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Yanick Heurteaux, Andrzej Stos. On measures driven by Markov chains. 2013. hal-00880320

# HAL Id: hal-00880320 https://hal.science/hal-00880320

Preprint submitted on 6 Nov 2013

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# ON MEASURES DRIVEN BY MARKOV CHAINS

#### YANICK HEURTEAUX AND ANDRZEJ STOS

ABSTRACT. We study measures on [0, 1] which are driven by a finite Markov chain and which generalize the famous Bernoulli products. We propose a hands-on approach to determine the structure function  $\tau$  and to prove that the multifractal formalism is satisfied. Formulas for the dimension of the measures and for the Hausdorff dimension of their supports are also provided.

## 1. INTRODUCTION

Multifractal measures on  $\mathbb{R}^d$  are measures m for which the level sets

$$E_{\alpha} = \left\{ x \in \mathbb{R}^d ; \lim_{r \to 0} \frac{\log \operatorname{m}(B(x, r))}{\log r} = \alpha \right\}$$

are non-trivial for at most two values of the real  $\alpha$ . In practice, it is impossible to completely describe the sets  $E_{\alpha}$ , but we can try to calculate their Hausdorff dimensions. To this end, Frisch and Parisi ([7]) were the first to use the Legendre transform of a structure function  $\tau$ . A mathematically rigorous approach was given by Brown, Michon and Peyrière in [2] and by Olsen in [9]. There are many situations in which the Legendre transform formula, now called multifractal formalism, is satisfied. For a comprehensive account see see e.g. [3], [5], [6], [11] or [8].

A fundamental model for multifractal measures is given by so called Bernoulli products, see for example Chapter 10 of [4]. Roughly speaking, if  $I_{\varepsilon_1\cdots\varepsilon_n}$  are the  $\ell$ -adic intervals of the  $n^{th}$  generation and  $(X_n)$  is an i.i.d. sequence of random variables, then Bernoulli product m can be defined by

$$m(I_{\varepsilon_1\cdots\varepsilon_n}) = P[X_1 = \varepsilon_1, \cdots, X_n = \varepsilon_n].$$

The purpose of this paper is to provide an explicit analysis of a natural generalization of this model. Instead of an i.i.d. sequence, we consider an irreducible homogeneous Markov chain. Consequently, the measure m satisfies the recurrence relation

$$\mathbf{m}(I_{\varepsilon_1\cdots\varepsilon_{n+1}}) = p_{\varepsilon_n\varepsilon_{n+1}} \mathbf{m}(I_{\varepsilon_1\cdots\varepsilon_n}),$$

where  $P = (p_{ij})$  is the transition matrix of  $X_n$  (see the next section for full details).

In Section 3, we identify a formula for the structure function  $\tau$  of such a measure and we compute its dimension. Section 4 contains a construction of auxiliary (Gibbs) measures. While it involves a nontrivial rescaling, we insist on the fact that our results don't require sophisticated tools but only some fundamental results of multifractal analysis and the use of Perron-Frobenius theorem. We are then able to prove that the multifractal formalism is satisfied and we give a formula for the Hausdorff dimension of the (closed) support of the measures. Finally, in Section 5

<sup>2010</sup> Mathematics Subject Classification. 28A80, 60J10, 28D05.

 $Key\ words\ and\ phrases.$  Multifractal formalism, Cantor sets, Hausdorff dimension, Markov chains.

we discuss ergodic properties and prove that for a given support K, the measure m with maximal dimension is essentially unique.

#### 2. Preliminaries

Set  $S = \{0, 1, 2, \ell - 1\}$ . Let  $\mathcal{W}_n$  be set of words of length n over the alphabet S. A concatenation of two words  $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$  and  $\delta = \delta_1 \cdots \delta_k \in \mathcal{W}_k$  will be denoted by  $\varepsilon \delta = \varepsilon_1 \cdots \varepsilon_n \delta_1 \cdots \delta_k$ . Let  $\mathcal{F}_0 = \{[0, 1)\}$  and for  $n \ge 1$ ,  $\mathcal{F}_n$  be the set of  $\ell$ -adic intervals of order n, that is the family of intervals of the form

$$I_{\varepsilon} = I_{\varepsilon_1 \cdots \varepsilon_n} = \left[\sum_{i=1}^n \frac{\varepsilon_i}{\ell^n}, \sum_{i=1}^n \frac{\varepsilon_i}{\ell^n} + \frac{1}{\ell^n}\right)$$

where  $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$ . If  $I = I_{\varepsilon} \in \mathcal{F}_n$  and  $J = I_{\delta} \in \mathcal{F}_k$ , we will write  $IJ = I_{\varepsilon\delta}$ . Note that  $IJ \subset I$ . Finally, we will denote by  $I_n(x)$  the unique interval  $I \in \mathcal{F}_n$  such that  $x \in I$  and |I| the length of the interval I.

Consider a discrete Markov chain  $X = (X_k)_{k\geq 1}$  on S with an initial distribution  $p_i = P(X_1 = i), i \in S$ , and the transition probabilities  $P = (p_{ij})_{i,j=0}^{\ell-1}$  where  $p_{ij} = P(X_{n+1} = j | X_n = i)$ . In order to exclude degenerated cases, we will suppose that  $p_{ij} \neq 1$  for any  $(i, j) \in S \times S$ . Note that for the sake of coherence with the definition of  $\mathcal{F}_n$ , the entries of matrices and vectors will be indexed by numbers starting from 0.

If necessary, we will assume that X is irreducible, that is for all  $i, j \in S$  there exists  $k \geq 2$  such that  $P(X_k = j | X_1 = i) > 0$ .

For a matrix  $P = (p_{ij})_{i,j\in\mathcal{S}}$  we denote by  $P^n$  the usual matrix power. This should not be confused with  $P_q$ ,  $q \in \mathbb{R}$ , which stands for the matrix whose entries are given by  $(p_{ij}^q)_{i,j\in\mathcal{S}}$ . In this context, by convention we will set  $0^q = 0$  for any  $q \in \mathbb{R}$ .

