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Mehdi Hage-Hassan

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An elementary introduction to Quantum mechanic

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Abstract

Quantum mechanics originates from the study of the spectrum of hydrogen atom and the black body radiation and the development of this theory has been very successful for the study of atoms, the nucleus and the classification of elementary particles using SU (3) group. But the course of quantum mechanics in the second year should be revised to take account of these developments not only in terms of program but also from an educational point of view, especially for students of mathematical orientation. In this work we give an idea about the origin of quantum mechanics and we summarize the Schrodinger, the Dirac formalisms of quantum mechanic and the path integral. After exposing the Dirac's method of the oscillator we treat the Schwinger's approach of angular momentum. We present the theory of the central potential, the Quarks model and the classification: of chemical elements and elementary particles. We present a summary of the approaches of group theory. We present the theory of SU (3) and the decomposition of tensor product of SU (3). To take account of level of students we use in the applications our generating function method for its simplicity.

Résumé

La mécanique quantique a pour origine l'étude du spectre de l'atome d'hydrogène et le rayonnement du corps noir mais le développement de cette théorie a eu un grand succès pour l'étude des atomes, du noyau et la classification des particules élémentaire à l'aide du groupe SU(3). Mais le cours de la mécanique quantique de la deuxième année doit être révisée pour tenir compte de ces développement, non seulement du point de vue programme mais aussi du point de vue pédagogique, surtout pour les étudiants d'orientation mathématique. Dans ce travail nous donnons une idée sur l'origine de la mécanique quantique et nous résumons les formalismes de la mécanique ondulatoire, le formalisme de Dirac et l'intégrale du chemin. Après avoir exposé l'oscillateur par la méthode de Dirac nous traitons la théorie du moment angulaire à l'aide de l'approche de Schwinger. Nous exposons la théorie du potentiel central, le modèle des Quarks et la classification : des éléments chimiques et des particules élémentaires. Nous faisons un résumé des approches de la théorie de groupe. Nous exposons la théorie et la décomposition du produit tensoriel de SU(3). Pour tenir compte du niveau des étudiants nous utilisons dans les applications notre méthode de la fonction génératrice pour sa simplicité.

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Introduction

The basis of scientific development and current technique are the classical and quantum mechanics. But quantum mechanics has objective to study and describe the phenomena of physical systems of the atomic and subatomic. It was developed to solve various problems such as the existence of the spectral lines of the hydrogen atom, the photoelectric effect, or black body radiation.

The development of quantum mechanics proved to be very fruitful in results and various applications. It allowed to clear up the mystery of the structure of the atom, the nucleus and they very important for the study of elementary particles and the quantum information. And more it is undoubtedly the basis for future technologies.

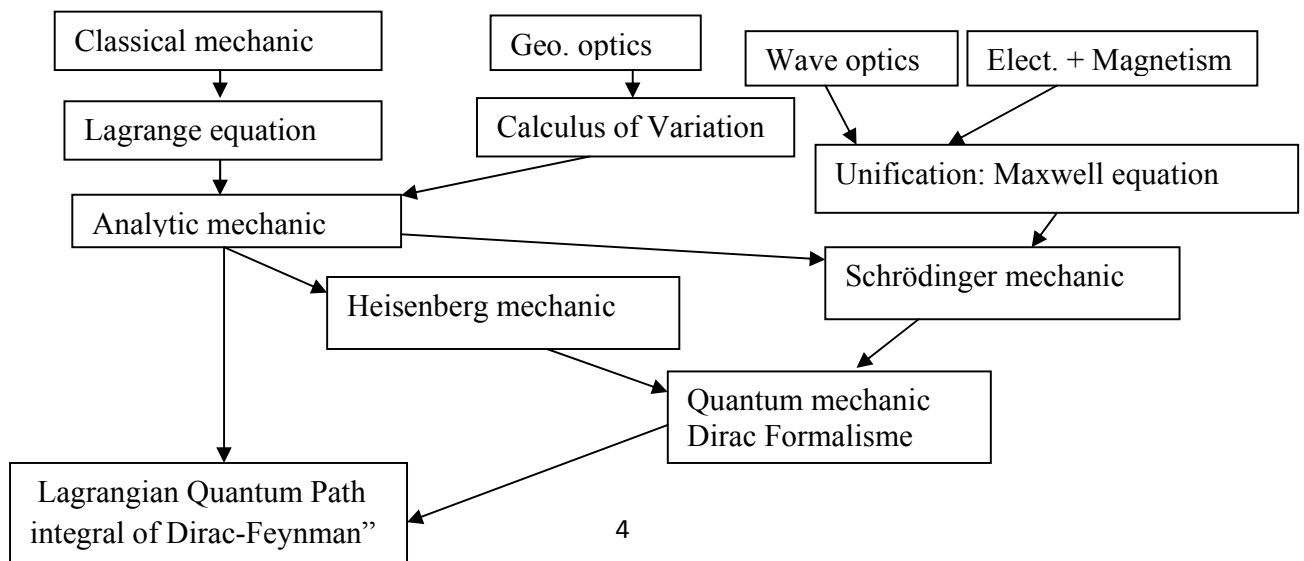
In this work we want to give the undergraduate students of the second year and especially who have mathematical orientations a simple overview of quantum mechanics and its connection with the mathematical theories.

The theoretical development of quantum mechanics had as its starting point the classical mechanics, optics and the analytical mechanics. And this theory has opened up new branches not only physical but also mathematical through Dirac formalism, distributions, the Fock space and the modern physical theories.

On the other hand the theory of groups: The Lie groups, the classical groups and groups of permutations have applications in the theory of angular momentum, nuclear physics and classification of chemical elements and elementary particles. therefore we expose a summary for the study of algebras of semi-simple Lie groups and the Weyl global method for unitary group. We are particularly interested in the study of the representation of $SU(3)$ and the tensor product of these groups.

We emphasize that this work is a revision and renovation of lecture notes taught to students in the second year in the 70s. we present a simple way to resolve many unresolved problems or solved by difficult methods using the generating function method that we have developed and using the octonion algebra. We emphasize that our method fits naturally in a simple way in all sections of the course of quantum mechanics.

Summary of the origins of quantum mechanics



I- Origins and formalisms of quantum theory

Understanding the movement of the stars and planets and a variety of fact have been the origin of the classical and quantum mechanics. In this section we will give a quick summary of these theories and the quarks model of elementary particles.

We represent the development of mechanics: classical, quantum and elementary particles by an astronaut who observed a pedestrian moves on the earth:

First: it is very far from the earth, he sees the pedestrian (particle) moves on a straight line or a curve.

Secondly: if approaching the earth again, he sees the pedestrian moves in a zigzag randomly.

Third: if the astronaut approaches the pedestrian, he sees the pedestrian characteristics: Eyes (spin), the head(charge), etc...

1. Classical and analytical mechanics

We present in this part a rapidly developing of the classical mechanics [1] and the calculation of Euler variations [2] which is the origin of the principle of least action and the analytical mechanics.

1.1 Origin of classical mechanics

1- We know that the origin of classical mechanics is the meeting between a rich young Tycho Brahe obsessed by astrophysics and a priest expelled from the church because Kepler doing mathematic during the confessions. After observing the planets during a long time Tycho Brahe died and Kepler found the Kepler's laws and more he has no resources and died of famine.

2- A young professor of mathematics Galileo good-looking and from a famous Italian family who loved dancing. One day when he was dancing in a very large and very clean saloon, "no-friction ", suddenly came a current of air and chandeliers start swinging without stop. Galileo deduced from the fact that the laws of nature are invariant under uniform translation. And then he defined the velocity and found the law of falling bodies.

3- Newton finally represented the force by a vector and introduced the acceleration and had a genius idea, "a brain-wave", that the laws of nature are represented by mathematics and by his famous formula.

undoubtedly Newton made a hole in the frozen ocean of knowledge and allowed other scientists to swim in it.

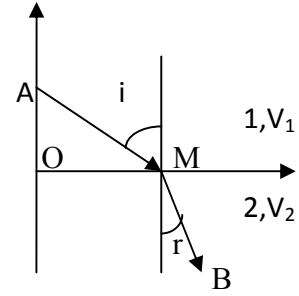
1.2 The Euler calculations of variations

a-The Fermat's principle of least time

The study of the movement of light is performed using the physical wave or using geometrical optics. But Fermat sought the minimum of time of light ray linking point A (0, a) in medium (1) to point B (b, c) of the medium (2) and passing through the point M (x, 0) with V_1 and V_2 are the velocities of the light in the first and the second medium.

$$t(x) = t_1 + t_2 = (|AM| / v_1) + (|MB| / v_2)$$

The time is minimum if: $\frac{\partial t(x)}{\partial x} = 0 \Rightarrow$
 $\sin(i) = n \sin(r)$, with $n = v_1 / v_2$



Fermat found the Descartes law of reflection of light. Descartes was very jealous and sent to Fermat a letter of displeasure.

b-Calculaton of Euler* variations

Generalizing the method of Fermat, Euler (Russia) has sought to find the minimum of the expression

$$I = \int_a^b f(q, q', t) dt \tag{1,1}$$

$$\delta I = 0, \text{ avec } t + \delta t, \delta q(a) = \delta q(b) = 0, \text{ and } \delta(q'(t)) = d(\delta(q))/dt$$

We find by integration by parts of Euler's formula:

$$\frac{d}{dt} \frac{\partial f}{\partial q'} = \frac{\partial f}{\partial q} \tag{1,2}$$

1.3 Analytical Mechanics and principle of least action

a- Lagrange formalism

Lagrange (France, friend of Napoleon) generalized Newton formula on a variety and applying D'Alembert principle of virtual work and found the formula

$$\frac{d}{dt} p = \frac{\partial L}{\partial q}, \text{ with } p = \frac{\partial L}{\partial \dot{q}} \tag{1,3}$$

L = T-V is the Lagrangian, T is the kinetic energy and V is the potential energy.

Comparing this formula and Euler's formula (1,2), Hamilton observed the identity of the two formulas if f=L and replacing I by the action S then we write:

$$S(q(t)) = \int_a^b L(q, q', t) dt \tag{1,4}$$

The principle of least action has various applications:

1 - It is applied in optics and gives the laws of dynamics which led Hamilton to predict a unifying theory "wave-matter."

2 - This principle has applications in various fields: economics, etc..

b- The Hamiltonian function H and the equations of mechanics

Starting from the Lagrangian $L (q , q ' , t)$

And
$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} \quad (1,5)$$

And using Lagrange's formula we find after a simple calculation

$$d (\dot{q} p - L) = dH = - \dot{p} dq + \dot{q} dp = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp \quad (1,6)$$

Hamilton has introduced the function H and found a new system of equations of analytical mechanics or the canonical system:

$$\dot{p} = - \frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad (1,7)$$

By definition, the function $H (q , q ' , t)$ is the Hamiltonian of the system.

And for a conservative system:

$$H (q , p , t) = P^2 / 2 m + V . \quad (1,8)$$

Then Jacobi continued the Hamilton work and found the known Hamilton-Jacobi equation:

$$H (q , \frac{\partial S}{\partial q} , t) + \frac{\partial S}{\partial t} = 0 . \quad (1,9)$$

With:

$$P = \partial S / \partial q .$$

Note that later Heisenberg found important equations in quantum mechanics similar to Hamilton's equations.

c- Physical quantities and the constants of motions

From the analytical mechanics we derive the physical quantities: the coordinates, momentum, the angular momentum and energy.

For a conservative system the angular momentum and energy are independent of time so are constants of motion.

$$\frac{\partial H}{\partial t} = 0, \quad \frac{\partial \vec{L}}{\partial t} = 0, \Rightarrow H = E, \quad \vec{L} = \vec{C} \quad (1,10)$$

Analytical mechanics has various applications including the harmonic oscillator and the Kepler problem with the equation is transformed into an equation of the oscillator by Binet transformation to solve this equation.

2. Quantum Mechanics

The second case is the latest approach to quantum mechanics or the path integral introduced by Dirac and developed by Feynman [3-4] but the methods of calculations by this approach are difficult and beyond the level of the 2nd year. In this work we give only a quick summary of this approach.

It is very important to give a summary of the Hamiltonian approaches of quantum mechanics and the Dirac formalism[5-11].

2.1 Origin of the quantum mechanic

We know that this approach has its origin in a true story:

In a very brighten day in Geneva in 1885, escaping from a balloon salesman's a number of balloon filled hydrogen and in the night he found an emission of radiation picked up from the balloons. Then a newspaper in Geneva "Geneva Gazette" published the wavelengths of radiation and a secondary teacher who has studied only the first year of the faculty found the series known Balmer series.

We must not forget the problem of black-body radiation (stove), which was the basis for the introduction of quanta of the light by Planck, Einstein and Bohr's work (table1-2) which were very important in the development of quantum mechanics.

2.2 Hamiltonian approaches

From the intense research has resulted in two equivalent approaches:

Heisenberg's matrix mechanics and wave mechanics of Schrödinger. The first requires only the appearance of physical quantities observed by experience and this requires the absence of electron orbits at the microscopic level and the search for Hermitian operators representing physical quantities. But the second approach is start from the work of Louis de Broglie who postulated the wave-particle duality, already predicted by Hamilton, as a general property of microscopic objects.

Schrödinger specialist of differential equations, and in a seminar presented a simple rule of correspondence

$$\vec{p} = \frac{\hbar}{i} \left(\frac{\partial}{\partial \vec{r}} \right) \quad (1,11)$$

Which allowed the derivation of the fundamental equation of quantum mechanics:

$$H \psi = i \hbar \left(\frac{\partial \psi}{\partial t} \right) \quad (1,12)$$

Thus Schrödinger find the energy levels of hydrogen atom.

Born gave a probabilistic interpretation of the wave function on the assumption that

$$|\psi(r, t)|^2 \quad (1,13)$$

Represents the probability of finding the particle at position \vec{r} at time t.

Schrödinger also showed the equivalence of the two approaches but Dirac established the general formalism of quantum mechanics.

Finally Dirac**, arrived on the scene as they say, and after two weeks of reading the work of Heisenberg noted that the coordinates and momentums are observables and do not commutes. This involves from physical point of view that we can not simultaneously measure the position and momentum and implies from mathematical point of view that we must have two functions: The first one is based on coordinated and the second is a function of momentum and are deduced from each other by Fourier transformation.

And this has also led Dirac to introduce: a new notation of the space and dual space, the state function of the physical system, the discovery of the delta function known by Dirac function and the presentation of a new formulation of quantum mechanics.

More, Dirac introduced the ladder operators (a^+ , a) or the raising and lowering operators, by analogy with complex numbers z and \bar{z} , for solving the equation of the harmonic oscillator which are the origin of the second quantization, the basis to study the many body problems and the theories of modern physics.

Heisenberg also introduced a new and different representation of the Schrödinger and showed that the transition from state (x_0, t_0) to the state (x, t) can be done using the

evolution operator $U(t, t_0) = e^{-\frac{i}{\hbar}(t-t_0)H}$ if H is independent of time.

It is easy to observe that the operator U satisfies the Schrödinger equation.

$$HU(t, t_0) = i\hbar \left(\frac{\partial U(t, t_0)}{\partial t} \right) \quad (1,14)$$

3. Quantum physics before 1950

In the 50s of last century the situation was as follows:

1 - *The spin*

Pauli introduced spin to explain the famous Stern-Gerlach experiment and defined The state of particles by the product of the spatial wavelength and the state of spin which is written in Dirac notation by $|s, s_z = \pm 1/2\rangle$.

2 - *The Pauli principle*

Pauli divided the particles into two categories:

a- The bosons: particles with integer spin

b- The fermions: particles with half integer spin.

c-The state of the physical system obeys the Pauli principle:

the wave function of bosons is symmetric and anti-symmetric for fermions.

→ The result is that two fermions can not coexist in the same state, in the same place, at the Same time and that ensures the existence of macroscopic matter.

3 - *The octonions algebra in physics*

The success of the analogy between the form of the Hamiltonian of the oscillator and the product of complex number and its conjugate led Dirac, in my opinion, to extend this idea to octonion algebra [12-13] to determine the relativistic equation of electron.

We must not forget that Maxwell wrote at the beginning his electromagnetism theory in terms of quaternions and more quaternions are very useful in classical mechanic (A4), Quantum Mechanics and Quantum Fields [12-18].

4 - The discovery of antimatter

Dirac also predicted on the basis of his famous equation of electron the existence of the anti-electron and the existence of antimatter, which was confirmed after that by the discovery of positrons etc.[8-9]. More matter and antimatter annihilate when they meet.

5 - Structure of the atom

We know that light is composed of photon spin 1. And atoms are made up of Z electrons (e^-) that revolve around the nucleus of Z protons with positive charges, and (A-Z) uncharged neutrons. And all these particles are fermions with spin $\frac{1}{2}$.

6 - Photons and the electromagnetic interaction

In the vacuum the interaction can not occur without exchange of particles:

The electromagnetic interaction has infinite range and which requires the particle exchange, virtual particles, zero mass and we deduce that:

The emission and absorption of photons by atoms is done by light quanta.

7- Electrons - nucleus

The connection between the electrons and the nucleus is due to the electromagnetic interaction and this interaction is due to the exchange of virtual photon with spin 1.

4. The Elementary Particles

Elementary particles [11,19-21] are numerous and have been detected in cosmic rays and from electron-proton reactions. Among these particles the protons and the electrons are stable and the others particles are resonance which decays and their lifetime is very short. And these particles are divided into three categories: hadrons or massive particles (proton, neutron, ...) and light particles or leptons (electron neutrino, ..) and photons, massless particle.

But the story of the discovery of these particles and their interactions is very long and complex, but we will give a brief idea about the interactions in the nucleus and between these elementary particles.

