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Francois Couchot

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# GAUSSIAN TRIVIAL RING EXTENSIONS AND FQP-RINGS

FRANÇOIS COUCHOT

ABSTRACT. Let  $A$  be a commutative ring and  $E$  a non-zero  $A$ -module. Necessary and sufficient conditions are given for the trivial ring extension  $R$  of  $A$  by  $E$  to be either arithmetical or Gaussian. The possibility for  $R$  to be Bézout is also studied, but a response is only given in the case where  $\text{pSpec } A$  (a quotient space of  $\text{Spec } A$ ) is totally disconnected. Trivial ring extensions which are fqp-rings are characterized only in the local case. To get a general result we introduce the class of fqf-rings satisfying a weaker property than fqp-ring. Moreover, it is proven that the finitistic weak dimension of a fqf-ring is 0, 1 or 2 and its global weak dimension is 0, 1 or  $\infty$ .

Trivial ring extensions are often used to give either examples or counterexamples of rings. One of the most famous is the chain ring  $R$  which is not factor of a valuation domain (see [8, X.6] and [7, Theorem 3.5]). This ring  $R$  is the trivial ring extension of a valuation domain  $D$  by a non-standard uniserial divisible  $D$ -module. This example gives a negative answer to a question posed by Kaplansky.

In [14], [15] and [1] there are many results on trivial ring extensions and many examples of such rings. In particular, necessary and sufficient conditions are given for the trivial ring extension of a ring  $A$  by an  $A$ -module  $E$  to be either arithmetical or Gaussian in the following cases: either  $A$  is a domain and  $K$  is its quotient field, or  $A$  is local and  $K$  is its residue field, and  $E$  is a  $K$ -vector space.

In our paper more general results are shown. For instance the trivial ring extension  $R$  of a ring  $A$  by a non-zero  $A$ -module  $E$  is a chain ring if and only if  $A$  is a valuation domain and  $E$  a divisible module, and  $R$  is Gaussian if and only if  $A$  is Gaussian and  $E$  verifies  $aE = a^2E$  for each  $a \in A$ . Complete characterizations of arithmetical trivial ring extensions are given too. But Bezout trivial ring extensions of a ring  $A$  are characterized only in the case where  $\text{pSpec } A$  (a quotient space of  $\text{Spec } A$ ) is totally disconnected.

We also study trivial ring extensions which are fqp-rings. The class of fqp-rings was introduced in [1] by Abuhdail, Jarrar and Kabbaj. We get a complete characterization of trivial ring extensions which are fqp-ring only in the local case. Each fqp-ring is locally fqp and the converse holds if it is coherent, but this is not generally true. We introduce the class of fqf-rings which satisfy the condition "each finitely generated ideal is flat modulo its annihilator", and it is exactly the class of locally fqp-rings. So, trivial fqf-ring extensions are completely characterized. This new class of rings contains strictly the class of fqp-rings.

In [6] it is proven that each arithmetical ring has a finitistic weak dimension equal to 0, 1 or 2. We show that any fqf-ring satisfy this property too, and its global weak dimension is 0, 1 or  $\infty$  as it is shown for fqp-rings in [1].

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All rings in this paper are associative and commutative with unity, and all modules are unital. We denote respectively  $\text{Spec } A$ ,  $\text{Max } A$  and  $\text{Min } A$ , the space of prime ideals, maximal ideals and minimal prime ideals of  $A$ , with the Zariski topology. If  $I$  a subset of  $R$ , then we denote

$$V(I) = \{P \in \text{Spec } A \mid I \subseteq P\} \quad \text{and} \quad D(I) = \text{Spec } A \setminus V(I).$$

Let  $A$  be a ring and  $E$  an  $A$ -module. The **trivial ring extension** of  $A$  by  $E$  (also called the idealization of  $E$  over  $A$ ) is the ring  $R := A \times E$  whose underlying group is  $A \times E$  with multiplication given by  $(a, e)(a', e') = (aa', ae' + a'e)$ .

## 1. ARITHMETICAL RINGS

An  $R$ -module  $M$  is said to be **uniserial** if its set of submodules is totally ordered by inclusion and  $R$  is a **chain ring**<sup>1</sup> if it is uniserial as  $R$ -module. Recall that a chain ring  $R$  is said to be **Archimedean** if its maximal ideal is the sole non-zero prime ideal.

**Proposition 1.1.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \times E$  the trivial ring extension of  $A$  by  $E$ . The following two conditions are equivalent:*

- (1)  $R$  is a chain ring;
- (2)  $A$  is a valuation domain and  $E$  is a uniserial divisible module.

