Feedback strategies for discrete-time linear-quadratic two-player descriptor games

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Abstract

This paper deals with discrete-time linear-quadratic descriptor two-player games. The two frameworks of Nash and Stackelberg strategies with feedback information structure are considered. The aim is to provide sufficient conditions ensuring, for any initial state, the existence and the uniqueness of a tri-trajectory, gathering the state and the controls of the two players, which reaches the equilibrium. The provided results are mainly based on a discrete-time matrix block formulation. The provided algorithms consist in an iterative backward in time procedure, issued from Dynamic Programming. Numerical examples illustrate this approach and the particular case of explicit systems is recovered for both kinds of strategies.

Keywords: Descriptor games, Game theory, feedback information structure, Nash strategy, Stackelberg strategy, matrix block formulation.
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1. Introduction

Game theory [1, 2] deals with interactive decision-making, where the outcome for each participant or player depends on the actions of all. In other

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words a player designs his/her strategy by taking into account the available information on the actions of the others. Therefore, the information structure, that means the set of information available for each player, has to be a priori defined [3, 4]. Among the different structures, we can cite the open-loop information structure (the strategy depends only on time and on a predefined state), the closed-loop ones (the strategy depends on the history of the state trajectory) and the memoryless closed-loop or feedback ones, where the strategy is a function of the time and the current state. Throughout the paper we will use the feedback information structure.

The notion of optimality from the classical control theory has no sense anymore in a game with at least two players. This notion is then replaced by compromises formulated as equilibria. The most famous equilibria are the Nash and Stackelberg ones. The Nash equilibrium [5, 6] is adequate to cope with noncooperative decision-making problems, where the roles of the different players are interchangeable. The Nash strategy offers a nice tool to design cautious actions: no player can improve his/her payoff or criterion by deviating unilaterally from his/her Nash strategy once the equilibrium is attained. Control design using Nash strategies has been formalized in [3, 4, 7]. The Stackelberg equilibrium [8, 9] assumes that the role of the players are not the same, in other words it assumes that there is a hierarchy between them. In a two-player game, one player is the leader and the other one is the follower. The leader knows the rational reaction of the follower and is able to communicate his/her strategy to the follower. The leader injects this information in his/her decision making to optimize his/her cost function. The Stackelberg strategy for control design were formalized in [10, 11, 12]. This concept was revisited in [13] for economics and control theory.

With an open-loop information structure (the controls are only functions of time), the necessary conditions for obtaining a Nash or a Stackelberg equilibrium are well known and are derived by applying the Pontryagin Maximum Principle. More details can be found in [14, 15] for the Nash equilibrium and in [16, 17, 18] for the Stackelberg one. In the linear-quadratic case, these necessary conditions can be reformulated into non-symmetric Riccati equations [19, 20, 21]. The framework of closed-loop information structure with memory is more difficult. Details can be found in [22, 20] for the Nash strategy and in [23, 24, 25, 26] for the Stackelberg one. When the feedback information structure is assumed, the sufficient conditions for the equilibria can be obtained by applying Dynamic Programming, via the Hamilton-Jacobi-Bellmann equation [2, 27].

All the contributions presented above consider only explicit systems. Descriptor (or implicit) systems consist in gathering a set of differential and algebraic equations to characterize the system [28, 29, 30]. Such a structure including a dynamic and a static part can be formulated as a dynamic involving a characteristic singular matrix. These systems are an important class of systems, because they model a large number of physical phenomena (Kirchhoff's laws in electricity, static relations in mechanics) or in economics (for instance Leontieff model). The main difficulty is that a trajectory satisfying the descriptor system may not exist for all initial conditions. In addition, if such a trajectory exists, it may be non unique. The standard properties of explicit systems, including solvability, reachability, stability, controllability, observability and regularization have been extended and strongly studied for descriptor systems from theoretic and numerical points of view. In addition of the already cited references, more details can be found in [31, 32, 33, 34, 35]. The variational calculus [36] and the linear quadratic optimal regulator [37, 38, 39, 40, 41, 42] have been proposed also for descriptor systems. The main approach is to use a canonical decomposition allowing to obtain an explicit system [43, 30]. Another way is to consider descriptor algebraic Riccati equations [44] or descriptor matrix pencils. To avoid a decomposition of the singular matrix and to avoid assumption about its index, a matrix block formulation has been provided in [45] and refined in [46].

To the best of our knowledge, in the case of linear quadratic descriptor games, there is only a few number of contributions. Among them, let us cite [47, 48] for zero-sum games and min-max strategy, [49] for Pareto strategy, [50] for closed-loop continuous–time Nash strategy and finally [51, 52] for a closed-loop Stackelberg strategy.

This paper is focused on proposing sufficient conditions, via a matrix block formulation based on Dynamic Programming, to solve Nash and Stackelberg strategies with a feedback information structure for discrete-time linear-quadratic two-player descriptor games. Roughly speaking, the aim of the paper is to provide sufficient conditions allowing the existence and the uniqueness of a tri-trajectory composed of the trajectory of the state and the two trajectories of the controls reaching a Nash or Stackelberg equilibrium with a feedback information structure. The induced iterative algorithms and the related properties will be detailed. Furthermore, links with both explicit games and infinite time horizon games will be emphasized.

The outline of the paper is as follows. Section 2 introduces the main

definitions and the required technical background to formulate the strategies design problem. Section 3 is focused on the feedback Nash strategy, for descriptor and explicit systems, while Section 4 is focused on the feedback Stackelberg strategy. The conclusions are summarized in Section 5.

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ the set of positive integers. Relative to a matrix M, M' denotes its transpose, M^{-1} its inverse and $\Lambda(M)$ its spectrum. I_{n_1} and $0_{n_1\times n_2}$ denote respectively the identity matrix of size n_1 and the null matrix of size $n_1 \times n_2$ and $0_{n_1} = 0_{n_1\times n_1}$.

2. Problem statement

Consider a discrete-time, two-player non-zero, descriptor game characterized by the following implicit linear dynamic, on the time horizon $\{0, \cdots, K\}$:

$$\begin{aligned} Ex_{k+1} &= Ax_k + B_1u_{1,k} + B_2u_{2,k}, \quad k \in \{0, \cdots, K-1\} = \mathcal{K}, & (1)\\ &= f(x_k, u_{1,k}, u_{2,k}), & (2)\end{aligned}$$

where $x_k \in \mathbb{R}^n$ and $u_{i,k} \in \mathbb{R}^{r_i}$ $(n, r_i \in \mathbb{N}^*)$ are respectively the state and the control of the player $i \in \mathcal{N} = \{1, 2\}$ at time k. The matrix $E \in \mathbb{R}^{q\times n}$ is either rectangular or singular. Thus, the existence of a trajectory from any initial condition $x_0 \in \mathbb{R}^n$ is not ensured. The matrices A, B_1 and B_2 have appropriate dimensions, that is $A \in \mathbb{R}^{q\times n}$, $B_1 \in \mathbb{R}^{q\times r_1}$ and $B_2 \in \mathbb{R}^{q\times r_2}$. The quadratic criterion associated with the player i is defined by

$$J_i(x_0, \{u_{1,k}\}_{k\in\mathcal{K}}, \{u_{2,k}\}_{k\in\mathcal{K}}) = \frac{1}{2}x'_K Q_{iK} x_K + \sum_{m=0}^{K-1} L_i(x_m, u_{1,m}, u_{2,m}), \quad (3)$$

where the term $\frac{1}{2}x'_K Q_{iK} x_K$ is called the final cost and the instantaneous criterion of the player i is defined by

$$L_i(x_m, u_{1,m}, u_{2,m}) = \frac{1}{2}\left(x'_m Q_i x_m + u'_{1,m} R_{i1} u_{1,m} + u'_{2,m} R_{i2} u_{2,m}\right). \quad (4)$$

All the weighting matrices in the criterion J_i are symmetric with $Q_i \geq 0$, $P_{iK} \geq 0$, $R_{ij} \geq 0$ and $R_{ii} > 0$, $\forall (i, j) \in \mathcal{N}^2$. The positive definiteness of the matrix R_{ii} implies the convexity of J_i with respect to the control $u_{i,k}$. For readability reasons, we introduce the notations $S_i = B_i R_{ii}^{-1} B'_i$ and $S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B'_j$, $\forall (i, j) \in \mathcal{N}^2$, $i \neq j$. In the sequel we recall several definitions specific to the game theory (see [2] for more details).

Definition 1 (Available information)**.** *At each time $k \in \mathcal{K}$, each player $i \in \mathcal{N}$ has access to a set of state values, noted $\eta_{i,k}$. Different information structures may be considered [2]:*

- *For an open-loop information structure:*

$$\eta_{i,k} = \{x_0\}. \tag{5}$$

- *For a memoryless closed-loop or feedback information structure:*

$$\eta_{i,k} = \{x_k\}. \tag{6}$$

- *For a closed-loop (with memory) information structure:*

$$\eta_{i,k} = \{x_0, x_1, \cdots, x_k\}. \tag{7}$$

In the sequel, only the memoryless closed-loop information structure will be investigated.

Definition 2 (Strategy)**.** *For the player $i \in \mathcal{N}$, a strategy, or a decision law γ_i is a mapping associating the available state $\eta_{i,k}$ and the time k to the action $u_{i,k}$:*

$$u_{i,k} = \gamma_i(\eta_{i,k}, k). \tag{8}$$

The set of all strategies of player i is called the strategy set and it is denoted by Γ_i. A couple of strategies $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ is called admissible *for the game (1) if the criteria J_1 and J_2 are finite. For the sake of clarity, we will denote by $J_i(x_0, \gamma_1, \gamma_2)$ the criterion (3) when the couple of strategies (γ_1, γ_2) is applied.*

Decisions made by the players are inter-dependent. The notion of optimality is thus not clear, since no one player completely designs the criteria [53]. To avoid this, game theory is helpful to define equilibria or compromises. Among the main equilibria, we will consider Nash and Stackelberg ones presented as follows. They involve the rational reaction set or the best answer set of a player.