The Hausdorff dimension (box dimension, respectively) of a set E will be denoted by  $\dim_H E$  ( $\dim_B E$ ). Recall the definitions of the lower and upper dimension of a probability measure  $\mu$  on  $\mathbb{R}^d$ :

$$\dim_* \mu = \inf\{\dim_H(E) : \mu(E) > 0\} = \sup\{s > 0 : \mu \ll \mathcal{H}^s\}$$
$$\dim^* \mu = \inf\{\dim_H(E) : \mu(E) = 1\} = \inf\{s > 0 : \mu \perp \mathcal{H}^s\}$$

where  $\mathcal{H}^s$  denotes the Hausdorff measure in dimension s. If these two quantities agree, the common value is called the dimension of  $\mu$  and denoted by dim  $\mu$ . In this case the measure is said to be *unidimensional*. For more details, we refer the reader to e.g. [8].

By c or C we will denote a generic positive constant whose exact value is not important and may change from line to line. For functions or expressions f and g, depending on a variable x, say, we will write  $f \approx g$  if there exists a constant c which does not depend on x and such that  $c^{-1}g(x) \leq f(x) \leq cg(x)$  for any admissible x.

## 3. The measures and the structure function $\tau$

Let  $(X_n)_{n\geq 1}$  be a Markov chain on S with initial distribution  $(p_i)_{i\in S}$  and transition matrix  $P = (p_{ij})$ . Define the measure m as follows:

$$m(I_{\varepsilon_1\cdots\varepsilon_n}) = P(X_1 = \varepsilon_1, \dots, X_n = \varepsilon_n).$$

Then  $m(I_i) = p_i, i \in S$ , and by the Markov property,

$$m(I_{\varepsilon_1\cdots\varepsilon_n}) = P(X_n = \varepsilon_n \mid X_{n-1} = \varepsilon_{n-1}\cdots X_1 = \varepsilon_1)P(X_{n-1} = \varepsilon_{n-1}\cdots X_1 = \varepsilon_1)$$
$$= p_{\varepsilon_{n-1}\varepsilon_n} m(I_{\varepsilon_1\cdots\varepsilon_{n-1}}).$$

Iterating, we get

(3.1) 
$$\mathbf{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = p_{\varepsilon_1}p_{\varepsilon_1\varepsilon_2}\cdots p_{\varepsilon_{n-1}\varepsilon_n}.$$

In other words, a finite trajectory of X selects an interval and assigns it a mass equal to the probability of the trajectory. The measure is well defined since for any given  $\ell$ -adic interval I there is exactly one path to it and its subintervals can be reached only through I.

Because of the additivity property  $m(I_{\varepsilon_1\cdots\varepsilon_n}) = \sum_{k=0}^{n-1} m(I_{\varepsilon_1\cdots\varepsilon_{n-1}k})$  and the property  $\lim_{n\to+\infty} m(I_{\varepsilon_1\cdots\varepsilon_n}) = 0$ , it is well known that formula (3.1) defines a probability Borel measure whose support is contained in [0,1] (see for example [12]). Moreover the measure m is such that  $m(\{x\}) = 0$  for any point x.

The construction proposed here can be viewed as a generalization of a classical Bernoulli measure. Our goal is to give a hands-on approach to the multifractal analysis of such measures.

**Example 3.1.** (1) Natural measure on the triadic Cantor set. Let

$$P = \frac{1}{2} \left( \begin{array}{rrr} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{array} \right)$$

and set the initial distribution  $(p_0, p_1, p_2) = (1/2, 0, 1/2)$ . Then  $\mathcal{C} = \text{supp m}$  is the ternary Cantor set and m is the normalized  $(\log_3 2)$ -Hausdorff measure on  $\mathcal{C}$ .

(2) Bernoulli measures. Let  $p = (p_0, \dots, p_{\ell-1})$  be a probability vector and suppose that  $(X_i)$  are i.i.d. random variables with  $P(X_1 = j) = p_j$ . By independence  $p_{ij} = p_j$  so that

$$P = \begin{pmatrix} p_0 & p_1 & \cdots & p_{\ell-1} \\ \vdots & & \vdots \\ \vdots & & & \vdots \\ p_0 & p_1 & \cdots & p_{\ell-1} \end{pmatrix}.$$

Hence, by (3.1) we get  $m(I_{\varepsilon_1\cdots\varepsilon_n}) = p_{\varepsilon_1}\cdots p_{\varepsilon_n}$ , and the measure m is the classical Bernoulli product.

(3) Let  $\ell = 2, p \neq 1/2$  and

$$P = \left(\begin{array}{cc} p & 1-p \\ 1-p & p \end{array}\right).$$

The associated measure m was introduced by Tukia in [13]. It is a doubling measure with dimension dim(m) =  $-(p \log_2 p + (1-p) \log_2(1-p)) < 1$ . The associated repartition function f(x) = m([0, x]) is a singular quasisymetric function.

(4) Random walk on  $\mathbb{Z}_{\ell}$ . Let  $p_{ij} = 1/2$  if |i - j| = 1 or  $(i, j) = (0, \ell - 1)$  or  $(i, j) = (\ell - 1, 0)$ . If, for example, n = 3,

$$P = \frac{1}{2} \left( \begin{array}{rrr} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right).$$

As we will see later, the associated measure is monofractal with dimension  $\frac{\log 2}{\log \ell}$ .

Define as usual the structure function  $\tau(q)$  by

(3.2) 
$$\tau(q) = \limsup_{n \to \infty} \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right),$$

with the usual convention  $0^q = 0$  for any  $q \in \mathbb{R}$ .

**Theorem 3.2.** Let m be a probability measure driven by a Markov chain with transition matrix P. Suppose that the matrix P is irreductible. For  $q \in \mathbb{R}$ , let  $\lambda_q$  be the spectral radius of  $P_q$ . Then  $\tau(q) = \log_{\ell}(\lambda_q)$  and the limit does exist in (3.2).