*Proof.* It is well known that (2)  $\Rightarrow$  (1) (for instance, see [7, Example I.1.9]). Conversely,  $A$  is a chain ring because it is a factor of  $R$ , and  $E$  is uniserial because it is isomorphic to an ideal of  $R$ . Let  $0 \neq a \in A$  and  $x \in E$ . It is obvious that  $(0, x) \in R(a, 0)$ . So,  $(0, x) = (a, 0)(b, y)$  for some  $y \in E$  and  $b \in A$ , whence  $x = ay$ . Hence  $E = aE$ . Let  $a, b$  be non-zero elements of  $A$ . Then  $E = aE = abE$ . We deduce that  $ab \neq 0$ . Hence  $A$  is a domain and  $E$  is divisible.  $\square$

Let  $E$  be a module over a ring  $R$ . We say that  $E$  has a **distributive** lattice of submodules if  $E$  satisfies one of the following two equivalent conditions:

- $(M + N) \cap P = (M \cap P) + (N \cap P)$  for any submodules  $M, N, P$  of  $E$ ;
- $(M \cap N) + P = (M + P) \cap (N + P)$  for any submodules  $M, N, P$  of  $E$ .

The following proposition can be proved as [13, Theorem 1].

**Proposition 1.2.** *Let  $E$  be a module over a ring  $R$ . The following conditions are equivalent:*

- (1)  $E$  has a distributive lattice of submodules;
- (2)  $E_P$  is a uniserial module for each maximal ideal  $P$  of  $R$ .

A ring (respectively domain)  $R$  is said to be **arithmetical** (respectively **Prüfer**) if its lattice of ideals is distributive. An  $R$ -module  $E$  is **FP-injective** if, for each finitely presented  $R$ -module  $F$ ,  $\text{Ext}_R^1(F, E) = 0$ . When  $R$  is a Prüfer domain then a module is FP-injective if and only if it is divisible.

**Lemma 1.3.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \times E$  the trivial ring extension of  $A$  by  $E$ . Let  $S$  be a multiplicative subset of  $R$  and  $S'$  the image of  $S$  by the map  $R \rightarrow A$  defined by  $(a, x) \mapsto a$ . Then  $S^{-1}R = S'^{-1} \times S'^{-1}E$ .*

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<sup>1</sup>we prefer “chain ring” to “valuation ring” to avoid confusion with “Manis valuation ring”.

*Proof.* If  $(a, x) \in R$  and  $(s, y) \in S$  then it is easy to check that

$$\frac{(a, x)}{(s, y)} = \left( \frac{a}{s}, \frac{sx - ay}{s^2} \right) = \frac{(sa, sx - ay)}{s^2} = \frac{(a, x)(s, -y)}{s^2}.$$

□

For any module  $E$  over ring  $R$  we define  $\text{Supp } E$  as:

$$\text{Supp } E = \{P \mid P \text{ prime ideal such that } E_P \neq 0\}.$$

An  $R$ -module  $E$  is **locally FP-injective** if for each maximal ideal  $P$ ,  $E_P$  is FP-injective.

From Propositions 1.1 and 1.2 and Lemma 1.3 we deduce the following corollary.

**Corollary 1.4.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . The following two conditions are equivalent:*

- (1)  $R$  is arithmetical;
- (2)  $A$  is arithmetical,  $A_P$  is an integral domain for each  $P \in \text{Supp } E$ , and  $E$  is locally FP-injective and has a distributive lattice of submodules.

**Remark 1.5.** Let us observe that  $E$  is a module over  $A/N$  where  $N$  is the nilradical of  $A$ . Let  $a \in N$  and  $P$  a maximal ideal of  $A$ . Either  $P \in \text{Supp } E$  and  $aR_P = 0$  or  $E_P = 0$ . So,  $aE = 0$ .

## 2. GAUSSIAN RINGS

Let  $R$  be a ring. For a polynomial  $f \in R[X]$ , denote by  $c(f)$  (the content of  $f$ ) the ideal of  $R$  generated by the coefficients of  $f$ . We say that  $R$  is **Gaussian** if  $c(fg) = c(f)c(g)$  for any two polynomials  $f$  and  $g$  in  $R[X]$ . By [19], a local ring  $R$  is Gaussian if and only if, for any ideal  $I$  generated by two elements  $a, b$ , in  $R$ , the following two properties hold:

- (1)  $I^2$  is generated by  $a^2$  or  $b^2$ ;
- (2) if  $I^2$  is generated by  $a^2$  and  $ab = 0$ , then  $b^2 = 0$ .

**Proposition 2.1.** *Let  $A$  be a local ring,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . The following two conditions are equivalent:*

- (1)  $R$  is Gaussian;
- (2)  $A$  is Gaussian and  $aE = a^2E$  for each  $a \in A$ .

