



# Problems in number theory from busy beaver competition

Pascal Michel

## ► To cite this version:

| Pascal Michel. Problems in number theory from busy beaver competition. 2013. hal-00879906

**HAL Id: hal-00879906**

**<https://hal.science/hal-00879906>**

Preprint submitted on 5 Nov 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Problems in number theory from busy beaver competition

Pascal MICHEL\*

Équipe de Logique Mathématique,  
Institut de Mathématiques de Jussieu – Paris Rive Gauche, UMR 7586,  
Bâtiment Sophie Germain, case 7012, 75205 Paris Cedex 13, France  
and Université de Cergy-Pontoise, IUFM, F-95000 Cergy-Pontoise, France  
michel@math.univ-paris-diderot.fr

November 5, 2013

## Abstract

By introducing the busy beaver competition of Turing machines, in 1962, Rado defined noncomputable functions on positive integers. The study of these functions and variants leads to many mathematical challenges. This article takes up the following one: How can a small Turing machine manage to produce very big numbers? It provides the following answer: mostly by simulating Collatz-like functions, that are generalizations of the famous  $3x+1$  function. These functions, like the  $3x+1$  function, lead to new unsolved problems in number theory.

*Keywords:* Turing machine, busy beaver, Collatz-like functions.

Mathematics Subject Classification (2000, 2010): *Primary* 03D10, *Secondary* 68Q05, 11B83.

## 1 Introduction

### 1.1 A well defined noncomputable function

It is easy to define a noncomputable function on nonnegative integers. Indeed, given a programming language, you produce a systematic list of the programs for functions, and, by diagonalization, you define a function whose output, on input  $n$ , is different from the output of the  $n$ th program. This simple definition raises many problems: Which programming language? How to list the programs? How to choose the output?

In 1962, Rado [38] gave a practical solution by defining the *busy beaver game*, also called now the *busy beaver competition*. Consider all Turing machines on one infinite tape, with  $n$

---

\*Corresponding address: 59 rue du Cardinal Lemoine, 75005 Paris, France.

states (plus a special halting state), and two symbols (1, and the blank symbol 0), and launch all of them on a blank tape. Define  $S(n)$  as the maximum number of computation steps made by such a machine before it stops, and define  $\Sigma(n)$  as the maximum number of symbols 1 left on the tape by a machine when it stops. Then functions  $S$  and  $\Sigma$  are noncomputable, and, moreover, grow faster than any computable function, that is, for any computable function  $f$ , there exists an integer  $N$  such that, for any  $n \geq N$ ,  $S(n) > \Sigma(n) > f(n)$ .

More than fifty years later, no better choice has been found for a practical noncomputable function. Only variants of Rado's definition have been proposed. So, in 1988, Brady [5] defined similar functions  $S(n, m)$  and  $\Sigma(n, m)$  for  $n \times m$  Turing machines, that is Turing machines with  $n$  states and  $m$  symbols. He also introduced analogous functions for two-dimensional Turing machines and "turNing machines", later resumed and expanded by Tim Hutton [13]. Bátfai [1, 2] relaxed the rule about head moving, by allowing the head to stand still.

In this article, we will consider functions  $S(n, m)$  and  $\Sigma(n, m)$ . Recall that  $S(n) = S(n, 2)$  and  $\Sigma(n) = \Sigma(n, 2)$ .

## 1.2 Computing the values of noncomputable functions

The busy beaver functions  $S$  and  $\Sigma$  are explicitly defined, and it is possible to compute  $S(n, m)$  and  $\Sigma(n, m)$  for small  $n$  and  $m$ . In the first article on busy beavers, Rado [38] gave  $\Sigma(2) = 2$  and  $\Sigma(3) \geq 6$ . These results show that two problems are at stake:

- **Problem 1:** To give lower bounds on  $S(n, m)$  and  $\Sigma(n, m)$  by finding Turing machines with high scores.
- **Problem 2:** To compute  $S(n, m)$  and  $\Sigma(n, m)$  by proving that no Turing machines do better than the best known ones.

Problem 1 can be tackled either by hand search, as did, for example, Green [11] and Lynn [26], or by computer search, using acceleration techniques of computation and, for example, simulated annealing, as did T. and S. Ligocki [21].

Solving Problem 2 requires more work to be done: clever enumeration of  $n \times m$  Turing machines, simulation of computation with acceleration techniques, proofs of non-halting for the machines that do not halt.

Currently, the following results are known (see Michel [32, 33] for a historical survey):

- $S(2) = 6$  and  $\Sigma(2) = 4$  (Rado [38]),
- $S(3) = 21$  and  $\Sigma(3) = 6$  (Lin and Rado [25]),
- $S(4) = 107$  and  $\Sigma(4) = 13$  (Brady [3, 4], Machlin and Stout [27]),
- $S(5) \geq 47, 176, 870$  and  $\Sigma(5) \geq 4098$  (Marxen and Buntrock [29]),
- $S(6) > 7.4 \times 10^{36534}$  and  $\Sigma(6) > 3.5 \times 10^{18267}$  (P. Kropitz in 2010),
- $S(2, 3) = 38$  and  $\Sigma(2, 3) = 9$  (Lafitte and Papazian [16]),
- $S(3, 3) > 1.1 \times 10^{17}$  and  $\Sigma(3, 3) \geq 347, 676, 383$  (T. and S. Ligocki in 2007),

- $S(4, 3) > 1.0 \times 10^{14072}$  and  $\Sigma(4, 3) > 1.3 \times 10^{7036}$  (T. and S. Ligocki in 2008),
- $S(2, 4) \geq 3,932,964$  and  $\Sigma(2, 4) \geq 2050$  (T. and S. Ligocki in 2005),
- $S(3, 4) > 5.2 \times 10^{13036}$  and  $\Sigma(3, 4) > 3.7 \times 10^{6518}$  (T. and S. Ligocki in 2007),
- $S(2, 5) > 1.9 \times 10^{704}$  and  $\Sigma(2, 5) > 1.7 \times 10^{352}$  (T. and S. Ligocki in 2007),
- $S(2, 6) > 2.4 \times 10^{9866}$  and  $\Sigma(2, 6) > 1.9 \times 10^{4933}$  (T. and S. Ligocki in 2008).

In order to achieve these results, many computational and mathematical challenges had to be taken up.

#### A. Computational challenges.

- A1.** To generate all  $n \times m$  Turing machines, or rather, to treat all cases without having to generate all  $n \times m$  Turing machines.
- A2.** To simulate the computation of a machine by using acceleration techniques (see Marxen and Buntrock [29], Marxen [28]).
- A3.** To gave automatic proofs that non-halting machines do not halt (see Brady [4], Marxen and Buntrock [29], Machlin and Stout [27], Hertel [12], Lafitte and Papazian [16]).

#### B. Mathematical challenges.

- B1.** To prove by hand that a non-halting machine that resists the computational proof does not halt.
- B2.** To understand how the Turing machines that reach high scores manage to do it.

### 1.3 Facing open problems in number theory

Let us come back to mathematical challenge *B1*. For example, the computational study of  $5 \times 2$  Turing machines by Marxen and Buntrock [29], Skelet [10] and Hertel [12] left holdouts that needed to be analyzed by hand. Marxen and Buntrock [29] gave an unsettled  $5 \times 2$  Turing machine, named #4, that turned out to never halt, by an intricate analysis.

Actually, the halting problem for Turing machines launched on a blank tape is m-complete, and this implies that this problem is as hard as the problem of the provability of a mathematical statement in a logical theory such as ZFC (Zermelo Fraenkel set theory with axiom of choice). So, when Turing machines with more and more states and symbols are studied, potentially all theorems of mathematics will be met. When more and more non-halting Turing machines are studied to be proved non-halting, one has to expect to face hard open problems in mathematics, that is problems that current mathematical knowledge can't settle.

Consider now mathematical challenge *B2*, which is the very subject of this article.

From 1983 to 1989, several  $5 \times 2$  Turing machines with high scores were discovered by Uwe Schult, by George Uhing, and by Heiner Marxen and Jürgen Buntrock. Michel [30, 31] analyzed some of these machines and found that their behavior is *Collatz-like*, which implies that the halting problems on general inputs for these machines are open problems in number theory (see Table 1).

From 2005 and 2007, many  $3 \times 3$ ,  $2 \times 4$  and  $2 \times 5$  Turing machines with high scores were discovered, mainly by two teams: the French one of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Collatz-like behavior of these champions seems to be the rule (see Tables 3, 4 and 5).

However, the behaviors of  $6 \times 2$  Turing machines display some variety. Many machines were discovered, from 1990 to 2010, by Heiner Marxen and Jürgen Buntrock, by Terry and Shawn Ligocki and by Pavel Kropitz. The analyses of some of these machines, by Robert Munafo, Clive Tooth, Shawn Ligocki and the author, show that the behaviors can be Collatz-like, exponential Collatz-like, loosely Collatz-like, or definitely not Collatz-like. Almost all of them raise open problems (see Table 2).

*Note:* The Turing machines listed in Tables 1–5 are those for which an analysis is known by the author. The machines without references for the study of behavior were analyzed by the author [32, 34]. Many other machines are waiting for their analyses.

## 1.4 Collatz functions, Collatz-like functions and other functions

The  $3x+1$  function, or *Collatz function*, is the function  $T$  on positive integers defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

This function is famous because, when it is iterated on a positive integer, it seems to lead to the loop  $2, 1, 2, 1, \dots$ . Is it always true? This is an open problem. See Lagarias [17, 18, 19, 20] for more information.

It is natural to generalize the definition of the  $3x+1$  function by replacing  $n$  even,  $n$  odd by  $n \equiv 0, \dots, d-1 \pmod{d}$ , and by replacing  $n/2$ ,  $(3n+1)/2$  by  $an+b$  for rational numbers  $a, b$ . Unfortunately, no name for such functions is currently taken for granted. Formal definitions were given by Rawsthorne [39], who proposed *Collatz-type iteration functions*, by Buttsworth and Matthews [6], who proposed *generalized Collatz mappings*, by Kaščák [14], who proposed *one-state linear operator algorithms (OLOA)*, and by Kohl [15], who proposed *residue-class-wise affine functions (RCWA)*. Without giving a formal definition, Lagarias [17] proposed *periodically linear functions*, and Wagon [41] proposed *Collatz-like functions*.

We will choose the following definitions.

**Definition 1.1** *A mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a generalized Collatz mapping if there exists an integer  $d \geq 2$  such that the following three equivalent conditions are satisfied:*

- (i) *(see [42, p.14]) There exist rational numbers  $q_0, \dots, q_{d-1}$ ,  $r_0, \dots, r_{d-1}$ , such that, for all  $i$ ,  $0 \leq i \leq d-1$ , we have  $q_i d \in \mathbb{Z}$ ,  $q_i i + r_i \in \mathbb{Z}$ , and, for all  $n \in \mathbb{Z}$ ,  $f(n) = q_i n + r_i$  if  $n \equiv i \pmod{d}$ .*
- (ii) *(see [6]) There exist integers  $m_0, \dots, m_{d-1}$ ,  $p_0, \dots, p_{d-1}$ , such that, for all  $i$ ,  $0 \leq i \leq d-1$ , we have  $p_i \equiv i m_i \pmod{d}$  and, for all  $n \in \mathbb{Z}$ ,  $f(n) = (m_i n - p_i)/d$  if  $n \equiv i \pmod{d}$ .*
- (iii) *There exist integers  $a_0, \dots, a_{d-1}$ ,  $b_0, \dots, b_{d-1}$ , such that we have, for all  $i$ ,  $0 \leq i \leq d-1$ , for all  $n \in \mathbb{Z}$ ,  $f(dn + i) = a_i n + b_i$ .*

These definitions are easily seen to be equivalent: we have  $a_i = m_i = q_i d$  and  $b_i = (im_i - p_i)/d = q_i i + r_i$ .

The definitions above concern total functions, but, in this article, we always deal with partial functions and functions with parameters, so we introduce the following definitions.

**Definition 1.2** *A partial function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a generalized Collatz function, or a pure Collatz-like function (without parameter) if, in the previous definition,  $f(dn + i)$  can be undefined for one or many  $i$ ,  $0 \leq i \leq d - 1$ .*

**Definition 1.3** *A partial function  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  is a pure Collatz-like function with parameter if there exist an integer  $d \geq 2$ , integers  $a_0, \dots, a_{d-1}$ ,  $b_0, \dots, b_{d-1}$ , a set  $S$  of integers and a function  $p : \{0, \dots, d-1\} \times S \rightarrow S$  such that, for all  $i$ ,  $0 \leq i \leq d-1$ , for all  $n \in \mathbb{Z}$ , for all  $s \in S$ ,  $f(dn + i, s) = (a_i n + b_i, p(i, s))$  or is undefined.*

**Definition 1.4** *If, in the definitions above,  $a_i = a$  for all  $i$ ,  $0 \leq i \leq d-1$ , we say that  $f$  is pure Collatz-like of type  $d \rightarrow a$ .*

We also need to define a new type of function, as follows.

**Definition 1.5** *A partial function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is an exponential Collatz-like function if there exist integers  $d, p \geq 2$ , integers  $a_0, \dots, a_{d-1}$ ,  $b_0, \dots, b_{d-1}$ ,  $c_0, \dots, c_{d-1}$ , such that, for all  $i$ ,  $0 \leq i \leq d-1$ , all  $n \in \mathbb{Z}$ ,  $f(dn + i) = (a_i p^n + b_i)/c_i$  or is undefined. In this definition, integers  $p$ ,  $a_i$ ,  $b_i$ ,  $c_i$  are chosen such that  $(a_i p^n + b_i)/c_i$  is an integer for all  $n \in \mathbb{Z}$ .*

Currently, no study of this type of function is known. Note that iterates  $f(n)$ ,  $f^2(n)$ ,  $\dots$  grow much faster than for pure Collatz-like functions.

## 1.5 From Collatz-like functions to high scores

The Turing machines studied in this article have behaviors modeled on iterations of functions, where halting configurations correspond to undefined values of functions.