Definition 3 (Rational reaction set)**.** *The rational reaction set of each player i is defined by all the strategies leading to the optimal value of the associated criterion J_i when the strategies of the other player is fixed.*

$$\mathcal{R}_1(\gamma_2) = \{\tilde{\gamma}_1 \in \Gamma_1 \mid J_1(x_0, \tilde{\gamma}_1, \gamma_2) \leq J_1(x_0, \gamma_1, \gamma_2),\ \forall \gamma_1 \in \Gamma_1\}, \tag{9}$$

$$\mathcal{R}_2(\gamma_1) = \{\tilde{\gamma}_2 \in \Gamma_2 \mid J_2(x_0, \gamma_1, \tilde{\gamma}_2) \leq J_2(x_0, \gamma_1, \gamma_2),\ \forall \gamma_2 \in \Gamma_2\}. \tag{10}$$

Definition 4 (Nash Equilibrium)**.** *A couple* $(\gamma_1^*(\eta_{1,k},k),\gamma_2^*(\eta_{2,k},k))$ *is a Nash strategy, if and only if the two coupled inclusions are verified*

$$\begin{cases} \gamma_1^* \in \mathcal{R}_1(\gamma_2^*), \\ \gamma_2^* \in \mathcal{R}_2(\gamma_1^*). \end{cases} \tag{11}$$

The couple (J_1^*, J_2^*)*, where* $J_i^* = J_i(x_0, \gamma_1^*, \gamma_2^*)$*,* $i \in \mathcal{N}$*, is known as a* Nash equilibrium *outcome.*

The Stackelberg strategies [9, 8, 54] are well adapted to cope with non-cooperative multicriteria optimization problems, where there is a hierarchy between the players. That is, there exists a leader (here the player 2) and a follower (here the player 1). The leader, knowing the follower's rational reaction set, is seeking a J_2-minimizing strategy γ_2^{**}, which is announced before the game starts. The follower will then minimize his cost functional J_1 with the strategy γ_1^{**}.

Definition 5 (Stackelberg Equilibrium)**.** *A couple* $(\gamma_1^{**}(\eta_{1,k},k),\gamma_2^{**}(\eta_{2,k},k))$ *is a Stackelberg strategy, if and only if it is a solution of*

$$\begin{cases} \gamma_1^{**} \in \mathcal{R}_1\left(\gamma_2^{**}\right) \\ \textit{and} \\ \max\limits_{\gamma_1 \in \mathcal{R}_1\left(\gamma_2^{**}\right)} J_2\left(\gamma_1, \gamma_2^{**}\right) \leq \max\limits_{\gamma_1 \in \mathcal{R}_1(\gamma_2)} J_2\left(\gamma_1, \gamma_2\right), \quad \forall \gamma_2 \in \Gamma_2. \end{cases} \tag{12}$$

The couple (J_1^{**}, J_2^{**})*, where* $J_i^{**} = J_i(x_0, \gamma_1^{**}, \gamma_2^{**})$*,* $i \in \mathcal{N}$*, is known as a* Stackelberg equilibrium *outcome.*

Remark 1. *The definition of a Stackelberg equilibrium proposed by the relation (12) may be qualified as* min–max *because it minimizes the criterion* J_2 *over the worst-case strategy of the follower belonging to the rational reaction set* $\mathcal{R}_1$*. It is noteworthy that a* min–min *counterpart exists [55]. The min–max version can be interpreted in term of robustness with respect to the rational reaction of the follower and the min–min one is related to a team cooperation between the leader and the follower to minimize the criterion of the leader. See [55, 56, 26] for more detailed discussion and interpretations. Note also that both definitions coincide when the rational reaction set is reduced to a singleton.*

When both Nash and Stackelberg strategies are concerned, we will denote by $\gamma_i^\bullet$ the related strategy. That is $\gamma_i^\bullet$ could be γ_i^* or γ_i^{**}.

In the framework of feedback information structure, the Nash and Stackelberg strategies are strongly time consistent [27, 15][2, Definition 5.14], which allows these strategies to satisfy the Bellmann principle and to apply Dynamic Programming. The strategies (γ_1^*, γ_2^*) and $(\gamma_1^{**}, \gamma_2^{**})$ are thus only functions of the time and the current state. This property helps to design the solution. One denotes by $J_{i,[k;K]}$ the cost associated with J_i, but restricted to the time horizon $[k; K]$, that is

$$J_{i,[k;K]} = \frac{1}{2} x_K' Q_{iK} x_K + \sum_{m=k}^{K-1} L_i(x_m, u_{1,m}, u_{2,m}). \tag{13}$$

In addition, $V_i(x_k, k)$ denotes the value of the criterion $J_{i,[k;K]}$ at a Nash or Stackelberg equilibrium for a game starting at time k and ending at time K. The quadratic class of criteria J_i allows to look for the values $V_i(x_k, k)$ as quadratic terms with respect to x_k that is $V_i(x_k, k) = \frac{1}{2} x_k' P_{i,k} x_k$. By definition, the outcomes of the Nash or Stackelberg strategies with a feedback information structure satisfy

$$J_i^\bullet = J_i(x_0, \gamma_1^\bullet, \gamma_2^\bullet) = V_i(x_0, 0). \tag{14}$$

The characterization of the best response set (9) requires the following lemmas.

Lemma 1. *Consider $E \in \mathbb{R}^{q \times n}$, $A \in \mathbb{R}^{q \times n}$, $B_1 \in \mathbb{R}^{q \times r}$ and $B_2 \in \mathbb{R}^{q \times r}$. For any $x_k \in \mathbb{R}^n$, any positive semidefinite matrices $P_1 \in \mathbb{R}^{n \times n}$, $Q_1 \in \mathbb{R}^{n \times n}$ and any positive definite matrix $R \in \mathbb{R}^{r \times r}$, if the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$ is surjective, then there exists at least one solution $(x_{k+1}, \lambda_{1,k}) \in \mathbb{R}^n \times \mathbb{R}^q$ of the set of equations:*

$$\begin{aligned} E x_{k+1} - B_1 R_{11}^{-1} B_1' \lambda_{1,k} &= A x_k + B_2 \gamma_2^\bullet(x_k), & (15) \\ P_1 x_{k+1} + E' \lambda_{1,k} &= 0_{n \times 1}. & (16) \end{aligned}$$

Proof. For the sake of clarity, let $M_1 = \begin{bmatrix} -B_1 R_{11}^{-1} B_1' & E \\ E' & P_1 \end{bmatrix}$. In order to show that there exists at least a solution of the set of equations (15)–(16), let us study the kernel of the matrix M_1. Consider $(\lambda', x')' \in \mathbb{R}^{q+n}$ such that

$$M_1 (\lambda', x')' = 0_{(q+n) \times 1}. \tag{17}$$

By developing Equation (17), we have

$$\begin{aligned} Ex &= B_1 R_{11}^{-1} B_1' \lambda, \quad (18) \\ E'\lambda &= -P_1 x. \quad (19) \end{aligned}$$

Next, multiply (18) from the left by λ' and (19) from the left by x' and transpose the latter equation. Subtracting the resulting equations yields

$$\lambda' B_1 R_{11}^{-1} B_1' \lambda + x' P_1 x = 0. \quad (20)$$

The positive (semi)definiteness of matrices P_1 and R_{11} induces that both terms, $\lambda' B_1 R_{11}^{-1} B_1' \lambda$ and $x' P_1 x$, are equal to zero. Finally it follows that $B_1'\lambda = 0_{r\times 1}$ and $P_1 x = 0_{n\times 1}$. Thanks to (19), $E'\lambda = 0_{n\times 1}$ yielding

$$\begin{pmatrix} B_1' \\ E' \end{pmatrix} \lambda = 0_{(r+n)\times 1}. \quad (21)$$

The surjectivity of the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$ leads to $\lambda = 0_{q\times 1}$, which implies

$$\mathrm{Ker}(M_1) \subset \mathrm{Ker} \begin{pmatrix} I_q & 0_n \end{pmatrix}. \quad (22)$$

The symmetry of the matrix M_1 leads to

$$\mathrm{Im} \begin{pmatrix} I_q \\ 0_n \end{pmatrix} \subset \mathrm{Im}(M_1') = \mathrm{Im}(M_1). \quad (23)$$

For any vector $x_k \in \mathbb{R}^n$,

$$\begin{pmatrix} Ax_k + B_2 \gamma_2^\bullet(x_k) \\ 0_{n\times 1} \end{pmatrix} \in \mathrm{Im} \begin{pmatrix} I_q \\ 0_n \end{pmatrix} \subset \mathrm{Im}(M_1). \quad (24)$$

The relation (24) ensures that there always exists at least a solution to the equations (15)-(16), for any vector $x_k \in \mathbb{R}^n$. □

Remark 2. *Under the assumptions of Lemma 1, the singular system (1) is well posed and admits a trajectory with the controls*

$$(u_{1,k}, u_{2,k}) = (R_{11}^{-1} B_1' \lambda_{1,k}, \gamma_2^\bullet(x_k)). \quad (25)$$

The surjectivity of the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$ *is not the weakest assumption to obtain this property, the surjectivity of* $\begin{pmatrix} E & B_1 & B_2 \end{pmatrix}$ *is sufficient for instance. Nevertheless this assumption allows in addition to satisfy the additional equation (16) which allows to reach the minimum of the cost of player 1, as formalized in Lemma 2.*