*Proof.* (1)  $\Rightarrow$  (2). As factor of  $R$   $A$  is Gaussian. Let  $0 \neq a \in A$  and  $x \in E$ . First assume  $a^2 \neq 0$ . Then  $(a, 0)^2 \neq 0$  and  $(0, x)^2 = 0$ . So,  $(a, 0)(0, x) = (0, ax) = (0, y)(a^2, 0) = (0, a^2y)$  for some  $y \in E$ . We get  $a^2y = ax$ . Now assume  $a^2 = 0$ . In this case  $(a, 0)^2 = 0$ . It follows that  $(a, 0)(0, x) = 0$ , whence  $ax = 0$ . In the two cases  $aE = a^2E$ .

(2)  $\Rightarrow$  (1). Let  $r = (a, x)$  and  $s = (b, y)$  two elements of  $R$ . Since  $A$  is Gaussian we may assume that  $ab = ca^2$  and  $b^2 = da^2$  for some  $c, d \in A$ . We shall prove that  $(Rr + Rs)^2 = Rr^2$  by showing there exist  $z$  and  $v$  in  $E$  satisfying the following two equations:

- (1)  $(a, x)(b, y) = (c, z)(a, x)^2$
- (2)  $(b, y)^2 = (d, v)(a, x)^2$

From (1) we deduce that  $a^2z = ay + bx - 2acx$ . Since  $bE = b^2E$  then  $bx = b^2m = da^2m$  for some  $m \in E$ . So,  $ay + bx - 2acx = a(y + adm - 2cx)$ . Now we use the equality  $aE = a^2E$  to conclude.

From (2) we deduce that  $a^2v = 2by - 2adx$ . Since  $bE = b^2E$  then  $by = b^2n = da^2n$  for some  $n \in E$ . So,  $2by - 2adx = a(2adn - 2dx)$ . Now we use the equality  $aE = a^2E$  to conclude.

If  $rs = 0$  then  $ab = 0$  and  $b^2 = 0$ . So,  $s^2 = (0, 2by)$ . But  $bE = b^2E = 0$ , whence  $by = 0$  and  $s^2 = 0$ .  $\square$

**Corollary 2.2.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \ltimes E$  the trivial ring extension of  $A$  by  $E$ . The following two conditions are equivalent:*

- (1)  $R$  is Gaussian;
- (2)  $A$  is Gaussian and  $aE = a^2E$  for each  $a \in A$ .

*Proof.* We deduce this corollary from Proposition 2.1 and from the fact that a ring  $A$  is Gaussian if and only if  $A_P$  is Gaussian for each maximal ideal  $P$  and that  $aE = a^2E$  if and only if  $aE_P = a^2E_P$  for each maximal ideal  $P$ .  $\square$

**Remark 2.3.** Let  $A$  be a Gaussian ring. Let  $E$  be an  $A$ -module such that  $aE = a^2E$  for each  $a \in A$ . If  $a \in N$ , where  $N$  is the nilradical of  $A$ , it is easy to see that  $aE = 0$ . So,  $E$  is a module over  $A/N$  which is arithmetical by [17, Corollary 7].

**Proposition 2.4.** *Let  $A$  be an integral domain and  $E$  an  $A$ -module such that  $aE = a^2E$  for each  $a \in A$ . Then:*

- (1)  $E$  is divisible if it is torsion-free;
- (2) when  $A$  is local,  $E$  is semisimple if it is finitely generated;
- (3) if  $A$  is an Archimedean valuation domain, then  $E$  is the extension of a divisible module by a semisimple module.

*Proof.* (1). Let  $0 \neq a \in A$  and  $x \in E$ . There exists  $y \in E$  such that  $ax = a^2y$ . Since  $E$  is torsion-free we get  $x = ay$ .

(2). Let  $P$  be the maximal ideal of  $A$  and  $a \in P$ . We have  $aE = a(aE)$ . By Nakayama Lemma  $aE = 0$ . So,  $E$  is an  $A/P$ -module.

(3). To do this we show that  $PE$  is divisible. Let  $x \in PE$  and  $0 \neq s \in A$ . Then  $x = ay$  for some  $a \in P$  and  $y \in E$ . It is easy to check that  $\bigcap_{n \in \mathbb{N}} Aa^n$  is a prime ideal. So, this intersection is 0. There exists an integer  $n$  such that  $s \notin Aa^n$ . We get that  $aE = saE = a^{n+1}E$ . Hence  $x \in saE \subseteq sPE$ .  $\square$

**Example 2.5.** *Let  $A$  be a valuation domain,  $\Lambda$  an index set and  $(L_\lambda)_{\lambda \in \Lambda}$  a family of prime ideals. For each  $\lambda \in \Lambda$  let  $E_\lambda$  be a non-zero divisible  $A/L_\lambda$ -module. Let  $E_1 = \bigoplus_{\lambda \in \Lambda} E_\lambda$  and  $E_2 = \prod_{\lambda \in \Lambda} E_\lambda$ . Then  $aE_i = a^2E_i$  for each  $a \in A$  and for  $i = 1, 2$ .*