*Proof.* Let  $\mathcal{W}_{n,k}$  be the subset of  $\mathcal{W}_n$  consisting of words that end with  $k \in \mathcal{S}$ . Set

$$s_{n,k} = \sum_{\varepsilon \in \mathcal{W}_{n,k}} \mathrm{m}(I_{\varepsilon})^q$$

and let  $S_n$  be the (line) vector  $(s_{n,0}, \ldots, s_{n,\ell-1})$ . In particular,  $\mathcal{W}_{1,k} = \{k\}$  and  $S_1 = (p_0^q, \ldots, p_{\ell-1}^q)$ . We claim that

(3.3) 
$$S_n P_q = S_{n+1}, \quad n \ge 1.$$

Indeed, the  $j^{\text{th}}$  coordinate of  $S_n P_q$  is given by

$$\sum_{k \in \mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} p_{kj}^q \operatorname{m}(I_{\varepsilon})^q$$

Using the Markov property, we have  $p_{kj} \operatorname{m}(I_{\varepsilon}) = \operatorname{m}(I_{\varepsilon j})$  when  $\varepsilon \in \mathcal{W}_{n,k}$ . It follows that

$$\sum_{k \in \mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} \mathbf{m}(I_{\varepsilon j})^q = \sum_{\varepsilon \in \mathcal{W}_{n+1,j}} \mathbf{m}(I_{\varepsilon})^q = s_{n+1,j}$$

which is the  $j^{\text{th}}$  coordinate of  $S_{n+1}$ . So that (3.3) follows. Iterating, we obtain

(3.4) 
$$S_n = S_1 \left( P_q \right)^{n-1}$$
.

Now, observe that

$$\sum_{\varepsilon \in \mathcal{W}_n} \mathbf{m}(I_{\varepsilon})^q = \sum_{k \in \mathcal{S}} s_{n,k} = \|S_n\|_1 = \|S_1 P_q^{n-1}\|_1$$

where  $\| \|_1$  is the  $\ell^1$  norm in  $\mathbb{R}^{\ell}$ . It follows that

$$\tau(q) = \limsup_{n \to \infty} \frac{1}{n} \log_{\ell} \left( \sum_{\varepsilon \in W_n} m(I_{\varepsilon})^q \right)$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log_{\ell} \|S_1 P_q^{n-1}\|_1$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log_{\ell} \|S_1 P_q^n\|_1$$

Let us now introduce the following notation. If  $a = (a_0, \dots, a_{\ell-1})$  and  $b = (b_0, \dots, b_{\ell-1})$  are two vectors in  $\mathbb{R}^{\ell}$ , we will write  $a \prec b$  when  $a_i \leq b_i$  for any value of *i*. Observe in particular that  $S_1 \succ 0$  and is not identically equal to 0. Let  $\lambda_q$  be the spectral radius of the matrix  $P_q$ , which is also the spectral radius of the transposed matrix  $P_q^t$ . The matrix  $P_q$  being positive and irreductible, Perron-Frobenius Theorem ensures the existence of an eigenvector  $\nu_q$  with strictly positive entries, satisfying  $\nu_q P_q = \lambda_q \nu_q$ . Therefore, there exists a constant C > 0 such that  $S_1 \prec C\nu_q$ . It follows that

$$||S_1 P_q^n||_1 \le C ||\nu_q P_q^n||_1 = C ||\nu_q||_1 \lambda_q^n.$$

On the other hand, using the irreductibility property of the matrix  $P_q$ , we can find an integer k such that the matrix  $I + P_q + \cdots + P_q^k$  has strictly positive entries. It follows that the line vector  $S_1 + S_1P_q + \cdots + S_1P_q^k$  has strictly positive entries and we can find a constant C > 0 such that

$$\nu_q \prec C \left( S_1 + S_1 P_q + \dots + S_1 P_q^k \right).$$

So

$$\begin{aligned} \|\nu_q P_q^n\|_1 &\leq C \|S_1 P_q^n + S_1 P_q^{n+1} + \dots + S_1 P_q^{n+k}\|_1 \\ &= C \| \left(S_1 P_q^n\right) \left(I + P_q + \dots + P_q^k\right) \|_1 \\ &\leq C' \|S_1 P_q^n\|_1. \end{aligned}$$

Finally,  $||S_1P_q^n||_1 \simeq \lambda_q^n$ . Taking the logarithm, we can conclude that  $\tau(q) = \log_\ell(\lambda_q)$  and that the limit exists.

## **Corollary 3.3.** The function $\tau$ is analytic on $\mathbb{R}$ .

*Proof.* This can be seen as a consequence of the Kato-Rellich theorem (see for example [10]). But in this finite dimensional context, there is an elementary proof. Let  $F(q, x) = \det(P_q - xI)$  be the characteristic polynomial of  $P_q$  and let  $q_0 \in \mathbb{R}$ . Observing that  $F(q_0, \lambda_{q_0}) = 0$  and  $\frac{\partial F}{\partial x}(q_0, \lambda_{q_0}) \neq 0$  (the eigenvalue  $\lambda_{q_0}$  is simple), the map  $q \mapsto \lambda_q$  is given arround  $q_0$  by the implicit functions theorem. Moreover, F being analytic in q and x, it is well known that the implicit function is analytic.  $\Box$ 

The existence of  $\tau'(1)$  ensures that the measure m is unidimensional (see e.g. [8], Theorem 3.1).

**Corollary 3.4.** The measure m is unidimensional with dimension dim(m) =  $-\tau'(1)$ .

Let us now describe some examples. Let  $h_{\ell}$  be the usual entropy function

$$h_{\ell}(p) = -\sum_{i=0}^{\ell-1} p_i \log_{\ell} p_i, \qquad p = (p_0, \dots, p_{\ell-1}) \text{ with } \sum_{i=0}^{\ell-1} p_i = 1.$$

In particular, set  $h(x) = h_2(x, 1-x) = -(x \log_2(x) + (1-x) \log_2(1-x)).$ 

**Example 3.5.** If m is the Bernoulli measure from Example 3.1 (2), then by Theorem 3.2 we get the well known formula for  $\tau$ 

$$\tau(q) = \log_{\ell} \left( p_0^q + \dots + p_{\ell-1}^q \right)$$

Furthermore, the dimension of the measure m is  $\dim(m) = -\tau'(1) = h_{\ell}(p)$ .