In order to apply Dynamic Programming to determine the minimum of $J_1(u_1, \gamma_2^\bullet(x_k))$ and to characterize $\mathcal{R}_1(\gamma_2^\bullet)$, let us define, for any $x_k \in \mathbb{R}^n$,

$$g_1(y,u) = \frac{1}{2}\left(y'P_1y + x_k'Q_1x_k + u'R_{11}u + (\gamma_2^\bullet(x_k))'R_{12}\gamma_2^\bullet(x_k)\right). \tag{26}$$

Lemma 2. *Under the assumptions of Lemma 1, any vector*

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} x_{k+1} \\ R_{11}^{-1}B_1'\lambda_{1,k} \end{pmatrix}, \tag{27}$$

where $(x_{k+1}, \lambda_{1,k})$ are the solutions of Equations (15)–(16), solves the constrained optimization problem

$$\min_{(y,u)\in\mathbb{R}^n\times\mathbb{R}^{r_1}} g_1(y,u) \tag{28}$$

subject to the singular system

$$Ey = Ax_k + B_1u + B_2\gamma_2^\bullet(x_k). \tag{29}$$

Proof. For any x_k and under the assumptions of Lemma 1, Equations (15)–(16) admit at least a solution $(x_{k+1}, \lambda_{1,k})$. Consider such a solution and any pair (y,u) satisfying the constraint (29). This leads to

$$E(y - x_{k+1}) = B_1(u - R_{11}^{-1}B_1'\lambda_{1,k}), \tag{30}$$

or

$$E\delta y = B_1\delta u, \tag{31}$$

with the notation $\delta y = y - x_{k+1}$ and $\delta u = u - R_{11}^{-1}B_1'\lambda_{1,k}$.

Let us study the difference $g_1(y,u) - g_1(x_{k+1}, R_{11}^{-1}B_1'\lambda_{1,k})$:

$$\begin{aligned}
&2g_1(y,u) - 2g_1(x_{k+1}, R_{11}^{-1}B_1'\lambda_{1,k}) && (32)\\
&\quad = 2g_1(x_{k+1}+\delta y, R_{11}^{-1}B_1'\lambda_{1,k}+\delta u) - 2g_1(x_{k+1}, R_{11}^{-1}B_1'\lambda_{1,k}) && (33)\\
&\quad = (x_{k+1}+\delta y)'P_1(x_{k+1}+\delta y) - x_{k+1}'P_1x_{k+1} && (34)\\
&\qquad + (R_{11}^{-1}B_1'\lambda_{1,k}+\delta u)'R_{11}(R_{11}^{-1}B_1'\lambda_{1,k}+\delta u) - \lambda_{1,k}'B_1R_{11}^{-1}B_1'\lambda_{1,k}. && (35)
\end{aligned}$$

Nevertheless, due to Equation (16), we have

$$\begin{aligned}
&(x_{k+1}+\delta y)'P_1(x_{k+1}+\delta y) - x_{k+1}'P_1x_{k+1}\\
&\qquad = 2\delta y'P_1x_{k+1} + \delta y'P_1\delta y = -2\delta y'E'\lambda_1 + \delta y'P_1\delta y
\end{aligned} \tag{36}$$

and thanks to Equation (31),

$$(R_{11}^{-1}B_1'\lambda_{1,k}+\delta u)'R_{11}(R_{11}^{-1}B_1'\lambda_{1,k}+\delta u)-\lambda_{1,k}'B_1R_{11}^{-1}B_1'\lambda_{1,k}$$
$$=2\delta u'B_1'\lambda_{1,k}+\delta u'R_{11}\delta u=2\delta y'E'\lambda_{1,k}+\delta u'R_{11}\delta u. \quad (37)$$

By summing Equations (36) and (37), we have

$$2g_1(y,u)-2g_1(x_{k+1},R_{11}^{-1}B_1'\lambda_{1,k})=\delta y'P_1\delta y+\delta u'R_{11}\delta u\geq 0. \quad (38)$$

It implies that $(x_{k+1},R_{11}^{-1}B_1'\lambda_{1,k})$ solves the constrained optimization problem and ends the proof. □

Let us characterize the rational reaction set of the player 1 (a player of the Nash strategy or the follower to the Stackelberg strategy) by the following proposition.

Proposition 1. *Assume R_{11} to be positive definite and the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$ to be surjective. Sufficient conditions for the rational reaction set for the player 1, $\gamma_1^\bullet \in \mathcal{R}(\gamma_2^\bullet)$, are given by the set of equations, $\forall k \in \mathcal{K}$,*

$$P_{1,K} = Q_{1K}, \quad (39)$$
$$E'\Psi_{1,k+1} = P_{1,k+1}x_{k+1}, \quad (40)$$
$$\gamma_1^\bullet(x_k,k) = -R_{11}^{-1}B_1'\Psi_{1,k+1}, \quad (41)$$
$$x_k'(P_{1,k}-Q_1)x_k = \begin{pmatrix} x_{k+1} \\ \gamma_1^\bullet \\ \gamma_2^\bullet \end{pmatrix}' \begin{bmatrix} P_{1,k+1} & 0 & 0 \\ 0 & R_{11} & 0 \\ 0 & 0 & R_{12} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ \gamma_1^\bullet \\ \gamma_2^\bullet \end{pmatrix}. \quad (42)$$

Proof. The minimization of the criterion J_1 is made under the dynamical constraint (1), when $u_2=\gamma_2^\bullet(x_k,k)$, that is

$$Ex_{k+1}=f(x_k,u_{1,k},\gamma_2^\bullet(x_k,k))=Ax_k+B_1u_{1,k}+B_2\gamma_2^\bullet(x_k,k). \quad (43)$$

Sufficient conditions are given by applying Dynamic Programming. Roughly speaking the transversality condition

$$V_1(x_K,K)=\frac{1}{2}x_K'Q_{1K}x_K, \quad (44)$$

is satisfied if the relation (39) is verified, and furthermore

$$V_1(x_k,k)=\min_{\substack{u_{1,k}\in\mathbb{R}^{r_1} \\ f(x_k,u_{1,k},\gamma_2^\bullet(x_k,k))=Ex_{k+1}}} \left(V_1(x_{k+1},k+1)+L_1(x_k,u_{1,k},\gamma_2^\bullet(x_k,k))\right) \quad (45)$$

with

$$\gamma_1^\bullet(x_k,k) = \arg \min_{\substack{u_{1,k}\in\mathbb{R}^{r_1} \\ f(x_k,u_{1,k},\gamma_2^\bullet(x_k,k))=Ex_{k+1}}} \big(V_1(x_{k+1},k{+}1){+}L_1(x_k,u_{1,k},\gamma_2^\bullet(x_k,k))\big). \tag{46}$$

By remarking on one hand that the strategy $\gamma_2^\bullet(x_k,k)$ is only a function of the state x_k (and incidentally of the time k) and on the other hand that the cost function to be minimized in (46) has the structure of the cost g_1 defined by (26), Lemmas 1 and 2 apply at each step $k \in \mathcal{K}$. By identifying $\lambda_k = -\Psi_{1,k}$, we obtain relations (40) and (41). The relation (45) reads as (42), which ends the proof. □

After these preliminaries, which are common to both Nash and Stackelberg strategies, the analysis will be devoted to the Nash strategy in Section 3 and to the Stackelberg one in Section 4.

3. The case of Nash strategy

The Nash strategy is investigated in this section. Only the case where E is singular, that is $q = n$, is considered in this section. More precisely, the subsection 3.1 treats the generic case of a descriptor game, while subsection 3.2 covers the specific case of explicit systems before illustrations in subsection 3.3.

3.1. Nash strategy for descriptor games

The role of the players are interchangeable in the case of a Nash strategy, due to the symmetry emphasized in Definition 4, and relations (11). When in addition R_{22} is positive definite and the pair $\begin{pmatrix} E & B_2 \end{pmatrix}$ is surjective, the sufficient conditions for characterizing the rational reaction set of the player 2 are given by Proposition 1, where the indices are switched, that is:

$$P_{2,K} = Q_{2K}, \tag{47}$$

$$E'\Psi_{2,k+1} = P_{2,k+1}x_{k+1}, \tag{48}$$

$$\gamma_2^*(x_k,k) = -R_{22}^{-1}B_2'\Psi_{2,k+1}, \tag{49}$$

$$x_k'(P_{2,k}-Q_2)x_k = \begin{pmatrix} x_{k+1} \\ \gamma_1^* \\ \gamma_2^* \end{pmatrix}' \begin{bmatrix} P_{2,k+1} & 0 & 0 \\ 0 & R_{21} & 0 \\ 0 & 0 & R_{22} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ \gamma_1^* \\ \gamma_2^* \end{pmatrix}. \tag{50}$$

The following theorem is the main result of this section. It provides a matrix block formulation to compute step by step the value functions for the both players associated with the feedback Nash strategy. A further assumption is introduced by considering that a block matrix is invertible at each step. This additional condition can be interpreted by the coupling between the both rational reaction sets occuring in a Nash strategy.