*Proof.* Let  $0 \neq a \in A$  and  $x = (x_\lambda)_{\lambda \in \Lambda} \in E_i$  where  $i = 1, 2$ . If  $ax_\lambda = 0$  then  $ax_\lambda = a^2y_\lambda$  with  $y_\lambda = 0$ . If  $ax_\lambda \neq 0$  then  $a \notin L_\lambda$ . Since  $E_\lambda$  is divisible over  $A/L_\lambda$  there exists  $y_\lambda \in E_\lambda$  such that  $ax_\lambda = a^2y_\lambda$ . So,  $ax = a^2y$  with  $y = (y_\lambda)_{\lambda \in \Lambda}$ .  $\square$

Let  $A$  be a valuation domain. A non-zero prime ideal  $L$  that is not the union of prime ideals properly contained in it is called **branched**.

**Proposition 2.6.** *Let  $A$  be a valuation domain. Assume that each non-zero prime ideal is branched. For any  $A$ -module  $E$  the following conditions are equivalent:*

- (1)  $aE = a^2E$  for each  $a \in A$ ;

- (2) for each prime ideal  $L$ ,  $LE/L'E$  is an  $A/L'$ -module divisible, where  $L'$  is the union of all prime ideals properly contained in  $L$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $s \in L \setminus L'$  and  $x \in LE$ . Then  $x = ay$  for some  $a \in L$  and  $y \in E$ . We may assume that  $a \notin L'$ . So,  $L' = \bigcap_{n \in \mathbb{N}} Aa^n$  and there exists  $n \in \mathbb{N}$  such that  $s \notin Aa^n$ . Whence  $aE = a^{n+1}E \subseteq saE \subseteq aE$ . Hence  $x \in sLE$ .

(2)  $\Rightarrow$  (1). Let  $0 \neq a$  a non-unit of  $A$ . Let  $L$  be the prime ideal which is the intersection of all prime ideals containing  $a$ . Then  $a \notin L'$ . Let  $x = ay \in aE$ . Since  $LE/L'E$  is divisible over  $A/L'$ , then  $x \in a^2E + L'E \subseteq a^2E$ .  $\square$

A chain ring is said to be **strongly discrete** if it contains no idempotent prime ideal. Each non-zero prime ideal of a strongly discrete valuation domain is branched.

### 3. BÉZOUT RINGS

A ring is a **Bézout ring** if every finitely generated ideal is principal. A ring  $R$  is **Hermite** if  $R$  satisfies the following property : for every  $(a, b) \in R^2$ , there exist  $d, a', b'$  in  $R$  such that  $a = da'$ ,  $b = db'$  and  $Ra' + Rb' = R$ . We say that  $R$  is an **elementary divisor ring** if for every matrix  $A$ , with entries in  $R$ , there exist a diagonal matrix  $D$  and invertible matrices  $P$  and  $Q$ , with entries in  $R$ , such that  $PAQ = D$ . Then we have the following implications:

elementary divisor ring  $\Rightarrow$  Hermite ring  $\Rightarrow$  Bézout ring  $\Rightarrow$  arithmetical ring;  
but these implications are not reversible: see [9] or [4].

**Proposition 3.1.** *Let  $A$  be a ring and  $N$  its nilradical. Assume that  $N$  is prime. Let  $E$  be a non-zero  $A$ -module and  $R = A \times E$  the trivial ring extension of  $A$  by  $E$ . The following three conditions are equivalent:*

- (1)  $R$  is Hermite;
- (2)  $R$  is Bézout;
- (3)  $A$  is Bézout,  $A_P$  is a domain for each  $P \in \text{Supp } E$ ,  $E$  is FP-injective and all its finitely generated submodules are cyclic.

*Proof.* (1)  $\Leftrightarrow$  (2). It is well known that each Hermite ring is Bézout. Since  $R$  contains a unique minimal prime ideal, then the converse holds by [12, Theorem 2].

(2)  $\Rightarrow$  (3). By Corollary 1.4  $E$  is FP-injective and  $A_P$  is a domain for each  $P \in \text{Supp } E$ . As factor of  $R$   $A$  is Bézout, and each finitely generated submodule of  $E$  is cyclic because  $E$  is isomorphic to an ideal of  $R$ .