**Example 3.6.** Actually, if  $\ell = 2$ , we can obtain an explicit formula for any given Markov chain. Suppose that  $a, b \in (0, 1)$  and

$$P = \left(\begin{array}{cc} 1-a & a \\ b & 1-b \end{array}\right).$$

Then

(3.5) 
$$\dim(\mathbf{m}) = \frac{b}{a+b}h(a) + \frac{a}{a+b}h(b).$$

Indeed, by Theorem 3.2 we get

$$\tau(q) = -1 + \log_2 \left( (1-a)^q + (1-b)^q + \sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q} \right).$$

Note that if q = 1, then  $\sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q}$  simplifies to a + b. Thus, we obtain

$$\tau'(1) = \frac{1}{2\log 2} \left( (1-a)\log(1-a) + (1-b)\log(1-b) + \frac{(b-a)((1-a)\log(1-a) - (1-b)\log(1-b)) + 2ab\log ab}{a+b} \right)$$

Rearranging, we get (3.5). Note that the coefficients in (3.5) come from the stationary distribution of X:

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right).$$

This observation will be generalized in Theorem 3.8 below.

**Example 3.7.** Let  $(a_0, \dots, a_{\ell-1})$  be a probability vector (with possibly some entries that are equal to 0) and P be an  $\ell \times \ell$  irreductible stochastic matrix with entries  $a_0, \dots, a_{\ell-1}$  in every row, but in an arbitrary order. Then  $\tau(q) = \log_{\ell} \left(a_0^q + \dots + a_{\ell-1}^q\right)$  and dim(m) =  $h_{\ell} (a_0, \dots, a_{\ell-1})$ . In particular, if  $\kappa$  is the number of  $a_i$ 's that are not equal to 0 and if each nonzero  $a_i$  is equal to  $1/\kappa$ , we get dim(m) =  $\frac{\log \kappa}{\log \ell}$  which is the maximal possible value and is also the dimension of the support of the measure m. Such a remark will be generalized below.

*Proof.* Set  $A_q = a_0^q + \cdots + a_{\ell-1}^q$ . We have

$$\sum_{\varepsilon \in \mathcal{W}_{n+1}} \mathbf{m}(I_{\varepsilon})^{q} = \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} \sum_{j \in \mathcal{S}} p_{kj}^{q} \mathbf{m}(I_{\varepsilon})^{q}$$
$$= A_{q} \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} \mathbf{m}(I_{\varepsilon})^{q}$$
$$= A_{q} \sum_{\varepsilon \in \mathcal{W}_{n}} \mathbf{m}(I_{\varepsilon})^{q}.$$

Iterating, we get

$$\sum_{\varepsilon \in \mathcal{W}_n} \mathbf{m}(I_{\varepsilon})^q = A_q^{n-1} \sum_{\varepsilon \in \mathcal{W}_1} \mathbf{m}(I_{\varepsilon})^q.$$

Consequently,

$$\tau(q) = \log_{\ell} \left( a_0^q + \ldots + a_{\ell-1}^q \right).$$

It follows that  $\tau'(1) = -h_{\ell}(a_0, \ldots, a_{\ell-1}).$ 

If there are only  $\kappa$  nonzero values in the probability vector  $(a_0, \dots, a_{\ell-1})$ , say e.g.  $a_0, \dots, a_{\kappa-1}$ , the formula turns to dim  $\mathbf{m} = -\sum_{j=0}^{\kappa-1} a_j \log_{\ell} a_j$  which is maximal when  $a_j = 1/\kappa$  for any j.

Let us finish this part with a general formula for the dimension of the measure m. That is the purpose of the following theorem.

**Theorem 3.8.** Denote by  $L_k$  the  $k^{th}$  line of the matrix P. Let  $H = (h_\ell(L_0), \ldots, h_\ell(L_{\ell-1}))$  be the vector of entropies of the lines of P. Then

$$\dim(\mathbf{m}) = <\pi \,|\, H>,$$

where  $\pi$  is the stationary distribution of the Markov chain X and  $\langle | \rangle$  is the canonical scalar product.

*Proof.* Let  $T_n = \sum_{I \in \mathcal{F}_n} m(I) \log_{\ell} m(I)$ , and  $H_n = \frac{-1}{n} T_n$  the entropy related to the partition  $\mathcal{F}_n$ . We know that  $\tau'(1)$  exists. It follows that  $-\tau'(1) = \lim_{n \to +\infty} H_n$  (see for example [8], Theorem 3.1). Further,

$$T_n = \sum_{k \in S} \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} \sum_{j \in S} \mathbf{m}(I_{\varepsilon}) p_{kj} \log_{\ell}(\mathbf{m}(I_{\varepsilon}) p_{kj})$$
  
=  $-\sum_{k \in S} h_{\ell}(L_k) \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} \mathbf{m}(I_{\varepsilon}) + \sum_{k \in S} \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} \mathbf{m}(I_{\varepsilon}) \log_{\ell}(\mathbf{m}(I_{\varepsilon}))$ 

Denote, as before,  $s_{n-1,k} = \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} m(I_{\varepsilon})$ . It follows that

$$T_n = -\sum_{k \in \mathcal{S}} h_\ell(L_k) s_{n-1,k} + T_{n-1}.$$

Iterating, we get

$$\tau'(1) = \lim_{n \to +\infty} \sum_{k \in S} h_{\ell}(L_k) \frac{1}{n} (s_{n-1,k} + s_{n-2,k} + \dots + s_{1,k})$$

Observe now that  $s_{n-1,k}$  is the  $k^{\text{th}}$  component of  $S_{n-1} = S_1 P^{n-2}$  (see (3.4)). It is well known that the Cesaro means converge to the stationary distribution  $\pi$  (even in the periodic case). The theorem follows.

Example 3.9. Let

$$P = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

An analytic formula for  $\tau(q)$  is complicated. Nevertheless, a numerical evaluation of  $\tau(0)$  (which is also the box dimension of the support of m) is possible and Theorem 3.8 allows us to estimate the dimension of the measure m. We find

dim m 
$$\approx 0.58$$
 and dim<sub>B</sub>(supp m)  $\approx 0.60$ .