In Section 3, we present a  $3 \times 3$  Turing machine  $M_1$  whose behavior is pure Collatz-like, of type  $8 \rightarrow 14$ . In Section 4, we present a  $2 \times 4$  Turing machine  $M_2$  whose behavior is pure Collatz-like with parameter, of type  $3 \rightarrow 5$ . In Section 5, we present a  $2 \times 5$  Turing machine  $M_3$  whose behavior is pure Collatz-like with parameter, of type  $2 \rightarrow 3$ . Thus, the halting problem for machines  $M_1$ ,  $M_2$  and  $M_3$  depends on an open problem about iterating Collatz-like functions.

In Section 6, we present a  $6 \times 2$  Turing machine  $M_4$  whose behavior is exponential Collatz-like.

In Section 7, we present a  $6 \times 2$  Turing machine  $M_5$  whose behavior depends on iterating a partial function  $g_5(n, p)$ . Without being Collatz-like, this function seems to share some properties with Collatz-like functions.

In Section 8, we present a  $6 \times 2$  Turing machine  $M_6$  whose behavior looks like a loosely Collatz-like behavior with parameter, of type  $2 \rightarrow 5$ . The novelty is that a potentially infinite set of rules seems to be necessary to describe the behavior of the machine on inputs  $00x$ ,  $x \in \{0, 1\}^*$ . A string  $x \in \{0, 1\}^*$  ending with symbol 1 can be taken as the binary writing of a number  $p$ , read in the opposite direction, so  $x = R(\text{bin}(p))$ , where  $\text{bin}(p)$  is the usual binary

writing of  $p$ , and  $R(w)$  is the reverse of string  $w$ , that is  $R(w_1 \dots w_n) = w_n \dots w_1$ . In Table 2 we write “ $R(\text{bin}(p))$ ” to indicate the machines with a behavior involving an infinite set of rules. Of course, only a finite subset of these rules are used when the machine is launched on a blank tape.

In Section 9, we present a  $6 \times 2$  Turing machine  $M_7$  whose behavior on the blank tape depends on configurations  $C(n)$  all of them provably leading to a halting configuration. We present such a machine to show how a Turing machine can take a long time to stop without calling for Collatz-like functions.

## 2 Preliminaries

A *Turing machine* involved in the busy beaver competition is defined as follows. It has a tape made of cells, infinite in both directions. Each cell contains a *symbol*, and a head can read and write a symbol on a cell. The Turing machine can be in a finite number of *states*. A computation of the Turing machine is a sequence of steps. In a step of computation, according to the current state and the symbol read by the head on the current cell, the head writes a symbol on the cell, moves to the next cell on the right side or on the left side, and the machine enters a new state.

Formally, a Turing machine  $M = (Q, \Gamma, \delta)$  has a finite set of states  $Q = \{A, B, \dots\}$ , a finite set of symbols  $\Gamma = \{0, 1, \dots\}$ , and a *transition function* (or *next move function*)  $\delta$ , which is a mapping

$$\delta : Q \times \Gamma \rightarrow (\Gamma \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$

If  $\delta(q, a) = (b, D, q')$ , then the Turing machine, when it is in state  $q$  reading symbol  $a$  on the current cell, writes symbol  $b$  instead of  $a$  on this cell, moves one cell left if  $D = L$ , one cell right if  $D = R$ , and comes in state  $q'$ . The transition function is usually given by a *transition table*.

There is a special state  $A$ , called the *initial state*, and a special symbol  $0$ , called the *blank symbol*. In the busy beaver competition, at the beginning of a computation, the Turing machine is in state  $A$ , and the tape is blank, that is all the cells of the tape contain the blank symbol. There is another state  $H$ , the *halting state*, not in the set  $Q$  of states. When a Turing machine comes in this state, the computation stops. We impose that, at the last step, the machine writes  $1$ , moves right, and comes in state  $H$ .

A *word* is a finite string of symbols. The set of words with symbols in the set  $\Gamma$  is denoted by  $\Gamma^*$ . The number of symbols in a word  $x \in \Gamma^*$  is called the *length* of  $x$  and is denoted by  $|x|$ . The *empty word* is the word of length zero, denoted by  $\lambda$ . If  $x \in \Gamma^*$ , and  $n \geq 0$ ,  $x^n$  is the word  $xx \dots x$ , where  $x$  is repeated  $n$  times, that is, formally:  $x^0 = \lambda$ ,  $x^1 = x$  and  $x^{n+1} = x^n x$ . An infinite to the left string of  $0$  is denoted by  ${}^\omega 0$ , and an infinite to the right string of  $0$  is denoted by  $0^\omega$ .

A *configuration* is a way to encode the symbols on the tape, the state, and the cell currently read by the head. The Turing machine is in configuration  ${}^\omega 0x(Sa)y0^\omega$ , with  $S \in Q \cup \{H\}$ ,  $a \in \Gamma$ ,  $x, y \in \Gamma^*$ , if the word  $xay$  is written on the tape, the state is  $S$ , and the head is reading symbol  $a$ . Since, at the beginning of the computation, the state is  $A$  and the tape is blank, the initial configuration is  ${}^\omega 0(A0)0^\omega$ . If the state is  $H$ , the configuration is halting. We also consider configurations  $x(Sa)y$  with finite length. If the computation from

| Machine   | Behavior   | Study of behavior                         |
|---|--|---|
| January 1983<br>Uwe Schult<br>$\sigma = 501$<br>$s = 134,467$             | Pure Collatz-like ( $4 \rightarrow 9$ )<br>without parameter<br>7 rules<br>7 transitions   | Robinson,<br>cited in [9].<br>Michel [30] |
| December 1984<br>George Uhing<br>$\sigma = 1915$<br>$s = 2,133,492$       | Pure Collatz-like ( $3 \rightarrow 8$ )<br>with parameter<br>5 rules<br>9 transitions      | Michel [30]                               |
| February 1986<br>George Uhing<br>$\sigma = 1471$<br>$s = 2,358,064$       | Pure Collatz-like ( $8 \rightarrow 15$ )<br>without parameter<br>5 rules<br>11 transitions |   |
| August 1989<br>Marxen, Buntrock<br>$\sigma = 4098$<br>$s = 11,798,826$    | Pure Collatz-like ( $3 \rightarrow 5$ )<br>without parameter<br>3 rules<br>15 transitions  | Michel [30]                               |
| September 1989<br>Marxen, Buntrock<br>$\sigma = 4097$<br>$s = 23,554,764$ | Pure Collatz-like ( $3 \rightarrow 5$ )<br>without parameter<br>3 rules<br>15 transitions  | Michel [30]                               |
| September 1989<br>Marxen, Buntrock<br>$\sigma = 4098$<br>$s = 47,176,870$ | Pure Collatz-like ( $3 \rightarrow 5$ )<br>without parameter<br>3 rules<br>15 transitions  | Michel [31]                               |

Table 1: Study of behavior of  $5 \times 2$  machines

| Machine   | Behavior   | Study of behavior            |
|---|--|------------------------------|
| September 1997<br>Marxen and Buntrock<br>$s > 8.6 \times 10^{15}$               | Pure Collatz-like ( $4 \rightarrow 10$ )<br>without parameter<br>5 rules<br>21 transitions | Munafo [36]                  |
| October 2000<br>Marxen and Buntrock<br>(machine o)<br>$s > 6.1 \times 10^{119}$ | $R(\text{bin}(p))$ ( $2 \rightarrow 3$ )<br><br>9 rules<br>337 transitions                 |                              |
| October 2000<br>Marxen and Buntrock<br>(machine q)<br>$s > 6.1 \times 10^{925}$ | All $C(n)$ stop<br><br>4 rules<br>5 transitions  | Munafo [37]<br>Section 9     |
| March 2001<br>Marxen and Buntrock<br>$s > 3.0 \times 10^{1730}$                 | $R(\text{bin}(p))$ ( $2 \rightarrow 3$ )<br><br>20 rules<br>4911 transitions               | Tooth [40]                   |
| November 2007<br>T. and S. Ligocki<br>$s > 8.9 \times 10^{1762}$                | $R(\text{bin}(p))$ ( $2 \rightarrow 5$ )<br><br>12 rules<br>3346 transitions               | Section 8                    |
| December 2007<br>T. and S. Ligocki<br>$s > 2.5 \times 10^{2879}$                | $R(\text{bin}(p))$ ( $4 \rightarrow 6$ )<br><br>18 rules<br>11026 transitions              |                              |
| May 2010<br>Pavel Kropitz<br>$s > 3.8 \times 10^{21132}$                        | Unclassifiable<br><br>6 rules<br>22158 transitions   | S. Ligocki [24]<br>Section 7 |
| June 2010<br>Pavel Kropitz<br>$s > 7.4 \times 10^{36534}$                       | Exponential Collatz-like<br>without parameter<br>4 rules<br>5 transitions                  | Section 6                    |

Table 2: Study of behavior of  $6 \times 2$  machines

| Machine  | Behavior   | Study of behavior |
|--|--|-------------------|
| December 2004<br>Brady<br>$s = 92,649,163$                         | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>5 rules<br>11 transitions     |                   |
| July 2005<br>Souris<br>$\sigma = 36089$<br>$s = 310,341,163$       | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>5 rules<br>12 transitions     |                   |
| July 2005<br>Souris<br>$s = 544,884,219$                           | Pure Collatz-like ( $3 \rightarrow 7$ )<br>with parameter<br>7 rules<br>12 transitions     |                   |
| August 2005<br>Lafitte and Papazian<br>$s > 4.9 \times 10^9$       | Pure collatz-like ( $4 \rightarrow 7$ )<br>with parameter<br>8 rules<br>21 transitions     |                   |
| September 2005<br>Lafitte and Papazian<br>$s > 9.8 \times 10^{11}$ | Pure Collatz-like ( $4 \rightarrow 7$ )<br>with parameter<br>7 rules<br>24 transitions     |                   |
| April 2006<br>Lafitte and Papazian<br>$s > 4.1 \times 10^{12}$     | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>5 rules<br>16 transitions     |                   |
| August 2006<br>T. and S. Ligocki<br>$s > 4.3 \times 10^{15}$       | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>4 rules<br>20 transitions     | S. Ligocki [23]   |
| November 2007<br>T. and S. Ligocki<br>$s > 1.1 \times 10^{17}$     | Pure Collatz-like ( $8 \rightarrow 14$ )<br>without parameter<br>9 rules<br>34 transitions | Section 3         |

Table 3: Study of behavior of  $3 \times 3$  machines

| Machine   | Behavior   | Study of behavior |
|---|--|-------------------|
| 1988<br>Brady<br><br>$s = 7195$                           | Pure Collatz-like ( $3 \rightarrow 5$ )<br>with parameter<br>6 rules<br>7 transitions  |                   |
| February 2005<br>T. and S. Ligocki<br><br>$s = 3,932,964$ | Pure Collatz-like ( $3 \rightarrow 5$ )<br>with parameter<br>7 rules<br>14 transitions | Section 4         |

Table 4: Study of behavior of  $2 \times 4$  machines

| Machine   | Behavior  | Study of behavior |
|---|---|-------------------|
| October 2005<br>Lafitte, Papazian<br><br>$s > 9.1 \times 10^{11}$   | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>7 rules<br>15 transitions    |                   |
| December 2005<br>Lafitte, Papazian<br><br>$s > 9.2 \times 10^{11}$  | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>5 rules<br>14 transitions    |                   |
| May 2006<br>Lafitte, Papazian<br><br>$s > 3.7 \times 10^{12}$       | Pure Collatz-like ( $3 \rightarrow 4$ )<br>with parameter<br>7 rules<br>45 transitions    |                   |
| June 2006<br>Lafitte, Papazian<br><br>$s > 1.4 \times 10^{13}$      | Pure Collatz-like ( $2 \rightarrow 3$ )<br>with parameter<br>9 rules<br>36 transitions    |                   |
| August 2006<br>T. and S. Ligocki<br><br>$s > 7.0 \times 10^{21}$    | Pure Collatz-like ( $2 \rightarrow 5$ )<br>with parameter<br>9 rules<br>30 transitions    | S. Ligocki [22]   |
| November 2007<br>T. and S. Ligocki<br><br>$s > 1.9 \times 10^{704}$ | Pure Collatz-like ( $2 \rightarrow 3$ )<br>with parameter<br>17 rules<br>2002 transitions | Section 5         |

Table 5: Study of behavior of  $2 \times 5$  machines

| $M_1$ | 0   | 1   | 2   |
|-------|-----|-----|-----|
| $A$   | 1RB | 2LA | 1LC |
| $B$   | 0LA | 2RB | 1LB |
| $C$   | 1RH | 1RA | 1RC |

Table 6: Machine  $M_1$  discovered in November 2007 by T. and S. Ligocki

configuration  $C_1$  to configuration  $C_2$  takes  $t$  steps, we write  $C_1 \vdash (t) C_2$ , and  $t$  is said to be the *time* taken by the machine to go from  $C_1$  to  $C_2$ . If  $C_2$  is a halting configuration, we also write  $C_1 \vdash (t) \text{END}$ . We write  $C_1 \vdash ( ) C_2$  if the time is not specified. If  $C_1$  and  $C_2$  are configurations with finite length, then they refer to the same part of the tape. For example,  $(A0)0 \vdash (1) 1(B0)$  if  $\delta(A, 0) = (1, R, B)$ .

A Turing machine  $M$  computes a partial function  $f_M : \Gamma^* \rightarrow \Gamma^*$  as follows. let  $x = x_1 \dots x_n \in \Gamma^*$ . Then  $x$  becomes an input for  $M$  by considering the computation of  $M$  on initial configuration  ${}^\omega 0(Ax_1)x_2 \dots x_n 0^\omega$ . If  $M$  never stops on this configuration, then  $f_M(x)$  is undefined. If  $M$  stops, in configuration  ${}^\omega 0y(Ha)z0^\omega$ , with  $a \in \Gamma$ ,  $y, z \in \Gamma^*$ , then the output  $f_M(x)$  is defined from this configuration by a suitable convention. The *halting set* is  $\{x \in \Gamma^* : f_M(x) \text{ is defined}\}$ . The *halting problem* for machine  $M$  is the problem consisting in determining the halting set. Note that the Turing machines with two symbols 0 and 1 are powerful enough to compute any computable function, and their halting sets can be any computably enumerable (also called recursively enumerable) set.