(3)  $\Rightarrow$  (2). Let  $a \in N$  and  $x \in E$ . For each  $P \in \text{Supp } E$   $aR_P = 0$ . It follows that  $(0 : a) \not\subseteq P$  and consequently  $\text{Supp } E \subseteq D((0 : a))$ . For each  $P \notin \text{Supp } E$   $R_P x = 0$ . It follows that  $(0 : x) \not\subseteq P$  and consequently  $D((0 : a)) \cup D((0 : x)) = \text{Spec } A$ . So,  $(0 : a) + (0 : x) = A$ , whence there exist  $b \in (0 : a)$  and  $c \in (0 : x)$  such that  $b + c = 1$ . Then  $a = ca$ ,  $x = bx$  and it is easy to check that  $R(a, 0) + R(0, x) = R(a, x)$ . Let  $(a, x)$  and  $(b, y)$  two elements of  $R$ . If  $(a, b) \in N \times N$  then there exist  $d \in A$  such that  $Ad = Aa + Ab$ , and  $z \in E$  such that  $Ax + Ay = Az$ . It follows that  $R(a, x) + R(b, y) = R(d, z)$ . Now assume that  $(a, b) \notin N \times N$ . There exist  $d, a', b', s, t \in A$  such that  $a = da'$ ,  $b = db'$  and  $sa + tb = d$ . Let  $z = sx + ty$ . Since  $E$  is divisible over  $A/N$  and  $d \notin N$ , there exist  $x', y' \in E$  such that  $x - a'z = dx'$  and  $y - b'z = dy'$ . Now it is easy to check that  $(a, x) = (d, z)(a', x')$ ,  $(b, y) = (d, z)(b', y')$  and  $(s, 0)(a, x) + (t, 0)(b, y) = (d, z)$ .  $\square$

If  $R$  is a ring, we consider on  $\text{Spec } R$  the equivalence relation  $\mathcal{R}$  defined by  $LRL'$  if there exists a finite sequence of prime ideals  $(L_k)_{1 \leq k \leq n}$  such that  $L = L_1$ ,  $L' = L_n$

and  $\forall k, 1 \leq k \leq (n-1)$ , either  $L_k \subseteq L_{k+1}$  or  $L_k \supseteq L_{k+1}$ . We denote by  $\text{pSpec } R$  the quotient space of  $\text{Spec } R$  modulo  $\mathcal{R}$  and by  $\lambda_R : \text{Spec } R \rightarrow \text{pSpec } R$  the natural map. The quasi-compactness of  $\text{Spec } R$  implies the one of  $\text{pSpec } R$ , but generally  $\text{pSpec } R$  is not  $T_1$ : see [16, Propositions 6.2 and 6.3]. A topological space is called **totally disconnected** if each of its connected components contains only one point. Every Hausdorff topological space  $X$  with a base of clopen neighbourhoods is totally disconnected and the converse holds if  $X$  is compact (see [11, Theorem 16.17]).

An ideal  $I$  of a ring  $A$  is **pure** if and only if  $A/I$  is a flat  $A$ -module.

**Theorem 3.2.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . Assume that  $\text{pSpec } A$  is totally disconnected. The following three conditions are equivalent:*

- (1)  $R$  is Hermite;
- (2)  $R$  is Bézout;
- (3)  $A$  is Bézout,  $A_P$  is a domain for each  $P \in \text{Supp } E$ ,  $E$  is locally FP-injective and all its finitely generated submodules are cyclic.

*Proof.* (1)  $\Rightarrow$  (2) is well known and we show (2)  $\Rightarrow$  (3) as in Proposition 3.1.

(3)  $\Rightarrow$  (1). By [5, Proposition 2.2]  $\text{pSpec } A$  is compact. Since  $A = R/J$  where  $J$  is contained in the nilradical of  $R$  then  $\text{pSpec } A$  and  $\text{pSpec } R$  are homeomorphic. Let  $x \in \text{pSpec } R$ ,  $I(x)$  the pure ideal of  $R$  for which  $x = V(I(x))$  and  $I' = I(x)/NI(x)$  (see [5, Lemma 2.5]). Thus  $I'$  is a pure ideal of  $A$  and it is contained in only one minimal prime ideal. So, the nilradical of  $A/I'$  is prime. If  $S = 1 + I(x)$  then  $S^{-1}R = R/I(x)$ . By using Lemma 1.3 we get that  $R/I(x) = A/I' \rtimes E/I'E$ . By Proposition 3.1  $R/I(x)$  is Bézout for each  $x \in \text{pSpec } R$ . We conclude that  $R$  is Hermite by [5, Theorem 3.1].  $\square$

Recall that a ring  $A$  is **coherent** (respectively **semihereditary**) if all its finitely generated ideals are finitely presented (respectively projective).

**Corollary 3.3.** *Let  $A$  be a coherent reduced ring,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . The following three conditions are equivalent:*

- (1)  $R$  is Hermite;
- (2)  $R$  is Bézout;
- (3)  $A$  is Bézout,  $E$  is FP-injective and all its finitely generated submodules are cyclic.