A- Interaction in nuclear physics

In nuclear physics, the interaction is short-range (10-12cm) and the binding energy of the nucleus is very high, for example the energy bonds of the deuteron is 2.2 MeV where the name of strong interactions.

1- Yukawa proposed by an empirical method a potential of nuclear interaction and by analogy with the electric field has predicted the existence of a particle as a quantum field of strong interaction ' the meson' with charge B , from the word baryon.

2 - The difference between proton and neutron masses is very low and it is due to the proton charge and this has led Heisenberg to introduce isospin and nucleons. This means that the proton and neutron are the same particle, the nucleon, with two different states and by analogy with the spin we write $|I, I_z = \pm 1/2\rangle$.

3 - In nuclear physics the interpretation of level energies of Mg poses a great difficulty but Elliot 58 proposed a model based on the SU (3) group by considering the existence of a quadrupole-quadrupole forces between particles and the general agreement with the

experimental levels is quite good [20]. Elliot was also calculated the basis of the representation of SU (3) group.

B-The Elementary Particles

The number of elementary particles exceeded the hundred and we must make a classification similar to the classification of atoms of Mendeleev. And after many unsuccessful trials Gell-Mann and Ne'eman proposed a model for hadrons based on the SU (3) group theory [19-25].

Then Gell-Mann proposed the model of Quarks to avoid the defects in the model. The Quarks model admits the existence of a common substructure of neutron and proton called Quarks. This model is based on well known research of mathematicians and physicists on group theory and these results led to the development of semi-simple lie groups and especially the unitary groups SU (n).

C-The Interactions

To our knowledge the interactions are:

4.1 The electromagnetic interactions:

The quantum of these fields is the photon with spin one and zero mass.

4.2 The strong interactions

Currently we assume that this interaction is due to the interaction between quarks and quantum of interaction of these fields is the gluon, particles with spin 1.

4.3 The weak-interaction

These interactions are due to decay of the neutron and radioactivity β . The quantum field are W^\pm and Z^0 predicted by the electroweak unification theory .

a-This interaction is very important for the creation of energy in stars.

b-This interaction does not respect the important property of left-right symmetry.

"Some researchers believe that this interaction is responsible of the sense of: rotation of the planets and rolling of DNA etc."

4.4 The interaction of gravitation.

4.5 Symmetry

Finally, we can not talk about the matter and the laws of nature without talking symmetries and invariances, especially the left and right symmetry, rotational invariance, the representation of groups especially semisimple Lie algebra [19-25]. Therefore we summarize the symmetry and the study of SU (3) in the chapters at the end of this work.

* Euler was the son of a Swiss baker and was sent to school early. After some months the child asked the teacher: wrote me a set of numbers and I give you on the sum and the product of these numbers. The teacher was amazed and at 3h30 am went to the bakery and told the father that your son is a genius and you are unable to educate him. I suggest that you send him to the Bernoulli family of mathematicians.

** English physicist and electrical engineer. He was very influenced by the work of Heisenberg and by the English School: Maxwell, Hamilton, Cayley, etc..

II- Schrödinger's theory and harmonic oscillator

1. The Schrödinger equation

1.1 de Broglie wave function and Schrödinger equation

The experience of the electron diffraction shows the wave aspect of particles and the classical conceptions are at an impasse. de Broglie suppose that the wave function of electrons is a monochromatic plane wave of the form:

$$\psi (r , t) = \exp[- i (\omega t - \vec{p} \cdot \vec{r} / \hbar)] \quad (2,1)$$

And which has a wavelength $\lambda = \hbar / p$

de Broglie put: $\hbar\omega = E = p^2 / 2m$

And the wave function is then written:

$$\psi (\vec{r} , t) = c \exp[- i (Et - \vec{p} \cdot \vec{r} / \hbar)]$$

It is simple to verify that

$$i \hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \Delta \psi \quad (2,2)$$

This equation is the quantum representation of movement in the absence of interaction. Schrödinger added the potential V and thus obtained the equation that bears his name.

$$i \hbar \frac{\partial \psi}{\partial t} = (T + V) \psi = H \Psi \quad (2,3)$$

Where T is the kinetic energy:

$$T = \vec{p}^2 / (2m), \quad \vec{p}^2 = -\hbar^2 \Delta \quad (2,4)$$

1.2 The functions of stationary states

a- If H is independent of time we write:

$$\psi (r , t) = \exp[- i (Et)] \varphi (r) \quad (2,5)$$

And the Schrodinger equation becomes

$$H \varphi = E \varphi$$

For an isolated system the states is described by the wave functions solutions of the equation :

$$H \varphi_n (r) = E_n \varphi_n (r) \quad (2,6)$$

And $\varphi_n (r)$ are called stationary states.

The set of the eigenvalues $\{E_n\}$ of H are the energy levels.

These wave functions are square integrable functions with:

$$(\varphi_m, \varphi_n) = \int \overline{\varphi_m (r)} \varphi_n (r) d^3 r = \delta_{m, n}, \quad (2,7)$$

And $d^3 r = dx dy dz$

- The set of wave functions $\{\varphi_n\}$ form the pre-Hilbert space H.

- Also be noted that any linear superposition of the stationary states is a wave function of the physical system.

2. Transformations and linear operators

2.1 Linear transformations and Euclidian space

2.1.1 We consider the linear transformations in $E_n(K)$

$$a : E_n \rightarrow E_n \quad (2,8)$$

With
$$a(e_i) = \sum_j a_{i,j} e_j$$

The matrix $A = (a_{i,j})$ is the matrix representation of the application a.

- The matrix A is hermitian if $(a_{i,j}) = {}^t(\bar{a}_{j,i})$.

2.1.2 Linear transformations which play an important role in physics are:

1-The special orthogonal groups SO_n or $SO(n)$ are linear transformations with:

$${}^t O_n O_n = I \quad \text{With } \det(O_n) = 1 \quad (2,9)$$

2-The unitary groups, SU_n , or $SU(n)$ are linear transformations with:

$${}^t \bar{U}_n U_n = I \quad \text{With } \det(U_n) = 1 \quad (2,10)$$

3 - The infinitesimal generators of the group

The elements A of the group G is composed of non-singular matrices of degree n and is expressed as a function of r continuous parameters [24]:

$$A = A(\alpha_1, \dots, \alpha_r).$$

We have the important formula of group theory:

$$Of(x) = f({}^t o(x)), \quad x \in E_n \quad \text{And} \quad Uf(z) = f({}^t u(z)), \quad z \in C_n \quad (2,11)$$

The infinitesimal generators of the group are:

$$X_k = \left(\partial A / \partial \alpha_k \right)_{\alpha=0} \quad (2,12)$$

From the conditions (2,9-10) we find that the number of parameters of $SO(n)$ is $n(n-1)/2$ and $SU(n)$ is n^2-1 .

2.2 Operators in Hilbert space

The commutator of the two operators A and B is defined by

$$[A, B] = AB - BA$$

We deduce that:

$$[A, B+C] = [A, B] + [A, C],$$

$$[A, BC] = [A, B]C + B[A, C].$$

$$[A, B] = -[B, A] = AB - BA \quad (2,13)$$

a- Property: if $[A, B] = 0$, then the operators A and B have the same eigenvectors.

b-If we consider the linear transformation A we have:

$$(\varphi, A \psi) = ({}^t A \varphi, \psi) \quad (2,14)$$

c-The operator A is Hermitian if

$$A = A^* = {}^t \bar{A} \quad (2,15)$$

d- We can simply prove the following theorem:

Theorem: The eigenvalues of a Hermitian operator are real and the eigenvectors are Orthogonal.

Casimir operators:

We denote by Casimir operators, the Hermitian operators that commute with the generators of SO (n) or SU (n).

e- **Lemma:** If λ is a real number around zero we write:

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \dots \quad (2,16)$$

3. Observable and Schrödinger quantization rules

3.1 Observable and the wave function

The results of experimental measurements of the position, energy, kinetic and angular moments are real numbers so the operators x, y, \dots, L_z are Hermitians and hence are observables.

a- We can not measure simultaneously observable unless they commute.

b- The proper function of all commuting observables, H, \dots , are the wave function of the physical system.

3.2 The Schrödinger quantization rules

3.2.1 Born interpreted the square of the modulus of the wave function $|\psi(x)|^2$ by probability of presence of the particle at the point x

3.2.2 If $x = \pm\infty$ $|\psi(\pm\infty)|^2 = 0$ the particle are not found in the infinity.

we write

$$\int_{-\infty}^{+\infty} \overline{\psi(x)} \left(\frac{d\psi(x)}{dx} \right) dx = - \int_{-\infty}^{+\infty} \frac{d\overline{\psi(x)}}{dx} \psi(x) dx + \overline{\psi(x)} \psi(x) \Big|_{-\infty}^{+\infty}$$

It is clear that the operator p_x must be Hermitian:

$$p_x = \frac{\hbar}{i} \frac{d}{dx}$$

If $\hbar = k$ we get the Schrödinger quantization rule.

3.3 The commutator operators of x and p

$$[x, p_x] f(r) = \frac{\hbar}{i} (x f'(r) - \frac{d}{dx} (x f(r))) = i \hbar f(r)$$

Then $[x, p_x] = i\hbar \quad (2,17)$

As a result, we can not simultaneously measure x and p_x .

Dirac deduced from this relationship the wave function in phase space.

3.4 The commutation of the operators L_x, L_y, L_z

Using the formula (2,17) we find:

$$[L_x, L_y] = i\hbar L_z, [L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y \quad (2,18)$$

The Casimir operator $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ commutes with L_x, L_y, L_z .

So if $V = V(|\vec{r}|)$ the wave function is eigenfunctions of \vec{L}^2, L_z and H.

3.5 The wave function of the physical system

The wavelength of a non-interacting physical system function is the tensor product of individual wave functions. But for interacting system the wave function is developed on the wave functions of non-interacting system.

4. The Heisenberg uncertainty relations

4.1 Average values of observables

The average value of the operator F is:

$$\begin{aligned} \langle F \rangle &= \int \bar{\psi}(r) F(x, y, z) \psi(r) d\vec{r} \\ \langle \Delta F \rangle^2 &= \langle (F - \langle F \rangle)^2 \rangle \end{aligned} \quad (2,19)$$

4.2 The uncertainty relation

We consider the integral:

$$I(\lambda) = \int \left| x\psi + \lambda \hbar \frac{d\psi}{dx} \right|^2 dx \geq 0, \quad \lambda \in R$$

The development of this integral gives

$$I(\lambda) = \langle x^2 \rangle + \lambda \hbar \int \left(\frac{d\bar{\psi}}{dx} x\psi + \bar{\psi} x \frac{d\psi}{dx} \right) dx + \langle p_x^2 \rangle \geq 0$$

using $[x, p_x] = i \hbar$ and after integration by parts we find:

$$I(\lambda) = \langle x^2 \rangle - \lambda \hbar + \lambda^2 \langle p_x^2 \rangle \geq 0$$

The discriminant of the quadratic equation in either negative or zero:

$$-4 \langle x^2 \rangle \langle p_x^2 \rangle + \hbar^2 \leq 0$$

If we replace x and p_x by the standard deviations we find:

$$\langle \Delta x \rangle^2 \langle \Delta p_x \rangle^2 \geq \hbar^2 / 4$$

We derive the Heisenberg uncertainty relations

$$\langle \Delta x \rangle \langle \Delta p_x \rangle = \hbar / 2, \dots \langle \Delta z \rangle \langle \Delta p_z \rangle = \hbar / 2 \quad (2,20)$$

-This relationship means that if the wave is close to x this implies that the wave in the phase space is very spread out.

-There is the uncertainty relation time -Energy:

$$\langle \Delta E \rangle \langle \Delta t \rangle = \hbar / 2 \quad (2,21)$$

5. The Harmonic oscillator

The Hamiltonian of the one dimension harmonic oscillator is:

$$H = \frac{p^2}{2m} + m\omega^2 x^2 \quad (2,22)$$

The Schrödinger equation is:

$$H \psi_n(x) = E_n \psi_n(x) . \quad (2,23)$$

Put $x = (\hbar / m\omega)q$, et $p = (m\omega\hbar)P$, $\alpha = m\omega / \hbar$, $E = \hbar \omega \epsilon$

Thus we find the following equation:

$$\widehat{H}u(q) = (P^2 + q^2)u(q) / 2 \text{ with } H = \hbar\omega \widehat{H} \quad (2,24)$$

If we make the change of function:

$$u(q) = f(q) e^{-q^2/2}$$

We find the equation:

$$\frac{d^2}{dq^2} f(q) - 2q \frac{d}{dq} f(q) + (2\epsilon - 1) f(q) = 0 \quad (2,25)$$

The resolution is done by writing f(q) in the form of a series:

$$f(q) = a_0 + a_1 q + a_2 q^2 + \dots \quad (2,26)$$

If the solutions are bounded at infinity we find that {f(q)} are the Hermite polynomials $H_n(q)$ [2,30] and the wave function is:

$$u_n(q) = (\sqrt{\pi} 2^n n!)^{-1/2} e^{-q^2/2} H_n(q)$$

And

$$\begin{aligned} u_n(x) &= ((m\omega) / \hbar)^{1/4} u_n(q) \\ u_n(-x) &= (-1)^n u_n(x) \end{aligned} \quad (2,27)$$

1 - These eigenfunctions form the Hilbert basis of $L_2(\mathbb{R})$ and $(-1)^n$ is the parity.

2 - The eigenvalues of the energy of the oscillator are:

$$E_n = \hbar \omega (n + 1 / 2) \quad (2,28)$$

6. Oscillator generating function and analytic Hilbert space

6.1 The generating function of the harmonic oscillator is known since long time [1-9] and is written as:

$$G(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} u_n(q) = \pi^{-1/4} \exp\{\sqrt{2}qz - \frac{q^2}{2} - \frac{z^2}{2}\} \quad (2,29)$$

6.2 If the parameter z is complex we find that the functions $f_n(z) = z^n / \sqrt{n!}$ form a basis of the analytic Hilbert space known by Fock or Fock-Bargmann space [21].

With

$$\langle f_n | f_m \rangle = \iint \overline{f_n(z)} f_m(z) d\mu(z) = \delta_{m,n} \quad (2,30)$$

$d\mu(z)$ Is the cylindrical measure:

$$d\mu(z) = e^{-(x^2+y^2)} dx dy / \pi, \quad z = x + iy \quad (2,31)$$

6.3 It is easy to show the following useful formulas [25]:

$$f(z) = \int f(z') e^{z\bar{z}'} d\mu(z') \text{ And } \int e^{\alpha\bar{z}} e^{\beta z} d\mu(z) = e^{\alpha\beta} \quad (2,32)$$

III- Dirac formalism of quantum mechanics

After reading the work of Heisenberg, Dirac observed several unresolved problems:

- 1 – Definition of the state of the physical system.
- 2 - x, p_x are non-commuting observables then we must find the eigenfunctions of x , and p_x .
- 3 - The state of the oscillator.
- 4 - The connection between quantum and classical mechanics.

Dirac introduced a new notation for the Euclidean space and the Hilbert space which leads to a new formalism of quantum mechanics and he find a new function: the delta function which is at the origin of the theory of distributions.

We present first the Dirac notation in the simplest way, starting from the Euclidean space, and then we extend it to the Hilbert space, space of Schrödinger wave functions then we introduce the delta function and the eigenfunctions of momentum p_x .

1. Euclidean Space and Dirac notation

The search for eigenfunctions of the observable (x) led Dirac to introduce a new notation for the vector space and the dual space from the scalar product notation. Using this new notation Dirac find the δ -function and the well known Dirac transformation.

1.1 Dirac's notation

The scalar product of two vectors is the bracket:

$$\vec{x} \cdot \vec{y} = (\vec{x}, \vec{y}) = \langle \vec{x} | \vec{y} \rangle \quad \text{bra} \downarrow \text{ket} \quad (3,1)$$

We cut the bracket in two word bra and ket.

We denote the vectors y of the space by $\text{ket } \{|y\rangle\}$ and the vectors x^* of the dual space by the bra $\{\langle x|\}$.

a) The Euclidean space E_3

In Dirac notation the basis vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of the three-dimensional Euclidean space E_3 may be written by $(|1\rangle, |2\rangle, |3\rangle)$.

b) The dual space E_3^*

By definition, the dual space is the space of linear applications E_3 in \mathbb{R} .

E_3^* Is the dual space and we have the basis $\vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*$.

In Dirac notation: $\vec{e}_1^* = \langle 1|, \vec{e}_2^* = \langle 2|, \vec{e}_3^* = \langle 3|$

If K is the field of the complex number \mathbb{C} , \vec{x} must be replaced by $\vec{\bar{x}}$.

c) The identity operator

1-The projector p_i is defined by: $p_i = |i\rangle\langle i|$ With $p_i^2 = p_i$ (3,2)

2- Completeness or the identity operator: $I = \sum_i |i\rangle\langle i|$ (3,3)

1.2 Applications:

a-Let $|x\rangle = \sum x_i |i\rangle$
 We write $|x\rangle = I|x\rangle = \sum_i \langle i|x\rangle |i\rangle$

b- In Dirac notation the linear transformation a is written:

$$a|i\rangle = I \times a|i\rangle = \sum \langle i|a|j\rangle |i\rangle,$$

With $x_i = \langle i|x\rangle$, and $a_{i,j} = \langle i|a|j\rangle$ (3,4)

1.3 Rotation in Euclidean space E (k = R or C)

1 - The rotation matrix which leaves invariant the scalar product in the $E_3(\mathbb{R})$ is known by the orthogonal matrix of $SO(3)$ with the Euler angles $(\Omega) = (\psi, \theta, \varphi)$.

$$R(\Omega) = e^{-i\psi S_z} e^{-i\theta S_y} e^{-i\varphi S_z} \quad (3,5)$$

The representation matrix of spin 1 or Pauli matrices [9] are:

$$\sigma_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (3,6)$$

With $[S_x, S_y] = i\hbar S_z$, $[S_y, S_z] = i\hbar S_x$, $[S_z, S_x] = i\hbar S_y$

2- The rotation matrix R_s which leaves invariant the scalar product in $E_2(\mathbb{C})$ is a unitary matrix of $SU(2)$ and the parameters are the Euler angles ψ, θ et φ with $\vec{s} = \frac{\vec{\sigma}}{2}$ is a spin half and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3,7)$$

2. Hilbert space and Dirac notation

2.1 Hilbert space and linear operators

Let H be a space of function with basis $\{\varphi_i\}$ and

$$(\varphi_i, \varphi_j) = \int_{\Delta} \varphi_i(x) \varphi_j(x) p(x) dx = \delta_{i,j} \quad (3,8)$$

$p(x)$ Is the weight function and the measure of integration is:

$$d\mu(x) = p(x) dx$$

The space of wave functions $\{\varphi_i(x)\}$ is a space of square integrable functions which is pre-Hilbert i.e. not complete.

