathcal{F}^\perp$, pour $i = 1, 2$

Complete (e_1, e_2) with sections of $T\mathcal{F}|_{U_\alpha}$ so that, for all $x \in U_\alpha$, $b(x) = (e_1(x), e_2(x), e_3(x), \dots, e_{d+2}(x))$ is a basis of $T_x M$. We have:

$$\text{mat}_{b(x)} g_0(x) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define the metric g_0^α in U_α as:

$$\text{mat}_{b(x)} g_0^\alpha(x) = \begin{pmatrix} C & 0 \\ 0 & B \end{pmatrix}$$

with $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, write $g = \sum_{\alpha} \lambda_{\alpha} g_0^{\alpha}$. The metric g satisfies all the wanted properties: indeed we defined g so that the projection p is horizontally conform. A modification of the metric transversely to the fibres do not change the fact that these fibres are minimal. \blacksquare

$p : (F, g|_F) \rightarrow (\Sigma, h)$ is horizontally conform and the fibres of p are minimal. So, according to theorem (3.3), p is a harmonic morphism. Consequently, using theorem (3.4), p is brownian path preserving. So, if $(B_t)_{t \geq 0}$ is a brownian motion starting from a point x_0 in F and stopped at the exit time $T = \inf \{t \text{ such that } B_t \in F^c\}$ of F . $p(B_{\sigma^{-1}(s)})$ is a brownian motion in Σ starting from $p(x_0)$ and stopped at time $\lim_{t \rightarrow T} \sigma_t$. We are going to prove that $\lim_{t \rightarrow T} \sigma_t = \infty$. It is an easy consequence of the following:

Lemma 4.2. *Let $\gamma : [0, +\infty[\rightarrow M$ be a continuous path such that $\gamma(0) \in F$. Denote $t_0 := \inf \{t \in [0, +\infty[; \gamma(t) \notin F\}$. Then $p \circ \gamma$ do not have limit when t tends to t_0 .*

Proof. Suppose on the contrary that $\lim_{t \rightarrow t_0} p \circ \gamma(t) = z_0 \in \Sigma$. Let U be a foliated chart defined in a neighborhood of $\gamma(t_0)$: U is then identified with $A \times B$ where A is an open set in \mathbb{R}^d and B is an open set in \mathbb{C} such that the plaques of the foliation are the sets $A \times \{z\}$. There exists $t_1 < t_0$ such that for all $t \geq t_1$, we have $\gamma(t) \in U$. Let x_0 be a point in $p^{-1}(\{z_0\})$ and let X be a basic normal vector field such that $X(x_0) \neq 0$. Let \tilde{X} and \tilde{Y} in $H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM))$ lifting X and iX . and consider the map:

$$\begin{aligned} \phi : \mathbb{C} \times M &\rightarrow M \\ (te^{i\theta}, x) &\mapsto \phi(te^{i\theta}, x) \end{aligned}$$

which associates to $(te^{i\theta}, x)$ the point $\phi(te^{i\theta}, x)$ of M obtained by flowing with the vector field $\tilde{X}_{\theta} := \tilde{X} \cos \theta + \tilde{Y} \sin \theta$. Using the previous identification of U and $A \times B$, for r small enough, we have:

$$\phi(\overline{D(0, r)} \times L_{x_0}) \cap U = \coprod_{i \in I} A \times V_i$$

where the V_i are closed topological discs pairwise disjoint. Remark also that the $\coprod_{i \in I} A \times V_i$ do not meet the Julia set and that it is possible that $I = \emptyset$. Denote $V = \phi(\overline{D(0, r)} \times L_{x_0})$ (L_{x_0} is the leaf through x_0) and $W = p(V)$: W is a neighborhood of z_0 . So, there exists $t_2 > t_1$ such that for all $t \in [t_2, t_0[$, we

have $p \circ \gamma(t) \in W$. So, for $t \in [t_2, t_0[, \gamma(t) \in p^{-1}(W) \cap U = V \cap U = \coprod_{i \in I} A \times V_i$. So, there is $i \in I$ such that for all $t \in [t_2, t_0[,$ we have $\gamma(t) \in A \times V_i$, which contradicts the fact that $\gamma(t_0)$ belongs to the Julia set. ■