This example shows that the "uniform" transition densities does not need to imply the maximality of dimm. Indeed, we will see in Corollary 4.2 that there always exists a choice of the transition matrix for which the dimension of the measure coincides with the dimension of its support.

#### 4. Multifractal analysis and dimension of the support

By the definition of  $\tau$  we have

$$\tau(0) = \limsup_{n \to \infty} \frac{\log N_n}{n \log \ell},$$

where  $N_n$  is the number of intervals from  $\mathcal{F}_n$  having positive measure. In our context the limit exists, so  $\tau(0) = \dim_B(\operatorname{supp} m)$ . Observe that the support of the measure m doesn't depend on the specific values of  $p_{ij}$  but only on the configuration of the nonzero entries in the matrix  $P_0$  and in the initial distribution  $p = (p_0, \cdots, p_{\ell-1})$ . More precisely, the support of the measure m is the compact set

$$K = \bigcap_{n \ge 1} \bigcup_{p_{\varepsilon_1} p_{\varepsilon_1} \varepsilon_2 \cdots p_{\varepsilon_{n-1} \varepsilon_n} > 0} \overline{I_{\varepsilon_1 \cdots \varepsilon_n}}.$$

Indeed, the construction of the support can be viewed as a Cantor-like removal process. Given an interval  $I_{\varepsilon_1\cdots\varepsilon_n}$  of the  $n^{\text{th}}$  generation with  $\varepsilon_n = i$ , its  $j^{\text{th}}$  subinterval will be removed if and only if  $p_{ij} = 0$  (cf. Example 3.1 (1)).

According to Theorem 3.2,  $\overline{\dim}_B(\operatorname{supp} m) = \log_\ell \lambda_0$  where  $\lambda_0$  is the spectral radius of the matrix  $P_0$ , so that the box dimension of supp m does not depend on the initial distribution p and only depends on the configuration of the nonzero entries of the matrix  $P_0$ .

This motivates the following questions. Given a configuration of nonzero entries of  $P_0$  and of the initial distribution p, which values of  $p_{ij}$  maximize dim m? Is the maximal value of dim m equal to the box dimension of the support ? Is this maximal measure unique?

In some cases one has an immediate answer. In particular, Example 3.7 says that if each row of the matrix P has the same number  $\kappa$  of nonzero entries, the maximum of dim m is obtained when each nonzero entry of the matrix P is equal to  $1/\kappa$  and is then equal to  $\frac{\log \kappa}{\log \ell}$ .

The general answer will be a consequence of the following result which says that the measure m satisfies the multifractal formalism.

**Theorem 4.1.** Let X be an irrductible Markov chain with transition matrix P and let m be the associated measure. Then m satisfies the multifractal formalism. More precisely, define

$$E_{\alpha} = \left\{ x \in [0,1] ; \lim_{n \to \infty} \frac{\log \operatorname{m}(I_n(x))}{\log |I_n(x)|} = \alpha \right\} .$$

Then, for any  $-\tau'(+\infty) < \alpha < -\tau'(\infty)$ ,

$$\dim(E_{\alpha}) = \tau^*(\alpha),$$

where  $\tau^*(\alpha) = \inf_q(\alpha q + \tau(q))$  is the Legendre transform of the function  $\tau$ .

*Proof.* We will prove the existence of a Gibbs measure at a given state q, that is an auxiliary measure  $m_q$  such that for any  $\ell$ -adic interval I one has

(4.1) 
$$\mathbf{m}_q(I) \asymp |I|^{\tau(q)} \, \mathbf{m}(I)^q.$$

Note that in (4.1), the constant may depend on q. Since the function  $\tau$  is differentiable, it is well known that the existence of such a measure at each state q implies the validity of the multifractal formalism for m (see for example [2] or [8]).

Now again, such a Gibbs measure will be obtained with an elementary construction. Note that P is irreductible if and only if  $P_q$  is. By Perron-Frobenius Theorem, the spectral radius  $\lambda_q$  of  $P_q$  is a simple eigenvalue and there exists a unique probability vector  $\pi_q$  with strictly positive entries and satisfying  $P_q \pi_q = \lambda_q \pi_q$ . Define  $D_q$  as the  $\ell \times \ell$ -matrix having the coordinates of  $\pi_q$  on the diagonal

Define  $D_q$  as the  $\ell \times \ell$ -matrix having the coordinates of  $\pi_q$  on the diagonal and zeros elsewhere. Set  $Q_q = \frac{1}{\lambda_q} D_q^{-1} P_q D_q$  and let **1** be the column vector  $\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{R}^{\ell}$ . Then we have

$$Q_q \mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q D_q \mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q \pi_q = D_q^{-1} \pi_q = \mathbf{1}$$

In other words,  $Q_q$  is a stochastic matrix and thus it can be associated to a Markov chain  $X^{(q)}$ . We may and do assume that this chain has the initial distribution  $q_i = \alpha p_i^q$ ,  $i \in S$ , where  $(p_0, \ldots, p_{\ell-1})$  is the initial distribution of X and  $\alpha = (p_0^q + \cdots + p_{\ell-1}^q)^{-1}$  is the normalization constant. Let  $m_q$  be the measure induced by  $X^{(q)}$ . Remark that if  $Q_q = (q_{ij}), D_q = (d_{ij})$ , then we have

$$q_{ij} = \frac{1}{\lambda_q} d_{ii}^{-1} p_{ij}^q d_{jj}, \qquad i, j \in \mathcal{S}$$

Cleraly,  $q_{ij} > 0$  if and only if  $p_{ij} > 0$  and  $q_i > 0$  if and only if  $p_i > 0$ . Therefore the measures  $m_q$  and m have the same support.