2.2 Hilbert space and Dirac notation

a-The Hilbert space in Dirac notation is: $\{|\varphi_i\rangle\}$

b The dual-space is: $\{\langle\varphi_i| \text{ and } \langle\varphi_i|\varphi_j\rangle = \delta_{i,j}\}$

c-The identity operator is: $\{ I = \sum_i | \varphi_i \rangle \langle \varphi_i | \}$ (completeness condition)

d-The operator which leaves invariant the scalar product in the Hilbert space is a unitary operator U has the form [24]:

$$U = e^{-i\alpha} u, \quad u = e^{-iF} \quad (3,9)$$

F is a Hermitian operator and $e^{-i\alpha}$ is the group U_1 and u belongs to the sub-unitary groups of operators $su(n)$.

- The irreducible space of the unitary groups are finite spaces, disjoint and invariant under unitary group $su(n)$.

3. Dirac transformation and δ -function

3.1 Continuous states of position and Dirac notation

a- Continuous states and Dirac notation

Dirac noted that the measurement of the position and the momentum $p_x = p$ implies that x and $p_x = p$ are the eigenvalues of the observable (x) and (p) that have continuous eigenvectors with:

$$\hat{x}|x\rangle = x|x\rangle, \text{ And } \hat{p}_x|p_x\rangle = p|p\rangle, \quad (3,10)$$

The identity operator I in the continuous case can be written as:

$$I = \int |x\rangle dx \langle x|, \quad I = \int p dp \langle p|, \quad (3,11)$$

b- Dirac distribution or Dirac function

The function Dirac $\delta(x'-x)$ is the normalization of the continuous space $\{|x\rangle\}$

$$\underline{\langle x'|x\rangle} = \delta(x'-x) \text{ with } \delta(x'-x) = \begin{cases} 0 & \text{if } x' \neq x \\ \infty & \text{if } x' = x \end{cases} \quad (3,12)$$

Dirac introduced this notation by analogy with Kronecker symbol δ_{ij} .

We will determine its expression in the next section.

The first important property:

$$\langle x'|\hat{x}|x\rangle = x' \langle x'|x\rangle = x \langle x'|x\rangle \quad (3,13)$$

As a result: $(x'-x)\delta(x'-x) \text{ or } \underline{\underline{x\delta(x) = 0}} \quad (3,14)$

3.2 The transformation of Dirac and its applications

We consider the scalar product

$$\langle g, f \rangle = \int_{-\infty}^{+\infty} \overline{g(x)} dx f(x)$$

In Dirac notation the scalar product is:

$$\langle g || f \rangle = \int_{-\infty}^{+\infty} \overline{g(x)} dx f(x) \quad (3,15)$$

Using the identity operator (3,11) we write:

$$(g, f) = \langle g | I | f \rangle = \int_{-\infty}^{+\infty} \langle g || x \rangle dx \langle x || f \rangle dx \quad (3,16)$$

After comparison of the two above expressions, Dirac has defined the function $f(x)$ as the application: $H \rightarrow R$.

$$\underline{\underline{\langle x | : | f \rangle \xrightarrow{\langle x |} \langle x || f \rangle = f(x) \in R}} \quad (3,17)$$

And
$$\underline{\underline{\langle g || x \rangle = \overline{g(x)}}} \quad (3,18)$$

We also deduce the very useful expressions:

$$\langle x | \hat{x} | f \rangle = xf(x), \quad \langle x | \hat{p}_x | f \rangle = \frac{\hbar}{i} f'(x) \quad (3,19)$$

** The state of the system is written by $|\varphi\rangle$ in Dirac notation.

3.2.1 Second important property of delta function:

We have:
$$f(x_0) = \langle x_0 | I | f \rangle = \int \langle x_0 || x \rangle dx \langle x || f \rangle$$

So
$$f(x_0) = \int \delta(x_0 - x) f(x) dx \quad (3,20)$$

3.2.2 The eigenfunctions of the momentum operators

We assume that the eigenfunctions of the momentum p is the continuous function $p(x)$. We write:

$$\langle x | \hat{p} | p \rangle = \frac{\hbar}{i} p'(x) = p \langle x || p \rangle = p p(x)$$

Therefore
$$p(x) = c \exp(ipx / \hbar) = c \exp(ikx) \quad (3,21)$$

3.2.3 The {p} representing

If the wave function of the system is $\varphi(x)$ we write:

$$\boxed{\langle x || \varphi \rangle = \varphi(x)} \quad (3,22)$$

The wave function in the {p} representation or the phase space can be written as:

$$\langle p || \varphi \rangle = \varphi(p) = \langle p | I | \varphi \rangle = \int_{-\infty}^{+\infty} \langle p || x \rangle dx \langle x || \varphi \rangle$$

But
$$\langle p || x \rangle = c e^{-ikx} \quad (3,23)$$

As a result we find:
$$\varphi(p) = c \int_{-\infty}^{+\infty} e^{-ikx} \varphi(x) dx \quad (3,24)$$

3.2.4 Analytical expression of Dirac function

We write
$$\langle x || x' \rangle = \langle x | I | x' \rangle = \int_{-\infty}^{+\infty} \langle x || p \rangle dp \langle p || x' \rangle \quad (3,25)$$

So we get:
$$\langle x || x' \rangle = \delta(x' - x) = |c|^2 \int_{-\infty}^{+\infty} e^{ip(x-x')} dp \quad (3,26)$$

We shall calculate the normalization constant c in chapter V.

IV- Heisenberg representation and the path integral

1. The evolution operator and Heisenberg representation

In this chapter we only want to give an idea of the path integral [3,26] and we assume that the linear superposition of states is preserved through time.

We write the Schrödinger representation by:

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \quad (4,1)$$

For a conservative system the evolution operator $U(t, t_0)$ satisfies the Schrödinger equation:

$$i\hbar \frac{d}{dt} U(t, t_0) = H U(t, t_0) \quad (4,2)$$

If we consider the inverse of the transformation, we write:

$$|\psi(t_0)\rangle = U^*(t, t_0)|\psi(t)\rangle \quad (4,3)$$

And any observable A in the Schrödinger representation “or picture” may be written in the Heisenberg picture of the form:

$$A_H = U^*(t, t_0) A U(t, t_0) \quad (4,4)$$

By a simple calculation we derive the Heisenberg equation

$$i\hbar \frac{dA_H}{dt} = [A_H, H_H] + i\hbar \frac{\partial A_H}{\partial t} \quad (4,5)$$

The Heisenberg equations for the coordinates $r = (x_1, x_2, \dots, x_n)$ and moments (p_1, p_2, \dots) is:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{1}{i\hbar} [x_i, H] = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} &= \frac{1}{i\hbar} [p_i, H] = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, \dots \end{aligned} \quad (4,6)$$

This system of equations is formally identical to the Hamilton's canonical system.

2. The path integral and the Feynman propagator

Using the unit operator $I = \sum |\psi_i\rangle \langle \psi_i|$ we write the Feynman propagator, or the Green's function, in the form:

$$K(r', t | r, t_0) = \langle r' | e^{-i(t-t_0)H} | r \rangle = \langle r' | e^{-i(t)H} I e^{-i(-t_0)H} | r \rangle \quad (4,7)$$

Thus we find:

$$K(r', t | r, t_0) = \sum \overline{\psi_i(r', t)} \psi_i(r, t_0) \quad (4,8)$$

Using the unitary operators (3,11) we find that

$$\int d^3 r' K(r, t | r', t') K(r', t' | r_0, t_0) = K(r, t | r_0, t_0) \quad (4,9)$$

This completeness condition is valid for $t > t'$, $t, t' \in [t_0, t_1]$.

The generalization of this relation to $n - 1$ intermediate point $t > t_{n-1} > t_{n-2} > \dots > t_1 > t_0$. Then we write:

$$K(r, t | r_0, t_0) = \int d^3 r \int d^3 r \dots \int d^3 r$$

$$K(r, t | r', t') K(r', t' | r_0, t_0) \dots K(r, t | r', t')$$
(4,10)

If the intermediate time is continuous we write the propagator using the symbols of the path integral in general as

$$K(r, t | r_0, t_0) = \int_{r(t_0)=r_0}^{r(t)=r} d[r(t)] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_n} L_c(r, \dot{r}, t) dt \right\} \quad (4,11)$$

- The Feynman conjecture

Feynman's conjecture is to introduce in the integral the classical Lagrangian and to identify the normalization factor by the coefficient of the propagator of the free particle (below) by putting:

$$d[r(t)] = \lim_{\substack{n \rightarrow \infty \\ \Delta t \rightarrow 0}} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{n-1}{2}} dx_1 dx_2 \dots dx_n \quad (4,12)$$

And $K(r_f, t_f | r_i, t_i) = \int_{r_i, t_i}^{r_f, t_f} d[r(t)] \exp \left\{ \frac{i}{\hbar} S(x(t)) \right\} dt \quad (4,13)$

$S(x(t))$ Is the classical Hamilton-Jacobi action.

3. The propagators of a free particle

The propagator of free particle is

$$K(x', t' | x, t) = \langle x' | \exp[\frac{i}{\hbar} (t' - t) H] | x \rangle \quad (4,14)$$

Using the unit operator

$$I = \int | p \rangle dp \langle p |$$

And $\langle x | p \rangle = \exp(i x p / \hbar) / \sqrt{2 \pi \hbar} \quad (4,15)$

We obtain the expression of the propagator:

$$K(x', t' | x, t) = \frac{1}{2 \pi \hbar} \int dp \exp \left\{ \frac{i}{\hbar} [p(x' - x) + (t' - t) p^2 / 2 m] \right\}$$

The evaluation of the integral is obtained by using the Gauss integral:

$$K(x', t' | x, t) = \sqrt{\frac{m}{2 \pi i \hbar \Delta t}} \exp \left\{ \frac{i}{\hbar} \frac{m (x' - x)^2}{2 \Delta t} \right\} \quad (4,16)$$

With $\Delta t = (t' - t)$.

V-The harmonic oscillator and the Dirac notation

We will present at the beginning Dirac method [8-9] which is based on the observation that the expression of the Hamiltonian has the form of product of a complex number and its complex conjugate. We find the expression of the generating function by a new method and then we derive the wave function of the harmonic oscillator.

Using the generating function and the Fock-Bargmann space we simply calculate the properties of the δ -function without recourse to the theory of distributions, the normalization of the free wave and the Feynman propagator of the harmonic oscillator.

1. The state of the harmonic oscillator

Dirac noted that the Hamiltonian of the oscillator:

$$H = \frac{p^2}{2m} + m\omega^2 x^2 \quad (5,1)$$

is analogous to the product $z \cdot \bar{z}$, $z \in \mathbb{C}$ and he posed:

$$a = \frac{\sqrt{2}}{2}(Q + iP), \quad a^+ = \frac{\sqrt{2}}{2}(Q - iP)$$

With
$$x = \left(\frac{\hbar}{m\omega}\right)^{1/2} Q \quad p = (m\hbar\omega)^{1/2} P$$

So
$$\hat{H}u_n(q) = (a^+a + 1/2)u_n(q) = E_n u_n(q) \quad (5,2)$$

a, a^+ Are adjoint operators with:
$$[a, a^+] = aa^+ - a^+a = 1. \quad (5.3)$$

1.1 The basis of the oscillator

It is well known that the basis of polynomials $P(z)$ is:

$$1, z, z^2, z^3, \dots, z^n, \dots$$

with
$$\frac{dz^n}{dz} = nz^{n-1} \quad \text{And} \quad \frac{d}{dz}1 = 0$$

Comparing $[a, a^+] = 1$ and $[(d/dz), z] = 1$ we simply deduct the basis of the harmonic oscillator:

$$|0\rangle, |1\rangle, \dots, |n\rangle = \frac{a^{+n}}{\sqrt{n!}}|0\rangle, \dots \quad \text{and} \quad a|0\rangle = 0. \quad (5,4)$$

With
$$a|n\rangle = n|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle \quad (5,5)$$

So a and a^+ are the creation and annihilation or the ladders operators of the harmonic oscillator.

The wave function is written using the Dirac transformation by:

$$u_n(q) = \langle q || n \rangle = \langle q | \frac{a^{+n}}{\sqrt{n!}} | n \rangle \quad (5,6)$$

1.2 From the isomorphism between the space of the oscillator and the Fock space $f_n(z) = z^n / \sqrt{n!}$ we deduce the correspondence:

$$a^+ \rightarrow z \text{ and } a \rightarrow \frac{d}{dz} \quad (5,7)$$

1.3 We also write formally: $af(a^+)|0\rangle = \frac{\partial f(a^+)}{\partial a^+}|0\rangle$ (5,8)

This formula is very useful for fast calculations in the oscillator basis.

2. The generating function of the harmonic oscillator

We will determine the generating function at the beginning by a new method [25] and we simply deduce the wave function of the oscillator in the coordinates representation. The generating function of the oscillator can be written with the help of (5,6) in the form:

$$G(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} u_n(q) = \sum_{n=0}^{\infty} \langle q | \frac{z^n (a^+)^n}{n!} | 0 \rangle = \langle q | e^{za^+} | 0 \rangle \quad (5,9)$$

Using the transformation of Dirac and (5,8) we find:

a-
$$\langle q | a e^{za^+} | 0 \rangle = \frac{1}{\sqrt{2}} \left(q + \frac{d}{dq} \right) G(z, q) \quad (5,10)$$

b-
$$\langle q | a e^{za^+} | 0 \rangle = z G(z, q) \quad (5,11)$$

Comparing the two expressions (a) and (b) we obtain:

$$\frac{d}{dq} G(z, q) = (\sqrt{2}z - q) G(z, q) \quad (5,12)$$

The solution of this equation is:

$$G(z, q) = c \exp \left\{ (\sqrt{2}qz - \frac{q^2}{2}) + \varphi(z) \right\} \quad (5,13)$$

To determine $\varphi(z)$ we use the above expressions and:

$$\langle q | a^+ e^{za^+} | 0 \rangle = \frac{1}{\sqrt{2}} \left(q - \frac{d}{dq} \right) G(z, q) = \frac{\partial}{\partial z} G(z, q) \quad (5,14)$$

We find: $\varphi'(z) = -z$

The solution of the above expression gives the generating function

$$G(z, q) = c \exp \left\{ \sqrt{2}qz - \frac{q^2}{2} - \frac{z^2}{2} \right\} \quad (5,15)$$

For $t = 0$ we find:

$$G(0, q) = \langle q | 0 \rangle = u_0(q) \quad (5,16)$$

The normalization of $u_0(q)$ is $c = \pi^{-1/4}$:

$$G(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} u_n(q) = \pi^{-1/4} \exp \left\{ \sqrt{2}qz - \frac{q^2}{2} - \frac{z^2}{2} \right\} \quad (5,17)$$

c-The development of the second member and the comparison with the first member gives the harmonic oscillator wave function (2,27)

$$u_n(q) = (\sqrt{\pi} 2^n n!)^{-1/2} e^{-\frac{q^2}{2}} H_n(q), \quad u_n(x) = ((m\omega)/\hbar)^{1/4} u_n(q) \quad (5,18)$$

d- From the transformations (2,29-31) we deduce:

$$\int G(z, q) u_n(q) = f_n(z), \quad \int G(z, q) \overline{f_n(z)} d\mu(z) = u_n(q) \quad (5,19)$$

And the kernel function is: $\langle G(z', q) | G(z, q) \rangle = e^{\bar{z}'z}$,

e- We note that the representation $\{p\}$ or phase space of the oscillator is simply calculated using the Gauss integral.

3. Properties of delta function

As a direct application of the generating function and the orthogonality of analytic Hilbert space, Fock or Fock-Bargmann, is the derivation of the properties of the delta function (Dirac distribution) and the normalization of the free wave:

We write

$$\langle q | q' \rangle = \langle q | I | q' \rangle = \sum_n \overline{u_n(q)} u_n(q') \quad (5,20)$$

This expression is invariant under the change $(-q, q')$ so the delta function is an even more clarifies the convergence in the function space H !!.

From the orthogonality of the Fock space (2,30) we find:

$$\langle q | q' \rangle = \int \sum_{i,j} \overline{u_i(q)} \frac{\bar{z}^i}{\sqrt{i!}} \frac{z^j}{\sqrt{j!}} u_j(q') d\mu(z) = \int \overline{G(z, q)} G(z, q') d\mu(z) \quad (5,21)$$

By performing the integration using the Gaussian integral we obtain:

$$\begin{aligned} \langle q | q' \rangle &= \delta(q - q') \\ &= \frac{1}{2\pi} \exp\left(-\frac{1}{4}(q - q')^2\right) \int \exp(+ik(q - q')) dk \end{aligned} \quad (5,22)$$

Integrating this expression gives:

$$a- \quad \int_{-\infty}^{+\infty} \delta(q - q') dq' = 1 \quad (5,23)$$

And using the expression $q \delta(q) = 0$ we find:

$$b- \quad \langle q || q' \rangle = \delta(q - q') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(q-q')k} dk \quad (5,24)$$

c- Then the constant normalization of the free wave is $c = 1/\sqrt{2\pi}$.