A Turing machine with  $n$  states and  $m$  symbols is called a  $n \times m$  machine. The set of  $n \times m$  machines is denoted by  $\text{TM}(n, m)$ . With our definition of the transition function, there are  $(2nm + 1)^{nm}$  machines in the set  $\text{TM}(n, m)$ . In the *busy beaver competition*, for fixed numbers of states  $n$  and symbols  $m$ , all the  $(2nm + 1)^{nm}$  Turing machines in  $\text{TM}(n, m)$  are launched on the blank tape. Some of them never stop. Those which stop are called *busy beaver*. Each busy beaver takes some time to stop, and leaves some non-blank symbols on the tape, so busy beavers are involved in two competitions: to take the longest time before stopping, and to leave the greatest number of non-blank symbols on the tape when stopping. The time taken by Turing machine  $M$  to stop is denoted by  $s(M)$ , and the number of non-blank symbols left by  $M$  when it stops is denoted by  $\sigma(M)$ . The *busy beaver functions* are defined by

$$S(n, m) = \max\{s(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols}\}$$

$$\Sigma(n, m) = \max\{\sigma(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols}\}$$

Rado [38] initially defined functions  $S(n) = S(n, 2)$  and  $\Sigma(n) = \Sigma(n, 2)$  for Turing machines with  $n$  states and two symbols.

### 3 Pure Collatz-like behavior

Let  $M_1$  be the  $3 \times 3$  Turing machine defined by Table 6

We have  $s(M_1) = 119,112,334,170,342,540$  and  $\sigma(M_1) = 374,676,383$ .

This machine is the current champion for the busy beaver competition for  $3 \times 3$  machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on

November, 9th) that they enumerated all the  $3 \times 3$  machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine  $M_1$  to reach a halting configuration from a blank tape.

**Theorem 3.1** *Let  $C(n) = {}^\omega 0(A0)2^n 0^\omega$ . Then*

- (a)  ${}^\omega 0(A0)0^\omega \vdash (3) \quad C(1)$ ,
- and, for all  $k \geq 0$ ,
- (b)  $C(8k+1) \vdash (112k^2 + 116k + 13) \quad C(14k+3)$ ,
- (c)  $C(8k+2) \vdash (112k^2 + 144k + 38) \quad C(14k+7)$ ,
- (d)  $C(8k+3) \vdash (112k^2 + 172k + 54) \quad C(14k+8)$ ,
- (e)  $C(8k+4) \vdash (112k^2 + 200k + 74) \quad C(14k+9)$ ,
- (f)  $C(8k+5) \vdash (112k^2 + 228k + 97) \quad {}^\omega 01(H1)2^{14k+9}0^\omega$ ,
- (g)  $C(8k+6) \vdash (112k^2 + 256k + 139) \quad C(14k+14)$ ,
- (h)  $C(8k+7) \vdash (112k^2 + 284k + 169) \quad C(14k+15)$ ,
- (i)  $C(8k+8) \vdash (112k^2 + 312k + 203) \quad C(14k+16)$ .

**Proof.** A direct inspection of the transition table gives

- (1)  $0(A0)0 \vdash (3) \quad (A0)20$ ,
- (2)  $0^3(A0)2^5 \vdash (53) \quad (B1)1^8$ ,
- (3)  $0(A1) \vdash (1) \quad (A0)2$ ,
- (4)  $1(A1) \vdash (1) \quad (A1)2$ ,
- (5)  $02(A1) \vdash (3) \quad 1(H1)2$ ,
- (6)  $12(A1) \vdash (4) \quad (A1)22$ ,
- (7)  $22(A1) \vdash (8) \quad (A1)22$ ,
- (8)  $2(B1)0^2 \vdash (7) \quad 11(A1)0$ ,
- (9)  $(B1)1 \vdash (1) \quad 2(B1)$ ,
- (10)  $(B1)2 \vdash (1) \quad 2(B2)$ ,
- (11)  $0^3(B2) \vdash (14) \quad 1^3(B1)$ ,
- (12)  $1(B2) \vdash (1) \quad (B1)1$ ,
- (13)  $2(B2) \vdash (1) \quad (B2)1$ .

From this point,  $k$  will be an integer,  $k \geq 0$ .

Iterating, respectively, (4), (7), (9) and (13) gives

- (14)  $1^k(A1) \vdash (k) \quad (A1)2^k$ ,
- (15)  $2^{2k}(A1) \vdash (8k) \quad (A1)2^{2k}$ ,
- (16)  $(B1)1^k \vdash (k) \quad 2^k(B1)$ ,
- (17)  $2^k(B2) \vdash (k) \quad (B2)1^k$ .

Using consecutively (16), (10), (17) and (12), we get

- (18)  $1(B1)1^k 2 \vdash (2k+3) \quad (B1)1^{k+2}$ .

Using (16), (10), (17) and (11), we get

- (19)  $0^3(B1)1^k 2 \vdash (2k+16) \quad 1^3(B1)1^{k+1}$ .

Using (19) and three times (18), we get

- (20)  $0^3(B1)1^k 2^4 \vdash (8k+43) \quad (B1)1^{k+7}$ .

For any  $n \geq 0$ , by induction on  $k$ , using (20), we get

- (21)  $0^{3k}(B1)1^n 2^{4k} \vdash (28k^2 + (8n+15)k) \quad (B1)1^{7k+n}$ .

By taking  $n = 8$  in (21), we get

$$(22) \quad 0^{3k}(B1)1^8 2^{4k} \vdash (28k^2 + 79k) \quad (B1)1^{7k+8}.$$

Using (8), (14) and (15), we get

$$(23) \quad 2^{2k+1}(B1)0^2 \vdash (8k + 9) \quad (A1)2^{2k+2}0.$$

We are now ready to prove the results of the theorem.

Using (2), (22) and (16), we get

$$(24) \quad 0^{3k+3}(A0)2^{4k+5} \vdash (28k^2 + 86k + 61) \quad 2^{7k+8}(B1).$$

Using (24), (23) and (5) we get

$$(25) \quad 0^{6k+4}(A0)2^{8k+5}0^2 \vdash (112k^2 + 228k + 97) \quad 1(H1)2^{14k+9}0.$$

Using (24), (23) and (3) we get

$$0^{6k+7}(A0)2^{8k+9}0^2 \vdash (112k^2 + 340k + 241) \quad (A0)2^{14k+17}0,$$

and this result is still true for  $k = -1$ , so we have

$$(26) \quad 0^{6k+1}(A0)2^{8k+1}0^2 \vdash (112k^2 + 116k + 13) \quad (A0)2^{14k+3}0.$$

Using (2), (22), (19) and (16) we get

$$(27) \quad 0^{3k+6}(A0)2^{4k+6} \vdash (28k^2 + 100k + 94) \quad 1^3 2^{7k+9}(B1).$$

Using (27), (23), (14) and (3) we get

$$(28) \quad 0^{6k+7}(A0)2^{8k+6}0^2 \vdash (112k^2 + 256k + 139) \quad (A0)2^{14k+14}0.$$

Using (27), (23), (6), (14) and (3) we get

$$0^{6k+10}(A0)2^{8k+10}0^2 \vdash (112k^2 + 368k + 294) \quad (A0)2^{14k+21}0,$$

and this result is still true for  $k = -1$ , so we have

$$(29) \quad 0^{6k+4}(A0)2^{8k+2}0^2 \vdash (112k^2 + 144k + 38) \quad (A0)2^{14k+7}0.$$

Using (2), (22), (19), (18), (16), (8) and (14) we get

$$(30) \quad 0^{3k+6}(A0)2^{4k+7}0^2 \vdash (28k^2 + 114k + 126) \quad 1^2 2^{7k+10}(A1)220.$$

Using (30), (15), (14) and (3) we get

$$(31) \quad 0^{6k+7}(A0)2^{8k+7}0^2 \vdash (112k^2 + 284k + 169) \quad (A0)2^{14k+15}0.$$

Using (30), (15), (6), (4) and (3) we get

$$0^{6k+10}(A0)2^{8k+11}0^2 \vdash (112k^2 + 396k + 338) \quad (A0)2^{14k+22}0,$$

and this result is still true for  $k = -1$ , so we have

$$(32) \quad 0^{6k+4}(A0)2^{8k+3}0^2 \vdash (112k^2 + 172k + 54) \quad (A0)2^{14k+8}0.$$

Using (2), (22), (19), (18), (18), (16), (8) and (14) we get

$$(33) \quad 0^{3k+6}(A0)2^{4k+8}0^2 \vdash (28k^2 + 128k + 153) \quad 12^{7k+12}(A1)220.$$

Using (33), (15), (4) and (3) we get

$$(34) \quad 0^{6k+7}(A0)2^{8k+8}0^2 \vdash (112k^2 + 312k + 203) \quad (A0)2^{14k+16}0.$$

Using (33), (15), (6) and (3) we get

$$0^{6k+10}(A0)2^{8k+12}0^2 \vdash (112k^2 + 424k + 386) \quad (A0)2^{14k+23}0,$$

and this result is still true for  $k = -1$ , so we have

$$(35) \quad 0^{6k+4}(A0)2^{8k+4}0^2 \vdash (112k^2 + 200k + 74) \quad (A0)2^{14k+9}0.$$

The results (1), (26), (29), (32), (35), (25), (28), (31) and (34) give, respectively, the results (a)–(i) of the theorem.  $\square$

Using the rules of this theorem, we have, in 34 transitions,

$$\omega 0(A0)0^\omega \vdash (3) \quad C(1) \vdash (13) \quad C(3) \vdash ( ) \cdots \vdash ( ) \quad \omega 01(H1)2^{374,676,381}0^\omega.$$

|       |     |     |     |     |
|-------|-----|-----|-----|-----|
| $M_2$ | 0   | 1   | 2   | 3   |
| $A$   | 1RB | 2LA | 1RA | 1RA |
| $B$   | 1LB | 1LA | 3RB | 1RH |

Table 7: Machine  $M_2$  discovered in February 2005 by T. and S. Ligocki

Let  $g_1$  be the pure Collatz-like function defined by: for  $k \geq 0$ ,

$$\begin{aligned}
g_1(8k+1) &= 14k+3, \\
g_1(8k+2) &= 14k+7, \\
g_1(8k+3) &= 14k+8, \\
g_1(8k+4) &= 14k+9, \\
g_1(8k+5) &= \text{undefined}, \\
g_1(8k+6) &= 14k+14, \\
g_1(8k+7) &= 14k+15, \\
g_1(8k+8) &= 14k+16.
\end{aligned}$$

Then  $g_1^{33}(1)$  is undefined.

The theorem gives immediately the following proposition.

**Proposition 3.2** *The behavior of Turing machine  $M_1$ , on inputs  $02^n$ ,  $n \geq 1$ , depends on the behavior of iterated  $g_1^k(n)$ ,  $k \geq 1$ .*

Since the behavior of iterated  $g_1^k(n)$  is an open problem in mathematics, the halting problem for Turing machine  $M_1$  is so.

Let  $h_1(n) = \min\{k : g_1^k(n) \text{ is undefined}\}$ . We have seen that  $h_1(1) = 33$ . We also have  $h_1(144) = 41$ ,  $h_1(270) = 51$ .

## 4 Collatz-like with parameter: first example

Let  $M_2$  be the  $2 \times 4$  Turing machine defined by Table 7.

We have  $s(M_2) = 3,932,964$  and  $\sigma(M_2) = 2050$ .

This machine is the current champion for the busy beaver competition for  $2 \times 4$  machines. It was discovered in February 2005 by Terry and Shawn Ligocki, who wrote (email on February, 13th) that they found this machine using simulated annealing.

The following theorem gives the rules that enable Turing machine  $M_2$  to reach a halting configuration from a blank tape.

**Theorem 4.1** *Let*

$$C(n, 1) = {}^\omega 0(A0)2^n 10^\omega,$$

$$C(n, 2) = {}^\omega 0(A0)2^n 110^\omega.$$

*Then*

$$(a) \quad {}^\omega 0(A0)0^\omega \vdash (6) \quad C(1, 2),$$

*and, for all  $k \geq 0$ ,*

- (b)  $C(3k, 1) \vdash (15k^2 + 9k + 3) \quad C(5k + 1, 1),$
- (c)  $C(3k + 1, 1) \vdash (15k^2 + 24k + 13) \quad {}^\omega 013^{5k+2}1(H1)0^\omega,$
- (d)  $C(3k + 2, 1) \vdash (15k^2 + 29k + 17) \quad C(5k + 4, 2),$
- (e)  $C(3k, 2) \vdash (15k^2 + 11k + 3) \quad C(5k + 1, 2),$
- (f)  $C(3k + 1, 2) \vdash (15k^2 + 21k + 7) \quad C(5k + 3, 1),$
- (g)  $C(3k + 2, 2) \vdash (15k^2 + 36k + 23) \quad {}^\omega 013^{5k+4}1(H1)0^\omega.$

**Proof.** A direct inspection of the transition table gives

- (1)  $0^2(A0)0 \vdash (6) \quad (A0)211,$
- (2)  $1(A0)0 \vdash (3) \quad (A1)11,$
- (3)  $(A0)2 \vdash (1) \quad 1(B2),$
- (4)  $0(A1) \vdash (1) \quad (A0)2,$
- (5)  $1(A1) \vdash (1) \quad (A1)2,$
- (6)  $3(A1) \vdash (2) \quad 1(A2),$
- (7)  $(A2)0 \vdash (1) \quad 1(A0),$
- (8)  $(A2)1 \vdash (1) \quad 1(A1),$
- (9)  $(A2)2 \vdash (1) \quad 1(A2),$
- (10)  $(B2)0 \vdash (3) \quad 1(H1),$
- (11)  $(B2)1 \vdash (4) \quad (A1)2,$
- (12)  $(B2)2 \vdash (1) \quad 3(B2).$

Iterating, respectively, (5), (9) and (12) gives

- (13)  $1^k(A1) \vdash (k) \quad (A1)2^k,$
- (14)  $(A2)2^k \vdash (k) \quad 1^k(A2),$
- (15)  $(B2)2^k \vdash (k) \quad 3^k(B2).$

Using (2), (13) and (4) we get

$$01^{k+1}(A0)0 \vdash (k+4) \quad (A0)2^{k+1}11,$$

and this result is still true for  $k = -1$ , so we have

$$(16) \quad 01^k(A0)0 \vdash (k+3) \quad (A0)2^k11.$$

Using (2), (13), (6), (14), (8), (13) and (4) we get

$$(17) \quad 0131^{k+1}(A0)0 \vdash (3k+10) \quad (A0)2^{k+4}1.$$

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13) and (4) we get

$$(18) \quad 01331^{k+1}(A0)0 \vdash (5k+19) \quad (A0)2^{k+6}.$$

Using (3), (15) and (10) we get

$$(19) \quad (A0)2^{k+1}0 \vdash (k+4) \quad 13^k1(H1).$$

Using (3), (15), (11), (6), (9) and (7) we get

$$(20) \quad (A0)2^{k+2}10 \vdash (k+10) \quad 13^k1^3(A0).$$

Using (3), (15), (11), (6), (9), (8), (13), (6), (14) and (7) we get

$$(21) \quad (A0)2^{k+3}110 \vdash (k+20) \quad 13^k1^5(A0).$$

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13), (6), (14) and (7) we get

$$(22) \quad 3^31^{k+1}(A0)0^2 \vdash (6k+24) \quad 1^{k+6}(A0).$$

By induction on  $k$ , from (22), we get

$$(23) \quad 3^{3k}1^{n+1}(A0)0^{2k} \vdash (15k^2 + 6nk + 9k) \quad 1^{5k+n+1}(A0),$$

so we have, for  $n = 2$  and  $n = 4$

$$\begin{aligned}
(24) \quad & 3^{3k}1^3(A0)0^{2k} \vdash (15k^2 + 21k) \quad 1^{5k+3}(A0), \\
(25) \quad & 3^{3k}1^5(A0)0^{2k} \vdash (15k^2 + 33k) \quad 1^{5k+5}(A0).
\end{aligned}$$

We are now ready to prove the theorem.