Let  $I = I_{\varepsilon_1 \cdots \varepsilon_n} \in \mathcal{F}_n$ . We have

$$m_q(I) = q_{\varepsilon_1} q_{\varepsilon_1 \varepsilon_2} \cdots q_{\varepsilon_{n-1} \varepsilon_n}$$

$$= \alpha p_{\varepsilon_1}^q \left( \frac{1}{\lambda_q} d_{\varepsilon_1 \varepsilon_1}^{-1} p_{\varepsilon_1 \varepsilon_2}^q d_{\varepsilon_2 \varepsilon_2} \right) \cdots \left( \frac{1}{\lambda_q} d_{\varepsilon_{n-1} \varepsilon_{n-1}}^{-1} p_{\varepsilon_{n-1} \varepsilon_n}^q d_{\varepsilon_n \varepsilon_n} \right)$$

$$= \frac{\alpha}{\lambda_q^{n-1}} d_{\varepsilon_1 \varepsilon_1}^{-1} m(I_{\varepsilon_1 \cdots \varepsilon_n})^q d_{\varepsilon_n \varepsilon_n}$$

Further, since the entries of the the eigenvector  $\pi_q$  are strictly positive, there is a constant c (possibly depending on q) such that

$$c^{-1} \le \frac{d_{ii}}{d_{jj}} \le c$$

for any  $i, j \in \mathcal{S}$ . This yields

$$m_q(I) \asymp \frac{1}{\lambda_q^{n-1}} m(I)^q.$$

Now, observe that by Theorem 3.2,  $\ell^{\tau(q)} = \lambda_q$  so that

$$|I|^{\tau(q)} = (\ell^{-n})^{\tau(q)} = \lambda_q^{-n}.$$

It follows that for any  $I \in \mathcal{F}_n$  we have

$$m_q(I) \asymp |I|^{\tau(q)} m(I)^q$$

Note that in the above estimate the implicit constants may depend on q but not on n and I. Therefore  $m_q$  is the needed Gibbs measure and the theorem follows.  $\Box$ 

**Corollary 4.2.** Let K be the support of the measure m. The measure  $\overline{\mathbf{m}} = \mathbf{m}_0$  associated to the matrix  $Q_0$  satisfies  $\operatorname{supp} \overline{\mathbf{m}} = K$ , is monofractal and strongly equivalent to the Hausdorff measure  $\mathcal{H}^{\tau(0)}$  on K. In particular,

$$\dim(\overline{\mathbf{m}}) = \dim_H(K) = \dim_B(K)$$

and  $\overline{\mathbf{m}}$  is a measure driven by a Markov chain, with support K and with maximal dimension.

*Proof.* Let  $\overline{\mathbf{m}} = \mathbf{m}_0$  from the previous proof. Then we have

$$\overline{\mathbf{m}}(I) \asymp |I|^{\tau(0)}$$

for any  $\ell$ -adic interval I such that m(I) > 0. In particular, for any  $x \in K$ ,  $\overline{m}(I_n(x)) \approx |I_n(x)|^{\tau(0)}$ . By Billingsley's theorem (see e.g. [4], Propositions 2.2 and 2.3), we conclude that  $\overline{m}$  is equivalent to  $\tau(0)$ -dimensional Hausdorff measure  $\mathcal{H}^{\tau(0)}$  on K. In particular  $\mathcal{H}^{\tau(0)}(K)$  is positive and finite. It follows that

$$\dim_H(K) = \tau(0) = \dim_B(K).$$

On the other hand, the structure function  $\overline{\tau}$  of the measure  $\overline{m}$  is  $\overline{\tau}(q) = \tau(0)(1-q)$ . It follows that the measure  $\overline{m}$  is monofractal, and that

$$\dim \overline{\mathbf{m}} = -\overline{\tau}'(1) = \tau(0) = \dim_B(K).$$

**Example 4.3.** Suppose that X is a random walk on  $\mathbb{Z}_{\ell}$  (cf. Example 3.1 (3)). Then  $\tau(q) = (1-q) \log_{\ell} 2$  and

$$\dim m = -\tau'(1) = \log_{\ell} 2 = \tau(0) = \dim_{B}(K).$$

Example 4.4. Let

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

It can be easily seen that P is irreducible (actually,  $P^2$  has only positive entries). We obtain

$$\tau(q) = -\log_3 2 + \log_3 \left( 2^{-q} + 3^{-q} + \sqrt{4^{-q} + 6^{1-q} + 9^{-q}} \right).$$

Hence

$$\dim_H(\text{supp m}) = \log_3(1 + \sqrt{2}) \approx 0.802 \text{ and } \dim m = \frac{1}{7}(3 + 4\log_3 2) \approx 0.789.$$

As observed in Example 3.9, the transitions are uniform row by row, but the measure m is not monofractal. The Markov chain inducing the maximal measure  $\overline{m}$  is associated to the following transition matrix :

$$Q_0 = (\sqrt{2} - 1) \times \begin{pmatrix} 1 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{pmatrix}.$$

The goal of this section is to discuss the uniqueness of measure  $\overline{\mathbf{m}}$  with maximal dimension given in Corollary 4.2. This is the object of Theorem 5.5.

We need to start with some preliminary results. Let us introduce the shift  $\sigma$  on [0,1) defined by  $\sigma(x) = \ell x - E(\ell x)$ , where E(y) is the integer part of y. Observe that  $\sigma(I_{\varepsilon_1\cdots\varepsilon_n}) = I_{\varepsilon_2\cdots\varepsilon_n}$  and that

$$\sigma(x) = \bigcap_{n} I_{\varepsilon_2 \cdots \varepsilon_n}$$
 if  $x = \bigcap_{n} I_{\varepsilon_1 \cdots \varepsilon_n}$ .

That is why  $\sigma$  is called the shift.

**Proposition 5.1.** Let P be an irreducible  $\ell \times \ell$  transition matrix,  $\nu$  be the (unique) probability vector such that  $\nu P = \nu$  and X be the Markov chain with transition P and initial law  $\nu$ . Set  $m_P$  to be the probability measure driven by the Markov chain X. Then,  $m_P$  is  $\sigma$ -invariant and ergodic.

Proof.

$$m_P(\sigma^{-1}(I_{\varepsilon_1\cdots\varepsilon_n})) = \sum_{j=0}^{\ell} m_P(I_{j\varepsilon_1\cdots\varepsilon_n})$$
$$= \sum_{j=0}^{\ell} \nu_j p_{j\varepsilon_1}\cdots p_{\varepsilon_{n-1}\varepsilon_n}$$
$$= \nu_{\varepsilon_1} p_{\varepsilon_1\varepsilon_2}\cdots p_{\varepsilon_{n-1}\varepsilon_n}$$
$$= m_P(I_{\varepsilon_1\cdots\varepsilon_n})$$

So, by the monotone class theorem, the measure  $m_P$  is  $\sigma$ -invariant.