So we found the properties of the delta without recourse to the theory of distributions function.

4. The Feynman propagator of the oscillator

The Feynman propagator of the oscillator was determined by several methods: the first one is the path integral, the second is the method of the Schwinger Green function, the third is the algebraic method and finally the method of direct calculation using the

formula of Mehler [4]. Calculations by these methods are complicated and all the books teachings give only the result.

We propose to do the calculations by an elementary method:

1- The Feynman propagator of the oscillator is:

$$\begin{aligned} K((x, t), (x', t_0)) &= \langle x | e^{-\frac{i}{\hbar}H(t-t_0)} | x' \rangle = \langle x | e^{-\frac{i}{\hbar}H(t-t_0)} I | x' \rangle \\ &= e^{-i\omega(t-t_0)/2} \sum_n \overline{u_n(x)} e^{-in\omega(t-t_0)} u_n(x') \end{aligned} \quad (5,25)$$

By using the Fock-Bargmann space as above (5,20) we write:

$$\begin{aligned} K((x, t), (x', t_0)) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-i\omega(t-t_0)/2} \times \\ &\int G(e^{-i\omega(t-t_0)/2} \bar{z}, q) G(e^{-i\omega(t-t_0)/2} z, q') d\mu(z) \end{aligned} \quad (5,26)$$

And using the expression of the generating function and asking we find:

$$\left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-i\alpha/2} \int \exp\left[-\frac{q^2 + q'^2}{2} + e^{-i\frac{\alpha}{2}} (\bar{z}q + zq')\sqrt{2} - \frac{\bar{z}^2 + z^2}{2} e^{-i\alpha}\right] d\mu(z)$$

2- After integration we obtain:

$$\begin{aligned} K((x, t), (x', t_0)) &= \frac{1}{\pi} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-i\alpha/2} \times \\ &\exp\left[\frac{i}{2\sin\alpha} [(q^2 + q'^2) \cos\alpha - 2xx']\right] \times E_1 \times E_2 \end{aligned}$$

$$\text{With } E_1 = \int \exp\left[-(2e^{-i\frac{\alpha}{2}} \cos\frac{\alpha}{2}) \left(x - \frac{\sqrt{2}}{4\cos\frac{\alpha}{2}} (q + q')\right)^2\right] dx \quad (5,27)$$

$$\text{And } E_2 = \int \exp\left[-(2ie^{-i\frac{\alpha}{2}} \sin\frac{\alpha}{2}) \left(y + \frac{\sqrt{2}}{4\sin\frac{\alpha}{2}} (-q + q')\right)^2\right] dy \quad (5,28)$$

$$\text{But } 2e^{-i\frac{\alpha}{2}} \cos(\alpha/2) = (1 + e^{-i\alpha}) \quad 2ie^{-i\frac{\alpha}{2}} \sin(\alpha/2) = (1 - e^{-i\alpha}),$$

$$\text{And } e^{-i\alpha/2} = 1/\sqrt{e^{+i\alpha}}.$$

The integral of Gauss is: $\int_{-\infty}^{+\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}}$, with $\text{Re}(a) > 0$

3- After integration of E_1 and E_2 we find:

$$\frac{1}{\sqrt{e^{+i\alpha}}} \times E_1 \times E_2 = \pi \sqrt{\frac{1}{2i\sin\alpha}} \text{ with } |1 \pm \cos\alpha| > 0 \quad (5,29)$$

Finally we get the Feynman propagator:

$$K((x, t), (x', t_0)) = \sqrt{\frac{m\omega}{2\pi\hbar i \sin\alpha}} \exp\left[\frac{i}{2\sin\alpha} [(q^2 + q'^2) \cos\alpha - 2qq']\right] \quad (5,30)$$

VI- Central potential and hydrogen atom

The problem of hydrogen (AH), has played a central role in the development of quantum mechanics [27-29]. Schrödinger has resolved this problem and found the wave function in term of the coordinate representation and the energy spectrum.

1. Central Potential

The Hamiltonian is of the form:

$$H = \frac{p^2}{2\mu} + V(r) = \frac{-\hbar^2}{2m} \Delta + V(r) \quad (6,1)$$

With μ is the reduced mass.

We calculate the expression of the Laplacian by using the formula of Laplace-Beltrami and the metric on a variety is:

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i,j} g_{i,j} dx^i dx^j \quad (6,2)$$

The expression of the Laplace-Beltrami is given by the formula

$$\Delta = \sum_{i,j} \frac{1}{\sqrt{|(g_{i,j})|}} \frac{\partial}{\partial x_i} \left(\sqrt{|(g^{i,j})|} \frac{\partial}{\partial x_j} \right) \quad (6,3)$$

$(g^{i,j})$ Is the inverse of the matrix $(g_{i,j})$ and $(x^1, x^2, x^3) = (\rho, \theta, \varphi)$.

A simple calculation gives: $ds^2 = r^2 \sin^2 \theta d\varphi^2 + dr^2 + r^2 d\theta^2$ (6,4)

$$\text{And} \quad H\psi(r) = \left[-\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{\bar{L}^2}{2\mu} + V(r) \right] \psi(r) = E\psi(r) \quad (6,5)$$

We seek a solution by the method of separation of variables or by searching the solutions of all the commuting observables.

. The invariance by rotation of the Hamiltonian implies that $[\bar{L}^2, H] = 0$ but $[\bar{L}^2, L_z] = 0$. so the observables are H, \bar{L}^2, L_z .

1.1 The eigenfunctions of \bar{L}^2, L_z

In spherical polar coordinated we write:

$$L_x = \frac{\hbar}{i} (-\cos \varphi \cot \theta \frac{\partial}{\partial \varphi} - \sin \varphi \frac{\partial}{\partial \theta}) \quad L_y = \frac{\hbar}{i} (-\cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \sin \varphi \frac{\partial}{\partial \theta}) \quad (6,6)$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \quad \bar{L}^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \quad (6,7)$$

The eigenfunctions of \bar{L}^2, L_z are the spherical harmonics

$$Y_{\ell m}(\theta, \varphi) = \langle \theta \varphi | \ell m \rangle = (-1)^m \left[\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!} \right]^{1/2} P_{\ell}^m(\cos \theta) e^{im\varphi} \quad (6,8)$$

Where $P_{\ell}^m(\cos \theta)$ are the associated Legendre functions [A6].

In Dirac notation we write:

$$\vec{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle, \quad L_z |lm\rangle = \hbar m |lm\rangle \quad (6,9)$$

l is the azimuthal quantum number and m the magnetic quantum number.

l and m are integers. We deduce the dimension of the space

$$E_l = \{|lm\rangle\}, \text{ and } \dim(E_l) = 2l + 1.$$

If we introduce the ladder operators L_+ and L_- or the raising and lowering operators:

$$L_+ = L_x + i L_y \quad \text{and} \quad L_- = L_x - i L_y \quad \text{with} \quad L_{\pm} |l \pm l\rangle = 0 \quad (6,10)$$

we find:

$$L_{\pm} |lm\rangle = \sqrt{l(l+1) - m(m \pm 1)} |lm \pm 1\rangle \quad (6,11)$$

1.2 The radial equation

The eigenvalue of the equation of energy becomes

$$H\psi(r) = \left[-\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) \right] R_l(r) = ER_l(r) \quad (6,12)$$

We note that the approximation of the potential in nuclear physics, interactions between nucleons (protons and neutrons), is the potential of the harmonic oscillator.

2. The hydrogen atom

The Hamiltonian for a particle in atomic physics can be approximated by the Hamiltonian of the hydrogen atom.

2.1 The wave function and the energy of the hydrogen atom

The potential of hydrogen is:

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} = \frac{k}{r}, \quad (6,13)$$

Since the charge of the proton is e^+ and the electron charge e^-
For the bound states $E < 0$ the wave function is

$$\psi_{nlm}(r) = R_{n,l}(r) Y_{lm}(\theta, \varphi) \quad (6,14)$$

With

$$R_{n,l}(r) = \frac{2}{n^2 a^{3/2}} \sqrt{\frac{(n-l-1)!}{[(n+1)!]}} \rho^l L_{n+l}^{2l+1}(\rho) e^{-\rho/2} \quad (6,15)$$

Energy is:

$$E_n = \frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2} \times \frac{1}{n^2}$$

and

$$a = \frac{4\pi\epsilon_0 \hbar^2}{\mu e^2} \quad \text{et} \quad \rho = \frac{2r}{na} \quad (6,16)$$

With

$$n = 1, 2, 3, \dots, \quad \text{et} \quad l \leq n-1$$

$L_{n+l}^{2l+1}(\rho)$ Is Laguerre polynomial [A6] of degree $n-l-1$.

($n-l$) and n are the radial and principal quantum numbers.

For $n=1$ we obtain the binding energy of the hydrogen atom $E_0 = -1/(2a)$. We obtain the discrete spectrum if $E < 0$ and the continuous spectrum if $E \geq 0$.

Parity: in the wave function if we change \vec{r} by $-\vec{r}$ we find:

$$\Psi_{nlm}(-\vec{r}) = (-)^l \Psi_{nlm}(\vec{r}) \quad (6,17)$$

$(-)^l$ Is the parity.

Degeneracy of the energy levels:

$$l = \ell = 0, 1, 2, 3, \dots, (n-1) \text{ and } m = -l, \dots, -1, 0, 1, \dots, l. \quad (6,18)$$

For a definite value of ℓ we find

$$\sum_{l=0}^{n-1} (2\ell + 1) = 1 + 3 + 5 + \dots + (2n-1) = n^2 \quad (6,19)$$

Classical notations of spectroscopy

$$\begin{array}{cccccc} \ell = & 0 & 1 & 2 & 3 & n = 1 & 2 & 3 & 4 & 5 \\ \text{Etat} & s & p & d & f & & K & L & M & N & O \end{array} \quad (6,20)$$

2.2 The $\{\vec{p}\}$ representation of hydrogen atom

The representation $\Psi(\vec{p})$ is the Fourier transform of the wave function $\Psi(\vec{r})$:

$$\Psi(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{p}\cdot\vec{r}} \Psi(\vec{r}) d\vec{r} \quad (6,21)$$

Using the interesting and useful formula [28-29]:

$$\int (e^{i(\vec{p}-\vec{p}')\cdot\vec{r}} / r) d^3 r = 4\pi \hbar^2 / |\vec{p} - \vec{p}'|^2 \quad (6,22)$$

Fock observed that the representation $\{\vec{r}\}$ is equivalent to the integral equation:

$$(\vec{p}^2 - 2mE)\Psi(\vec{p}) = \frac{mk}{\pi^2 \hbar} \int \frac{\Psi(\vec{r})}{|\vec{p} - \vec{p}'|^2} d^3 p' \quad (6,23)$$

Calculations of $\{p\}$ - representation by this method is complicated but we present in the Appendix 5 a new, simple and direct method using our generating function method.

3. Connections hydrogen atom and harmonic oscillator

In classical mechanics solving the equation of the hydrogen is done by transforming this equation in a harmonic oscillator equation using Binet transformation. In quantum mechanics we use the octonions quadratic transformations* [A3] for the connection of the hydrogen atom and the harmonic oscillator. These quadratic transformations and hypercomplex numbers originate from the old problem of sum squares:

Find the solution of the equation:

$$(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2) = (\sum_{i=1}^n z_i^2) \quad (6,24)$$

With $\{z_i\}$ are the quadratic forms $z_i = \sum_{jk} a_{jk} x_j y_k$

3.1 The quadratic transformation and hypercomplex numbers

We will present a new and simple recurrence method:

3.1.1 The transformation of Levi-Civita $R^2 \rightarrow R^2$.

We begin with the well known formulas:

$$\cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2)$$

and $\sin \theta = 2 \cos(\theta/2) \sin(\theta/2)$

Multiplying these expressions by r then we write:

$$x_1 = 2u_1 u_2 \quad x_2 = u_1^2 - u_2^2 \quad (6,25)$$

Avec $x_1 = r \sin \theta, \quad x_2 = r \cos \theta,$

And $u_1 = \sqrt{r} \cos(\theta/2), \quad u_2 = \sqrt{r} \sin(\theta/2)$

We write (6.25) in matrix form as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (H_2)(U_2) \quad (6,26)$$

The orthogonal and anti symmetric matrix H_2 has the properties:

$$H_2 = u_1 I + J u_2, \quad J^2 = -1, \text{ and } {}^t H_2 H_2 = u_1^2 + u_2^2 \quad (6,27)$$

3.1.2 The generalization of the transformation (6,26) is obtained by setting:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = 2(H_2)(U_2') \quad (6,28)$$

We find (6,24) if we put: $x_3 = (u_1^2 + u_2^2) - (u_3^2 + u_4^2)$

We find the octonion quadratic, or Kustaanheimo-Stiefel, transformation $R^4 \rightarrow R^3$
 $(u_1, u_2, u_3, u_4) \rightarrow (x_1, x_2, x_3, 0)$, and it's generalization in the appendix (A3,2).

Also it is simple to prove that:

$$\Delta_3(x) = \Delta_4(u) / \bar{u}^2, \quad r = \bar{u}^2. \quad (6,29)$$

3.2 The equation hydrogen atom in the basis of the oscillator

The Schrodinger equation for the hydrogen atom is written in $\{u\}$ representation as:

$$\left(-\frac{\hbar^2}{2\mu} \Delta_4(u) - 4E \bar{u}^2\right) f(u) = 4\beta f(u) \quad (6,30)$$

With $\frac{\mu\omega^2}{2} = -4E, \quad 4\beta = \hbar\omega(2n+2), \quad \beta = \frac{e^2}{4\pi\epsilon_0},$ (6,31)

After solving the equation (6.28) we find:

$$E_n = -2\mu((\beta/(n+2))^2) \quad (6,32)$$

*Note: The term octonions was added to specify these quadratic transformations.

4. The energy spectrum and the permitted transitions of hydrogen atom

1-Energy Levels of hydrogen atom

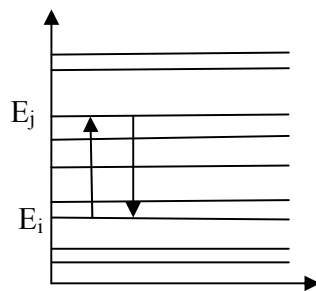
The wavelengths of spectral lines of hydrogen

1-The Balmer formula (1885): $\lambda = Bn^2/(n^2-4)$, $n=3,4,5,6$

2- The Rydberg formula: $\nu = \frac{1}{\lambda} = \frac{4}{B} \left(\frac{1}{2^2} - \frac{1}{n^2} \right)$ $n=3, 4, 5,6$

3 - The Ritz formula: $\nu = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$, $R = \frac{4}{B}$
 $= T_m - T_n$

2-The energy spectrum



-Einstein introduced the concept of "quantum radiation" called photon, Particle for a light with mass zero and pulse or frequency $\nu = \omega / 2\pi$ has energy.

$$E = h\nu/2\pi$$

Planck's constant $h = 6.6262 \cdot 10^{-34}$ joule seconds

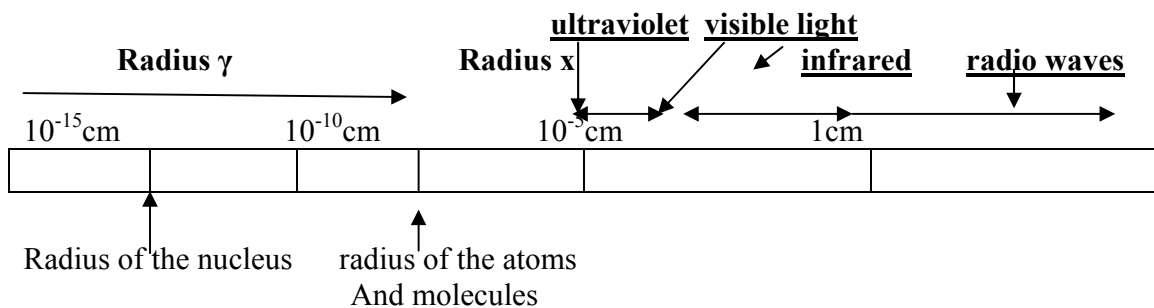
-The energies of atoms adopt discrete values.

-The emission or absorption of light by atoms is by very specific light frequencies:

$$\nu_{if} = | E_i - E_j | / h$$

E_i and E_j are the energies of the system before and after the emission or absorption of the light .