Using (20), (24) and (17) we get

$$0(A0)2^{3k+3}10^{2k+2} \vdash (15k^2 + 39k + 27) \quad (A0)2^{5k+6}1,$$

and the result is still true for  $k = -1$ , so we have

$$(26) \quad 0(A0)2^{3k}10^{2k} \vdash (15k^2 + 9k + 3) \quad (A0)2^{5k+1}1.$$

Using (20), (24), (18) and (19) we get

$$0(A0)2^{3k+4}10^{2k+3} \vdash (15k^2 + 54k + 52) \quad 13^{5k+7}1(H1),$$

and the result is still true for  $k = -1$ , so we have

$$(27) \quad 0(A0)2^{3k+1}10^{2k+1} \vdash (15k^2 + 24k + 13) \quad 13^{5k+2}1(H1).$$

Using (20), (24) and (16) we get

$$(28) \quad 0(A0)2^{3k+2}10^{2k+2} \vdash (15k^2 + 29k + 17) \quad (A0)2^{5k+4}11.$$

Using (21), (25) and (16) we get

$$0(A0)2^{3k+3}110^{2k+2} \vdash (15k^2 + 41k + 29) \quad (A0)2^{5k+6}11,$$

and the result is still true for  $k = -1$ , so we have

$$(29) \quad 0(A0)2^{3k}110^{2k} \vdash (15k^2 + 11k + 3) \quad (A0)2^{5k+1}11.$$

Using (21), (25) and (17) we get

$$0(A0)2^{3k+4}110^{2k+2} \vdash (15k^2 + 51k + 43) \quad (A0)2^{5k+8}1,$$

and the result is still true for  $k = -1$ , so we have

$$(30) \quad 0(A0)2^{3k+1}110^{2k} \vdash (15k^2 + 21k + 7) \quad (A0)2^{5k+3}1.$$

Using (21), (25), (18) and (19) we get

$$0(A0)2^{3k+5}110^{2k+3} \vdash (15k^2 + 66k + 74) \quad 13^{5k+9}1(H1),$$

and the result is still true for  $k = -1$ , so we have

$$(31) \quad 0(A0)2^{3k+2}110^{2k+1} \vdash (15k^2 + 36k + 23) \quad 13^{5k+4}1(H1).$$

The theorem comes from results (1) and (26)–(31).  $\square$

Using the rules of this theorem, we have, in 14 transitions,

$$\omega 0(A0)0^\omega \vdash (6) \quad C(1, 2) \vdash (7) \quad C(3, 1) \vdash ( ) \cdots \vdash ( ) \quad \omega 013^{2047}1(H1)0^\omega.$$

Let  $g_2$  be the pure Collatz-like function with parameter defined by: for  $k \geq 0$ ,

$$\begin{aligned}
g_2(3k, 1) &= (5k + 1, 1), \\
g_2(3k + 1, 1) &\text{undefined}, \\
g_2(3k + 2, 1) &= (5k + 4, 2), \\
g_2(3k, 2) &= (5k + 1, 2), \\
g_2(3k + 1, 2) &= (5k + 3, 1), \\
g_2(3k + 2, 2) &\text{undefined}.
\end{aligned}$$

Then  $g_2^{13}(1, 2)$  is undefined.

The theorem gives immediately the following proposition.

**Proposition 4.2** *The behavior of Turing machine  $M_2$ , on inputs  $02^n1^i$ ,  $n \geq 1$ ,  $i \in \{1, 2\}$  depends on the behavior of iterated  $g_2^k(n, i)$ ,  $k \geq 1$ .*

|       |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|
| $M_3$ | 0   | 1   | 2   | 3   | 4   |
| $A$   | 1RB | 2LA | 1RA | 2LB | 2LA |
| $B$   | 0LA | 2RB | 3RB | 4RA | 1RH |

Table 8: Machine  $M_3$  discovered in November 2007 by T. and S. Ligocki

Since the behavior of iterated  $g_2^k(n, i)$  is an open problem in mathematics, the halting problem for Turing machine  $M_2$  is so.

Let  $h_2(n, i) = \min\{k : g_2^k(n, i) \text{ is undefined}\}$ . We have seen that  $h_2(1, 2) = 13$ . We also have  $h_2(137, 1) = 16$ ,  $h_2(210, 2) = 20$ .

## 5 Collatz-like with parameter: second example

Let  $M_3$  be the  $2 \times 5$  Turing machine defined by Table 8.

We have  $s(M_3) > 1.9 \times 10^{704}$  and  $\sigma(M_3) > 1.7 \times 10^{352}$ .

This machine is the current champion for the busy beaver competition for  $2 \times 5$  machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on November, 9th) that, as they did for  $3 \times 3$  machine  $M_1$ , they enumerated all the  $2 \times 5$  machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine  $M_3$  to reach a halting configuration from a blank tape.

**Theorem 5.1** *Let*

$$\begin{aligned}
C(n, 1) &= {}^\omega 013^n(B0)0^\omega, \\
C(n, 2) &= {}^\omega 023^n(B0)0^\omega, \\
C(n, 3) &= {}^\omega 03^n(B0)0^\omega, \\
C(n, 4) &= {}^\omega 04113^n(B0)0^\omega, \\
C(n, 5) &= {}^\omega 04123^n(B0)0^\omega, \\
C(n, 6) &= {}^\omega 0413^n(B0)0^\omega, \\
C(n, 7) &= {}^\omega 0423^n(B0)0^\omega, \\
C(n, 8) &= {}^\omega 043^n(B0)0^\omega.
\end{aligned}$$

*Then*

$$(a) \quad {}^\omega 0(A0)0^\omega \vdash (1) \quad C(0, 1),$$

*and, for all  $k \geq 0$ ,*

|     |                |                            |                                |
|-----|----------------|----------------------------|--------------------------------|
| (b) | $C(2k, 1)$     | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 1, 1),$                |
| (c) | $C(2k, 2)$     | $\vdash (3k^2 + 14k + 9)$  | $C(3k + 2, 1),$                |
| (d) | $C(2k, 3)$     | $\vdash (3k^2 + 8k + 2)$   | $C(3k, 1),$                    |
| (e) | $C(2k, 4)$     | $\vdash (3k^2 + 8k + 8)$   | $C(3k + 3, 1),$                |
| (f) | $C(2k, 5)$     | $\vdash (3k^2 + 14k + 13)$ | $C(3k + 4, 1),$                |
| (g) | $C(2k, 6)$     | $\vdash (3k^2 + 8k + 6)$   | $C(3k + 2, 1),$                |
| (h) | $C(2k, 7)$     | $\vdash (3k^2 + 14k + 11)$ | $C(3k + 3, 1),$                |
| (i) | $C(2k, 8)$     | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 1, 1),$                |
| (j) | $C(2k + 1, 1)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 1, 2),$                |
| (k) | $C(2k + 1, 2)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 2, 3),$                |
| (l) | $C(2k + 1, 3)$ | $\vdash (3k^2 + 8k + 22)$  | $C(3k + 1, 4),$                |
| (m) | $C(2k + 1, 4)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 1, 5),$                |
| (n) | $C(2k + 1, 5)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 2, 6),$                |
| (o) | $C(2k + 1, 6)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 1, 7),$                |
| (p) | $C(2k + 1, 7)$ | $\vdash (3k^2 + 8k + 4)$   | $C(3k + 2, 8),$                |
| (q) | $C(2k + 1, 8)$ | $\vdash (3k^2 + 5k + 3)$   | $\omega 01(H2)2^{3k}0^\omega.$ |

**Proof.** A direct inspection of the transition table gives

- (1)  $(A0)0 \vdash (1) \quad 1(B0),$
- (2)  $0(A0)0 \vdash (17) \quad 41(A0),$
- (3)  $(A0)2 \vdash (1) \quad 1(B2),$
- (4)  $0(A1) \vdash (1) \quad (A0)2,$
- (5)  $1(A1) \vdash (1) \quad (A1)2,$
- (6)  $4(A1) \vdash (1) \quad (A4)2,$
- (7)  $(A2)0^2 \vdash (2) \quad 1^2(B0),$
- (8)  $(A2)2 \vdash (1) \quad 1(A2),$
- (9)  $1(B0) \vdash (1) \quad (A1)0,$
- (10)  $3^2(B0)0 \vdash (5) \quad 41^2(B0),$
- (11)  $(B2)0 \vdash (1) \quad 3(B0),$
- (12)  $(B2)2 \vdash (1) \quad 3(B2),$
- (13)  $0(A4) \vdash (1) \quad (A0)2,$
- (14)  $1(A4) \vdash (1) \quad (A1)2,$
- (15)  $2(A4) \vdash (1) \quad (A2)2,$
- (16)  $0^23(A4) \vdash (3) \quad (A0)02^2,$
- (17)  $13(A4) \vdash (4) \quad 23(B2),$
- (18)  $23(A4) \vdash (4) \quad 33(B2),$
- (19)  $3^2(A4) \vdash (4) \quad 41(A2),$
- (20)  $43(A4) \vdash (3) \quad 1(H2)2,$
- (21)  $04(A4) \vdash (2) \quad (A0)2^2.$

Iterating, respectively, (5), (8) and (12) gives

- (22)  $1^k(A1) \vdash (k) \quad (A1)2^k,$
- (23)  $(A2)2^k \vdash (k) \quad 1^k(A2),$
- (24)  $(B2)2^k \vdash (k) \quad 3^k(B2).$

Using (9), (22), (6), (19), (23) and (7) we get

$$3^2 41^{k+1}(B0)0 \vdash (2k + 9) \quad 41^{k+4}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(25) \quad 3^2 41^k(B0)0 \vdash (2k+7) \quad 41^{k+3}(B0).$$

For any  $n \geq 0$ , by induction on  $k$ , using (25), we get

$$(26) \quad 3^{2k} 41^n(B0)0^k \vdash (3k^2 + (2n+4)k) \quad 41^{3k+n}(B0),$$

so we have, for  $n = 2$  in (26),

$$(27) \quad 3^{2k} 41^2(B0)0^k \vdash (3k^2 + 8k) \quad 41^{3k+2}(B0).$$

Using (10), (27), (9), (22) and (6), we get

$$(28) \quad 3^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 11k + 8) \quad (A4)2^{3k+2}0.$$

Using (3), (24) and (11) we get

$$(A0)2^{k+1}0 \vdash (k+2) \quad 13^{k+1}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(29) \quad (A0)2^k0 \vdash (k+1) \quad 13^k(B0).$$

Using (15), (23), (7), (9) and (22), we get

$$(30) \quad 2(A4)2^k0^2 \vdash (2k+7) \quad (A1)2^{k+2}0.$$

We are now ready to prove the theorem.