*Proof.* Since  $A$  is Bézout, reduced and coherent then  $A$  is semihereditary. By [18, Proposition 10]  $\text{Min } A$  is compact, and by [5, Proposition 2.2]  $\text{pSpec } A$  is homeomorphic to  $\text{Min } A$ , and consequently it is totally disconnected. Since  $A$  is coherent, each FP-injective module is locally FP-injective. We conclude by Theorem 3.2.  $\square$

**Corollary 3.4.** *Let  $A$  be a ring,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . Assume that  $\text{pSpec } A$  is totally disconnected. The following two conditions are equivalent:*

- (1)  $R$  is an elementary divisor ring;
- (2)  $A$  is an elementary divisor ring,  $A_P$  is a domain for each  $P \in \text{Supp } E$ ,  $E$  is locally FP-injective and all its finitely generated submodules are cyclic.

*Proof.* It is a consequence of Theorem 3.2 and the following. By [10, Theorem 6] a ring  $R$  is an elementary divisor ring if and only if  $R$  is Hermite and for any  $a, b, c \in R$

such that  $Ra + Rb + Rc = R$  there exist  $p, q \in R$  such that  $Rpa + R(pb + qc) = R$ . By [12, Theorem 3] a Hermite ring  $R$  is an elementary divisor ring if and only if so is  $R/N$ , where  $N$  is the nilradical of  $R$ .  $\square$

**Corollary 3.5.** *Let  $A$  be a coherent reduced Bézout ring,  $E$  a non-zero FP-injective  $A$ -module. Assume that each finitely generated submodule of  $E$  is cyclic. Then, for any  $x, y \in E$  there exist  $z \in E$  and an invertible  $2 \times 2$  matrix  $B$  with coefficients in  $A$  such that  $\begin{pmatrix} z \\ 0 \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}$ .*

*Proof.* Let  $R = A \rtimes E$ . Let  $x, y \in A$ . By Corollary 3.3  $R$  is Hermite. So, there exist  $r \in R$  and an invertible  $2 \times 2$  matrix  $C$  with entries in  $R$  such that  $\begin{pmatrix} r \\ 0 \end{pmatrix} = C \begin{pmatrix} (0, x) \\ (0, y) \end{pmatrix}$ . It is obvious that  $r = (0, z)$  for some  $z \in E$ . From  $C$  we deduce an invertible  $2 \times 2$  matrix  $B$  with entries in  $A$  satisfying  $\begin{pmatrix} z \\ 0 \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}$ .  $\square$

Contrary to the two first sections we do not get general results, even in the case where  $A$  is reduced. In [20] there are two examples of reduced Bézout rings which are not semihereditary. For the first example  $A$  ([20, Example 1.3b])  $\text{pSpec } A$  is totally disconnected, so, if  $E$  is an  $A$ -module satisfying the conditions of Theorem 3.2 then  $A \rtimes E$  is Bézout. But for the second example  $A$  ([20, Example 6.2]),  $\text{pSpec } A$  is connected and infinite, so, we do not know if  $A$  admits Bézout proper trivial ring extensions.

#### 4. FQP-RINGS AND FQF-RINGS

Let  $A$  be a ring,  $M$  an  $A$ -module. An  $A$ -module  $V$  is  **$M$ -projective** if the natural homomorphism  $\text{Hom}_A(V, M) \rightarrow \text{Hom}_A(V, M/X)$  is surjective for every submodule  $X$  of  $M$ . We say that  $V$  is **quasi-projective** if  $V$  is  $V$ -projective. A ring  $A$  is said to be an **fqp-ring** if every finitely generated ideal of  $A$  is quasi-projective.

**Theorem 4.1.** *Let  $A$  a local ring and  $N$  its nilradical. Then  $A$  is a fqp-ring if and only if either  $A$  is a chain ring or  $A/N$  is a valuation domain and  $N$  is a divisible torsionfree  $A/N$ -module.*

*Proof.* Assume that  $A$  is a fqp-ring but not a chain ring. By [1, Lemmas 3.12 and 4.5],  $N^2 = 0$  and every zero-divisor belongs to  $N$ . So,  $N$  is prime. From [1, Lemma 3.8] it follows that two elements of  $A$  which are not in  $N$  are comparable. This implies that  $A/N$  is a valuation domain. Let  $a \notin N$  and  $b \in N$ . By using again [1, Lemma 3.8] we get that  $b \in Aa$ . Hence  $N$  is divisible over  $A/N$  and it is torsionfree since each element in  $A \setminus N$  is regular.

Conversely, it is easy to see that each chain ring is fqp. So, we may assume that  $A$  is not a chain ring. Let  $I$  a finitely generated ideal of  $A$ . If  $I \subseteq N$  then  $I$  is a free module over  $A/N$ . Consequently  $I$  is quasi-projective. Now, suppose that  $I \not\subseteq N$ . Thus  $(I + N)/N$  is principal, and from the fact that  $N$  is divisible over  $A/N$  we deduce that  $I$  is principal too. Hence  $I$  is quasi-projective.  $\square$

**Theorem 4.2.** *Let  $A$  be a local ring,  $N$  its nilradical,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$  the trivial ring extension of  $A$  by  $E$ . Then  $R$  is a fqp-ring if and only if  $A$  is a fqp-ring and one of the following conditions holds:*

- (1)  $A$  is a valuation domain and  $E$  is divisible and uniserial;



- (2)  $E$  is divisible and torsionfree over  $A/N$ , each zero-divisor of  $A$  belongs to  $N$  and  $N^2 = 0$ .