Theorem 1. *Consider the descriptor two-player game (1), associated with the criteria (3). Assume that R_{11} and R_{22} are positive definite and the pairs $\begin{pmatrix} E & B_1 \end{pmatrix}$ and $\begin{pmatrix} E & B_2 \end{pmatrix}$ are surjective. There exists one and only one tri-trajectory (the trajectory of the state and the trajectory of each player's control input) related to the feedback Nash strategy, if the following recurrence relations lead to invertible matrices $\mathcal{M}_{k+1}$ (defined below by (53)) at each step.*

$$P_{i,K} = Q_{iK}, \quad \forall i \in \mathcal{N}, \tag{51}$$

$$P_{i,k} = Q_i + \begin{bmatrix} A \\ 0_n \\ 0_n \end{bmatrix}' (\mathcal{M}_{k+1}^{-1})' \mathcal{Q}_{i,k+1} \mathcal{M}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \end{bmatrix}, \ \forall (i,k) \in \mathcal{N} \times \{0; \cdots; K-1\} \tag{52}$$

where

$$\mathcal{M}_{k+1} = \begin{bmatrix} E & S_1 & S_2 \\ P_{1,k+1} & -E' & 0_n \\ P_{2,k+1} & 0_n & -E' \end{bmatrix} \in \mathbb{R}^{3n \times 3n} \tag{53}$$

and

$$\mathcal{Q}_{i,k+1} = \begin{bmatrix} P_{i,k+1} & 0_n & 0_n \\ 0_n & S_{i1} & 0_n \\ 0_n & 0_n & S_{i2} \end{bmatrix} \in \mathbb{R}^{3n \times 3n}. \tag{54}$$

Proof. By applying the expression of γ_1^* and γ_2^* (given by (41) and (49) respectively) to the dynamical system (1), we have

$$Ex_{k+1} = Ax_k - S_1 \Psi_{1,k+1} - S_2 \Psi_{2,k+1}. \tag{55}$$

By collecting the sufficient conditions (40), (48) and (55), one obtains

$$\mathcal{M}_{k+1} \begin{pmatrix} x_{k+1} \\ \Psi_{1,k+1} \\ \Psi_{2,k+1} \end{pmatrix} = \begin{bmatrix} A \\ 0_n \\ 0_n \end{bmatrix} x_k. \tag{56}$$

It is noteworthy that, the relation (56) gathers the sufficient conditions associated with the two rational reaction sets (9) and (10). The surjectivity of the pairs $\begin{pmatrix} E & B_1 \end{pmatrix}$ and $\begin{pmatrix} E & B_2 \end{pmatrix}$ allows the minimization of the cost functions with possibly different trajectories. The invertibility of the matrix $\mathcal{M}_{k+1}$ ensures the coupling between the two rational reaction sets. By assuming that $\mathcal{M}_{k+1}$ is invertible at the step $k+1$, there exists one and only one solution of (56) for any vector $x_k \in \mathbb{R}^n$ and we obtain

$$\begin{aligned}\begin{pmatrix} x_{k+1} \\ \gamma_1^*(x_k,k) \\ \gamma_2^*(x_k,k) \end{pmatrix} &= \begin{bmatrix} I_n & 0_{n\times q} & 0_{n\times q} \\ 0_{r_1\times n} & -R_{11}^{-1}B_1' & 0_{r_1\times q} \\ 0_{r_2\times n} & 0_{r_2\times q} & -R_{22}^{-1}B_2' \end{bmatrix} \begin{pmatrix} x_{k+1} \\ \Psi_{1,k+1} \\ \Psi_{2,k+1} \end{pmatrix} \\ &= \begin{bmatrix} I_n & 0_{n\times q} & 0_{n\times q} \\ 0_{r_1\times n} & -R_{11}^{-1}B_1' & 0_{r_1\times q} \\ 0_{r_2\times n} & 0_{r_2\times n} & -R_{22}^{-1}B_2' \end{bmatrix} \mathcal{M}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \end{bmatrix} x_k. \quad (57)\end{aligned}$$

Injecting the relation (57) in the equation (42) and (50), we obtain the required relations (52). The uniqueness is verified because the matrices $\mathcal{M}_{k+1}$ are assumed to be invertible. □

Remark 3. *The matrix $\mathcal{M}_{k+1}$ has singular matrices E and $-E'$ on its diagonal. This prevents the use of Schur Complement to compute the inverse of $\mathcal{M}_{k+1}$, [43, 57]. Nevertheless it does not imply that $\mathcal{M}_{k+1}$ is singular (see [58] for a generic discussion on this point). This is in one of the main justification of our approach.*

Remark 4. *It should be emphasized that the proposed matrix block formulation in Theorem 1 avoids to decompose the matrix E into canonical forms, like the Weierstrass canonical form (see [29, 43, 30] for more details). No assumption on the matrix E is made in Theorem 1 (in particular on its index). Nevertheless, the matrix E is involved implicitly in the assumption related to the invertibility of the matrix $\mathcal{M}_{k+1}$.*

Remark 5. *Due to the symmetry and the semi-definiteness of the weighting matrices Q_{iK}, Q_i, and R_{ij}, the matrices $P_{i,k}$ are also symmetric and positive definite, as expected from the definition of the value function (optimal value of the criteria J_i).*

Remark 6. *The structure of the matrix $\mathcal{M}_{k+1}$ has already been encountered, but in the framework of the two-player Nash strategy with open-loop information structure, see for instance [20, 14, 59]. Contrary to appearances, such a*

structure has no particular properties, like symmetry of its eigenvalues. To the best of our knowledge, there are no sufficient conditions to ensure the invertibility of $\mathcal{M}_{k+1}$, however it is possible to point out several particular cases, where this matrix is not invertible and thus Theorem 1 could not be applied. For example, a pure algebraic system, that is $E = 0_n$ leads to $\mathcal{M}_K$ being singular, because $[S_1 \ S_2]$ is not full column rank. Furthermore, in the case of quasi-cooperation *between the two players [14, 59], that is when there exists a couple $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $(\alpha_1, \alpha_2) \neq 0$, such that $\alpha_1 Q_{1K} = \alpha_2 Q_{2K}$; or $\alpha_1 S_1 = \alpha_2 S_2$, then $\Lambda(-E) \subset \Lambda(\mathcal{M}_K)$, implying that $\mathcal{M}_K$ is singular.*

Remark 7. *When the matrix $\mathcal{M}_{m+1}$ is not invertible, the time $K-(m+1)$ is called a* finite escape time *(by convention taking into account the backward integration in time), extending the finite escape time of standard Riccati equations to the case of descriptor systems [20].*

Remark 8. *Even if the structure of the matrix $\mathcal{M}_{k+1}$ has only a few symmetry properties, it is possible to emphasize necessary conditions for its invertibility. Precisely the pairs $\begin{pmatrix} -E & B_i \end{pmatrix}$, and $\begin{pmatrix} -E' & P_{i,k+1} \end{pmatrix}$ should be surjective. One more time, these are not sufficient conditions of invertibility. When $\mathcal{M}_{k+1}$ is not invertible, Theorem 1 does not apply. This may be then explained, for instance, by the fact that the system cannot be regularized, or by the fact that the tri-trajectory does not exist for any initial state x_0.*

Remark 9. *When $E \in \mathbb{R}^{q \times n}$ is rectangular, with $q \neq n$, the matrix $\mathcal{M}_{k+1}$ belongs to the set $\mathbb{R}^{(q+2n)\times(n+2q)}$ and is thus rectangular and cannot be invertible. This comment justifies why E is assumed to be square in this section.*

As in the case of non-symmetric Riccati-type equations, we can also define coupled algebraic Riccati-type equations for descriptor games by replacing $P_{i,k}$ and $P_{i,k+1}$ by a time independent value $\tilde{P}_i$ in the relation (52). A natural method to find these algebraic solutions is to integrate backward in time the relation (52), from the initial condition (51), for a time horizon tending to infinity ($K \to +\infty$):

$$\tilde{P}_i = \lim_{K \to +\infty} P_{i,0}. \tag{58}$$

Of course, in the generic case, the algebraic solutions are not necessarily unique. The solutions depend on a basin of attraction over the matrices Q_{iK}.

3.2. Recovering the case of explicit systems

The result obtained in Theorem 1 includes the case of explicit systems, where the matrix E is invertible. In this framework, the adequate change of variables ($A \leftarrow E^{-1}A$, $B_i \leftarrow E^{-1}B_i$, $i \in \mathcal{N}$) allows to consider $E = I_n$ without loss of generality in this subsection. As a first consequence, the pairs $\begin{pmatrix} E & B_i \end{pmatrix} = \begin{pmatrix} I_n & B_i \end{pmatrix}$ and $\begin{pmatrix} E & B_i \end{pmatrix} = \begin{pmatrix} I_n & B_i \end{pmatrix}$ are surjective. The assumptions of Lemmas 1 and 2 are verified obviously. The objective here is to derive the matrix block approach to recover the coupled difference Riccati-type equations available in the literature for the feedback Nash strategy [2, Chapter 6][20, 60].