Corolary 4.3. *Almost surely $\lim_{t \rightarrow T} \sigma_t = \infty$*

Proof. Almost surely,

$$\begin{aligned} B & : [0, T[\rightarrow M \\ t & \mapsto B_t \end{aligned}$$

is continuous. So, according to the previous lemma, almost surely $p \circ B_t$ do not have limit when t tends to T . Using the fact that harmonic morphisms are brownian path preserving, there is a brownian motion B'_s in Σ such that for all $t \in [0, T[,$ we have: $p \circ B_t = B'_{\sigma_t}$. So, almost surely, B'_{σ_t} do not have limit when t goes to T . So, $\lim_{t \rightarrow T} \sigma_t = \infty$ ■

Denote μ the exit measure of F : for A borel set in ∂F , we have $\mu(A) = \mathbb{P}_{x_0}(B_T \in A)$. We have to prove that for μ -almost every $a \in \partial F$, for all $x \in F, a \in \overline{L_x}$.

As Σ is a Riemann surface of finite type, $B'_s := (p(B_{\sigma^{-1}(s)}))_{0 \leq s \leq \sigma(T)=+\infty}$ is reccurent in Σ . Take $x \in F$, a neighborhood U_x of x , a point $a \in \text{supp}(\mu)$ and a neighborhood V_a of a in M . We are going to prove that $\text{sat}(U_x) \cap V_a \neq \emptyset$. As a belongs to the support of μ , we have $\mu(V_a) \neq 0$. Denote $A = \{\omega \text{ tel que } B_{T(\omega)}(\omega) \in V_a\}$, we have $\mathbb{P}_{x_0}(A) = \mu(V_a) > 0$. And so, for all $\omega \in A, B_t(\omega) \in V_a$ for t close enough to $T(\omega)$. As B'_s is reccurent in Σ , for almost every $\omega \in \Omega$, there is a sequence s_n tending to infinity such that $B'_{s_n}(\omega) \in p(U_x)$. So, for almost every $\omega \in \Omega$, there is a sequence t_n converging to $T(\omega)$ such that $B_{t_n}(\omega) \in p^{-1}(p(U_x)) = \text{sat}(U_x)$. So, for almost every $\omega \in A$ (ie in a set with strictly positive probability), there is a sequence t_n converging to $T(\omega)$ such that $B_{t_n}(\omega) \in \text{sat}(U_x) \cap V_a$. So, $\text{sat}(U_x) \cap V_a \neq \emptyset$.

Why does this imply that $a \in \overline{L_x}$? for any neighborhood U_x of x and any neighborhood V_a of a , we have: $\text{sat}(U_x) \cap V_a \neq \emptyset$. So, there is a sequence x_n with $x_n \rightarrow x$ and a sequence y_n with $y_n \rightarrow a$ such that $y_n \in L_{x_n}$. Let X_n be a sequence of basic normal vector fields in $H^0(M, C_{\mathcal{F}}(\nu^{1,0}))$ with $X_n(x_n) \neq 0$ and \tilde{X}_n in $H^0(M, C_{\mathcal{F}}^{\epsilon \log \epsilon}(TM))$ lifting X_n and satisfying $\Phi_n(1, x_n) = x$ where $\Phi_n : \mathbb{R} \times M \rightarrow M$ is the flow associated to the vector field \tilde{X}_n . As $y_n \rightarrow a \in \text{Julia}$, $\tilde{X}_n(y_n) \rightarrow 0$. So $\Phi_n(1, y_n) \rightarrow a$. As $\Phi_n(1, \cdot) : M \rightarrow M$ preserve the foliation, we have that $\Phi_n(1, y_n) \in L_x$. So, we have proved that $a \in \overline{L_x}$.

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