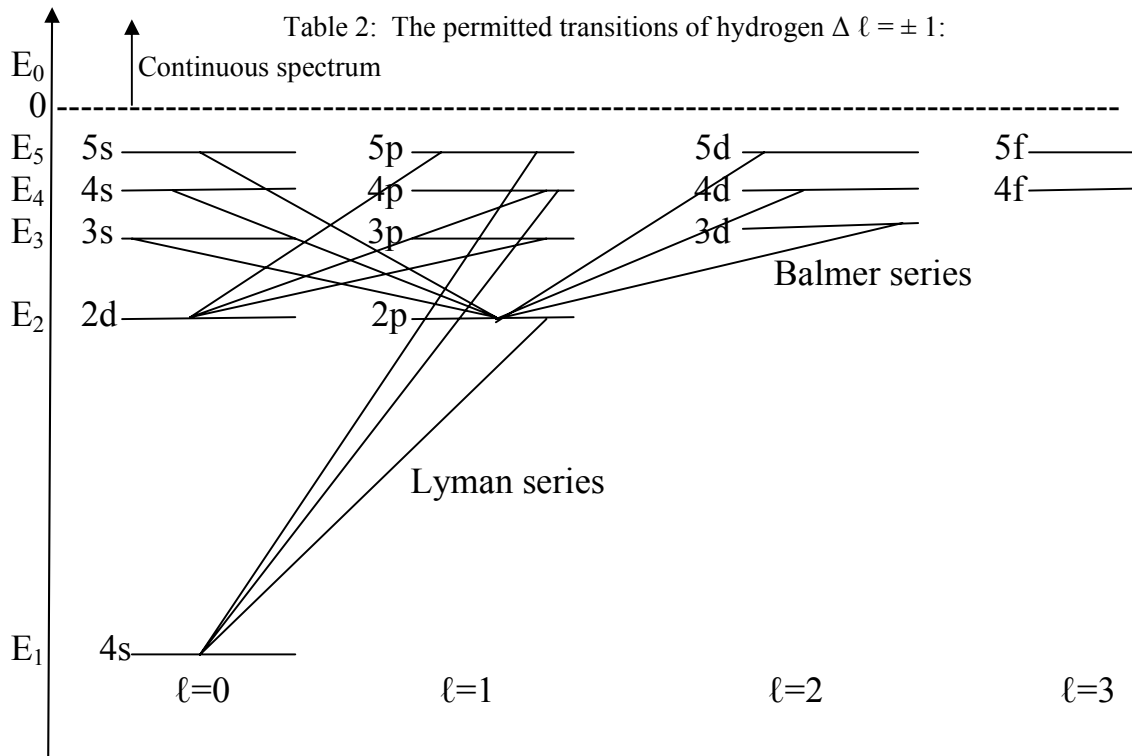
3-Le spectre des ondes électromagnétique



Echelle of wavelengths of the radiation. [9]

Table 1: states of the hydrogen atom

Principal quantum number	Energy	states	ℓ	m	Degeneration	The discrete spectrum
n=1	E_1	1s	0	0	1	
n=2	$E_1/4$	2s	0	0	4	
		2p	1	1		
			1	0		
1	-1					
n=3	$E_1/9$	3s	0	0	9	
		3p	1	1		
			1	0		
			1	-1		
		3d	2	2		
			2	1		
			2	0		
			2	-1		
			2	-2		



VII-The theory of angular momentum

The symmetry is very important in physics and especially the symmetry by rotation. The invariance of the Hamiltonian by rotation involves to study the angular momentum and the calculation of the coupling coefficients of the total angular momentum. Thus the system states is eigenfunctions of the square of the total angular momentum and its projection on the z-axis.

But quantum theory of the atom does not taking account for all phenomena such as Stern-Gerlach experiment. And after a lot of work to analyze the spectrum, the spin of the electron was introduced and the total angular momentum to solve some of these difficulties.

In this part we follow the Schwinger approach [31-32] for its simplicity and we can generalize it to the semi-simple groups specially unitary group.

1. The electron spin

1.1 The magnetic moment

We consider an atom with Z electrons and the nucleus is the center and the Hamiltonian of the system is H_0 . According to the theory of electromagnetism if the system is subjected to a magnetic field \vec{H}_1 , the Hamiltonian is written:

$$H = H_0 - \frac{e}{2mc} (\vec{H}_1 \cdot \vec{L}) \quad (7,1)$$

Everything happens as all the electrons circulating in their orbit, each electron induces a magnetic moment:

$$\vec{\mu} = \frac{e}{2mc} \vec{l} \quad (7,2)$$

The magnetic moment of the atom is the sum of the individual magnetic moments Z .

$$\vec{M} = \frac{e}{2mc} \vec{L}, \quad \vec{L} = \sum_{i=1}^Z \vec{l}_i \quad (7,3)$$

1.2 The hypothesis of electron spin

The Stern-Gerlach experiment is to measure the deviation of silver atoms in a region where reigns a strongly inhomogeneous magnetic field. Results contradict the predictions: the beam, instead of expanding, splits into two separate beams giving two separate spots of identical intensities.

Normally division levels in a magnetic field produces an odd number of levels ($2L+1$) but often appears an even number of levels.

The explanation was simple but revolutionary for the first time appear a purely quantum quantity: the spin $1/2$.

These phenomena have led Goudsmit and Uhlenbeck to admit in 1925 that the electron was a anime rotation on itself, which corresponded own angular momentum called spin $1/2$. Pauli introduced the function of two components:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \psi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_+ |+\rangle + \psi_- |-\rangle \quad (7,4)$$

With
$$\int (|\psi_+|^2 + |\psi_-|^2) d^3r = 1$$

By analogy with the angular momentum we write

$$\vec{S} = \vec{\sigma} / 2, \text{ And } \vec{S}^2 = S_x^2 + S_y^2 + S_z^2 \quad (7,5)$$

with $[S_x, S_y] = i\hbar S_z, [S_y, S_z] = i\hbar S_x, [S_z, S_x] = i\hbar S_y$

The state of spin $1/2$ particles are then written

$$|sm_s\rangle, \quad (s = 1/2, \quad m_s = \pm 1/2), \text{ or } \uparrow \downarrow$$

$$\vec{S}^2 |sm_s\rangle = s(s+1) |sm_s\rangle = \frac{3}{4} |sm_s\rangle, \quad S_z |sm_s\rangle = m_s |sm_s\rangle \quad (7,6)$$

To resolve all the difficulties of these phenomena Goudsmit and Uhlenbeck introduced also the spin magnetic moment.

$$\vec{\mu}_s = \frac{e}{2mc} \vec{S} \quad (7,7)$$

2. The Schwinger approach of angular momentum

2.1 The generators of SU (2) and the basis of the oscillator

Schwinger noted that the orbital moment is written according to the operators of creations and annihilations of the oscillator and spin1 as:

$$\vec{L} = \vec{r} \times \vec{p} = (\mathbf{a}^+)^t (\vec{\sigma}) (\mathbf{a}) \quad (7,8)$$

With $(\mathbf{a}^+)^t = (a_x^+, a_y^+, a_z^+)$

Schwinger in his work "on angular momentum" has replaced the matrices of spin1 by the matrices of spin 1/2 and found the generators of SU (2), (J_1, J_2 and J_3), in terms of the creations and annihilations operators of two dimensional harmonic oscillator:

$$\text{We write : } \quad \vec{J} = \begin{pmatrix} a_1^+ & a_2^+ \end{pmatrix} \begin{pmatrix} \vec{\sigma} \\ 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (7,9)$$

$\vec{\sigma}$ are the Pauli matrices of spin $1/2$, $\vec{S} = \vec{\sigma} / 2$.

We find easily the ladders operators J_+, J_- and J_3 :

$$J_+ = J_1 + iJ_2 = a_1^+ a_2, \quad J_- = J_1 - iJ_2 = a_2^+ a_1, \quad J_3 = [a_1^+ a_1 - a_2^+ a_2] / 2$$

$$[J_3, J_{\pm}] = J_{\pm}, \quad [J_+, J_-] = 2J_3$$

$$\text{And } \quad N = [a_1^+ a_1 + a_2^+ a_2] / 2 \text{ et } \vec{J}^2 = N(N+1) \quad (7,10)$$

2.2 The representation of SU (2) in the basis of the oscillator

We formally write:

$$N = [a_1^+ \partial / \partial a_1 + a_2^+ \partial / \partial a_2] / 2, \quad J_3 = [a_1^+ \partial / \partial a_1 - a_2^+ \partial / \partial a_2] / 2 \quad (7,11)$$

According to Euler's theorem on the homogeneous functions, the eigenfunctions of J_3, N et \vec{J}^2 are the homogeneous functions:

$$|jm\rangle = \frac{a_1^{+(j+m)} a_2^{+(j-m)}}{\sqrt{(j+m)!(j-m)!}} |0,0\rangle = \phi_{jm}(a^+) |0,0\rangle \quad (7,12)$$

The one-dimensional subspace E_j is $(2j+1)$.

$$E_j = \{ |jm\rangle, -j \leq m \leq j, j = 0, 1/2, 3/2, \dots \}$$

1- We also find that: $\vec{J}^2 |jm\rangle = j(j+1) |jm\rangle, J_3 |jm\rangle = m |jm\rangle$

$$J_{\pm} |jm\rangle = \sqrt{j(j+1) - m(m \pm 1)} |jm \pm 1\rangle \quad (7,13)$$

2.3 The SU (2) Wigner matrix D and its generating function

The rotation matrix according to the Euler angles is well known in classical mechanics. And in quantum mechanics the matrix elements in space $\{|jm\rangle\}$ or D-Wigner are:

$$\begin{aligned} D_{(m',m)}^j(\Omega) &= \langle jm' | R^J | jm \rangle = \langle jm' | e^{-i\psi J_3} e^{-i\theta J_2} e^{-i\varphi J_3} | jm \rangle \\ &= e^{-i\psi m'} d_{(m',m)}^j(\theta) e^{-i\varphi m} \end{aligned} \quad (7,14)$$

And
$$D_{(m,0)}^j(\Omega) = \left(\frac{4\pi}{2l+1} \right)^{1/2} Y_{lm}^*(\theta, \varphi) \quad (7,15)$$

Multiply by $\phi_{jm'}(u)\phi_{jm}(v)r^{2j}$ and perform the summation then we find the generating function of the matrix elements of rotations.

$$\begin{aligned} \sum_{jmm'} \phi_{jm'}(u) D_{(m',m)}^j(z) \phi_{jm}(v) &= \exp[{}^t(u)(\rho R_s)(v)] \\ &= \exp\left[\begin{pmatrix} \zeta & \eta \end{pmatrix} \begin{pmatrix} z_1 & \bar{z}_2 \\ -z_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} \chi \\ \lambda \end{pmatrix} \right] = \phi(u, v, z) \end{aligned} \quad (7,16)$$

With $u = (\zeta, \eta), v = (\chi, \lambda)$ and $z = (z_1, z_2),$

$$z_1 = \rho a_1 = \rho \exp(\varphi_1) \cos\left(\frac{\theta}{2}\right), \quad z_2 = \rho a_2 = \rho \exp(\varphi_2) \sin\left(\frac{\theta}{2}\right) \quad (7,17)$$

And $0 \leq \rho \leq \infty, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \psi \leq 2\pi$

$$\Omega = (\psi\theta\varphi), \quad \varphi_1 = \left(\frac{\psi + \varphi}{2}\right), \quad \varphi_2 = \left(\frac{\psi - \varphi}{2}\right) \quad (7,18)$$

1 - Expression of matrix elements of rotations

From the above generating function we deduce after development the expression of matrix elements of rotations in terms of Jacobi polynomials.

$$d_{(m',m)}^j(\theta) = N_{(m',m)}^j (\cos(\theta/2))^{m'+m} (\sin(\theta/2))^{m'-m} P_{j-m'}^{(m'-m, m'+m)}(\cos\theta) \quad (7,19)$$

With
$$N_{(m',m)}^j = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{-1/2}$$

And
$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{\Gamma(\alpha+\beta+1)}{n!\Gamma(\beta+1)} {}_2F_1(n+\alpha+\beta+1, -n; +1; \frac{I+x}{2}) \quad (7,20)$$

Using (7,16) we derive the orthogonality of the matrix-D:

$$\frac{1}{8\pi^2} \int (D_{(m_1, m_1)}^{j_1})^*(\Omega) D_{(m_2, m_2)}^{j_2}(\Omega) d(\Omega) = \frac{1}{2j_1+1} \delta_{m_1 m_2} \delta_{m_1 m_2} \delta_{j_1 j_2} \quad (7,21)$$

2 - The projection operator

We find by a simple calculation the projection operator

$$P_{(m,k)}^j = \frac{2j+1}{8\pi^2} \int (D_{(m,k)}^j)^*(\Omega) R(\Omega) d(\Omega) \quad (7,22)$$

The projection operator and our generalization of Cramer's rule for linear systems, formula (A6), allow us to calculate the spectrum of rotation of the nucleus [25].

3 - The characters of the group SU (2)

$$\chi^j = \sum_{m=-j}^j D_{(m,m)}^j(\Omega) = \sum_{m=-j}^j e^{-im\gamma} = \frac{\sin(j+1/2)\gamma}{\sin(\gamma/2)} \quad (7,23)$$

With $tr(R_s) = 2 \cos(\gamma/2) = 2 \cos(\theta/2) \cos((\varphi + \psi)/2)$

3. The couplings in the theory of angular momentum

3.1 The addition of two angular momentum

In a system of two particles the total angular momentum is the sum of the individual angular momentum we write:

$$\vec{J}_3 = \vec{J}_1 + \vec{J}_2 \quad (7,24)$$

$$\begin{aligned} \text{a-} \quad J_1^2 |j_1 m_1\rangle &= j_1(j_1+1) |j_1 m_1\rangle, \quad J_2^2 |j_2 m_2\rangle = j_2(j_2+1) |j_2 m_2\rangle \\ J_3^2 |(j_1 j_2) j_3 m_3\rangle &= j_3(j_3+1) |(j_1 j_2) j_3 m_3\rangle \end{aligned} \quad (7,25)$$

$$\begin{aligned} \text{b-} \quad |(j_1 j_2) j_3 m\rangle &= \sum_{m_1, m_2} \langle j_1 m_1, j_2 m_2 | (j_1 j_2) j_3 m \rangle |j_1 m_1\rangle |j_2 m_2\rangle \\ |j_1 m_1\rangle |j_2 m_2\rangle &= \sum_j \langle j_1 m_1, j_2 m_2 | (j_1 j_2) j_3 m \rangle |(j_1 j_2) j_3 m \rangle \end{aligned} \quad (7,26)$$

The coefficient $\langle j_1 m_1, j_2 m_2 | (j_1 j_2) j_3 m \rangle$ Is the Clebsch-Gordan.

c- We can use also the notations:

$$|(j_1 j_2) j_3 m\rangle = |(j_1 j_2) j_3 (m'_3, m_3)\rangle, \quad m'_3 = j_1 - j_2, \quad m_3 = m_1 + m_2$$

$$\text{d-} \quad D_{(m',m)}^j(\Omega) = \langle (j_1 j_2) j m' | R^J | (j_1 j_2) j m \rangle = e^{-im'\psi} d_{(m',m)}^j(\theta) e^{-im\varphi} \quad (7,27)$$

e- We define the Wigner symbols by:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_2} (2j_3+1)^{-1/2} \langle j_1 m_1, j_2 m_2 | (j_1 j_2) j_3 -m_3 \rangle \quad (7,28)$$

3.2 Expression of the integral of the product of three D

Several methods have been proposed for the calculation of Wigner symbols:

a- Racah has applied the method of raising and lowering operators.

- b- The method that has as its starting point the integral of the product of three D
c-The Wan der Waerden method, or Weyl, that we present in the following.

a- Racah method or the infinitesimal method

Put
$$\vec{J}_{3\pm} = \vec{J}_{1\pm} + \vec{J}_{2\pm}$$

Using (7.13) we find the recurrences formula of Clebsh-Gordan coefficients:

$$\begin{aligned} [(j_3 \mp m_3)(j_3 \pm m_3 + 1)] \langle (j_1 j_2) j_3 m_3 | j_1 j_2 m_1 m_2 \rangle = \\ [(j_1 \mp m_1)(j_1 \pm m_1 + 1)] \langle (j_1 j_2) j_3 m_3 | j_1 j_2 m_1 m_2 \rangle + \\ [(j_2 \mp m_2)(j_2 \pm m_2 + 1)] \langle (j_1 j_2) j_3 m_3 | j_1 j_2 m_1 m_2 \rangle \end{aligned} \quad (7,29)$$

From this expression Racah deducted the Clebsh-Gordan coefficients but Wander Waerden method is much more simpler.

b- The method of product of three D or the global method

$$\begin{aligned} \langle j_1 m_1 | \langle j_2 m_2 | R^{J_3} | j_1 m_1' \rangle | j_2 m_2' \rangle = D_{(m_1, m_1')}^{j_1}(\Omega) D_{(m_2, m_2')}^{j_2}(\Omega) = \\ \sum_{j_m} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} D_{(m_3, m_3')}^{*j_3}(\Omega) \end{aligned} \quad (7,30)$$

with $m = -m_3 = m_1 + m_2$ and $m' = -m_3' = m_1' + m_2'$

3.3 The invariants of 3-j symbols

Van der Wearden [8,25,37] determined from elementary invariants of SU (2) the invariant of SU (2) and from which he determined the 3-j symbols.

But we will use the generating function of the matrix elements of rotations for the determination of these symbols.

The entire product of these three generating functions is:

$$G^3 = \int \prod_{i=1}^3 [\phi(x^i, y^i, z^i)] d\mu(z_1) d\mu(z_2) = \sum_{j_1 j_2 j_3} H_{(j_1 j_2 j_3)}(x) H_{(j_1 j_2 j_3)}(y) \quad (7,31)$$

Using the expression $\int \exp[\alpha z + \beta \bar{z}] d\mu(z) = e^{\alpha\beta}$ we find:

$$G^3 = \exp \left[[x^1, x^2] [y^1, y^2] + [x^1, x^3] [y^1, y^3] + [x^2, x^3] [y^2, y^3] \right] \quad (7,32)$$

With $x^i = (\xi_i, \eta_i), y^i = (u_i, v_i)$ and $x = (x^1, x^2)$

After development and identification of (7,32) we find the invariants:

$$H_{(j_1 j_2 j_3)}(x) = \frac{[x^2, x^3]^{(J-2j_1)} [x^3, x^1]^{(J-2j_2)} [x^1, x^2]^{(J-2j_3)}}{\sqrt{(J+1)!(J-2j_1)!(J-2j_2)!(J-2j_3)}} \quad (7,33)$$

And
$$H_{(j_1 j_2 j_3)}(x) = \sum_{m_i} \left[\prod_{i=1}^3 \phi_{j_i, m_i}(x^i) \right] \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad (7,34)$$

$H_{(j_1 j_2 j_3)}(\mathbf{x})$ Is the Van der Wearden invariant of 3-j symbols.
 $[x^2, x^3]$, $[x^3, x^1]$, $[x^1, x^2]$ Are the elementary invariants of SU (2).