Lemma 3. *The matrix $\mathcal{M}_{k+1}$, defined by (53), rewrites for $E = I_n$*

$$\mathcal{M}_{k+1} = \begin{bmatrix} I_n & S_1 & S_2 \\ P_{1,k+1} & -I_n & 0_n \\ P_{2,k+1} & 0_n & -I_n \end{bmatrix}. \tag{59}$$

$\mathcal{M}_{k+1}$ is invertible if and only if

$$\Phi_{k+1} = I_n + S_1 P_{1,k+1} + S_2 P_{2,k+1} \tag{60}$$

is invertible. In this case, the recurrence relation (52) becomes the coupled difference Riccati-type equations, $\forall i \in \mathcal{N}$

$$P_{i,k} = Q_i + A' P_{i,k+1} \Phi_{k+1}^{-1} A + A'(\Phi_{k+1}^{-1})' \Big(P_{1,k+1}(S_{i1} P_{1,k+1} - S_1 P_{i,k+1}) + P_{2,k+1}(S_{i2} P_{2,k+1} - S_2 P_{i,k+1}) \Big) \Phi_{k+1}^{-1} A. \tag{61}$$

Proof. The matrix $\mathcal{M}_{k+1}$ containing identity matrices on its diagonal, allows to consider the Schur Complement to obtain the inverse of $\mathcal{M}_{k+1}$. We have the Schur Complement as follows

$$I_n - \begin{bmatrix} S_1 & S_2 \end{bmatrix} (-I_n) \begin{bmatrix} P_{1,k+1} \\ P_{2,k+1} \end{bmatrix} = I_n + S_1 P_{1,k+1} + S_2 P_{2,k+1} = \Phi_{k+1}. \tag{62}$$

The invertibility condition of Φ_{k+1} leads to the definition of the finite escape time [20]. If $\mathcal{M}_{k+1}$ is invertible, its inverse verifies the following relation

$$\mathcal{M}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \end{bmatrix} = \begin{bmatrix} I_n \\ P_{1,k+1} \\ P_{2,k+1} \end{bmatrix} \Phi_{k+1}^{-1} A \tag{63}$$

because

$$\mathcal{M}_{k+1}\begin{bmatrix} I_n \\ P_{1,k+1} \\ P_{2,k+1} \end{bmatrix} = \begin{bmatrix} \Phi_{k+1} \\ 0_n \\ 0_n \end{bmatrix}. \tag{64}$$

Injecting the relation (63) in the recurrence relation yields

$$P_{i,k} = Q_i + A'(\Phi_{k+1}^{-1})' \left(P_{i,k+1} + P_{1,k+1}S_{i1}P_{1,k+1} + P_{2,k+1}S_{i2}P_{2,k+1}\right)\Phi_{k+1}^{-1}A. \tag{65}$$

Thanks to the matrix inversion lemma,

$$\begin{aligned} (\Phi_{k+1}^{-1})'P_{i,k+1} &= (\Phi_{k+1}^{-1})'\left(\Phi_{k+1}' - P_{1,k+1}S_1 - P_{2,k+1}S_2\right)P_{i,k+1} && (66)\\ &= P_{i,k+1} - (\Phi_{k+1}^{-1})'\left(P_{1,k+1}S_1 + P_{2,k+1}S_2\right)P_{i,k+1}, && (67)\end{aligned}$$

the relation (65) can be rewritten into the desired relation (61). □

The strategy of each player is then explicitly given by

$$\gamma_i^*(x_k,k) = -R_{ii}^{-1}B_i'P_{i,k+1}\Phi_{k+1}^{-1}Ax_k, \tag{68}$$

with Φ_{k+1} defined by (60) and the closed-loop dynamic

$$x_{k+1} = \Phi_{k+1}^{-1}Ax_k. \tag{69}$$

3.3. Illustrations

We present here an example to illustrate the new approach in this paper concerning the feedback Nash strategy. One considers a game on a finite time horizon with $K = 10$. The dimension of the state is $n = 3$.

$$E = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 0.9 & 0.1 & 0.5 \\ 0.7 & 0.8 & 1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad R_{11} = 2; \quad R_{12} = 1;$$

$$Q_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}; \quad R_{21} = 1; \quad R_{22} = 3;$$

$$Q_{1K} = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}; \quad Q_{2K} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}; \quad x_0 = \begin{bmatrix} 1 \\ 0.2 \\ 2 \end{bmatrix}.$$

The matrices $P_{1,k}$ and $P_{1,2}$ are computed backward in time by the iterative relation (52) in Theorem 1. Then it is possible to compute forwardly the state x_{k+1} (drawn in Fig. 1) and the costate vectors $\Psi_{1,k+1}$ and $\Psi_{2,k+1}$. These costate vectors lead to the strategies $\gamma_1^*(x_k, k)$ and $\gamma_2^*(x_k, k)$, which are depicted in Fig. 2. The value functions $V_1(x_k, k)$ and $V_2(x_k, k)$ are depicted in Fig. 3.

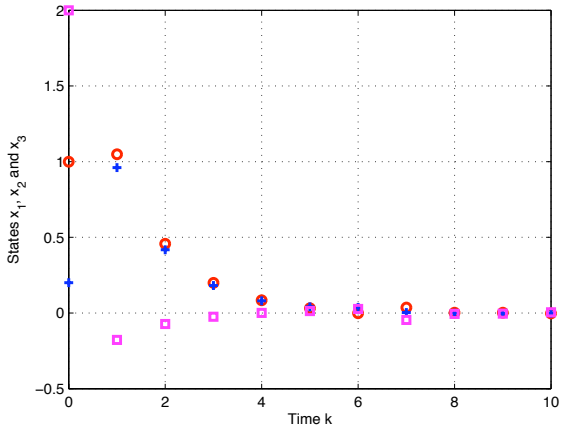

Figure 1: State components of x_k (first component ($\circ$ in red), second one ($+$ in blue) and third one ($\square$ in magenta).

The global criteria for the feedback Nash strategy are

$$V_1(x_0, 0) = \frac{1}{2} x_0' P_{1,0} x_0 = 10.20; \qquad V_2(x_0, 0) = \frac{1}{2} x_0' P_{2,0} x_0 = 22.20.$$

As a verification, one can notice that the value functions $V_1(x_k, k)$ and $V_2(x_k, k)$ are decreasing functions with respect to the discrete time k.

Finally, by considering the backward integration method, we obtain an algebraic solution of the equations (52)

$$\tilde{P}_1 = \begin{bmatrix} 2.0624 & -0.0088 & 0.6067 \\ -0.0088 & 1.0838 & 0.0711 \\ 0.6067 & 0.0711 & 1.4177 \end{bmatrix}; \; \tilde{P}_2 = \begin{bmatrix} 0.0231 & -0.1958 & -0.1506 \\ 1.1707 & 0.6966 & 1.2585 \\ -0.4629 & -0.0382 & -0.3007 \end{bmatrix}.$$

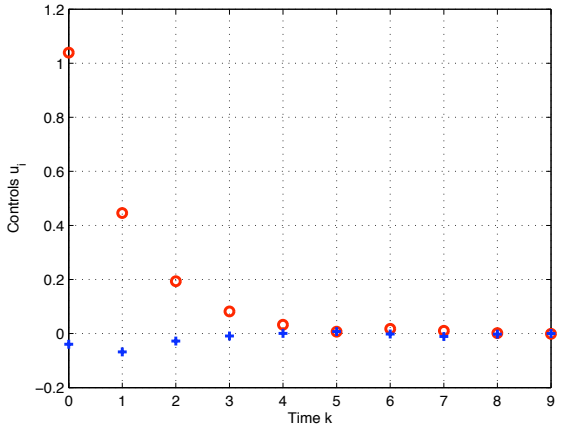

Figure 2: Strategies $\gamma_1^*(x_k, k)$ ($\circ$ in red) and $\gamma_2^*(x_k, k)$ ($+$ in blue).

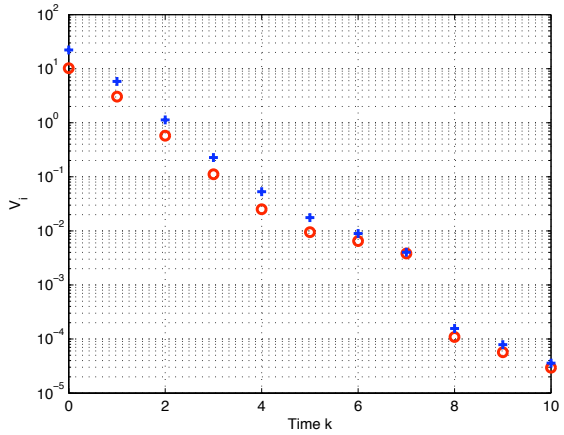

Figure 3: Value functions $V_1(x_k, k)$ ($\circ$ in red) and $V_2(x_k, k)$ ($+$ in blue).

4. The case of Stackelberg strategy

This section is devoted to the Stackelberg strategy, in the generic case where the matrix $E \in \mathbb{R}^{q \times m}$ is rectangular. The sufficient conditions for the strategy of the leader (player 2) will be firstly obtained. The sufficient conditions for the both players will be then gathered in the main theorem of this section. The particular case of explicit systems will be finally considered

to recover results from the literature [61]. In addition of the framework of the Nash strategy, we will assumed in the sequel that the weighting matrix R_{21} is positive definite, for technical reasons seen in the following.

4.1. Stackelberg strategy for descriptor games

Based on the definition (Definition 5) of a Stackelberg equilibrium, the sufficient conditions for the strategy of the leader are given in the next proposition.