Using (28), (14), (4) and (29) we get

$$013^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \quad 13^{3k+4}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(31) \quad 013^{2k}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 13^{3k+1}(B0).$$

Using (28), (30), (4) and (29) we get

$$023^{2k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 26) \quad 13^{3k+5}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(32) \quad 023^{2k}(B0)0^{k+1} \vdash (3k^2 + 14k + 9) \quad 13^{3k+2}(B0).$$

Using (28), (13) and (29) we get

$$03^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 13) \quad 13^{3k+3}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(33) \quad 03^{2k}(B0)0^k \vdash (3k^2 + 8k + 2) \quad 13^{3k}(B0).$$

Using (28), (14), (5), (6), (13) and (29) we get

$$04113^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 19) \quad 13^{3k+6}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(34) \quad 04113^{2k}(B0)0^k \vdash (3k^2 + 8k + 8) \quad 13^{3k+3}(B0).$$

Using (28), (30), (5), (6), (13) and (29) we get

$$04123^{2k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 30) \quad 13^{3k+7}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(35) \quad 04123^{2k}(B0)0^{k+1} \vdash (3k^2 + 14k + 13) \quad 13^{3k+4}(B0).$$

Using (28), (14), (6), (13) and (29) we get

$$0413^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 17) \quad 13^{3k+5}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(36) \quad 0413^{2k}(B0)0^k \vdash (3k^2 + 8k + 6) \quad 13^{3k+2}(B0).$$

Using (28), (30), (6), (13) and (29) we get

$$0423^{2k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 28) \quad 13^{3k+6}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(37) \quad 0423^{2k}(B0)0^{k+1} \vdash (3k^2 + 14k + 11) \quad 13^{3k+3}(B0).$$

Using (28), (21) and (29) we get

$$043^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \quad 13^{3k+4}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(38) \quad 043^{2k}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 13^{3k+1}(B0).$$

Using (28), (17), (24) and (11) we get

$$13^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \quad 23^{3k+4}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(39) \quad 13^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 23^{3k+1}(B0).$$

Using (28), (18), (24) and (11) we get

$$23^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \quad 3^{3k+5}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(40) \quad 23^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 3^{3k+2}(B0).$$

Using (28), (16), (2) and (29) we get

$$0^3 3^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 33) \quad 41^2 3^{3k+4}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(41) \quad 0^3 3^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 22) \quad 41^2 3^{3k+1}(B0).$$

Using (39) we get

$$(42) \quad 04113^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 04123^{3k+1}(B0).$$

Using (40) we get

$$(43) \quad 04123^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 0413^{3k+2}(B0).$$

Using (39) we get

$$(44) \quad 0413^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 0423^{3k+1}(B0).$$

Using (40) we get

$$(45) \quad 0423^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 043^{3k+2}(B0).$$

Using (28) and (20) we get

$$43^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 11k + 11) \quad 1(H2)2^{3k+3}0,$$

and the result is still true for  $k = -1$ , so we have

$$(46) \quad 43^{2k+1}(B0)0^k \vdash (3k^2 + 5k + 3) \quad 1(H2)2^{3k}0.$$

Results (1) and (31)–(46) give results (a)–(p) of the theorem.  $\square$

Using the rules of this theorem, we have, in 2002 transitions,

$${}^\omega 0(A0)0^\omega \vdash (1) \quad C(0,1) \vdash (4) \quad C(1,1) \vdash ( ) \cdots \vdash ( ) \quad \text{END}.$$

Let  $g_3$  be the pure Collatz-like function with parameter defined by: for  $k \geq 0$ ,

$$\begin{array}{ll|ll} g_3(2k, 1) & = & (3k + 1, 1) & g_3(2k + 1, 1) & = & (3k + 1, 2) \\ g_3(2k, 2) & = & (3k + 2, 1) & g_3(2k + 1, 2) & = & (3k + 2, 3) \\ g_3(2k, 3) & = & (3k, 1) & g_3(2k + 1, 3) & = & (3k + 1, 4) \\ g_3(2k, 4) & = & (3k + 3, 1) & g_3(2k + 1, 4) & = & (3k + 1, 5) \\ g_3(2k, 5) & = & (3k + 4, 1) & g_3(2k + 1, 5) & = & (3k + 2, 6) \\ g_3(2k, 6) & = & (3k + 2, 1) & g_3(2k + 1, 6) & = & (3k + 1, 7) \\ g_3(2k, 7) & = & (3k + 3, 1) & g_3(2k + 1, 7) & = & (3k + 2, 8) \\ g_3(2k, 8) & = & (3k + 1, 1) & g_3(2k + 1, 8) & = & \text{undefined} \end{array}$$

Then  $g_3^{2001}(0, 1)$  is undefined.

| $M_4$ | 0   | 1   |
|-------|-----|-----|
| $A$   | 1RB | 1LE |
| $B$   | 1RC | 1RF |
| $C$   | 1LD | 0RB |
| $D$   | 1RE | 0LC |
| $E$   | 1LA | 0RD |
| $F$   | 1RH | 1RC |

Table 9: Machine  $M_4$  discovered in June 2010 by P. Kropitz

**Proposition 5.2** *The behavior of Turing machine  $M_3$ , on inputs  $02^n$ ,  $n \geq 1$ , depends on the behavior of iterated  $g_3^k(n, 1)$ ,  $k \geq 1$ .*

**Proof.** We have  ${}^\omega 0(A0)2^n 0^\omega \vdash (n+1) \quad {}^\omega 013^n(B0)0^\omega = C(n, 1)$ .  $\square$

Since the behavior of iterated  $g_3^k(n, 1)$  is an open problem in mathematics, the halting problem for Turing machine  $M_3$  is so.

Note that the way by which a high score is obtained is particularly clear for machine  $M_3$ . The parameter  $p$ ,  $1 \leq p \leq 8$ , can be seen as a state. If  $n$  is odd,  $g_3(n, p) = (n', p+1)$ , the state goes from  $p$  to  $p+1$ , and the computation stops when state 8 is reached. If  $n$  is even,  $g_3(n, p) = (n', 1)$ , and the state goes back to 1.

## 6 Exponential Collatz-like

Let  $M_4$  be the  $6 \times 2$  Turing machine defined by Table 9.

We have  $s(M_4) > 7.4 \times 10^{36534}$  and  $\sigma(M_4) > 3.5 \times 10^{18267}$ .

This machine is the current champion for the busy beaver competition for  $6 \times 2$  machines. It was discovered in June 2010 by Pavel Kropitz.

The following theorem gives the rules observed by Turing machine  $M_4$ .

**Theorem 6.1** *Let  $C(n) = {}^\omega 0(A0)1^n 0^\omega$ . Then*

- (a)  $C(0) \vdash (29) \quad C(9)$ ,
- (b)  $C(2) \vdash (36) \quad C(11)$ ,
- (c)  $C(3) \vdash (48) \quad C(13)$ ,

and, for all  $k \geq 0$ ,

- (d)  $C(3k+1) \vdash (3k+3) \quad {}^\omega 0111(011)^k(H0)0^\omega$ ,
- (e)  $C(9k+5) \vdash ((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 25362)/270) \quad C((98 \times 4^k - 11)/3)$ ,
- (f)  $C(9k+6) \vdash ((125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27) \quad C((50 \times 4^{k+1} - 59)/3)$ ,
- (g)  $C(9k+8) \vdash ((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270) \quad C((98 \times 4^k + 1)/3)$ ,
- (h)  $C(9k+9) \vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27) \quad C((50 \times 4^{k+1} - 11)/3)$ ,
- (i)  $C(9k+11) \vdash ((4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270) \quad C((98 \times 4^{k+1} - 59)/3)$ ,
- (j)  $C(9k+12) \vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27) \quad C((50 \times 4^{k+1} + 1)/3)$ .

*Note that the behavior of this Turing machine on the blank tape involves only items (a), (d), (h) and (j).*

**Proof.** A direct inspection of the transition table gives

- (1)  $0^3(A0)0^6 \vdash (29) \quad (A0)1^9,$
- (2)  $0^4(A0)1^20^5 \vdash (36) \quad (A0)1^{11},$
- (3)  $0^4(A0)1^30^6 \vdash (48) \quad (A0)1^{13},$
- (4)  $(A0)1 \vdash (1) \quad 1(B1),$
- (5)  $01(E0) \vdash (2) \quad (E0)11,$
- (6)  $0(E0) \vdash (1) \quad (A0)1,$
- (7)  $11(E0) \vdash (2) \quad (E1)11,$
- (8)  $01(C0) \vdash (2) \quad (C0)01,$
- (9)  $11(C0) \vdash (2) \quad (C1)01,$
- (10)  $0(C0) \vdash (2) \quad 1(E1),$
- (11)  $(E1)01 \vdash (2) \quad 01(E1),$
- (12)  $(E1)00 \vdash (2) \quad 01(E0),$
- (13)  $(E1)1 \vdash (2) \quad (C0)0,$
- (14)  $(B1)1^3 \vdash (3) \quad 110(B1),$
- (15)  $(B1)00 \vdash (2) \quad 11(H0),$
- (16)  $(B1)10 \vdash (6) \quad 01(C1),$
- (17)  $(B1)1100 \vdash (12) \quad (01)^2(C1),$
- (18)  $(C1)01 \vdash (2) \quad 01(C1),$
- (19)  $(C1)00 \vdash (2) \quad 01(C0),$
- (20)  $(C1)1^60^6 \vdash (44) \quad 01(E1)1^{10},$
- (21)  $(C1)1^80^{11} \vdash (113) \quad 1(01)^5(E1)1^8.$

Iterating, respectively, (5), (8), (11), (14) and (18) gives

- (22)  $(01)^k(E0) \vdash (2k) \quad (E0)1^{2k},$
- (23)  $(01)^k(C0) \vdash (2k) \quad (C0)(01)^k,$
- (24)  $(E1)(01)^k \vdash (2k) \quad (01)^k(E1),$
- (25)  $(B1)1^{3k} \vdash (3k) \quad (110)^k(B1),$
- (26)  $(C1)(01)^k \vdash (2k) \quad (01)^k(C1).$

Using (19), (23), (10) and (24), we get

- (27)  $0(01)^k(C1)00 \vdash (4k+8) \quad 1(01)^{k+1}(E1).$

Using (12), (22) and (6), we get

- (28)  $0(01)^k(E1)00 \vdash (2k+5) \quad (A0)1^{2k+3}.$

Using (12), (22) and (7), we get

- (29)  $11(01)^k(E1)00 \vdash (2k+6) \quad (E1)1^{2k+4}.$

Using (13), (23), (10) and (24), we get

- (30)  $0(01)^k(E1)1 \vdash (4k+4) \quad 1(01)^k(E1)0.$

Using (13), (23), (9) and (26), we get

- (31)  $11(01)^k(E1)1 \vdash (4k+6) \quad (01)^{k+1}(C1)0.$

Using (20), (30), (11), (31) and (18), we get

- (32)  $10(01)^k(C1)1^60^6 \vdash (8k+70) \quad (01)^{k+4}(C1)1^6.$

By induction on  $n$ , using (32), we get

- (33)  $(10)^n(01)^k(C1)1^60^{6n} \vdash (16n^2 + 8kn + 54n) \quad (01)^{4n+k}(C1)1^6.$

Using (30) and (11), we get

- (34)  $00(01)^k(E1)11 \vdash (4k+6) \quad (01)^{k+2}(E1).$

By induction on  $n$ , using (34), we get

$$(35) \quad 0^{2n}(01)^k(E1)1^{2n} \vdash (4n^2 + 4kn + 2n) \quad (01)^{2n+k}(E1).$$

Using (20), (31), (18), (21), (31), (18) and (33), we get

$$(36) \quad 11(01)^k(C1)1^6 0^{6k+29} \vdash (16k^2 + 178k + 481) \quad 0(01)^{4k+15}(C1)1^6.$$

Using (20), (35) and (28), we get

$$(37) \quad 0^{11}(01)^k(C1)1^6 0^8 \vdash (22k + 201) \quad (A0)1^{2k+25}.$$

Using (20), (35), (31), (18), (32) and (36), we get

$$(38) \quad (110)^3 0(01)^k(C1)1^6 0^{6k+101} \vdash (16k^2 + 514k + 4045) \quad 0(01)^{4k+55}(C1)1^6.$$

By induction on  $k$ , using (38) we get

$$(39) \quad (110)^{3k} 0(01)^n(C1)1^6 0^a \vdash (T) \quad 0(01)^b(C1)1^6,$$

with  $a = (2(3n+55)4^k - 6n - 27k - 110)/3$ ,  $b = ((3n+55)4^k - 55)/3$ , and  $T = \frac{16(3n+55)^2}{135}16^k - \frac{218(3n+55)}{27}4^k - \frac{5}{9}k - \frac{16(3n+55)^2}{135} + \frac{218(3n+55)}{27}$ .

Using (4), (25), (17), (27), (29), (31), (18), (21), (31), (18) and (33), we get

$$(40) \quad (A0)1^{3k+9} 0^{29} \vdash (3k + 484) \quad 1(110)^k 0(01)^{15}(C1)1^6.$$

Using (40), (39), (32) and (37), we get

$$(41) \quad 0^{11}(A0)1^{9k+9} 0^a \vdash (T) \quad (A0)1^b,$$

with  $a = (50 \times 4^{k+1} - 11)/3 - 9k - 20$ ,  $b = (50 \times 4^{k+1} - 11)/3$ , and  $T = (125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27$ .

Using (40), (39), (20), (35), (31), (18) and (37), we get

$$(42) \quad 0^{12}(A0)1^{9k+12} 0^a \vdash (T) \quad (A0)1^b,$$

with  $a = (50 \times 4^{k+1} + 1)/3 - 9k - 24$ ,  $b = (50 \times 4^{k+1} + 1)/3$ , and  $T = (125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27$ .

Using (4), (25) and (15), we get

$$(43) \quad (A0)1^{3k+1} 0^0 \vdash (3k + 3) \quad 111(011)^k(H0).$$

Results (1), (43), (41) and (42) are results (a), (d), (h) and (j) of the theorem. They are sufficient to analyze the behavior of the Turing machine on a blank tape. The following gives its behavior from configurations  $\omega 0(A0)1^n 0^\omega$ , where  $n = 9k + m$ ,  $m \in \{5, 6, 8, 11\}$ .

Using (4), (25), (16), (27), (29), (31), (18), (32) and (36), we get

$$(44) \quad (A0)1^{3k+14} 0^{88} \vdash (3k + 3076) \quad 1(110)^k 0(01)^{47}(C1)1^6.$$

Using (44), (39), (32) and (37), we get

$$0^{11}(A0)1^{9k+14} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^{k+1} - 11)/3$  and  $T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k+1) - 25362)/270$ , and this result is still true for  $k = -1$ , since

$$0^{11}(A0)1^5 0^{13} \vdash (285) \quad (A0)1^{29},$$

so we get

$$(45) \quad 0^{11}(A0)1^{9k+5} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^k - 11)/3$  and  $T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 25362)/270$ .

Using (20), (35), (31), (18), (32), (36) and (37), we get

$$(46) \quad 0^{10} 1(110)^2 0(01)^k(C1)1^6 0^{6k+103} \vdash (16k^2 + 570k + 4886) \quad (A0)1^{8k+127}.$$

Using (40), (39) and (46), we get

$$0^{10}(A0)1^{9k+15} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (50 \times 4^{k+2} - 59)/3$  and  $T = (125 \times 16^{k+3} - 575 \times 4^{k+3} + 228(k+1) - 2226)/27$ , and this result is still true for  $k = -1$ , since

$$0^{10}(A0)1^6 0^{31} \vdash (762) \quad (A0)1^{47},$$

so we get

$$(47) \quad 0^{10}(A0)1^{9k+6} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (50 \times 4^{k+1} - 59)/3$  and  $T = (125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27$ .