3.4 Applications:

1-The powers of the invariants (7,33) are positive, it follows that:

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2 \text{ and } j_3 \geq |m_3| \quad (7,35)$$

2 - The decomposition of the tensor product of the states of spin $\frac{1}{2}$ is:

$$(2) \otimes (2) = (1) \oplus (3) \quad (7,36)$$

a- The singlet state is:

$$|00\rangle = \frac{1}{\sqrt{2}} \left[|1/2, 1/2\rangle_1 |1/2, -1/2\rangle_2 - |1/2, -1/2\rangle_1 |1/2, 1/2\rangle_2 \right]$$

b- The triplet state is:

$$\begin{cases} |11\rangle = |1/2, 1/2\rangle_1 |1/2, 1/2\rangle_2 \\ |10\rangle = \frac{1}{\sqrt{2}} \left[|1/2, m_1\rangle_1 |1/2, m_2\rangle_2 + |1/2, m_1\rangle_1 |1/2, m_2\rangle_2 \right] \\ |1-1\rangle = |1/2, -1/2\rangle_1 |1/2, -1/2\rangle_2 \end{cases} \quad (7,37)$$

3.5 The Van der Wearden formula for 3j symbols

The Van der Wearden formula of 3j symbols can be derived simply form (7,34):

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{2(l_2 - l_1)} \Delta(l, m) \frac{(l - l_1 + l_2)!(l - m_3)!(l_2 - m_2)!}{(-l + l_1 + l_2)!(l - l_2 + m_1)!(l - l_1 - m_2)!} \times {}_3F_2(-l_2 - m_2, -l_1 + m_1, l - l_1 - m_2; l - l_1 - m_2 + 1, l - l_2 + m_1 + 1; 1) \quad (7,38)$$

${}_3F_2$ is a generalized hypergeometric series (A6) and

$$\Delta(l, m) = (-1)^{l_2 + m_2} \sqrt{\frac{(l + l_1 - l_2)!(-l + l_1 + l_2)!(l + m_3)!(l_1 - m_1)!}{(l - l_1 + l_2)!(l + l_1 + l_2 + l)!(l - m_3)!(l_1 + m_1)!(l_2 + m_2)!(l_2 - m_2)!}}$$

The method of invariants has been the subject of many studies [25, 39] but the generalization of this method to SU (n) for $n > 3$ is given in [25] .

3.5 The Schwinger approach of the coupling of angular momentum

Using (7, 33) we find the generating function of SU (2) is:

$$\exp[\tau_3[x^1, x^2] + \tau_2[x^1, x^3] + \tau_1[x^2, x^3]] = \sum \Phi_{j_1 j_2 j_3}(\tau) H_{(j_1 j_2 j_3)}(\mathbf{x}) \quad (7,39)$$

$$\text{With } \Phi_{(j_1 j_2 j_3)}(\tau) = \frac{\sqrt{(J+1)!} \tau_1^{(J-2j_1)} \tau_2^{(J-2j_2)} \tau_3^{(J-2j_3)}}{\sqrt{(J-2j_1)!(J-2j_2)!(J-2j_3)}}$$

The generating function of the coupling of two angular momentum is obtained by using (7,28) and changing $[x^1, x^3]$, $[x^2, x^3]$ by (x^3, x^1) , (x^3, x^2) in above the expression.

We deduce the Schwinger's formula:

$$\begin{aligned}
G(\alpha, z) &= \exp\{\alpha_3[x^1, x^2] + \alpha_1(z, x^2) + \alpha_2(x^1, z)\} = \\
&\exp\left\{\alpha_3\left[\frac{\partial}{u}, \frac{\partial}{v}\right] + \alpha_2\left[z_1 \frac{\partial}{u_1} + z_2 \frac{\partial}{u_2}\right] + \alpha_1\left[z_1 \frac{\partial}{v_1} + z_2 \frac{\partial}{v_2}\right]\right\} \exp[(u, x^1) + (v, x^2)] \\
&= \sum (2j+1)^{-1/2} \Phi_{j_1 j_2 j_3}(\alpha) \varphi_m^j(z) {}_{(m)}F_{(m)}^j(x) \quad (7,40)
\end{aligned}$$

With ${}_{(m)}F_{(m)}^j(x)$ is the coupling of two angular momentums in the Fock space and :

$$j_3 = j, \quad m' = j_1 - j_2, \quad m = m_3 = m_1 + m_2$$

The formula (7,40) allows simply the determination of the coupling of several angular momentum and the representation of SU (3) (chapter X).

4. The tensor operators

The introduction of the tensor operator is very important for the calculation of matrix elements of the transitions.

The eigenfunctions of \vec{J}^2 and J_3 are $\{|jm\rangle\}$, ($m = -j, \dots, j$) and the subspace $E_j = \{|jm\rangle\}$ is invariant by rotation and as a result:

$$R |jm\rangle = \sum_m D_{(m,m')}^j(\Omega) |jm'\rangle \quad (7,41)$$

1-Definition: T_m^j Is a tensor operator if:

$$RT_m^j R^{-1} = \sum_m D_{(m,m')}^j(\Omega) T_m^j \quad (7,42)$$

Where an equivalent way:

$$[J_{\pm}, T_m^j] = \sqrt{j(j+1) - m(m \pm 1)} T_{m \pm 1}^j \quad \text{And } [J_3, T_m^j] = m T_m^j \quad (7,43)$$

2-Wigner-Eckart theorem:

A representation in which the basis vectors $\{|pjm\rangle\}$ are eigenfunctions \vec{J}^2 and J_z . The matrix elements of the tensor operator are defined by:

$$\langle pjm | T_q^k | \rho' j' m' \rangle = \frac{1}{\sqrt{2j+1}} \langle \rho j | T^k | \rho' j' \rangle \langle j' m', kq | (j' k) jm \rangle \quad (7,44)$$

$\langle \rho j | T^k | \rho' j' \rangle$ Is a coefficient which depends of the tensor operator.

3-The rules of selection of the operator tensor:

The matrix elements of the tensor operator is different from zero if:

$$|j - j'| \leq k \leq j + j', \quad \text{with } q = m - m'. \quad (7,45)$$

Applications: The transitions of hydrogen atom are given in the table 2.

VIII- Classification of elementary particles and Quarks model

1. Introduction

Dirac predicted the existence of antiparticles starting from his famous equation of the electron [A1] and in nuclear physics Elliot approximate in 58 the nuclear interaction by the quadrupole-quadrupole interaction [20] to interpret the energy levels of Mg and found that the wave function is developed on the basis of the representation of SU (3).

But the discovery of new particles and resonance particle decays quickly led to the classifications of particles performed by Gell-Mann and Ne'eman [33] with the group SU (3) in 64.

But the study of the scattering of electrons by protons and neutrons show that these particles have a complex structure and the discovery of strange particles has caused confusion and to break the impasse Gell-Mann was assumed that the elementary particles are formed from particles called quarks. And the introduction of the Quarks model by Gell-Mann has helped to solve the defects of this model.

At first we want to give a reminder of the work of Yukawa and Heisenberg in nuclear physics as an introduction to the quark model knowing that protons and neutrons are elementary particles and are part of baryons family.

2. Yukawa potential and the particle state in nuclear physics

We know that the gravitational and electromagnetic interactions have infinite range (potential $1 / r$) and the binding energy of hydrogen atoms is low which means that electromagnetic interactions are also low.

a- In nuclear physics, the interaction is short-range (10-12cm) and the binding energy of nuclei is very high, for example, the binding energy of deuteron energy is 2.2 MeV where the name of strong interactions.

b- Yukawa proposed a potential of the strong interaction and predicted the existence of the particle (meson) assumed as the quantum field of strong interaction.

c- Experience shows that the intensity its range of interaction between proton and neutron are the same and this has led Heisenberg to introduce the nucleon and the isospin.

2.1 Yukawa nuclear potential

By analogy with the derivation of the Coulomb potential and that consists the spherical solution of the known equation

$$\Delta \varphi (r , t) = - 4 \pi \rho \quad (9,1)$$

ρ is the density of the charges.

The solution is :

$$\varphi (r) = q / r \quad (9,2)$$

Where is $q = \int \rho(r)dv$ the charge of the source

Yukawa start from the Klein-Gordon equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu^2\right)\varphi(r, t) = 0, \quad (9,3)$$

The potential must be independent of time and directions so we seek solution of the radial part of

$$(\Delta - \mu^2)\varphi(r, t) = 0 \rightarrow \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\varphi}{dr} \right) - \mu^2 \varphi(r, t) = 0 \quad (9,4)$$

Solving the last equation leads to a physically acceptable solution and that is the well known Yukawa nuclear potential:

$$\varphi(r) = g \frac{e^{-\mu r}}{r} \quad (9,5)$$

Where μ is the mass with ($c = \hbar = 1$) and $1 / \mu$ is range.

2.2 The state of a particle in the shell model of nuclei

Heisenberg by analogy with the theory of spin considered that protons and neutrons are two aspects of a single particle, the nucleon and thus he introduced the isospin.

The notations in isospin particle $\vec{I} = (I_x, I_y, I_z)$, and (I, I_z) .

Nucleons are in two isospin states:

$|I, I_z = 1/2\rangle$ is the proton state.

$|I, I_z = -1/2\rangle$ is the neutron state.

So the nucleons are fermions and the basis are the products of: The basis of the oscillator, the spin state and isospin

$$\left\{ |n_x n_y n_z\rangle |s, m_s\rangle |I, I_z\rangle \right\} \text{ with } |n_x n_y n_z\rangle = |n_x\rangle |n_y\rangle |n_z\rangle \quad (9,6)$$

And $\Psi_{[n_x, n_y, n_z]}(x, y, z) = \Psi_{n_x}(x) \Psi_{n_y}(y) \Psi_{n_z}(z)$

The energy is:

$$E_N = \hbar\omega(N + 3/2), \quad N = n_x + n_y + n_z \quad (9,7)$$

The total degeneracy of a state energy E_N is given by:

$$d_N = (N + 1)(N + 2) \quad (9,8)$$

The spherical basis of the harmonic oscillator is

$$\Psi_{nlm}(r, \theta, \varphi) = N_{nl} \rho^l L_n^{l+1/2}(\rho^2) \exp\left(-\frac{\rho^2}{2}\right) Y_{lm}(\theta, \varphi) \quad (9,9)$$

With $\rho = r\sqrt{m\omega/\hbar}$, $N_{nl} = \left[\frac{2\Gamma(n+1)}{\Gamma(n+l+3/2)} \right]^{1/2}$

The principal quantum number N is connected to the radial quantum number n and azimuthal quantum number ℓ by $N = 2(n - 1) + \ell$.

3. The Quarks model

A-The elementary Particles:

Elementary particles are numerous and can be classified as follows:

1-The hadrons are particles capable of strong interactions and consist of particles called quarks.

Hadrons are: The Baryons particles with spin half (p, n, ...)

The mesons Particles with integer spin (π_{\pm}, π_0, \dots)

2-The leptons are particles capable of weak and electromagnetic interactions.

These particles are:

The electron e^{-} , muon μ^{-} , tau τ^{-} (charge negative)

And neutrinos $\nu_e, \nu_{\mu}, \nu_{\tau}$ (no charge)

3- Photon massless and spin 1

4 - Gravitons and massless particle of spin 2

5- Antiparticles:

At any charged particle is a particle associated with the same mass and opposite charge.

B-Quarks: The Quarks are characterized by the quantum numbers are summarized in the table:

Quark	u	d	c	s	t	b
Charge Q	2/3	-1/3	2/3	-1/3	2/3	-1/3
Charm C	0	0	+1	0	0	0
strange S	0	0	0	-1	0	0
Top T	0	0	0	0	+1	0
Bottom B	0	0	0	0	0	-1

Table 3

4. The quantum numbers of Quarks

In what follows we will be interested in the quantum numbers of quarks [33].

1- Strangeness and hypercharge

It is well known that the electromagnetic interaction is due to the electric charge Q of the particle and by analogy the field of strong interaction is due to a baryonic charge B.

Similarly, the baryon charge should be additive such as the electric charge.

We note that the charge:

For nucleons:

$$Q=I_z+1/2 \text{ for the proton}$$

$$Q=I_z-1/2 \text{ for the neutron}$$

For mesons:

$$Q=I_z=1 \text{ for } \pi^+$$

$$Q=I_z=0 \text{ for } \pi^0$$

$$Q=I_z=-1 \text{ for } \pi^{-}$$

Gell-Mann proposed an additive quantum number S, which he called strangeness S is equal zero for mesons and nucleons, but $S = -1$ for $K^{-}, K^0, K^+, \Sigma^+, \Sigma^0, \Sigma^{-}$.

The corresponding antiparticles have opposite strangeness $S = +1$.

$$S = Y - B \tag{9,10}$$

Y is the hypercharge.

So the nucleons and mesons are non-strange particles.
We can also deduce the relation of Gell-Mann-Nishijima:

$$Q = I_z + Y / 2 \tag{9,11}$$

2- parity

Attributing intrinsic parity appears as an experimental necessity under the assumption of parity conservation in strong interaction processes.
- The parity of fermions and anti fermions is negative.

3- color

Baryons consist of three quarks with spin $\frac{1}{2}$ and the wave function must obey the Pauli principle, but this is not the case.
To eliminate this contradiction color was introduced as a new quantum number to save the Pauli principle.
The color has since become the basis of the modern theory of strong interactions between Quarks.
The quanta of these fields are called gluons whose numbers is eight.

4- Quarks, leptons and fundamental interactions

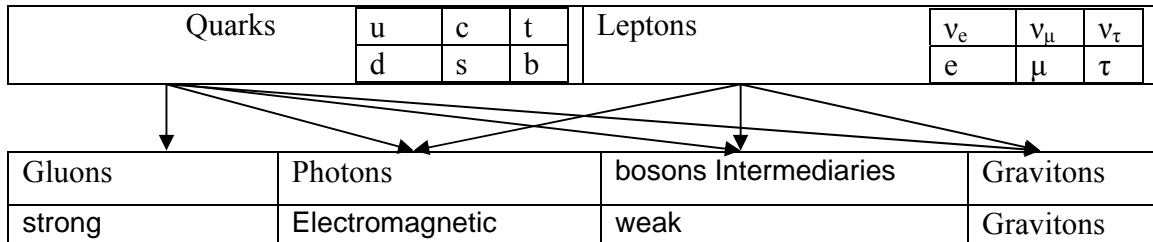


Table 4

5- The symmetries Laws

All the interactions conserve energy, momentum, angular momentum, charge, the number of quarks, the baryon number and lepton numbers.
But the weak interactions do not conserve the number of quarks.

6- The mass formula of Gell-Mann-Okubo

The mass of the formula:

$$M = m_0 + aY + b[I(I+1) - Y^2/4]. \tag{9,12}$$

7- The search for Higgs bosons

The theories of electroweak and strong interactions are called standard model. But this theory predicts zero mass to all particles and Higgs proposed a mechanism to give mass to these particles, but this mechanism provides for the existence of a heavy particle the Higgs boson.