Proposition 2. *Assume R_{11} to be positive definite and the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$ to be surjective. Sufficient conditions for the strategy of the leader, γ_2^{**}, are given by the set of equations*

$$\begin{aligned}
P_{2,K} &= Q_{2K}, && (70)\\
E'\Psi_{2,k+1} &= P_{2,k+1}x_{k+1} + P_{1,k+1}\mu_{k+1}, && (71)\\
\gamma_2^{**}(x_k,k) &= -R_{22}^{-1}B_2'\Psi_{2,k+1}, && (72)\\
E\mu_{k+1} &= S_{21}\Psi_{1,k+1} - S_1\Psi_{2,k+1}, && (73)\\
x_k'(P_{2,k}-Q_2)x_k &= z_{k+1}' \begin{bmatrix} P_{2,k+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_2 & 0 \\ 0 & 0 & 0 & S_{21} \end{bmatrix} z_{k+1}, && (74)
\end{aligned}$$

with the extended state

$$z_{k+1} = \begin{pmatrix} x_{k+1} \\ \mu_{k+1} \\ \Psi_{2,k+1} \\ \Psi_{1,k+1} \end{pmatrix} \in \mathbb{R}^{2n+2q}. \quad (75)$$

Proof. The minimization of the criterion J_2 is made under the dynamical constraint (1), when $u_1 = \gamma_1^{**}(x_k,k)$, that is

$$Ex_{k+1} = f(x_k, \gamma_1^{**}(x_k,k), u_{2,k}) = Ax_k + B_1\gamma_1^{**}(x_k,k) + B_2u_{2,k}. \quad (76)$$

By applying Dynamic Programming, the transversality condition

$$V_2(x_K,K) = \frac{1}{2}x_K'Q_{2K}x_K \quad (77)$$

is satisfied due to the equation (70). Furthermore the Hamilton-Jacobi-Bellmann equation writes as

$$V_2(x_k,k) = \min_{u_{2,k}\in\mathbb{R}^{r_2}} \max_{\substack{Ex_{k+1}=f(x_k,\gamma_1^{**}(x_k,k),u_{2,k}) \\ \gamma_1^{**}(x_k,k)\in\mathcal{R}_1(u_{2,k})}} \left(V_2(x_{k+1},k+1) + L_2(x_k,\gamma_1^{**}(x_k,k),u_{2,k})\right) \tag{78}$$

and $\gamma_2^{**}(x_k,k)$ is the argument of this minimum.

Because R_{11} is assumed to be positive definite, the strategy γ_1^{**} may be reformulated by the relation (41) by using $\Psi_{1,k+1}$. The relation (40) is a necessary condition to the relation $\gamma_1^{**}(x_k,k) \in \mathcal{R}_1(u_{2,k})$. However, the cost function L_1 being strictly convex with respect to the strategy γ_1^{**}, this is also a sufficient condition [62]. The optimization problem is reformulated into

$$V_2(x_k,k) = \min_{u_{2,k}\in\mathbb{R}^{r_2}} \max_{\substack{Ex_{k+1}=f(x_k,\gamma_1^{**}(x_k,k),u_{2,k}) \\ \frac{\partial V_1}{\partial x_{k+1}}(x_{k+1},k+1)=E'\Psi_{1,k+1}}} \left(V_2(x_{k+1},k+1) + L_2(x_k,\gamma_1^{**}(x_k,k),u_{2,k})\right), \tag{79}$$

with $\gamma_1^{**}(x_k,k) = -R_{11}^{-1}B_1'\Psi_{1,k+1}$. It is noteworthy that the strict convexity of L_1 with respect to $\gamma_1^{**}(x_k,k)$ ($R_{11} > 0$) allows the equivalence between the both formulations, nevertheless due to the singularity of the system, the strategy γ_1^{**} is not necessarily unique, only the value of the optimal cost function is.

There may exist multiple solutions to the system of equations in the constraints of the optimization problem (79). This justifies roughly speaking Remark 1 about the definition of the Stackelberg strategy. Under the surjectivity of the pair $\begin{pmatrix} E & B_1 \end{pmatrix}$, Lemma 1 can be applied and the solution of this system of equations exists and is unique. The maximum in the optimization problem (79) may be avoided. That yields to

$$V_2(x_k,k) = \min_{\substack{u_{2,k}\in\mathbb{R}^{r_2} \\ Ex_{k+1}=f(x_k,\gamma_1^{**}(x_k,k),u_{2,k}) \\ \frac{\partial V_1}{\partial x_{k+1}}(x_{k+1},k+1)=E'\Psi_{1,k+1}}} \left(V_2(x_{k+1},k+1) + L_2(x_k,\gamma_1^{**}(x_k,k),u_{2,k})\right). \tag{80}$$

The minimization in equation (80) is constrained by the dynamical system (76) and the equation (40), characterizing the strategy of the follower.

That is then equivalent to the minimization of the next modified cost function

$$V_2(x_{k+1},k+1)+L_2(x_k,\gamma_1^{**},u_{2,k})+\Psi'_{2,k+1}[f(x_k,\gamma_1^{**}(x_k,k),u_{2,k})-Ex_{k+1}] \\ +\mu'_{k+1}\left[\frac{\partial V_1}{\partial x_{k+1}}(x_{k+1},k+1)-E'\Psi_{1,k+1}\right], \quad (81)$$

where $\Psi_{2,k+1}\in\mathbb{R}^q$ and $\mu_{k+1}\in\mathbb{R}^n$ are the costate vectors associated with the constraints (76) and (40) respectively. The term to be minimized in (81) should thus be stationary with respect to x_{k+1}, $u_{2,k}$ and $\Psi_{1,k+1}$. It leads to the following sufficient conditions

$$\frac{\partial V_2}{\partial x_{k+1}}(x_{k+1},k+1)-E'\Psi_{2,k+1}+\frac{\partial^2 V_1}{\partial x_{k+1}^2}\mu_{k+1}=0, \quad (82)$$

$$\frac{\partial L_2}{\partial u_{2,k}}(x_k,\gamma_1^{**}(x_k,k),u_{2,k})+\left(\frac{\partial f}{\partial u_{2,k}}(x_k,\gamma_1^{**}(x_k,k),u_{2,k})\right)'\Psi_{2,k+1}=0 \quad (83)$$

and

$$\frac{\partial\gamma_1^{**}}{\partial\Psi_{1,k+1}}\left[\frac{\partial L_2}{\partial\gamma_1^{**}}(x_k,\gamma_1^{**},u_{2,k})+\frac{\partial f}{\partial\gamma_1^{**}}(x_k,\gamma_1^{**}(x_k,k),u_{2,k})\Psi_{2,k+1}\right]-E\mu_{k+1}=0. \quad (84)$$

In our linear-quadratic game framework, the equations (82), (83) and (84) read as (71), (72) and (73) respectively. The equation (74) is then obtained by injecting these sufficient conditions in the Hamilton-Jacobi-Bellmann equation (80). □

The next theorem gathers the sufficient conditions for the leader and the follower.

Theorem 2. *Consider the descriptor two-player game (1), associated with the criteria (3). Assume that R_{11}, R_{21} and R_{22} are positive definite and the pairs $\begin{pmatrix} E & B_1 \end{pmatrix}$ and $\begin{pmatrix} E & B_2 \end{pmatrix}$ are surjective. There exists one and only one feedback Stackelberg strategy if the following recurrence relations lead to invertible matrices $\mathcal{T}_{k+1}$ at each step.*

$$P_{i,K}=Q_{iK}, \quad \forall i\in\mathcal{N}, \quad (85)$$

$$P_{i,k}=Q_i+\begin{bmatrix} A \\ 0_{q\times n} \\ 0_n \\ 0_n \end{bmatrix}'(\mathcal{T}_{k+1}^{-1})'\mathcal{U}_{i,k+1}\mathcal{T}_{k+1}^{-1}\begin{bmatrix} A \\ 0_{q\times n} \\ 0_n \\ 0_n \end{bmatrix}, \ \forall(i,k)\in\mathcal{N}\times\{0;\cdots;K-1\} \quad (86)$$

where

$$\mathcal{T}_{k+1} = \begin{bmatrix} E & 0_{q\times n} & S_2 & S_1 \\ 0_{q\times n} & E & S_1 & -S_{21} \\ P_{2,k+1} & P_{1,k+1} & -E' & 0_{n\times q} \\ P_{1,k+1} & 0_{n\times n} & 0_{n\times q} & -E' \end{bmatrix} \in \mathbb{R}^{(2q+2n)\times(2q+2n)} \tag{87}$$

and

$$\mathcal{U}_{i,k+1} = \begin{bmatrix} P_{i,k+1} & 0_n & 0_{n\times q} & 0_{n\times q} \\ 0_n & 0_n & 0_{n\times q} & 0_{n\times q} \\ 0_{q\times n} & 0_{q\times n} & S_{i2} & 0_q \\ 0_{q\times n} & 0_{q\times n} & 0_q & S_{i1} \end{bmatrix}. \tag{88}$$

Proof. By applying the expression of γ_1^{**} and γ_2^{**} (given by (41) and (49) respectively) to the dynamical system (1), we have

$$Ex_{k+1} = Ax_k - S_1\Psi_{1,k+1} - S_2\Psi_{2,k+1}. \tag{89}$$

Collecting the sufficient conditions (40), (71), (73) and (89), yields

$$\mathcal{T}_{k+1}\begin{pmatrix} x_{k+1} \\ \mu_{k+1} \\ \Psi_{2,k+1} \\ \Psi_{1,k+1} \end{pmatrix} = \begin{bmatrix} A \\ 0_{q\times n} \\ 0_n \\ 0_n \end{bmatrix} x_k. \tag{90}$$

By assuming that $\mathcal{T}_{k+1}$ is invertible at the step $k+1$, we obtain

$$\begin{aligned} \begin{pmatrix} x_{k+1} \\ \mu_{k+1} \\ \gamma_2^*(x_k,k) \\ \gamma_1^*(x_k,k) \end{pmatrix} &= \begin{bmatrix} I_n & 0_n & 0_{n\times q} & 0_{n\times q} \\ 0_n & I_n & 0_{n\times q} & 0_{n\times q} \\ 0_{r_2\times n} & 0_{r_2\times n} & -R_{22}^{-1}B_2' & 0_{r_2\times q} \\ 0_{r_1\times n} & 0_{r_1\times n} & 0_{r_1\times q} & -R_{11}^{-1}B_1' \end{bmatrix} \begin{pmatrix} x_{k+1} \\ \mu_{k+1} \\ \Psi_{2,k+1} \\ \Psi_{1,k+1} \end{pmatrix} \\ &= \begin{bmatrix} I_n & 0_n & 0_{n\times q} & 0_{n\times q} \\ 0_n & I_n & 0_{n\times q} & 0_{n\times q} \\ 0_{r_2\times n} & 0_{r_2\times n} & -R_{22}^{-1}B_2' & 0_{r_2\times q} \\ 0_{r_1\times n} & 0_{r_1\times n} & 0_{r_1\times q} & -R_{11}^{-1}B_1' \end{bmatrix} \mathcal{T}_{k+1}^{-1} \begin{bmatrix} A \\ 0_{q\times n} \\ 0_n \\ 0_n \end{bmatrix} x_k. \end{aligned} \tag{91}$$

Injecting the relation (91) in the equation (42) and (74), we obtain the required relations (86). The uniqueness is verified because the matrices $\mathcal{T}_{k+1}$ are assumed to be invertible. □

The matrix $\mathcal{T}_{k+1}$ has a particular structure. Such a structure will be used in the sequel in order to determine sufficient conditions allowing the invertibility of $\mathcal{T}_{k+1}$. When the matrix E is singular ($q = n$), the matrix $\mathcal{T}_{k+1}$ is Hamiltonian [63]. In the generic case, when E is rectangular, we can use the following lemma.