Let k be an integer such that  $P + P^2 + \cdots + P^k$  has strictly positive entries. We claim that there exists a constant C > 0 such that for any  $I, J \in \bigcup_n \mathcal{F}_n$ , we have

(5.1) 
$$\frac{1}{C} \operatorname{m}_{P}(I) \operatorname{m}_{P}(J) \leq \sum_{j=0}^{k-1} \sum_{K \in \mathcal{F}_{j}} \operatorname{m}_{P}(IKJ) \leq C \operatorname{m}_{P}(I) \operatorname{m}_{P}(J).$$

Indeed, if  $I = I_{\varepsilon_1 \cdots \varepsilon_n}$ ,  $J = I_{\delta_1 \cdots \delta_m}$ , and if  $(\pi_{ij})_{i,j}$  denote the coefficients of the matrix  $P + P^2 + \cdots + P^k$ , it is easy to check that

$$\sum_{j=0}^{k-1} \sum_{K \in \mathcal{F}_j} \mathbf{m}_P(IKJ) = \nu_{\varepsilon_1} p_{\varepsilon_1 \varepsilon_2} \cdots p_{\varepsilon_{n-1} \varepsilon_n} \times \pi_{\varepsilon_n \delta_1} p_{\delta_1 \delta_2} \cdots p_{\delta_{m-1} \delta_m}$$

and the claim follows.

Inequality (5.1) can be rewritten as

(5.2) 
$$\sum_{j=0}^{k-1} \mathbf{m}_P \left( I \cap \sigma^{-(n+j)}(J) \right) \asymp \mathbf{m}_P(I) \times \mathbf{m}_P(J)$$

where n is the generation of I. If we observe that any open set is a countable union of disjoint intervals in  $\cup_n \mathcal{F}_n$ , inequality (5.2) remains true when J is an open set.

Finaly, by regularity of the measure  $m_p$ , it is also true for any Borel set J. In particular, if E is a  $\sigma$ -invariant Borel set, we get

$$\forall I \in \bigcup_{n} \mathcal{F}_{n}, \qquad k \operatorname{m}_{P}(I \cap E) \asymp \operatorname{m}_{P}(I) \times \operatorname{m}_{P}(E).$$

Again, it remains true when I is an arbitrary Borel set. In particular,

$$\mathbf{m}_P\left(([0,1]\setminus E)\cap E\right) \asymp \mathbf{m}_P\left([0,1]\setminus E\right) \times \mathbf{m}_P(E)$$

which proves that  $m_P(E) = 0$  or  $m_P([0,1] \setminus E) = 0$ .

*Remark* 5.2. Inequality 5.2 is a particular case of the so called weak quasi-Bernoulli property which was introduced by B. Testud in [11].

**Corollary 5.3.** Let P and  $\tilde{P}$  be two different irreducible  $\ell \times \ell$  transition matrices. Then  $m_P$  is singular with respect to  $m_{\tilde{P}}$ .

*Proof.* According to the ergodic theorem, it suffices to show that  $m_P \neq m_{\tilde{P}}$ . Let  $\nu$  and  $\tilde{\nu}$  be the invariant distributions of the stochastic matrix P and  $\tilde{P}$ . If  $\nu_i \neq \tilde{\nu}_i$  for some i, then  $m_P(I_i) \neq m_{\tilde{P}}(I_i)$ . If  $\nu = \tilde{\nu}$  and if  $p_{ij} \neq \tilde{p}_{ij}$ , we can write :

$$\mathbf{m}_P(I_{ij}) = \nu_i p_{ij} \neq \tilde{\nu}_i \tilde{p}_{ij} = \mathbf{m}_{\tilde{P}}(I_{ij}).$$

**Corollary 5.4.** Let P,  $\tilde{P}$  be two irreducible  $\ell \times \ell$  transition matrices, p and  $\tilde{p}$  to be two probability vectors. Set m and  $\tilde{m}$  be the associated measures. Suppose that  $supp(m) = supp(\tilde{m})$ . Then, there are only two possible cases :

- (1)  $P = \tilde{P}$  and the measures m and  $\tilde{m}$  are strongly equivalent (i.e.  $m \asymp \tilde{m}$ )
- (2)  $P \neq \tilde{P}$  and the measures m and m are mutually singular.

*Proof.* Let  $\mathcal{A} \subset \mathcal{S}$  be the set of ranges of the nonzero entries of p (which is also the set of ranges of nonzero entries of  $\tilde{p}$ ). Let  $F = \bigcup_{\varepsilon \in \mathcal{A}} \overline{I_{\varepsilon}}$ . We claim that m is strongly equivalent to the measure  $m_P$  restricted to F. This is an easy consequence of the fact that the invariant probability vector  $\nu$  (satisfying  $\nu P = \nu$ ) has strictly positive entries. In the same way,  $\tilde{m}$  is strongly equivalent to the measure  $m_{\tilde{P}}$  restricted to F. Corollary 5.4 is then a consequence of Corollary 5.3.

Now, we are able to prove the following theorem on the measure  $\overline{\mathbf{m}}$  given by Corollary 4.2.

**Theorem 5.5.** Let  $P_0 = (p_{ij}^0)$  be an irreductible  $\ell \times \ell$  matrix such that  $p_{ij}^0 \in \{0, 1\}$  for any ij and let  $p^0 = (p_0^0, \dots, p_{\ell-1}^0)$  be a line vector such that  $p_i^0 \in \{0, 1\}$  for any i. Suppose that  $p^0 \neq (0, \dots, 0)$  and define the compact set K by

$$K = \bigcap_{n \ge 1} \bigcup_{\substack{p_{\varepsilon_1}^0 p_{\varepsilon_1 \varepsilon_2}^0 \cdots p_{\varepsilon_{n-1} \varepsilon_n}^0 = 1}} \overline{I_{\varepsilon_1 \cdots \varepsilon_n}}.$$

Let  $\delta = \dim_H(K)$  and let m be a measure with support K, driven by a Markov chain X with irreductible transition matrix P. Then, dim  $m = \delta$  if and only if

$$P = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$$

where  $\lambda_0$  is the spectral radius of  $P_0$  and  $D_0$  is the diagonal matrix whose diagonal entries are the coordinates of the (unique) probability vector  $\pi_0$  satisfying  $P_0\pi_0 =$ 

 $\lambda_0 \pi_0$ . Moreover, the case  $P = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$  is the only case where the measure m is monofractal.