*Proof.* First assume that  $R$  is fqp. By [1, Proposition 4.1] so is  $A$ . If  $R$  is a chain ring we use Proposition 1.1. If not, since  $R$  is Gaussian by [1, Theorem 3.2] then  $E$  is an  $A/N$ -module by Remark 2.3. Let  $N'$  be the nilradical of  $R$ . It is easy to see that  $R/N' \cong A/N$  and  $N' \cong N \oplus E$ . By Theorem 4.1,  $N'^2 = 0$ ,  $N^2 = 0$  and  $E$  is divisible and torsionfree over  $A/N$ .

Conversely, condition (1) implies that  $R$  is a chain ring. Now, assume that condition (2) holds. As in the first part of the proof,  $R/N' \cong A/N$ ,  $N' \cong N \oplus E$  and  $N'^2 = 0$ . So,  $N'$  is divisible and torsionfree over  $R/N'$ . We conclude by Theorem 4.1.  $\square$

**Corollary 4.3.** *Let  $A$  be a valuation domain which is not a field,  $E$  a non-zero divisible  $A$ -module and  $R = A \rtimes E$ . Then:*

- (1) if  $E$  is uniserial then  $R$  is a chain ring;
- (2) if  $E$  is torsionfree but not uniserial then  $R$  is a fqp-ring which is not arithmetical;
- (3) if  $E$  is neither torsionfree nor uniserial then  $R$  is Gaussian but not a fqp-ring.

**Proposition 4.4.** *Let  $A$  be a coherent ring. Then  $A$  is a fqp-ring if and only so is  $A_P$  for each maximal ideal  $P$  of  $A$ .*

*Proof.* Let  $I$  be a finitely generated ideal. Then  $(\text{End}(I))_P \cong \text{End}(I_P)$  since  $I$  is finitely presented. We use [1, Lemma 3.7] to conclude.  $\square$

We say that a ring  $A$  is a **fqf-ring** if each finitely generated ideal  $I$  is flat modulo its annihilator, i.e  $I$  is a flat  $A/(0 : I)$ -module. It is obvious that every fqp-ring  $A$  is a fqf-ring (by [1, Lemma 2.2] a finitely generated module is quasi-projective if and only if it is projective modulo its annihilator). By Proposition 4.4 the converse holds if  $A$  is coherent. But we shall see that it is not generally true.

**Corollary 4.5.** *Let  $A$  be a ring,  $N$  its nilradical,  $E$  a non-zero  $A$ -module and  $R = A \rtimes E$ . Then the following conditions are equivalent:*

- (1)  $R$  is a fqf-ring;
- (2)  $A$  is a fqf-ring and for each  $P \in \text{Supp } E$  either  $N_P = 0$  and  $E_P$  is a uniserial divisible  $A_P$ -module or  $N_P^2 = 0$  and  $E_P$  and  $N_P$  are divisible and torsionfree over  $A_P/N_P$ .

*Proof.* Over a local ring each finitely generated flat module is free. So, a local ring is a fqf-ring if and only if it is a fqp-ring (by [1, Lemma 2.2]). As a module is flat if and only if it is locally flat, then a ring  $A$  is a fqf-ring if and only if so is  $A_P$  for each maximal ideal  $P$  of  $A$ . So, we use Theorem 4.2 to conclude.  $\square$

**Example 4.6.** *Let  $A$  be a von Neumann regular ring which is not self-injective,  $H$  the injective hull of  $A$ ,  $x \in H \setminus A$ ,  $E = A + Ax$  and  $R = A \rtimes E$ . Then  $R$  is a fqf-ring which is not a fqp-ring.*

*Proof.* Clearly  $A$  is a reduced fqp-ring, and, for each maximal ideal  $P$ ,  $E_P$  is injective and torsionfree over the field  $A_P$ . By corollary 4.5  $R$  is a fqf-ring. On the other hand  $E$  is isomorphic to a finitely generated ideal  $J$  of  $R$ . It is obvious that  $E$  is a faithful  $A$ -module, so its annihilator as  $R$ -module is  $J$ . If  $P \in \text{Supp } E/A$

then  $E_P$  is of rank 2 and if  $P \notin \text{Supp } E/A$  it is of rank one. But  $(A : x)$  is an essential ideal of  $A$  and consequently it is not generated by an idempotent. So,  $\text{Supp } E/A = V((A : x))$  is not open. It follows that the map  $\text{Spec } A \rightarrow \mathbb{N}$  defined by  $P \mapsto \text{rank}_{A_P}(E_P)$  is not locally constant. Hence  $E$  is not a projective  $A$ -module by [3, Théorème II.§5.2.1]. By [1, Lemma 2.2] it is not quasi-projective over  $R$ , whence  $R$  is not a fqp-ring.  $\square$