Lemma 4. *If* $\begin{pmatrix} E & B_1 \end{pmatrix}$ *is surjective,* $\begin{pmatrix} E \\ P_{1,k+1} \end{pmatrix}$ *and* $\begin{pmatrix} E \\ P_{2,k+1} \end{pmatrix}$ *are injective and* R_{21} *is positive definite, then the matrix* $\mathcal{T}_{k+1}$ *is invertible.*

Proof. A matrix is invertible if and only if its kernel is reduced to the trivial singleton. Assume that, $y_1 \in \mathbb{R}^n$, $y_2 \in \mathbb{R}^n$, $y_3 \in \mathbb{R}^q$ and $y_4 \in \mathbb{R}^q$,

$$\mathcal{T}_{k+1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0_{(2n+2q)\times 1}. \tag{92}$$

By developping the equation (92), we have

$$\begin{aligned} Ey_1 + S_2y_3 + S_1y_4 &= 0_{q\times 1}, & (93)\\ Ey_2 + S_1y_3 - S_{21}y_4 &= 0_{q\times 1}, & (94)\\ P_{2,k+1}y_1 + P_{1,k+1}y_2 - E'y_3 &= 0_{n\times 1}, & (95)\\ P_{1,k+1}y_1 - E'y_4 &= 0_{n\times 1}. & (96) \end{aligned}$$

Combining the left multiplication of Equations (93), (94), (95) and (96)) by y_3', y_4', y_1' and y_2', respectively, yields

$$y_1'P_{2,k+1}y_1 + y_3'S_2y_3 + y_4'S_{21}y_4 = 0. \tag{97}$$

The matrix $P_{2,k+1}$ being positive semi-definite and the matrices R_{22} and R_{21} being positive definite, one gets

$$P_{2,k+1}y_1 = 0_{n\times 1}; \quad B_2'y_3 = 0_{r_2\times 1}; \quad B_1'y_4 = 0_{r_1\times 1}. \tag{98}$$

It follows $\begin{pmatrix} E \\ P_{2,k+1} \end{pmatrix} y_1 = 0_{2n\times 1}$ inducing $y_1 = 0_{n\times 1}$. Thus $\begin{pmatrix} E' \\ B_1' \end{pmatrix} y_4 = 0_{(n+r_1)\times 1}$, that is $y_4 = 0_{q\times 1}$. Equations (94) and (95) become

$$\begin{aligned} Ey_2 + S_1y_3 &= 0_{q\times 1}, & (99)\\ P_{1,k+1}y_2 - E'y_3 &= 0_{n\times 1}. & (100) \end{aligned}$$

By left multiplication of Equations (99) and (100) by y_3' and y_2', respectively, one obtains

$$y_3' S_1 y_3 + y_2' P_{1,k+1} y_2 = 0, \tag{101}$$

which implies, by the positive definiteness of R_{11} and the positive semi-definiteness of $P_{1,k+1}$ that

$$B_1' y_3 = 0_{r_1 \times 1}; \quad P_{1,k+1} y_2 = 0_{n \times 1}. \tag{102}$$

Due to these relations, we conclude that $\begin{pmatrix} E \\ P_{1,k+1} \end{pmatrix} y_2 = 0_{(n+q)\times 1}$, that is $y_2 = 0_{n\times 1}$. It results that $\begin{pmatrix} E' \\ B_1' \end{pmatrix} y_3 = 0_{(n+r_1)\times 1}$, and $y_3 = 0_{q\times 1}$. The matrix $\mathcal{T}_{k+1}$ is then invertible. □

Thus, it is possible to provide sufficient conditions, based only on the matrices of the system, for the existence of feedback Stackelberg strategies. These conditions are presented in the following lemma.

Lemma 5. *Assume that R_{21} is positive definite, that $\begin{pmatrix} E \\ Q_{iK} \end{pmatrix}$ and $\begin{pmatrix} E \\ Q_i \end{pmatrix}$ $(i \in \mathcal{N})$ are injective and that $\begin{pmatrix} E & B_1 \end{pmatrix}$ is surjective, then for any $k \in \{0, \cdots, K\}$, $P_{1,k}$ and $P_{2,k}$ exist, are symmetric and are positive definite. In addition $\begin{pmatrix} E \\ P_{i,k} \end{pmatrix}$, $(i \in \mathcal{N})$ are injective.*

Proof. For $k = K$, the lemma is true due to Equation (85). Let us prove the lemma step by step backward in time. Assume that the property is verified at time $(k+1)$, then by using Lemma 4, the matrix $\mathcal{T}_{k+1}$ is invertible and $P_{i,k}$ are defined by the relation (86), that is they are symmetric and positive semi-definite. Consider now a vector $y \in \mathbb{R}^n$ such that $\begin{pmatrix} E \\ P_{i,k} \end{pmatrix} y = 0$, that is in other words $Ey = 0$ and $P_{i,k} y = 0$. Let us prove that this implies that $y = 0$. We have $y' P_{i,k} y = 0$. Due to the definition of $P_{i,k}$ given by (86), $y' Q_i y = 0$, which induces that $Q_i y = 0$ and $\begin{pmatrix} E \\ Q_i \end{pmatrix} y = 0$. The assumption allows to obtain $y = 0$, that ends the proof. □

The following subsection investigates the particular case of explicit systems.

4.2. Recovering the case of explicit systems

As previously mentioned, the case of explicit systems, where the matrix E is square and invertible can be considered as a particular case of descriptor systems. We propose here to recover the coupled difference Riccati-type equations issued from the literature for the feedback Stackelberg strategy [2, Section 7.3] for games with $E = I_n$. The matrix block formulation for explicit systems has been proposed for feedback Stackelberg strategy in [61].

First of all, it should be noticed that the assumptions of Lemma 5 are reduced to R_{21} positive definite. Actually with $E = I_n$, the matrix $\begin{pmatrix} I_n & B_1 \end{pmatrix}$ is surjective and the matrices $\begin{pmatrix} I_n \\ Q_{iK} \end{pmatrix}$ and $\begin{pmatrix} I_n \\ Q_i \end{pmatrix}$, $(i \in \mathcal{N})$ are injective, regardless the values of matrices B_1, Q_1, Q_2, Q_{1K} and Q_{2K}. That is there always exists a feedback Stackelberg strategy for an explicit system because the matrix $\mathcal{T}_{k+1}$ is always invertible.

Lemma 6. *The matrix $\mathcal{T}_{k+1}$, defined by (87), writes for $E = I_n$,*

$$\mathcal{T}_{k+1} = \begin{bmatrix} I_n & 0_n & S_2 & S_1 \\ 0_n & I_n & S_1 & -S_{21} \\ P_{2,k+1} & P_{1,k+1} & -I_n & 0_n \\ P_{1,k+1} & 0_n & 0_n & -I_n \end{bmatrix}. \tag{103}$$

Even if the matrix $\mathcal{T}_{k+1}$ is invertible, exhibiting its inverse may induce some difficulties. In fact some generic methods, as for example the Schur complement, require some technical assumptions to be applied [58]. Here we made the assumption that these requirements are fullfiled. Roughly speaking, we assume that $(I_n + P_{1,k+1}S_1)$ and $(I_n + S_1P_{1,k+1} + S_2(I_n + P_{1,k+1}S_1)^{-1}(P_{2,k+1} + P_{1,k+1}S_{21}P_{1,k+1}))$ are invertible. In this case, the recurrence relation (86) becomes the coupled difference Riccati-type equations [2, Section 7.3], $\forall i \in \mathcal{N}$

$$P_{i,k} = Q_i + A'(\Theta_{k+1}^{-1})'(P_{i,k+1} + \Delta_{k+1}'S_{i2}\Delta_{k+1} + P_{1,k+1}S_{i1}P_{1,k+1})\Theta_{k+1}^{-1}A. \tag{104}$$

Proof. The inverse of $\mathcal{T}_{k+1}$ verifies

$$\mathcal{T}_{k+1}\begin{bmatrix} I_n \\ \Omega_{k+1} \\ \Delta_{k+1} \\ P_{1,k+1} \end{bmatrix} = \begin{bmatrix} \Theta_{k+1} \\ 0_n \\ 0_n \\ 0_n \end{bmatrix}, \tag{105}$$

where

$$\begin{aligned}
\Theta_{k+1} &= I_n + S_1 P_{1,k+1} + S_2 (I_n + P_{1,k+1} S_1)^{-1} (P_{2,k+1} S_{21} P_{1,k+1}), \quad (106)\\
\Delta_{k+1} &= (I_n + P_{1,k+1} S_1)^{-1} (P_{2,k+1} S_{21} P_{1,k+1}), \quad (107)\\
\Omega_{k+1} &= S_{21} P_{1,k+1} - S_1 \Delta_{k+1}. \quad (108)
\end{aligned}$$