*Proof.* Assume for simplicity that  $p_i^0 = 1$  for any *i*. The general case is a standard modification. Remember that the support of the measure m only depends on the positions of the non-zero entries of the matrix *P*. It follows the non-zero entries of the matrices *P* and *P*<sub>0</sub> have located at the same places. Let  $Q_0 = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$ .

Suppose that  $P = Q_0$ . According to Corollary 4.2 and Corollary 5.4, the measure m is strongly equivalent to  $\overline{\mathbf{m}} = \mathbf{m}_0$  which satisfies  $\overline{\mathbf{m}}(I_n(x)) \simeq |I_n(x)|^{\delta}$  for any  $x \in K$ . In particular, m is such that dim  $\mathbf{m} = \delta$  and is monofractal.

Suppose now that  $P \neq Q_0$ . Corollary 5.4 says that m is singular with respect to  $m_0$  and we have to prove that dim  $m < \delta$ . Let

$$\tau(q) = \lim_{n \to +\infty} \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} \mathbf{m}(I)^q \right).$$

Recall that  $\tau$  is analytic and such that  $\tau(0) = \delta$  and  $\tau(1) = 0$ . Using the convexity of  $\tau$ , it is clear that

$$\tau'(1) = -\delta \Longleftrightarrow \forall q \in [0,1], \ \tau(q) = \delta(1-q) \Longleftrightarrow \forall q \in \mathbb{R}, \ \tau(q) = \delta(1-q).$$

In order to prove that dim  $m < \delta$ , it is then sufficient to establish that  $\tau(2) > -\delta$ . Denote by  $I_0, \dots, I_{\ell-1}$  the intervals of the first generation  $\mathcal{F}_1$ . If  $j \in \{0, \dots, \ell-1\}$ 

1} and  $n \ge 1$ , let  $\mathcal{F}_n(j)$  be the intervals of  $\mathcal{F}_n$  that are included in  $I_j$ . The measures m and  $m_0$  being mutually singular, we know that for dm-almost every  $x \in K$ ,

$$\lim_{n \to +\infty} \frac{\mathrm{m}_0(I_n(x))}{\mathrm{m}(I_n(x))} = 0,$$

which can be rewritten as

$$\lim_{n \to +\infty} \frac{\ell^{-n\delta}}{\mathrm{m}(I_n(x))} = 0.$$

Using Egoroff's theorem in each  $I_j$ , we can find a set  $A \subset K$  such that  $m(A \cap I_j) \ge \frac{1}{2}m(I_j)$  for any  $j \in \{0, \dots, \ell-1\}$  and satisfying

$$\forall \varepsilon > 0, \quad \exists n_0 \ge 1; \quad \forall n \ge n_0, \ \forall x \in A, \quad \mathrm{m}(I_n(x)) \ge \frac{1}{\varepsilon} \ell^{-n\delta}.$$

It follows that

$$\sum_{\in \mathcal{F}_{n_0}(j)} \mathbf{m}(J)^2 \geq \sum_{J \in \mathcal{F}_{n_0}(j) \ ; \ J \cap A \neq \emptyset} \frac{1}{\varepsilon} \ell^{-n_0 \delta} \, \mathbf{m}(J) \geq \frac{1}{2\varepsilon} \ell^{-n_0 \delta} \, \mathbf{m}(I_j).$$

Now, let  $I \in \mathcal{F}_k$  and suppose that  $I_{\varepsilon_1,\dots,\varepsilon_k}$  with  $\varepsilon_k = i$ . Observe that if  $J \in \mathcal{F}_n(j)$ , then  $m(IJ) = \frac{p_{ij}}{m(I_j)} m(I) m(J)$ . If we choose  $j_i \in \{0,\dots,\ell-1\}$  such that  $p_{ij_i} \neq 0$ , we get

$$\begin{split} \sum_{I \in \mathcal{F}_{n_0}} \mathbf{m}(IJ)^2 &\geq \sum_{J \in \mathcal{F}_{n_0}(j_i)} \mathbf{m}(IJ)^2 \\ &\geq \frac{p_{ij_i}^2}{\mathbf{m}(I_{j_i})^2} \mathbf{m}(I)^2 \sum_{J \in \mathcal{F}_{n_0}(j_i)} \mathbf{m}(J)^2 \\ &\geq \frac{p_{ij_i}^2}{2\varepsilon \mathbf{m}(I_{j_i})} \ell^{-n_0\delta} \mathbf{m}(I)^2. \end{split}$$

Let  $\varepsilon = \inf_i \left( \frac{p_{i_j}^2}{4 \operatorname{m}(I_{j_i})} \right)$  and the corresponding  $n_0$ . If  $\eta$  is such that  $\ell^{n_0 \eta} = 2$ , we can rewrite the last inequality as

$$\sum_{I \in \mathcal{F}_{n_0}} \mathbf{m}(IJ)^2 \ge \ell^{-n_0(\delta - \eta)} \mathbf{m}(I)^2.$$

If we sum this inequality on every interval I of the same generation and iterate the process, we get for any  $p\geq 1$ 

$$\sum_{I \in \mathcal{F}_{pn_0}} \mathbf{m}(I)^2 \ge \ell^{-(p-1)n_0(\delta-\eta)} \sum_{I \in \mathcal{F}_{n_0}} \mathbf{m}(I)^2 = C\ell^{-pn_0(\delta-\eta)}$$

which gives

$$\tau(2) \ge -(\delta - \eta) > -\delta.$$

Moreover, it is clear that  $\tau(\mathbb{R})$  is not reduced to a single point. If follows that the measure m is multifractal.

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