Using (20), (30), (11), (31), (18) and (37), we get

$$(48) \quad 0^{12}11100(01)^k(C1)1^6 0^{14} \vdash (30k + 407) \quad (A0)1^{2k+37}.$$

Using (44), (39) and (48), we get

$$0^{12}(A0)1^{9k+17} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^{k+1} + 1)/3$  and  $T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k+1) - 11592)/270$ , and this result is still true for  $k = -1$ , since

$$0^{12}(A0)1^8 0^{13} \vdash (336) \quad (A0)1^{33},$$

so we get

$$(49) \quad 0^{12}(A0)1^{9k+8} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^k + 1)/3$  and  $T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270$ .

Using (44), (39) and (46), we get

$$0^{10}(A0)1^{9k+20} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^{k+2} - 59)/3$  and  $T = (4802 \times 16^{k+3} - 11270 \times 4^{k+3} + 2280(k+1) - 22452)/270$ , and this result is still true for  $k = -1$ , since

$$0^{10}(A0)1^{11} 0^{90} \vdash (3802) \quad (A0)1^{111},$$

so we get

$$(50) \quad 0^{10}(A0)1^{9k+11} 0^a \vdash (T) \quad (A0)1^b,$$

with  $b = (98 \times 4^{k+1} - 59)/3$  and  $T = (4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270$ .

Results (45), (47), (49) and (50) are results (e), (f), (g) and (i) of the theorem.

Results (1), (2) and (3) give special cases (a), (b) and (c) of the theorem.  $\square$

Using the rules of this theorem, we have,

$$\omega 0(A0)0^\omega \vdash (29) \quad C(9) \vdash (1293) \quad C(63) \vdash (19, 884, 896, 677)$$

$$C(273063) \vdash (T_1) \quad C((50 \times 4^{30340} + 1)/3) \vdash (T_2) \quad \omega 0111(011)^K(H0)0^\omega,$$

with  $T_1 = (125 \times 16^{30341} + 325 \times 4^{30341} + 6916380)/27$ ,  $T_2 = (50 \times 4^{30340} + 7)/3$ ,  $K = (50 \times 4^{30340} - 2)/9$ .

The total time is  $s(M_4) = (125 \times 16^{30341} + 1750 \times 4^{30340} + 15)/27 + 19, 885, 154, 163$ , and the final number of symbols 1 is  $\sigma(M_4) = (25 \times 4^{30341} + 23)/9$ .

Let  $g_4$  be the exponential Collatz-like function defined by: for  $k \geq 0$ ,

| $M_5$ | 0   | 1   |
|-------|-----|-----|
| $A$   | 1RB | 0LD |
| $B$   | 1RC | 0RF |
| $C$   | 1LC | 1LA |
| $D$   | 0LE | 1RH |
| $E$   | 1LA | 0RB |
| $F$   | 0RC | 0RE |

Table 10: Machine  $M_5$  discovered in May 2010 by P. Kropitz

$$\begin{aligned}
g_4(0) &= 9, \\
g_4(2) &= 11, \\
g_4(3) &= 13, \\
g_4(3k+1) &\text{ undefined,} \\
g_4(9k+5) &= (98 \times 4^k - 11)/3, \\
g_4(9k+6) &= (50 \times 4^{k+1} - 59)/3, \\
g_4(9k+8) &= (98 \times 4^k + 1)/3, \\
g_4(9k+9) &= (50 \times 4^{k+1} - 11)/3, \\
g_4(9k+11) &= (98 \times 4^{k+1} - 59)/3, \\
g_4(9k+12) &= (50 \times 4^{k+1} + 1)/3.
\end{aligned}$$

Then  $g_4^5(0)$  is undefined.

The theorem gives immediately the following proposition.

**Proposition 6.2** *The behavior of Turing machine  $M_4$ , on inputs  $01^n$ ,  $n \geq 0$ , depends on the behavior of iterated  $g_4^k(n)$ ,  $k \geq 1$ .*

Since the behavior of iterated  $g_4^k(n)$  is an open problem in mathematics, the halting problem for Turing machine  $M_4$  is so.

Let  $h_4(n) = \min\{k : g_4^k(n) \text{ is undefined}\}$ . We have seen that  $h_4(0) = 5$ . We also have  $h_4(2) = 8$ , and  $C(2) \vdash (T) \text{ END}$  with  $T > 10^{10^{10^{18641000}}}$ . We also have  $h_4(36) = 15$ .

## 7 Unclassifiable machine

Let  $M_5$  be the  $6 \times 2$  Turing machine defined by Table 10.

We have  $s(M_5) > 3.8 \times 10^{21132}$  and  $\sigma(M_5) > 3.1 \times 10^{10566}$ .

This machine was discovered in May 2010 by Pavel Kropitz. It was the champion for the busy beaver competition for  $6 \times 2$  machines from May to June 2010.

The following theorem is adapted from an analysis of S. Ligocki [24]. It gives the rules that enable Turing machine  $M_5$  to reach a halting configuration from a blank tape.

**Theorem 7.1** *Let  $C(k, n) = \omega 010^n 1(C1)1^{3k} 0^\omega$ . Then*

$$(a) \quad \omega 0(A0)0^\omega \vdash (47) \quad C(2, 5),$$

*and, for all  $k \geq 0$ ,*

- |     |               |                               |                                     |
|-----|---------------|-------------------------------|-------------------------------------|
| (b) | $C(k, 0)$     | $\vdash (3)$                  | ${}^\omega 01(H0)1^{3k+1}0^\omega,$ |
| (c) | $C(k, 1)$     | $\vdash (3k + 37)$            | $C(2, 3k + 2),$                     |
| (d) | $C(k, 2)$     | $\vdash (12k + 44)$           | $C(k + 2, 4),$                      |
| (e) | $C(k, 3)$     | $\vdash (3k + 57)$            | $C(2, 3k + 8),$                     |
| (f) | $C(k, n + 4)$ | $\vdash (27k^2 + 105k + 112)$ | $C(3k + 5, n).$                     |

**Proof.** A direct inspection of the transition table gives

- |      |                  |               |                    |
|------|------------------|---------------|--------------------|
| (1)  | $0^4(A0)0^9$     | $\vdash (47)$ | $10^5 1(C1)1^6,$   |
| (2)  | $0(C0)$          | $\vdash (1)$  | $(C0)1,$           |
| (3)  | $1(C0)$          | $\vdash (1)$  | $(C1)1,$           |
| (4)  | $(B1)00$         | $\vdash (2)$  | $0^2(C0),$         |
| (5)  | $(B1)01$         | $\vdash (4)$  | $01(B1),$          |
| (6)  | $(B1)10$         | $\vdash (4)$  | $01(B1),$          |
| (7)  | $(B1)10^3$       | $\vdash (14)$ | $10^3(B1),$        |
| (8)  | $(B1)1^2 0^2$    | $\vdash (7)$  | $0^2 1(B1)1,$      |
| (9)  | $(B1)1^3$        | $\vdash (3)$  | $0^3(B1),$         |
| (10) | $0(C1)$          | $\vdash (2)$  | $1(B1),$           |
| (11) | $11(C1)$         | $\vdash (3)$  | $1(H0)1,$          |
| (12) | $0^3 1(C1)$      | $\vdash (10)$ | $(C1)1^4,$         |
| (13) | $0^3 10^2 1(C1)$ | $\vdash (8)$  | $1(B1)0^2 10^2 1,$ |
| (14) | $101(C1)$        | $\vdash (8)$  | $1(B1)1^2.$        |

Iterating, respectively, (2) and (9) gives

- |      |              |               |               |
|------|--------------|---------------|---------------|
| (15) | $0^k(C0)$    | $\vdash (k)$  | $(C0)1^k,$    |
| (16) | $(B1)1^{3k}$ | $\vdash (3k)$ | $0^{3k}(B1).$ |

Using (4), (15) and (3), we get

- |      |              |                  |                |
|------|--------------|------------------|----------------|
| (17) | $10^k(B1)00$ | $\vdash (k + 5)$ | $(C1)1^{k+3}.$ |
|------|--------------|------------------|----------------|

Using (16), (17) and (10), we get

- |      |                  |                   |                  |
|------|------------------|-------------------|------------------|
| (18) | $01(B1)1^{3k}00$ | $\vdash (6k + 7)$ | $1(B1)1^{3k+3}.$ |
|------|------------------|-------------------|------------------|

By induction on  $k$ , using (18), we get

- |      |                   |                      |                |
|------|-------------------|----------------------|----------------|
| (19) | $0^k 1(B1)0^{2k}$ | $\vdash (3k^2 + 4k)$ | $1(B1)1^{3k}.$ |
|------|-------------------|----------------------|----------------|

Using (12), (10), (16), (6), (19), (16) and (17), we get

- |      |                             |                               |                   |
|------|-----------------------------|-------------------------------|-------------------|
| (20) | $0^4 1(C1)1^{3k} 0^{6k+11}$ | $\vdash (27k^2 + 105k + 112)$ | $1(C1)1^{9k+15}.$ |
|------|-----------------------------|-------------------------------|-------------------|

Using (14), (16), (8), (7) and (17), we get

- |      |                     |                    |                        |
|------|---------------------|--------------------|------------------------|
| (21) | $101(C1)1^{3k} 0^7$ | $\vdash (3k + 37)$ | $10^{3k+21} 1(C1)1^6.$ |
|------|---------------------|--------------------|------------------------|

Using (13), (17), (10), (9), (6), (5), (16), (17), (10), (16) and (17), we get

- |      |                            |                     |                       |
|------|----------------------------|---------------------|-----------------------|
| (22) | $0^4 10^2 1(C1)1^{3k} 0^4$ | $\vdash (12k + 44)$ | $10^4 1(C1)1^{3k+6}.$ |
|------|----------------------------|---------------------|-----------------------|

Using (12), (12), (10), (16), (8), (6), (17), (10), (9) and (17), we get

- |      |                            |                    |                       |
|------|----------------------------|--------------------|-----------------------|
| (23) | $0^4 10^3 1(C1)1^{3k} 0^7$ | $\vdash (3k + 57)$ | $10^{3k+8} 1(C1)1^6.$ |
|------|----------------------------|--------------------|-----------------------|

Results (1), (11), (21), (22), (23) and (20) give results (a)–(f) of the theorem.  $\square$

Using the rules of this theorem, we have, in 22158 transitions,

$${}^\omega 0(A0)0^\omega \vdash (47) \quad C(2, 5) \vdash (430) \quad C(11, 1) \vdash ( ) \cdots \vdash ( ) \quad \text{END.}$$

Let  $g_5$  be the partial function defined by: for  $k, n \geq 0$ ,

$$\begin{aligned}
g_5(k, 0) & \quad \text{undefined,} \\
g_5(k, 1) & = (2, 3k + 2), \\
g_5(k, 2) & = (k + 2, 4), \\
g_5(k, 3) & = (2, 3k + 8), \\
g_5(k, n + 4) & = (3k + 5, n).
\end{aligned}$$

Then  $g_5^{2^{2157}}(2, 5)$  is undefined.

**Proposition 7.2** *The behavior of Turing machine  $M_5$ , on inputs  $01^{3n+3}$ ,  $n \geq 0$ , depends on the behavior of iterated  $g_5^k(n, p)$ ,  $k \geq 1$ ,  $n, p \geq 0$ .*

**Proof.** We have  ${}^\omega 0(A0)1^{3n+3}0^\omega \vdash (3n + 30) \quad {}^\omega 01^{3n+2}1(C1)1^60^\omega = C(2, 3n + 2)$ .  $\square$

Since the behavior of iterated  $g_5^k(n, p)$  is an open problem in mathematics, the halting problem for Turing machine  $M_5$  is so.

The following proposition shows that some configurations take a long time to halt.

**Proposition 7.3**  $C(9, 1) \vdash (T) \quad \text{END with } T > 10^{10^{10^{10^{3520}}}}$ .

**Proof.** By induction on  $n$ , using Theorem 7.1 (f), it is easy to prove that, if  $n \geq 0$ ,  $0 \leq r \leq 3$ , we have

$$C(2, 4n + r) \vdash (t_n) \quad C(u_n, r),$$

with  $u_n = (3^{n+2} - 5)/2$  and  $t_n = (3 \times 9^{n+3} - 80 \times 3^{n+3} + 584n - 27)/32$ .

By induction on  $k$ , it is easy to prove that, if  $k \geq 2$ , we have

$$3^{2^{k-1}} \equiv 2^{k+1} + 1 \pmod{2^{k+2}}$$

so the multiplicative order of 3 modulo  $2^{k+2}$  is  $2^k$  for  $k \geq 1$ . Thus we can prove that, for  $k \geq 1$ ,  $n, m \geq 0$ , we have

$$n \equiv m \pmod{2^k} \iff u_n \equiv u_m \pmod{2^{k+1}}.$$

Now, suppose that, for  $a \in \{1, 3\}$ ,  $n, n' \geq 1$ ,  $q, q' \geq 1$ ,  $0 \leq r, r' \leq 3$ , we have

$$C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3n + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),$$

and

$$C(n', a) \vdash (3n' + 27 + 10a) \quad C(2, 3n' + 3a - 1) = C(2, 4q' + r') \vdash (t_{q'}) \quad C(u_{q'}, r'),$$

and let  $k \geq 2$  such that  $n \equiv n' \pmod{2^{k+1}}$ . Then it is easy to prove that  $r = r'$  and  $u_q \equiv u_{q'} \pmod{2^k}$ . So the behavior of configurations  $C(n, a)$  is mirrored by the behavior of configurations  $C(n', a)$  with  $n' \leq 2^k$  for suitable  $k$ .