8- Spectrum masses of the 8 baryons $J^P=1/2^+$ and the 10 baryons $J^P=3/2^+$

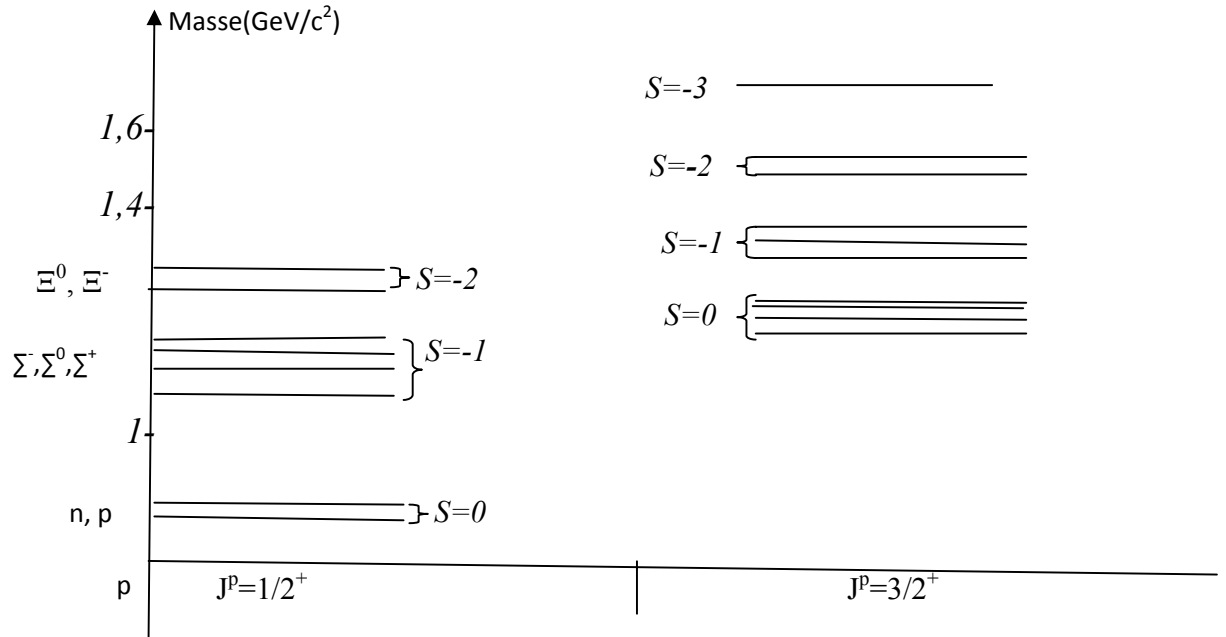


Table 5

5- Tables of Hadrons

Baryon $J^P=1/2^+$							
	Q	S	I	Quarks content	Mass MeV/c^2	Lifetime s	Decay modes
p	+1	0	1/2	uud	938,27	∞	
n	0	0	1/2	udd	938,57	888,6	$p\pi^-$
Λ	0	-1	0	uds	1115,7	$2,63 \cdot 10^{-10}$	$p\pi^-, n\pi^0$
Σ^+	+1	-1	1	uus	1189,4	$0,810 \cdot 10^{-19}$	$n\pi^+, p\pi^0$
Σ^0	0	-1	1	uds	1192,5	$7,410 \cdot 10^{-20}$	$\Lambda\gamma$
Σ^-	-1	-1	1	dds	1197,4	$1,510 \cdot 10^{-10}$	$n\pi^-$
Ξ^0	0	-2	1/2	ssu	1314,9	$2,9 \cdot 10^{-10}$	$\Lambda\pi^0$
Ξ^-	-1	-2	1/2	ssd	1321,3	$1,610 \cdot 10^{-10}$	$\Lambda\pi^-$
Baryon $J^P=3/2^+$							

	Q	S	I	Quarks content	Mass MeV/c ²	Lifetime s	Decay modes
Δ^{++}	2	0	3/2	uuu	1231	$0,6 \cdot 10^{-23}$	$p\pi^+$
Δ^+	+1	0	3/2	uud	1235	$0,6 \cdot 10^{-23}$	$p\pi^0, n\pi^+$
Δ^0	0	0	3/2	udd	1234	$0,6 \cdot 10^{-23}$	$p\pi^-, n\pi^0$
Δ^-	-1	0	3/2	ddd	1234	$0,6 \cdot 10^{-23}$	$n\pi^-$
Σ^{++}	+1	-1	1	uus	1383	$2 \cdot 10^{-23}$	$\Lambda\pi^+$
Σ^{*0}	0	-1	1	uds	1384	$2 \cdot 10^{-23}$	$\Lambda\pi^0$
Σ^{*-}	-1	-1	1	dds	1387	$2 \cdot 10^{-23}$	$\Lambda\pi^-$
Ξ^{*0}	0	-2	1/2	ssu	1532	$7 \cdot 10^{-23}$	$\Xi^-\pi^+, \Xi^0\pi^0$
Ξ^{*-}	-1	-2	1/2	ssd	1535	$7 \cdot 10^{-23}$	$\Xi^-\pi^0, \Xi^0\pi^-$
Ω	-1	-3	0	sss	1672	$0,8 \cdot 10^{-23}$	$\Lambda K^-, \Xi^0\pi^-, \Xi^-\pi^0$

Table 6 : Baryons

Mesons $J^P=0^-$							
	Q	S	I	Quarks content	Mass MeV/c ²	Lifetime s	Decay modes
π^+, π^-	+1,-1	0	1	$u\bar{d}, \bar{u}d$	139,57	$2,6 \cdot 10^{-8}$	$\mu\nu, e\nu$
π^0	0	0	1	$(u\bar{u}-d\bar{d})/\sqrt{2}$	134,97	$8,4 \cdot 10^{-17}$	$\gamma\gamma$
η	0	0	0	$(u\bar{u}-d\bar{d}-2s\bar{s})/\sqrt{6}$	548,8	$7 \cdot 10^{-19}$	$\gamma\gamma, \pi^+\pi^-, \pi^+\pi^-\gamma$
η'	0	0	0	$(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{6}$	957,5	$3 \cdot 10^{-21}$	$\eta\pi^+\pi^-, \pi^+\pi^-\gamma$
K^+, K^-	+1,-1	+1,-1	1/2	$\bar{s}u, s\bar{u}$	493,65	$1,24 \cdot 10^{-8}$	$\mu\nu, \pi^\pm\pi^0, \pi^\pm\pi^+\pi^-$
K^0, \bar{K}^0	0	+1,-1	1/2	$\bar{s}d, s\bar{d}$	497,67	$K_S^0 0,89 \cdot 10^{-10}$ $K_L^0 5,17 \cdot 10^{-8}$	$\pi^+\pi^-, \pi^0\pi^0$ $\pi^0 e\nu, \pi^0\mu\nu, \pi\pi\pi$

Mesons $J^P=1^-$							
	Q	S	I	Quarks content	Mass MeV/c ²	Lifetime s	Decay modes
ρ^+, ρ^-	+1,-1	0	1	$u\bar{d}, \bar{u}d$	774	$0,4 \cdot 10^{-23}$	$\pi^\pm\pi^0$
ρ^0	0	0	1	$(u\bar{u}-d\bar{d})/\sqrt{2}$	774	$0,4 \cdot 10^{-23}$	$\pi^+\pi^-$
ω	0	0	0	$(u\bar{u}+d\bar{d})/\sqrt{6}$	781,9	$7 \cdot 10^{-23}$	$\pi^+\pi^-\pi^0, \pi^0\gamma$
ϕ	0	0	0	$s\bar{s}$	1019,4	$20 \cdot 10^{-23}$	$K^+K^-, K^0\bar{K}^0$
K^{*+}, K^{*-}	+1,-1	+1,-1	1/2	$\bar{s}u, s\bar{u}$	891,8	$1 \cdot 10^{-23}$	$K^\pm\pi^0, K^0\pi^+, K^0\pi^-$
K^{*0}, \bar{K}^{*0}	0	+1,-1	1/2	$\bar{s}d, s\bar{d}$	896,1	$1 \cdot 10^{-23}$	$K^\pm\pi^0, K^0\pi^0, K^0\pi^0$

Table 7: mesons

IX-The N-fermions states and the classification of chemicals elements

We start from the Pauli principle, the theory of second quantization and the determination of one particle state to study the many-body system. But the approximation of the single-particle state by the hydrogen atom is very interesting for the study and the classification of chemical elements.

We also give an idea of the shell model and the states of a system of fermions in the notations of the second quantization.

- The one-particle states in atomic physics

The particle state is the tensor product of the wave function and the state of spin so the wave function of the i -th particle is:

$$\Psi_{[E_i]}(i) = \Psi_{n_i, l_i, m_i}(r_i) |sm_i\rangle, \quad (8,1)$$

$\Psi_{n_i, l_i, m_i}(r_i)$ Is the spatial part and the $|sm_i\rangle$ is the spinor part.

b-In Dirac notation we write:

$$|\Psi_{nlm, sm_s}\rangle = |nlm\rangle |sm_s\rangle, \quad (8,2)$$

d- Usually appears in the Hamiltonian the term $\vec{L} \cdot \vec{S}$ of the spin-orbit interaction and the wave function takes the form.:

$$|\Psi_{n(ls)jm}(r, s)\rangle = \sum_{m_l, m_s} \langle lm_l, sm_s || (ls) jm \rangle \Psi_{nlm}(r) |sm_s\rangle \quad (8,3)$$

1. Pauli principle

The Hamiltonian of a system of n-identical particles, such as electrons, is invariant under permutation.

In the case of a two-particle system we write:

$$H\Psi(1,2) = E \Psi(1,2) \quad et \quad H\Psi(2,1) = E \Psi(2,1) \quad (8,4)$$

The operator of permutations is defined by

$$P\Psi(2,1) = \Psi(1,2)$$

The probability of presence can be written

$$|\Psi(1,2)|^2 = |\Psi(2,1)|^2 = |P\Psi(1,2)|^2 = P^2 |\Psi(1,2)|^2 \quad (8,5)$$

So $P = \pm 1$ so we find the normalized wave function must be symmetric or antisymmetric:

$$\Psi^s = \frac{1}{\sqrt{2}} [\Psi(1,2) + \Psi(2,1)] \quad \Psi^A = \frac{1}{\sqrt{2}} [\Psi(1,2) - \Psi(2,1)] \quad (8,6)$$

Pauli principle:

- The bosons have symmetric wave functions
- Fermions have anti-symmetric wave functions
- The space of bosons have integer spin: 1, 2,3, ...
- The space of fermions have half-integer spin 1/2, 3/2, 5/2, ...
- Bosons obey Bose-Einstein statistics
- Fermions obey Fermi-Dirac statistics

2. The wave function of the system of N-fermions

We assume that the one-particle states are a solution one particle Hamiltonian h , e.g. Hartree-Fock, with:

$$h \varphi_E = E \varphi_E \quad (8,7)$$

And we assume that $E_1 < E_2 < E_3 \dots < E_N, \dots$

For a system of N fermions was

$$E_0 = \sum_{i=1}^N E_i \quad (8,8)$$

We call Fermi energy is the highest energy E_N .

The wave functions are Slater determinants:

-For a system of two particles $N = 2$, $E = \{E_1 E_2\}$

$$\Psi_{[E]}^A(1,2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \varphi_{E_1}(1) & \varphi_{E_1}(2) \\ \varphi_{E_2}(1) & \varphi_{E_2}(2) \end{vmatrix} \quad (8,9)$$

-For a system of N particles the ground state can be written using the Slater determinant, with $E = [E_1 E_2 E_3 \dots]$ by:

$$\Psi_{[E]}^A(1,2,3,\dots,N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_{E_1}(1) & \varphi_{E_1}(2) & \dots & \varphi_{E_1}(N) \\ \varphi_{E_2}(1) & \varphi_{E_2}(2) & \dots & \varphi_{E_2}(N) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{E_N}(1) & \varphi_{E_N}(2) & \dots & \varphi_{E_N}(N) \end{vmatrix} \quad (8,10)$$

3. The state of the system and the notations of second quantization

3.1 Fock space and ground state

By analogy with the harmonic oscillator we consider for example the fundamental N-particle state:

$$\Psi_{[E]}^A(1,2,\dots,N) = |1 \ 1 \ 1 \ 1 \ \dots \ 1 \ 0 \ 0 \dots \ 0 \dots\rangle, [E] = [E_1 E_2 E_3 \dots E_N] \quad (8,11)$$

The quantum vacuum is the state or there has only zeros.

$$|0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \dots\rangle = |0\rangle \quad (8,12)$$

a- The particle-hole or p-h states are the excited states of replacing the states i with $i \leq N$ by 0 and for the states j with $j > N$ by 1:

b- The 2p-2h states is excited state with two holes two particles.

c- We can continue 3p-3h etc..

The set of states (fundamental, p-h, 2p-2h ...), is the Dirac-Fock or Fock space.

3.2 The fermions operators of creation and annihilation

We define an operator called a_m : annihilations operators of a particle in the state m using the formula:

$$a_m |\dots N_m \dots\rangle = \begin{cases} 0 & \text{si } N_m = 0 \\ (-1)^v |\dots 0 \dots\rangle & \text{si } N_m = 1 \end{cases} \quad (8,13)$$

(v) is the number of occupied states with lexicographical index lower than m , we deduce the action of the creation operator by:

$$a_m^+ |\dots N_m \dots\rangle = \begin{cases} (-1)^v |\dots 1 \dots\rangle & \text{si } N_m = 0 \\ 0 & \text{si } N_m = 1 \end{cases} \quad (8,14)$$

Anticommutations the following rules are easily verified:

$$[a_i^+, a_j]_+ = (a_i^+ a_j + a_j a_i^+) = \delta_{i,j}, \quad [a_i, a_j]_+ = [a_i^+, a_j^+]_+ = 0 \quad (8,15)$$

The ground state can be written in this notation by:

$$a_1^+ a_2^+ \dots a_N^+ |0\rangle = |1 \ 1 \ 1 \ \dots \ \dots \ 1 \ 0 \ \dots\rangle. \quad (8,16)$$

3.3 Expression of the Hamiltonian

We assume that the fermions indistinguishable and the Hamiltonian is written:

$$H = \sum_{ij} \langle i|T|j\rangle a_i^+ a_j + \frac{1}{4} \sum_{ijkl} \langle i, j|\tilde{V}|k, l\rangle a_i^+ a_j^+ a_l a_k \quad (8,17)$$

with $\langle i, j|\tilde{V}|k, l\rangle = \langle i, j|V|k, l\rangle - \langle i, j|V|l, k\rangle$

In general we choose for effective calculus a woods- Saxon or Gaussian potential plus other terms such as $\vec{L} \cdot \vec{S}$.

We can find the one-particle Hamiltonian h and the eigenfunctions using Hartree-Fock variationelle method [25] seeking the minimum of the energy

$$E = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad (8,18)$$

In this course we want only to give an idea as a simple introduction to the Hartree-Fock theory.

4. Classification of chemical elements

4.1 The electron energy levels

To study the energy levels of many-electron of atoms, it was usually approximate the central field by the equation of hydrogen with the quantum number n, ℓ, m . Because of the influence of the electron energy levels depends on both n and ℓ . For a given value of n , the energy levels increases as ℓ increases. This is because the electrons with low ℓ pass near the nucleus, that is to say, in a region where the potential is similar to that created by the nucleus of charge Ze .

On the other hand, an electron whose "path" would be virtually external to those of other electrons have an energy approximately equal to that of a corresponding level of the hydrogen atom.

4.2 Distribution of electrons of an atom

We will discuss the distribution of electrons of an atom on the different levels. Under the Pauli principle, we know that for every quantum number n, ℓ, m , so it can not correspond two anti-parallel spin electrons $m_s = \pm(1/2)$.

Each pair of values n, ℓ , may correspond $2(2\ell + 1)$ electrons.

The existence of quantum numbers can leave the electrons of an atom in shell and subshell. A shell of electrons corresponds to the same value of n , when it is complete they have $2n^2$ electrons, for $n = 1, 2, 3, \dots$, is a $2n^2 = 2, 8, 18, 32, \dots$. May then divide this shell into n subshell each thereof corresponding to a given value of ℓ ;

A completes subshell are $2(2\ell + 1)$ electrons, which gives 2, 6, 10, ... for $\ell = 0, 1, 2, 3, \dots$

To classify the different elements, we must study how to fill the shell and subshell of successive atoms with the number of electrons increases. The levels are filled in order of increasing energy.

4.3 Configuration of different atoms in their ground state

One can study the electron configuration of different atoms in their ground state. This configuration is expressed by giving levels occupied by electrons. These levels are assigned an exponent indicating the number of electrons that are (Appendix 2).

For example, the ground states of following atoms are:

He	C	Na
$1s^2$	$1s^2 2s^2 2p^2$	$1s^2 2s^2 2p^6 3s$

4.4 The shell model in nuclear physics

a- The shell model in nuclear physics is partly analogous to the atomic shell model. More the shells for protons and for neutrons are independent of each other .

b- In nuclear physics, a magic number is a number of protons or neutrons in a nucleus which is particularly stable. The seven magic numbers experimentally verified are: 2, 8, 20, 28, 50, 82, 126. But the three-dimensional harmonic oscillator gives the following magic numbers 2, 8, 20, 40, 70, 112 which only agree with the first three entries which implies making calculations with more realistic potentials.

X- The representation theory of groups (Symmetry)

The left-right symmetry is visible in all every living and particles, more the spatial symmetry and rotational invariance are very important in physics and chemistry. Weyl in his book "symmetry" shows that the symmetry in art then we can say: there is no: mater, art, beauty or love without symmetry then the symmetry is the secret of the existence.

In quantum mechanics we are interested in transformations of groups and Lie algebra Especially the classical groups: rotations $O(n)$, unitary $U(n)$ and symplectic $SP(n)$. The theory of unitary groups is of fundamental importance in quantum physics and we want to do an introduction for students to follow more advanced courses on the representation theory of groups and Lie algebra.

But the study of this theory began in mathematics and this explains the diversity of the proposed methods [34-40]: the Cartan-Weyl infinitesimal method, the Global method of Young, Frobenius and Weyl. Therefore we will give only a quick summary of semi-simple Lie group and the generalization of Euler's angles to $SO(n)$ and $SU(n)$ saw its applications in quantum and in quantum information theory [41].

1. Introduction to the semi-simple Lie algebra

1.1 infinitesimal method

This is the algebraic method that was developed by Cartan ,Killing, Weyl, Chevalley, and Dynkin etc. And it takes its starting point the study of the properties of infinitesimal operators of the groups. And it is widely used by physicists because it is of great interest from a practical and theoretical point of view.

1.2 The global method

The global method of Weyl originates the connection between the representation of groups of permutations and unitary groups. This method has been the subject of intense study by mathematicians: Gel'fand, Godement and Zelobenko ... [37] and by physicists [36] as Biedenharn, Moshinsky and others. But physicists have worked particularly in determining the basis of irreducible [35-37] representations of unitary groups in the oscillator basis as an extension of Schwinger's work on the theory of angular momentum.

1.3 The method of Schur function

Finally the method of Schur function is the least used and least known in physics.

2. The infinitesimal method and the Lie algebra

If we consider the Lie group of r -parameters and r -infinitesimal operators (2,11) and the commutation relations:

$$[X_\rho, X_\sigma] = \sum_\tau C_{\rho,\sigma}^\tau X_\tau, \text{ with } C_{\rho,\sigma}^\tau = C_{\sigma,\rho}^\tau \quad (10,1)$$

We define the metric tensor and the Killing form by:

$$g_{\sigma\lambda} = g_{\lambda\sigma} = \sum_{\tau,\rho} C_{\sigma,\rho}^\tau C_{\lambda,\tau}^\rho \quad (10,2)$$

Note that Weyl determined infinitesimal generators of unitary groups $U(n)$ and they are denoted by E_{ij} , ($i, j = 1 \dots n$)

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \quad (10,3)$$

These generators are expressed [36] using the operators of creation and annihilation of n -dimensional harmonic oscillator:

$$E_{ij} = \sum_{ij} a_i^+ a_j \quad (10,4)$$

2.1 The Cartan theorems

Theorem: The Lie algebra is semisimple if and only if $\det(g_{\alpha\gamma}) \neq 0$.