## 5. FINITISTIC AND GLOBAL WEAK DIMENSIONS OF FQP-RINGS

Let  $R$  be a ring. If  $M$  is an  $R$ -module, we denote by  $\text{w.d.}(M)$  its **weak dimension**. Recall that  $\text{w.d.}(M) \leq n$  if  $\text{Tor}_{n+1}^R(M, N) = 0$  for each  $R$ -module  $N$ . For any ring  $R$ , its **global weak dimension**  $\text{w.gl.d.}(R)$  is the supremum of  $\text{w.d.}(M)$  where  $M$  ranges over all (finitely presented cyclic)  $R$ -modules. Its **finitistic weak dimension**  $\text{f.w.d.}(R)$  is the supremum of  $\text{w.d.}(M)$  where  $M$  ranges over all  $R$ -modules of finite weak dimension.

We shall extend [1, Theorem 3.11] and [6, Theorem 1] to fqp-rings by showing the following theorem:

**Theorem 5.1.** *The global weak dimension of a fqp-ring is 0, 1 or  $\infty$  and its finitistic weak dimension is 0, 1 or 2.*

*Proof.* The first assertion can be proven as [1, Theorem 3.11].

Let  $R$  be a fqp-ring. It is well known that an  $R$ -module  $M$  is flat if and only if so is  $M_P$  for each  $P \in \text{Max } R$ . So, we may assume that  $R$  is local. By Theorem 4.2 either  $R$  is a chain ring and we use [6, Theorem 2] or  $R$  satisfies the assumptions of the following proposition and we conclude by using it.  $\square$

**Proposition 5.2.** *Let  $R$  be a local ring and  $N$  its nilradical. Assume that  $R/N$  is a valuation domain,  $N^2 = 0$  and  $N$  is divisible and torsionfree over  $R/N$ . Then:*

- (1) *if  $N$  is the maximal ideal then  $\text{f.w.d.}(R) = 0$ ;*
- (2) *if  $N$  is not maximal then  $\text{f.w.d.}(R) = 1$ .*

*Proof.* (1). In this case  $R$  is a primary ring. We conclude by [2, Theorems P and 6.3].

(2). Let  $Q$  be the quotient ring of  $R$ . Since each element of  $R$  which is not in  $N$  is regular then  $Q = R/N$  and since  $N$  is divisible and torsionfree over  $R/N$  then  $N$  is a  $Q$ -module and it is the maximal ideal of  $Q$ . From (1) we deduce that  $\text{f.w.d.}(Q) = 0$ .

Let  $M$  be an  $R$ -module with  $\text{w.d.}(M) < \infty$ . Then we have  $\text{w.d.}(M_N) < \infty$ . So,  $M_N$  is flat. Let  $I$  be a finitely generated proper ideal. Then either  $I \not\subseteq N$  and  $I = Ra$  for some  $a \in R \setminus N$ , or  $I \subseteq N$  and  $I \cong (R/N)^n$  for some integer  $n \geq 1$ .

In the first case we have  $\text{w.d.}(R/I) = 1$  since  $a$  is a regular element, whence  $\text{Tor}_2^R(R/I, M) = 0$ . In the other case we have the following exact sequence:

$$0 \rightarrow NF \rightarrow F \rightarrow I \rightarrow 0,$$

where  $F$  is a free  $R$ -module of rank  $n$ . We deduce that the sequence

$$0 \rightarrow NF_N \rightarrow F_N \rightarrow I_N \rightarrow 0 \quad \text{is exact.}$$

We have  $\mathrm{Tor}_1^Q(I_N, M_N) = 0$ . Since  $N$  is a  $Q$ -module,  $NF \cong NF_N$ . So, in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Tor}_1^R(I, M) & \rightarrow & NF \otimes_R M & \rightarrow & F \otimes_R M \\ & & & & \downarrow & & \downarrow \\ & & 0 & \rightarrow & NF_N \otimes_Q M_N & \rightarrow & F_N \otimes_Q M_N \end{array}$$

the left vertical map is an isomorphism. Then  $NF \otimes_R M \rightarrow F \otimes_R M$  is a monomorphism. We successively deduce that  $\mathrm{Tor}_1^R(I, M) = 0$  and  $\mathrm{Tor}_2^R(R/I, M) = 0$ . Hence  $\mathrm{w.d.}(M) \leq 1$ .  $\square$

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UNIVERSITÉ DE CAEN BASSE-NORMANDIE, CNRS UMR 6139 LMNO, F-14032 CAEN, FRANCE  
*E-mail address:* francois.couchot@unicaen.fr