In the following computation on  $C(9, 1)$ :

$$\begin{array}{llll}
C(9, 1) & \vdash ( ) & C(2, 4 \times 7 + 1) & \vdash (t_7) \\
C(9839, 1) & \vdash ( ) & C(2, 4 \times 7379 + 3) & \vdash (t_{7379}) \\
C(u_{7379}, 3) & \vdash ( ) & C(2, 4 \times q_3 + r_3) & \vdash (t_{q_3}) \\
C(u_{q_3}, r_3) & \vdash ( ) & C(2, 4 \times q_4 + r_4) & \vdash (t_{q_4}) \\
C(u_{q_4}, r_4) & \vdash ( ) & C(2, 4 \times q_5 + r_5) & \vdash (t_{q_5}) \\
C(u_{q_5}, r_5) & \vdash ( ) & C(2, 4 \times q_6 + r_6) & \vdash (t_{q_6}) \\
C(u_{q_6}, r_6) & \vdash (3) & \text{END} & 
\end{array}$$

| $M_6$ | 0   | 1   |
|-------|-----|-----|
| $A$   | 1RB | 0RF |
| $B$   | 0LB | 1LC |
| $C$   | 1LD | 0RC |
| $D$   | 1LE | 1RH |
| $E$   | 1LF | 0LD |
| $F$   | 1RA | 0LE |

Table 11: Machine  $M_6$  discovered in November 2007 by T. and S. Ligocki

we know that  $r_6 = 0$  because we have

$$\begin{aligned}
u_{q_1} = u_7 &\equiv 47 \pmod{64}, & (3 \times 47) + 2 &= (4 \times 35) + 3, & q'_2 &= 35, & r_2 &= 3, \\
u_{q_2} &\equiv u_{q'_2} \equiv 23 \pmod{32}, & (3 \times 23) + 8 &= (4 \times 19) + 1, & q'_3 &= 19, & r_3 &= 1, \\
u_{q_3} &\equiv u_{q'_3} \equiv 7 \pmod{16}, & (3 \times 7) + 2 &= (4 \times 5) + 3, & q'_4 &= 5, & r_4 &= 3, \\
u_{q_4} &\equiv u_{q'_4} \equiv 3 \pmod{8}, & (3 \times 3) + 8 &= (4 \times 4) + 1, & q'_5 &= 4, & r_5 &= 1, \\
u_{q_5} &\equiv u_{q'_5} \equiv 2 \pmod{4}, & (3 \times 2) + 2 &= (4 \times 2) + 0, & q'_6 &= 2, & r_6 &= 0.
\end{aligned}$$

It is easy to see that, if  $a \in \{1, 3\}$ ,  $n \geq 0$ , if

$$C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3n + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),$$

then  $q \geq (3n - 1)/4$  and  $u_q > (3^{3/4})^n > 2^n$ .

And we also have  $n \geq 5 \Rightarrow t_n > 68 \times 9^n$ , so, if  $C(9, 1) \vdash (T)$  END, we have

$$T > t_{q_6} > 9^{q_6} > 9^{3u_{q_5}/4} > 5^{u_{q_5}},$$

and  $u_{q_5} > 2^{u_{q_4}}$ ,  $u_{q_4} > 2^{u_{q_3}}$ ,  $u_{q_3} > 2^{u_{q_2}} = 2^{u_{7379}}$ , so  $T > 5^{2^{2^{u_{7379}}}}$ .

Using  $u_{7379} > 10^{3521}$ , and, for  $x \geq 1$ ,  $2^{10^x} > 10^{10^{x-.53}}$ ,  $2^{10^{10^x}} > 10^{10^{10^{x-.03}}}$ ,  $2^{10^{10^{10^x}}} > 10^{10^{10^{10^{x-.03}}}}$  and  $5^{10^{10^{10^{10^x}}}} > 10^{10^{10^{10^{10^{x-.03}}}}}$ , we are done.  $\square$

## 8 An infinite set of rules

Let  $M_6$  be the  $6 \times 2$  Turing machine defined by Table 11.

We have  $s(M_6) > 8.9 \times 10^{1762}$  and  $\sigma(M_6) > 2.5 \times 10^{881}$ .

This machine was discovered in November 2007 by Terry and Shawn Ligocki. It was the champion for the busy beaver competition for  $6 \times 2$  machines from November to December 2007.

The following theorem gives the rules that enable Turing machine  $M_6$  to reach a halting configuration from a blank tape.

Recall that  $\text{bin}(p)$  is the usual binary writing of number  $p$ , and  $R(w_1 \dots w_n) = w_n \dots w_1$ .

**Theorem 8.1** *Let  $C(n, p) = {}^\omega 0(F0)(10)^n R(\text{bin}(p))0^\omega$ , so that  $C(k, 4m + 1) = C(k + 1, m)$ . Then*

$$(a) \quad {}^\omega 0(A0)0^\omega \vdash (6) \quad C(0, 15),$$

and, for all  $k, m \geq 0$ ,

|     |                        |                             |   |
|-----|------------------------|-----------------------------|---|
| (b) | $C(k, 4m + 3)$         | $\vdash (4k + 6)$           | $C(k + 2, m),$  |
| (c) | $C(2k, 4m)$            | $\vdash (30k^2 + 20k + 15)$ | $C(5k + 2, 2m + 1),$                                  |
| (d) | $C(2k + 1, 4m)$        | $\vdash (30k^2 + 40k + 25)$ | $C(5k + 2, 32m + 20),$                                |
| (e) | $C(k, 8m + 2)$         | $\vdash (8k + 20)$          | $C(k + 3, 2m + 1),$                                   |
| (f) | $C(2k, 16m + 6)$       | $\vdash (30k^2 + 40k + 23)$ | $C(5k + 2, 32m + 20),$                                |
| (g) | $C(2k + 1, 16m + 6)$   | $\vdash (30k^2 + 80k + 63)$ | $C(5k + 7, 2m + 1),$                                  |
| (h) | $C(k, 32m + 14)$       | $\vdash (4k + 18)$          | $C(k + 3, 2m + 1),$                                   |
| (i) | $C(2k, 128m + 94)$     | $\vdash (30k^2 + 40k + 39)$ | $C(5k + 2, 256m + 84),$                               |
| (j) | $C(2k + 1, 128m + 94)$ | $\vdash (30k^2 + 80k + 79)$ | $C(5k + 9, m),$                                       |
| (k) | $C(k, 256m + 190)$     | $\vdash (4k + 34)$          | $C(k + 5, m),$  |
| (l) | $C(k, 512m + 30)$      | $\vdash (2k + 43)$          | $\omega 0(10)^k 1(H0)(10)^2(01)^2 R(bin(m))0^\omega.$ |

**Proof.** A direct inspection of the transition table gives

- (1)  $0^4(A0)0 \vdash (6) (F0)1^40,$
- (2)  $0(F0)00 \vdash (9) (F0)1^3,$
- (3)  $(F0)10 \vdash (2) 10(F0),$
- (4)  $(F0)11 \vdash (4) (F1)10,$
- (5)  $(F0)01 \vdash (4) 10(C1),$
- (6)  $10(E1) \vdash (2) (E1)10,$
- (7)  $1(E1) \vdash (2) 1(H0),$
- (8)  $0^3(E1) \vdash (3) (F0)110,$
- (9)  $00(F1) \vdash (2) (F0)10,$
- (10)  $10(F1) \vdash (2) (F1)10,$
- (11)  $1(F1) \vdash (1) (E1)0,$
- (12)  $(C1)1 \vdash (1) 0(C1),$
- (13)  $(C1)0 \vdash (1) 0(C0),$
- (14)  $0^3(C0) \vdash (3) (F0)1^3,$
- (15)  $100(C0) \vdash (3) (F1)1^3.$

Iterating, respectively, (3), (10), (6) and (12) gives

- (16)  $(F0)(10)^k \vdash (2k) (10)^k(F0),$
- (17)  $(10)^k(F1) \vdash (2k) (F1)(10)^k,$
- (18)  $(10)^k(E1) \vdash (2k) (E1)(10)^k,$
- (19)  $(C1)1^k \vdash (k) 0^k(C1).$

Using (5), (19) and (13), we get

- (20)  $(F0)01^{k+1}0 \vdash (k+5) 10^{k+2}(C0).$

Using (16), (4), (17) and (9), we get

- (21)  $0^2(F0)(10)^k 11 \vdash (4k+6) (F0)(10)^{k+2}.$

Using (16), (20), (15), (17), (9) and (21), we get

- (22)  $0^4(F0)(10)^k 010 \vdash (8k+20) (F0)(10)^{k+3}1.$

Using (22) and (21), we get

- (23)  $0^6(F0)(10)^k(01)^2 \vdash (12k+38) (F0)(10)^{k+5}.$

By induction on  $n$ , using (23), we get

- (24)  $0^{6n}(F0)(10)^k(01)^{2n} \vdash (30n^2 + 12kn + 8n) (F0)(10)^{5n+k}.$

Using (24), with  $k = 2$ , we get

$$(25) \quad 0^{6k}(F0)(10)^2(01)^{2k} \vdash (30k^2 + 32k) \quad (F0)(10)^{5k+2}.$$

Using (16), (2), (4), (11), (18), (8) and (21), we get

$$(26) \quad 0^5(F0)(10)^{k+1}0^2 \vdash (4k + 25) \quad (F0)(10)^2(01)^k00101.$$

Using (26) and (25), we get

$$(27) \quad 0^{6k+5}(F0)(10)^{2k+1}0^2 \vdash (30k^2 + 40k + 25) \quad (F0)(10)^{5k+2}00101.$$

Using (25) and (22), we get

$$(28) \quad 0^{6k+4}(F0)(10)^2(01)^{2k+1}0 \vdash (30k^2 + 72k + 36) \quad (F0)(10)^{5k+5}1.$$

Using (26) and (28), we get

$$0^{6k+9}(F0)(10)^{2k+2}0^2 \vdash (30k^2 + 80k + 65) \quad (F0)(10)^{5k+7}1,$$

and the result is still true for  $k = -1$ , so we have

$$(29) \quad 0^{6k+3}(F0)(10)^{2k}0^2 \vdash (30k^2 + 20k + 15) \quad (F0)(10)^{5k+2}1.$$

Using (16), (20), (14), (4), (11), (18), (8) and (21), we get

$$(30) \quad 0^5(F0)(10)^k01^20 \vdash (4k + 23) \quad (F0)(10)^2(01)^k00101.$$

Using (30) and (25), we get

$$(31) \quad 0^{6k+5}(F0)(10)^{2k}01^20 \vdash (30k^2 + 40k + 23) \quad (F0)(10)^{5k+2}00101.$$

Using (30) and (28), we get

$$(32) \quad 0^{6k+9}(F0)(10)^{2k+1}01^20 \vdash (30k^2 + 80k + 63) \quad (F0)(10)^{5k+7}1.$$

Using (16), (20), (14), (4), (17) and (9), we get

$$(33) \quad 0^2(F0)(10)^k01^30 \vdash (4k + 18) \quad (F0)(10)^{k+3}1.$$

Using (16), (20), (14), (4), (9) and (16), we get

$$(34) \quad (F0)(10)^k01^40 \vdash (2k + 21) \quad (10)^k1(10)^2(F0)1.$$

Using (34), (4), (17), (11), (18), (8) and (21), we get

$$(35) \quad 0^5(F0)(10)^k01^401 \vdash (4k + 39) \quad (F0)(10)^2(01)^k00(10)^3.$$

Using (35) and (25), we get

$$(36) \quad 0^{6k+5}(F0)(10)^{2k}01^401 \vdash (30k^2 + 40k + 39) \quad (F0)(10)^{5k+2}0^2(10)^3.$$

Using (35), (25) and (22), we get

$$(37) \quad 0^{6k+9}(F0)(10)^{2k+1}01^401 \vdash (30k^2 + 80k + 79) \quad (F0)(10)^{5k+9}.$$

Using (16), (20), (14), (4), (9) and (16), we get

$$(38) \quad (F0)(01)^k01^50 \vdash (2k + 22) \quad (10)^{k+3}(F0)1.$$

Using (38), (4), (17) and (9), we get

$$(39) \quad 0^2(F0)(10)^k01^501 \vdash (4k + 34) \quad (F0)(10)^{k+5}.$$

Using (34), (3), (2), (4), (11), (18) and (7), we get

$$(40) \quad (F0)(10)^k01^40^4 \vdash (2k + 43) \quad (10)^k1(H0)(10)^2(01)^2.$$

Results (1), (21), (29), (27), (22), (31), (32), (33), (36), (37), (39) and (40) give results (a)–(l) of the theorem.  $\square$

Note that the rules (a)–(l) are written in their order of occurrence in the computation of Turing machine  $M_6$  on the blank tape.

Using the rules of this theorem, we have, in 3346 transitions,

$$\omega 0(A0)0^\omega \vdash (6) \quad C(0, 15) \vdash (6) \quad C(2, 3) \vdash ( ) \cdots \vdash ( ) \quad \text{END}.$$

| $M_7$ | 0   | 1   |
|-------|-----|-----|
| $A$   | 1RB | 0LB |
| $B$   | 0RC | 1LB |
| $C$   | 1RD | 0LA |
| $D$   | 1LE | 1LF |
| $E$   | 1LA | 0LD |
| $F$   | 1RH | 1LE |

Table 12: Machine  $M_7$  discovered in October 2000 by Marxen and Buntrock

We have

$$\begin{aligned} & \omega 0(A0)0R(\text{bin}(p))0^\omega \vdash (6) \quad \omega 0(F0)1^4 0R(\text{bin}(p))0^\omega \\ & = C(0, 32p + 15) \vdash (6) \quad C(2, 8p + 3) \vdash (14) \quad C(4, 2p), \end{aligned}$$

so the behavior of Turing machine  $M_6$  on inputs  $00x$ ,  $x \in \{0, 1\}^*$ , depends on the behavior of configurations  $C(n, p)$ , and the halting problem for Turing machine  $M_6$  depends on this behavior.

## 9 Configurations provably stopping

Let  $M_7$  be the  $6 \times 2$  Turing machine defined by Table 12.

We have  $s(M_7) > 6.1 \times 10^{925}$  and  $\sigma(M_7) > 6.4 \times 10^{462}$ .

This machine was discovered in October 2000 by Heiner Marxen and Jürgen Buntrock. It was the champion for the busy beaver competition for  $6 \times 2$  machines from October 2000 to March 2001.

The following theorem was initially obtained by Munafo [37]. It gives the rules observed by Turing machine  $M_7$ .

**Theorem 9.1** *Let  $C(n) = \omega 01^n(B0)0^\omega$ . Then*

- (a)  $\omega 0(A0)0^\omega \vdash (1) \quad C(1)$ ,
- and, for all  $k \geq 0$ ,
- (b)  $C(3k) \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86) \quad C(9 \times 2^{k+1} - 8)$ ,
- (c)  $C(3k + 1) \vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792) \quad C(2^{k+5} - 8)$ ,
- (d)  $C(3k + 2) \vdash (3k + 8) \quad \omega 01(H1)(011)^k 01010^\omega$ .