Due to the equation (105), we have

$$\mathcal{T}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \\ 0_n \end{bmatrix} = \begin{bmatrix} I_n \\ \Omega_{k+1} \\ \Delta_{k+1} \\ P_{1,k+1} \end{bmatrix} \Theta_{k+1}^{-1} A. \quad (109)$$

Injecting this last equation into Equation (86) yields equations (104). □

The strategy of each player is then explicitly given by

$$\begin{aligned}
\gamma_1^{**}(x_k, k) &= -R_{11}^{-1} B_1' P_{1,k+1} \Theta_{k+1}^{-1} A x_k, \quad (110)\\
\gamma_2^{**}(x_k, k) &= -R_{22}^{-1} B_2' P_{2,k+1} \Delta_{k+1} \Theta_{k+1}^{-1} A x_k, \quad (111)\\
\mu_{k+1} &= \Omega_{k+1} \Theta_{k+1}^{-1} A x_k. \quad (112)
\end{aligned}$$

Thus the closed-loop dynamic is

$$x_{k+1} = \Theta_{k+1}^{-1} A x_k. \quad (113)$$

4.3. Illustrations

In this section, two academic examples are presented: one with the matrix E square and one with the matrix E rectangular. Let us consider firstly the following example to illustrate our matrix block approach to solve the feedback Stackelberg strategy, with E square. The time-horizon is defined by $K = 10$.

$$E = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}; \quad A = \begin{bmatrix} 0.9 & 1 \\ 0.7 & 0.8 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad R_{11} = 2; \quad R_{12} = 1;$$

$$Q_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}; \quad R_{21} = 0.5; \quad R_{22} = 1;$$

$$Q_{1K} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}; \quad Q_{2K} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}; \quad x_0 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}.$$

The matrices $P_{1,k}$ and $P_{2,k}$ are computed backward in time by the iterative relation (86). When it is done, the state trajectory is computed forwardly, in addition of the costate vectors. The state trajectory is depicted in Fig. 4, and the strategies $\gamma_1^{**}(x_k, k)$ and $\gamma_2^{**}(x_k, k)$ in Fig. 5. The value functions $V_1(x_k, k)$ and $V_2(x_k, k)$ are represented in Fig. 6.

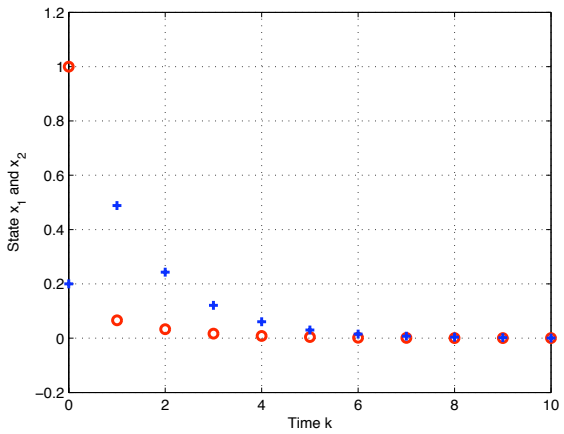

Figure 4: State components in function of time (first component of x_k in red $\circ$; second component of x_k in blue $+$).

The global criteria for memoryless closed-loop Stackelberg strategy are

$$V_1(x_0, 0) = \frac{1}{2} x_0' P_{1,0} x_0 = 3.33; \qquad V_2(x_0, 0) = \frac{1}{2} x_0' P_{2,0} x_0 = 3.53.$$

As a verification, one can notice that the value functions $V_1(x_k, k)$ and $V_2(x_k, k)$ are decreasing functions with respect to the discrete time k.

As for the Nash strategy, it is also possible to obtain a solution of the algebraic equation related to the recursive equation (86), that is where $P_{i,k} = P_{i,k+1}$. By a backward integration starting from $K = 50$, we find numerically

$$\tilde{P}_1 = \begin{bmatrix} 2.5401 & 1.6101 \\ 1.6101 & 3.6891 \end{bmatrix}; \qquad \tilde{P}_2 = \begin{bmatrix} 2.7418 & 1.7872 \\ 1.7872 & 1.8376 \end{bmatrix},$$

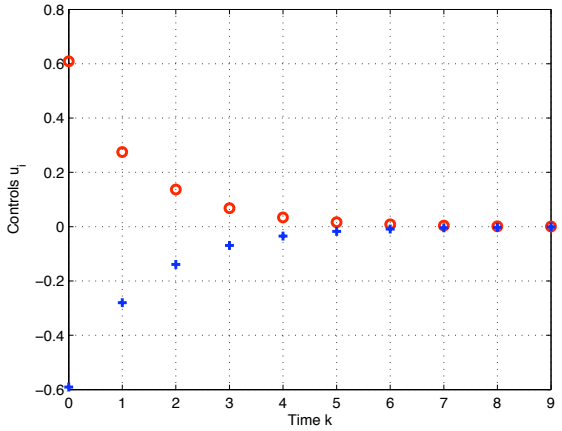

Figure 5: Strategies $\gamma_1^{**}(x_k, k)$ (in red $\circ$) and $\gamma_2^{**}(x_k, k)$ (in blue $+$).

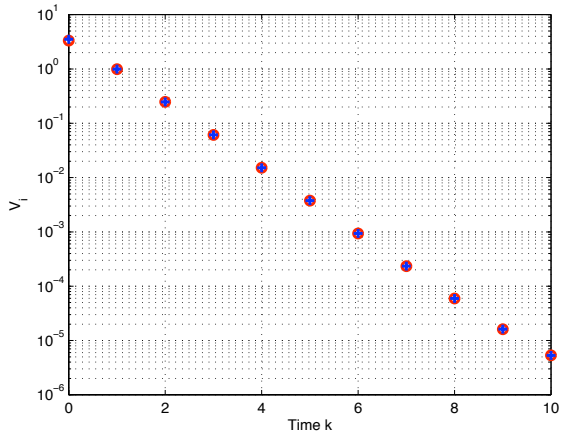

Figure 6: Value functions $V_1(x_k, k)$ (in red $\circ$) and $V_2(x_k, k)$ (in blue $+$).

which naturally depends on the choice of the initial conditions Q_{iK}.

Now let us consider an example with E rectangular. Here $n = 2$, $q = r_1 = r_2 = 1$ and

$$E = \begin{bmatrix} 1 & 0.2 \end{bmatrix}; \quad A = \begin{bmatrix} -5 & -4 \end{bmatrix}; \quad B_1 = \begin{bmatrix} -1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 1.5 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad R_{11} = 10; \quad R_{12} = 1;$$

$$Q_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}; \quad R_{21} = 0.5; \quad R_{22} = 20;$$

$$Q_{1K} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}; \quad Q_{2K} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}; \quad x_0 = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}.$$

The assumptions of Lemmas 4 and 5 are fulfilled. By applying the provided methodology related to Theorem 2, one obtains a (unique) tri-trajectory, starting from one arbitrary initial state, depicted in Fig. 7 and the inputs trajectories, depicted in Fig. 8.

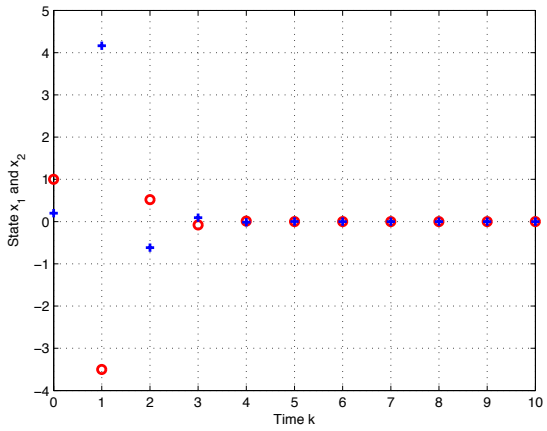

Figure 7: State components in function of time (first component of x_k in red $\circ$; second component of x_k in blue $+$).

5. Conclusion

Feedback Nash and Stackelberg strategies for discrete-time linear-quadratic descriptor two-player games have been investigated in this paper. Sufficient conditions have been provided for the existence and the uniqueness of a tri-trajectory, consisting of the state-trajectory and of the two control inputs trajectories. These sufficient conditions are based on Dynamic Programming

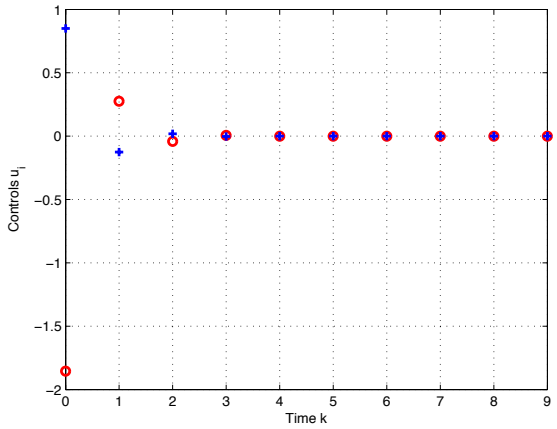

Figure 8: Strategies $\gamma_1^{**}(x_k, k)$ (in red $\circ$) and $\gamma_2^{**}(x_k, k)$ (in blue $+$).

and a matrix block formulation regularizing the singular dynamic. Properties of the characteristic matrices related to the matrix block formulation have been underlined. The standard case of explicit systems and the associated sufficient conditions available in the literature have been recovered. Examples have been proposed to illustrate this matrix block formulation for descriptor games.

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