**Proof.** A direct inspection of the transition table gives

- (1)  $11(B0)00 \vdash (6) \quad (D1)0101$ ,
- (2)  $00(B0)01 \vdash (8) \quad (B0)1^4$ ,
- (3)  $11(B0)01 \vdash (6) \quad (B1)01^3$ ,
- (4)  $(B0)1 \vdash (3) \quad 1(B0)$ ,
- (5)  $0(B1) \vdash (1) \quad (B0)1$ ,
- (6)  $1(B1) \vdash (1) \quad (B1)1$ ,
- (7)  $0(D1) \vdash (2) \quad 1(H1)$ ,
- (8)  $0^3 1(D1) \vdash (6) \quad (B0)1^4$ ,
- (9)  $0^4 1^2(D1) \vdash (8) \quad (B0)1^3 01^2$ ,
- (10)  $1^3(D1) \vdash (3) \quad (D1)011$ .

Iterating, respectively, (4), (6) and (10) gives

$$\begin{aligned} (11) \quad & (B0)1^k \vdash (3k) \quad 1^k(B0), \\ (12) \quad & 1^k(B1) \vdash (k) \quad (B1)1^k, \\ (13) \quad & 1^{3k}(D1) \vdash (3k) \quad (D1)(011)^k. \end{aligned}$$

Using (3), (12), (5) and (11), we get

$$01^{k+2}(B0)01 \vdash (4k+10) \quad 1^{k+1}(B0)01^3,$$

and the result is still true for  $k = -1$ , so we have

$$(14) \quad 01^{k+1}(B0)01 \vdash (4k+6) \quad 1^k(B0)01^3.$$

By induction on  $k$ , using (14), we get

$$(15) \quad 0^k 1^k(B0)01 \vdash (2k^2 + 4k) \quad (B0)01^{2k+1}.$$

Using (1), (13) and (7), we get

$$(16) \quad 01^{3k+2}(B0)0^2 \vdash (3k+8) \quad 1(H1)(011)^k 0101.$$

Using (1), (13) and (9), we get

$$0^4 1^{3k+4}(B0)0^2 \vdash (3k+14) \quad (B0)1^3(011)^{k+1} 0101,$$

and the result is still true for  $k = -1$ , so we have

$$(17) \quad 0^4 1^{3k+1}(B0)0^2 \vdash (3k+11) \quad (B0)1^3(011)^k 0101.$$

Using (1), (13) and (8), we get

$$(18) \quad 0^3 1^{3k+3}(B0)0^2 \vdash (3k+12) \quad (B0)1^4(011)^k 0101.$$

Using (11), (15) and (2), we get

$$(19) \quad 0^{k+2}(B0)1^k 01 \vdash (2k^2 + 7k + 8) \quad (B0)1^{2k+4}.$$

By induction on  $k$ , using (19), we get

$$(20) \quad 0^{2^k(n+5)-5-n-3k}(B0)1^n(011)^k \vdash (T) \quad (B0)1^{2^k(n+5)-5},$$

with  $T = 2(n+5)^2(4^k-1)/3 - 13(n+5)(2^k-1) + 23k$ .

Using (20), for  $n = 3$  and  $n = 4$ , we get respectively

$$\begin{aligned} (21) \quad & 0^{2^{k+3}-3k-8}(B0)1^3(011)^k \vdash (128(4^k-1)/3 - 13 \times 2^{k+3} + 23k + 104) \quad (B0)1^{2^{k+3}-5}, \\ (22) \quad & 0^{9 \times 2^k-3k-9}(B0)1^4(011)^k \vdash (54 \times 4^k - 117 \times 2^k + 23k + 63) \quad (B0)1^{9 \times 2^k-5}. \end{aligned}$$

Using (11), (15), (2), (11), (15), (2) and (11), we get

$$(23) \quad 0^{3k+8}(B0)1^k 0101 \vdash (10k^2 + 65k + 112) \quad 1^{4k+12}(B0).$$

Using (17), (21) and (23), we get

$$(24) \quad 0^{2^{k+5}-3k-11} 1^{3k+1}(B0)0^2 \vdash (2048 \times (4^k-1)/3 - 3 \times 2^{k+7} + 26k + 792) \quad 1^{2^{k+5}-8}(B0).$$

Using (18), (22) and (23), we get

$$0^{9 \times 2^{k+2}-3k-13} 1^{3k+3}(B0)0^2 \vdash (54 \times 4^{k+2} - 27 \times 2^{k+4} + 26k + 112) \quad 1^{9 \times 2^{k+2}-8}(B0),$$

and the result is still true for  $k = -1$ , so we have

$$(25) \quad 0^{9 \times 2^{k+1}-3k-10} 1^{3k}(B0)0^2 \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86) \quad 1^{9 \times 2^{k+1}-8}(B0).$$

Results (25), (24) and (16) give results (b)–(d) of the theorem.  $\square$

Using the rules of this theorem, we have

$$\omega 0(A0)0^\omega \vdash (1) \quad C(1) \vdash (408) \quad C(24) \vdash (14100774)$$

$$C(4600) \vdash (T) \quad C(2^{1538} - 8) \vdash (2^{1538} - 2) \quad \omega 01(H1)(011)^p 01010^\omega,$$

with  $T = 2048 \times (4^{1533} - 1)/3 - 3 \times 2^{1540} + 40650$  and  $p = (2^{1538} - 10)/3$ .

So the total time is  $s(M_7) = 2048 \times (4^{1533} - 1)/3 - 11 \times 2^{1538} + 14141831$ , and the final number of symbols 1 is  $\sigma(M_7) = 2 \times (2^{1538} - 10)/3 + 4$ .

Note that

$$C(6k+1) \vdash ( ) \quad C(3m) \vdash ( ) \quad C(6p+4) \vdash ( ) \quad C(3q+2) \vdash ( ) \quad \text{END},$$

with  $m = (2^{2k+5} - 8)/3$ ,  $p = 3 \times 2^m - 2$ ,  $q = (2^{2p+6} - 10)/3$ . So all configurations  $C(n)$  lead to a halting configuration. Those taking the most time are  $C(6k+1)$ . For example,  $C(7) \vdash (t)$  END with  $t > 10^{3.9 \times 10^{12}}$ . More generally,  $C(6k+1) \vdash (t(k))$  END with  $t(k) > 10^{10^{10^{(3k+2)/5}}}$ .

## 10 Conclusion

We discuss two questions as a conclusion to this article.

**A.** *How simulating Collatz-like functions allows Turing machines to achieve high scores?*

Lagarias [17] noted that the successive iterates of the  $3x+1$  function  $T$  have an irregular behavior. For example, 7 iterations of function  $T$  on  $n = 26$  lead to the value 1, but 70 iterations are necessary on  $n = 27$ . It seems that many Collatz-like functions have the same irregular behavior. Iterating them on small numbers may produce very long runs before stopping.

Adding parameters may increase the number of iterations by allowing the iterated values to range the set of parameters before stopping. The pure Collatz-like function with parameter  $g_3(n, p)$  presented in Section 5 is particularly illustrative.

Another way to high scores is given by exponential Collatz-like functions such as function  $g_4$  in Section 6. Only five iterations are performed on a blank tape, but exponential growth ensures a high score.

Irregular behavior is a condition for a Collatz-like function to be eligible to the busy beaver competition. Another condition is, of course, being computable by a very small Turing machine.

**B.** *Are some universal devices more natural than others?*

Conway [7] proved that there is no algorithm that, given as inputs a Collatz-like function  $g$  and two integers  $n, p$ , outputs an answer yes or no to the question: Does there exist a positive integer  $k$  such that  $g^k(n) = p$ ? Conway [7, 8] also proved that Collatz-like functions can be used to simulate all computable (also called recursive) functions. These properties can be summed up by writing that Collatz-like functions provide a universal model of computation with a m-complete decision problem.

Many universal models of computation are known: Turing machines, tag-systems, cellular automata, Diophantine equations, etc. (see [35]). Of course, any universal model can simulate and be simulated by any other universal model. But it is Collatz-like functions, and not another model, that appear naturally in this study. Their unexpectedly pervasive presence leads to wonder about the significance of their status among mathematical beings.

## References

- [1] N. Bátfai, On the running time of the shortest programs,  
<http://arxiv.org/abs/0908.1159>.

- [2] N. Bátfai, Recombinations of busy beaver machines, <http://arxiv.org/abs/0908.4013>.
- [3] A.H. Brady, The conjectured highest scoring machines for Rado's  $\Sigma(k)$  for the value  $k = 4$ , *IEEE Transactions on Electronic Computers*, EC-15, October 1966, 802–803.
- [4] A.H. Brady, The determination of the value of Rado's noncomputable function  $\Sigma(k)$  for four-state Turing machines, *Mathematics of Computation*, 40 (162), April 1983, 647–665.
- [5] A.H. Brady, The busy beaver game and the meaning of life, in: The Universal Turing machine: A Half-Century Survey, R. Herken (Ed.), Oxford University Press, 1988, 259–277.
- [6] R.N. Buttsworth and K.R. Matthews, On some Markov matrices arising from the generalized Collatz mapping, *Acta Arithmetica* 55, 1990, 43–57.
- [7] J.H. Conway, Unpredictable iterations, in: *Proc. 1972 Number Theory Conference*, University of Colorado, Boulder, CO, 1972, 49–52, and also in [20], 219–221.
- [8] J.H. Conway, FRACTAN: a simple universal programming language for arithmetic, in: *Open problems in Communication and Computation* (T.M. Cover and B. Gopinath, Eds.), Springer, 1987, 4–26, and also in [20], 249–264.
- [9] A.K. Dewdney, Computer recreations, *Scientific American* 251 (5), November 1984, 27.
- [10] G. Georgiev (Skelet), <http://skelet.ludost.net/bb>.
- [11] M.W. Green, A lower bound on Rado's sigma function for binary Turing machines, in: *Proceedings of the 5th IEEE Annual Symposium on Switching Circuit Theory and Logical Design*, November 1964, 91–94.
- [12] J. Hertel, Computing the uncomputable Rado sigma function, *The Mathematica Journal* 11 (2), 2009, 270–283.
- [13] T. Hutton, Two dimensional Turing machines, <http://code.google.com/p/ruletablerepository/wiki/TwoDimensionalTuringMachines>.
- [14] F. Kaščák, Small universal one-state linear operator algorithm, Proc. MFCS'92, *Lecture Notes in Computer Science* 629, 1992, 327–335.
- [15] S. Kohl, Wildness of iteration of certain residue-class-wise affine mappings, *Advances in Applied Mathematics* 39 (3), 2007, 322–328.
- [16] G. Lafitte and C. Papazian, The fabric of small Turing machines, in: *Computation and Logic in the Real World, Proceedings of the Third Conference on Computability in Europe*, June 2007, 219–227.
- [17] J.C. Lagarias, The  $3x+1$  problem and its generalizations, *Amer. Math. Monthly* 92 (1), 1985, 3–23, and also in [20], 31–54.
- [18] J.C. Lagarias, The  $3x+1$  problem: an annotated bibliography (1963–1999), in [20], 267–341, and also <http://arxiv.org/abs/math/0309224>.

- [19] J.C. Lagarias, The  $3x+1$  problem: an annotated bibliography (2000–), <http://arxiv.org/abs/math/0608208>.
- [20] J.C. Lagarias (Ed.), The Ultimate Challenge: The  $3x+1$  Problem, AMS, 2010.
- [21] T. Ligocki, Communicated by email on 13 February 2005.
- [22] S. Ligocki, Communicated by email on 15 August 2006.
- [23] S. Ligocki, Communicated by email on 20 August 2006.
- [24] S. Ligocki, Communicated by email on 8 May 2010.
- [25] S. Lin and T. Rado, Computer studies of Turing machine problems, *Journal of the ACM* **12** (2), April 1965, 196–212.
- [26] D.S. Lynn, New results for Rado’s sigma function for binary Turing machines, *IEEE Transactions on Computers*, C-**21** (8), August 1972, 894–896.
- [27] R. Machlin and Q.F. Stout, The complex behavior of simple machines, *Physica D* **42**, June 1990, 85–98.
- [28] H. Marxen, <http://www.drb.insel.de/~heiner/BB/macro.html>.
- [29] H. Marxen and J. Buntrock, Attacking the Busy Beaver 5, *Bulletin of the EATCS*, No 40, February 1990, 247–251.
- [30] P. Michel, Étude de machines de Turing et complexité algorithmique, Thèse Université Paris 7, 1992.
- [31] P. Michel, Busy beaver competition and Collatz-like problems, *Arch. Math. Logic* **32** (5), 1993, 351–367.
- [32] P. Michel, The busy beaver competition: a historical survey, <http://arxiv.org/abs/0906.3749>.
- [33] P. Michel, Historical survey of busy beavers, <http://www.logique.jussieu.fr/~michel/ha.html>.
- [34] P. Michel, Behavior of busy beavers, <http://www.logique.jussieu.fr/~michel/beh.html>.
- [35] P. Michel and M. Margenstern, Generalized  $3x+1$  functions and the theory of computation, in [20], 105–128.
- [36] R. Munafo, <http://mrob.com/pub/math/ln-notes1-4.html#mb-bb-1>.
- [37] R. Munafo, <http://mrob.com/pub/math/ln-notes1-5.html#mb6q>, and <http://mrob.com/pub/math/ln-mb6q.html>.
- [38] T. Rado, On non-computable functions, *Bell Syst. Tech. J.* **41** (3), May 1962, 877–884.
- [39] D.A. Rawsthorne, Imitation of an iteration, *Mathematics Magazine* **58** (3), May 1985, 172–176.

- [40] C. Tooth, Google Groups sci.math>the Turing machine known as #r, June 28, 2002.
- [41] S. Wagon, The Collatz problem, *The Mathematical Intelligencer* **7** (1), 1985, 72–76.
- [42] G.J. Wirsching, The Dynamical System Generated by the  $3n+1$  Function, *Lecture Notes in Mathematics* **1681**, Springer, 1998.