The inverse matrix of $(g_{\alpha\lambda})$ is denoted by $(g^{\alpha\lambda})$ (6,3).

The first theorem of Cartan:

The necessary and sufficient condition for a semi-simple group is compact if the Killing form is negative definite [23].

2.2 Classification of semi-simple Lie algebras

Cartan's method is to find the roots of the equation solutions with eigenvalues denoted by (ρ)

$$[A, X] = \rho X, \quad (10,5)$$

with $A = \sum_{\mu} a^{\mu} X_{\mu}$ et $X = \sum_{\nu} b^{\nu} X_{\nu}$

The vectors $\{X_{\nu}\}$ are independent then:

$$(a^{\mu} c^{\tau}_{\mu,\nu} - \rho \delta^{\tau}_{\nu}) b^{\nu} = 0$$

We deduce the secular equation

$$\det(a^{\mu} c^{\tau}_{\mu,\nu} - \rho \delta^{\tau}_{\nu}) = 0 \quad (10,6)$$

The second theorem of Cartan:

If the Lie algebra G is semisimple and if we choose for A the element of G which has the most distinct eigenvalues, then one's own degenerate zero value. The degree ℓ of degeneracy of this eigenvalue, $[A, H_i] = \rho = 0$, is characteristic of the algebra then ℓ is called the rank of the semisimple algebra and $A = \sum_i H_i$.

The set $\{H_i\}$ span an ℓ -dimensional subspace of the r -dimensional vector space, and the dimension of $\{E_{\alpha}\}$, $[A, E_{\alpha}] = \alpha E_{\alpha}$, is $r - \ell$.

2.3 The Cartan-Weyl basis

We only give the very important results of Cartan and Weyl:

$$\begin{aligned} [H_i, H_k] &= 0, & [H_i, E_{\alpha}] &= \alpha_i E_{\alpha}, & (i, k = 1, \dots, \ell) \\ [E_{\alpha}, E_{\beta}] &= N_{\alpha, \beta}, & [E_{\alpha}, E_{-\alpha}] &= \alpha^i E_{\alpha} \end{aligned} \quad (10,7)$$

With $N_{\alpha, \beta} = C_{\alpha, \beta}^{\alpha + \beta}$, $\alpha^i = \sum_j g^{ij} \alpha_j$ (10,8)

Note that this basis is frequently used in physics. And it has been demonstrated starting from this basis that the semi-simple groups are the classical groups: the orthogonal, the unitary and symplectic and exceptional groups.

Note that the classification of these algebras can be easily carried by the Dynkin diagrams, diagrams roots [23].

2.4 Cartan theorem on the irreducible representations of classical groups.

For classical groups G of order n there are n fundamental weights:

$$[h]_n = [h_{1n}, h_{2n}, \dots, h_{nn}] \tag{10,9}$$

And any maximum weight corresponding to an irreducible representation there are n positive numbers m_i as:

$$\begin{aligned} H_i | [h]_n \rangle &= h_{in} | [h]_n \rangle \\ m_i &= h_{in} - h_{(i+1)n}, \quad i=1,2,\dots,n, \end{aligned} \tag{10,10}$$

The index h_{nn} is zero if G is the group SU (n).

2.5 Racah theorem:

The number of Casimir operators of G is n.

2.6 The fundamental representations of unitary groups

The irreducible representations of G are functions of fundamental representations for which $h_{in} = 1, i \leq n$.

E. Cartan showed that the irreducible representations of U (n) are functions of subspaces of fundamental representations [21].

These basic representations are subspace $[h]_n$.

$$\{\Delta_{n,1}\} = [100\dots 0]_n, \{\Delta_{n,2}\} = [110\dots 0]_n, \dots, \{\Delta_{n,n}\} = [111,\dots,11]_n \tag{10,11}$$

The dimension of the subspace $[1, \overbrace{1,1,\dots,1}^p, 0, \dots, 0] = \Delta_{n,p}$, or F (n, p), is C_n^p then the dimension F (n, n) is 1. And it is easy to verify that the sum of the dimensions of the fundamental representations of U (n) is $2^n - 1$ and $2^n - 2$ of SU (n).

2.7 The fundamental basis of binary representations:

We observe that the vectors of the bases of fundamental representations can be expressed by binary numbers that we call the binary basis of fundamental representations. And afterwards we make the correspondence with the Gel'fand basis of U (n) .

We write as examples:

$\Delta_{(2,1)}$	1	0	0	0	1	0	0	0	0						
$\Delta_{(3,1)}$	1	0	0	0	0	1	0	0	0	1					
$\Delta_{(3,2)}$	0	1	1	0	0	1	0	1	1	0					
$\Delta_{(4,1)}$	1	0	0	0	0	1	0	0	0	0	1	1	1	0	
$\Delta_{(4,2)}$	1	1	0	0	1	0	1	0	1	0	0	0	1	0	
$\Delta_{(4,2)}$	0	0	1	1	0	1	0	1	0	1	1	0	0	0	
$\Delta_{(4,3)}$	0	1	1	1	1	0	1	1	1	0	1	0	0	0	1

Table 8

3. The global method: representations of permutations and unitary groups

The global method of Weyl has the starting point the connection between the representations of permutations groups and unitary groups. Weyl introduced the characters of classical groups and he deduced the branching law and the formula of the dimension known by Weyl formula.

3.1 The Weyl branching law

If we consider the restriction of $U(n)$ to $U(n-1)$, with $[h]_n$ and $[h]_{n-1}$ are the highest weights. Weyl found the important result:

$$h_{1,n} \geq h_{1,n-1} \geq h_{2,n} \geq h_{2,n-1} \geq \dots \geq h_{n-1,n-1} \geq h_{n,n} \cdot \quad (10,12)$$

- a- From this formula Gel'fand and Zeitlin deduced first in an intuitive way the representation of $U(n)$ represented by an array of index ≥ 0 .
- b- The search for the explicit form of Gel'fand basis, using the bosons operators or Fock space, was found by Bargmann and Moshinsky, Biedenharn and others. But Nagel and Moshinsky expressed this basis in terms of the raising and lowering operators. All these methods are an inextricable complexity for $n \geq 3$ for this we have developed the method of generating function to solve this difficult problem [25].

3.2 Representations of permutations groups

3.2.1 Consider the Schrödinger equation of a system of n - fermions (electrons or nucleons) for example

$$H(r_1, r_2, \dots, r_n) \Psi(r_1, r_2, \dots, r_n) = E \Psi(r_1, r_2, \dots, r_n) \quad (10,13)$$

The Hamiltonian H must be invariant under a permutation of the variables r_i . The solution of the equation must be transformed according to an irreducible representation of these groups and these representations are symmetric or antisymmetric if Ψ function of the variables r_i and the spins.

3.2.2 The irreducible representations of the permutation group S_n are determined by the decomposition of n into positive integers:

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_r, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r, \quad r \leq n \quad (10,14)$$

A table or Young diagram $(\lambda_1, \dots, \lambda_r)$ is a Young tableau in which inscribed the numbers 1 to n . The boxes are arranged in a line $\lambda_1, \lambda_2, \dots, \lambda_k$ by line because the lines are arranged in ascending order.

We fill the boxes of the Young diagram with numbers ψ_i functions for $n = 11$

1	2	3	4	5
6	7	8		
9	10			
11				

The number of permutations is $n!$. This corresponds to $n!$ Young tableaux that are not linearly independent of where the introduction of the standard tableaux.

The standard tableaux is the distribution in ascending order numbers from left to right and from top to bottom columns built by the rule of recurrences [38-39]. We will give these tables by recurrence method [38]:

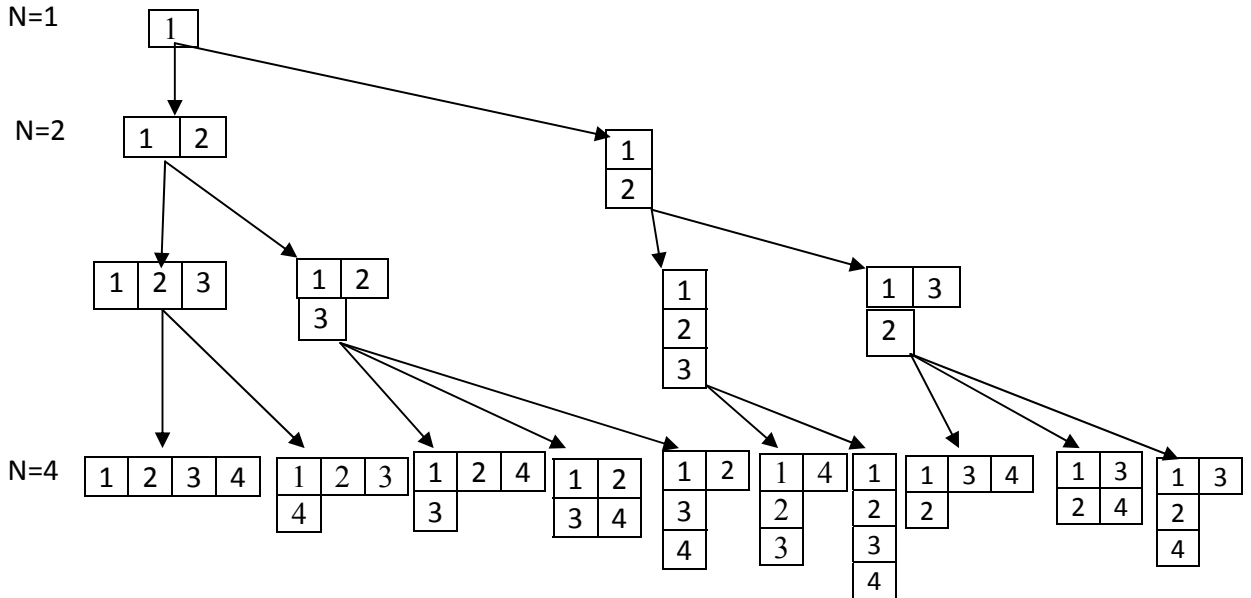


Table 9

3.2.3 We note that: the first by (4), the second (3,1), fourth $((2,2)=(2^2))$, fifth $(2,1,1) = (2,1,1)=(2,1^2)$, and seventh (1^4) .

3.2.4 The conjugates partitions: The conjugates partitions are partitions that can be deduced from each other by the exchange of rows and columns by rows and columns and to each partition there is a conjugate partition.

3.2.5 Representations: We note that the decomposition of irreducible representations can be done with the help of Young operators [36-39] and the Young diagrams.

3.3 The dimension of the representation of S_n

The dimension of the representation of S_n associated with the partition $(\lambda_1, \dots, \lambda_r)$ is:

$$d_n = n! \times \left[\frac{\left(\prod_{j=1}^r (\lambda_j - \lambda_j + j - i) \right)}{\left(\prod_{i=1}^r (\lambda_i + r - i) \right)!} \right] \quad (10,15)$$

3.4 Weyl representation of the unitary groups

Weyl showed that representations of unitary groups [36] can be deduced from its patterns by repeating numbers and we will give the representation of $U(2)$ and semi maximum representation of $U(3)$, and then we deduce the Weyl branching law.

N=2 $\begin{bmatrix} 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} (\varphi(1)\varphi(2) - \varphi(2)\varphi(1)) \Rightarrow \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow), [1 \ 1] = \varphi(1)\varphi(1) \Rightarrow \uparrow\uparrow$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}} (\varphi(1)\varphi(2) + \varphi(2)\varphi(1)) \Rightarrow \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow), [2 \ 2] = \varphi(2)\varphi(2) \Rightarrow \downarrow\downarrow$$

a- Representation of U(2)

	h_{11}	h_{22}	h_{12}
11...	11..	222....	
222..			

We have: $h_{11} \leq h_{22} \leq h_{12}$ (10,16)

b- Representation de U(3)

	h_{23}	h_{22}	h_{23}	h_{12}	h_{13}
11...	11...	11.....1	33.....	
22...	2.....	3...			
33.....					

We have also: $h_{23} \leq h_{22} \leq h_{23} \leq h_{12} \leq h_{13}$ (10,17)

c- The Weyl branching law

The generalization of expressions (10,17) allows us to deduce the restriction of weight U (n) to U (n-1). The maximum weight are $[h]_n$ and $[h]_{n-1}$:

$$h_{1,n} \geq h_{1,n-1} \geq h_{2,n} \geq h_{2,n-1} \geq \dots \geq h_{n-1,n-1} \geq h_{n,n} . \quad (10,18)$$

4. The Gel'fand basis of unitary groups

4.1 The Gel'fand basis of unitary groups

Based on the law of the Weyl branching Gel'fand and Zeitlin found by intuition the orthogonal basis of U (n).

The Gel'fand basis or Γ_n is:

$$|(h)_n\rangle = \left| \begin{array}{cccc} h_{1n} & h_{2n} & \dots & h_{nn} \\ \dots & \dots & \dots & \dots \\ & h_{12} & & h_{22} \\ & & & h_{11} \end{array} \right\rangle = \left| \begin{array}{c} [h]_n \\ (h)_{n-1} \end{array} \right\rangle = \left| \begin{array}{c} [h]_n \\ [h]_{n-1} \\ (h)_{n-2} \end{array} \right\rangle \dots \quad (10,19)$$

And $h_{i,j} \geq h_{i,j-1} \geq h_{i+1,j}$ avec $j = 2, \dots, n$ et $i = 1, \dots, j$

4.2 The Weyl dimension formula

The dimension of the subspace of $U(n)$ is given by the Weyl formula:

$$d_{[h_{uv}]} = d[h_{1n}, \dots, h_{nn}] = \left[\prod_{i < j} (p_{in} - p_{jn}) / [1!2! \dots (n-1)!] \right] \quad (10,20)$$

with $p_{in} = h_{in} + n - i$ and $h_{nn} = 0$ for the subgroups $SU(n)$.

4.3 The maximal and minimal states

We associate with any vector a weight vector that has the components:

$$\omega(h) = (\omega_{1n}(h), \omega_{2n}(h), \dots, \omega_{nn}(h)). \quad (10,21)$$

with

$$\omega_{in} = \left(\sum_{j=1}^i h_{j,i} - \sum_{j=1}^{i-1} h_{j,i-1} \right)$$

A weight $\omega(h')$ is higher than the weight $\omega(h)$ if the difference of the first component $\omega(h') - \omega(h)$ is positive.

We deduce the existence of a minimal and maximal vectors.

4.4 Explicit expression of Gel'fand basis

By analogy with the $SU(2)$, Nagel and Moshinsky [35] showed that the basis can be deduced from the minimum and maximum state by applying ladders operators $L_\lambda^\mu, R_\lambda^\mu$. We write:

$$|(h)_n\rangle = N \prod_{\lambda=2}^n \prod_{\mu=1}^{k-1} (L_\lambda^\mu)^{L_\lambda^\mu} \left| \begin{matrix} [h]_n \\ (\max)_{n-1} \end{matrix} \right\rangle = N' \prod_{\lambda=2}^n \prod_{\mu=1}^{k-1} (R_\lambda^\mu)^{R_\lambda^\mu} \left| \begin{matrix} [h]_n \\ (\min)_{n-1} \end{matrix} \right\rangle$$

With $L_\lambda^\mu = h_{\mu,\lambda} - h_{\mu,\lambda-1}$, $R_\lambda^\mu = h_{\mu,\lambda-1} - h_{\mu+1,\lambda}$ (10,22)

N and N' are the normalization constants.

4.5 The binary representations of the analytic Hilbert space

(z_i^j) , $i, j = 1, \dots, n$ Is a matrix defined on the field of complex numbers [25,37]. We associate to any minors $\Delta_{i_1 \dots i_k}^{12 \dots k}$ of (z_i^j) , $i, j = 1, \dots, n$ a binary number of length n and ordered. This number ones for the numbers i_1, i_2, \dots, i_k and zero elsewhere.

$${}_n \Delta_{[i]}^k = {}_n \Delta_{i_1 \dots i_k}^{12 \dots k} = \begin{matrix} 1 & 2 & \dots & i_1 & \dots & i_k & \dots & n \\ \hline 0 & 0 & \dots & 1 & \dots & 1 & \dots & 0 \end{matrix} \quad (10,23)$$

We note in the following the basis vectors of Fock-Bargmann space by:

$$\Delta_{n,k}^i(z) = \left\{ {}_n \Delta_{[i]}^k = {}_n \Delta_{i_1 \dots i_k}^{12 \dots k}(z) \right\} \quad (10,24)$$

The Gel'fand representation is:

$$\Gamma_n(\Delta(z)) = \langle (h)_n \| \Delta(z) \rangle \quad (10,25)$$

5. Generalization of Euler's angles to SO(n) and SU(n)

We establish recurrences formulas of the order of the classical groups that allow us to find the generalization of Euler's angles for the classical groups [25, 41-44].

5.1 Generalization of the Euler parameterization of SO(3)

The rotation may be written by

$$R(\psi\theta\varphi) = e^{-i\psi L_z} e^{-i\theta L_y} e^{-i\varphi L_z} \quad (10,26)$$

By analogy we write the finite transformation of classical groups in the form:

$$A_n^m = A_{n-1}^m B_n^m A_{n-1}^m \quad (10,27)$$

With $m = 0, 1$ and 2 for orthogonal, unitary and symplectic groups.

We design by $N_m(n)$ the number of parameters of the classical groups:

$$N_0(n) = n(n-1)/2, \quad N_1(n) = n^2 - 1, \quad N_2(n) = n(2n+1). \quad (10,28)$$

In the following we derive two kinds of recurrences formulas

a- The first relation:

$$N_m(n) = N_m(n-1) + 2^m n - 1, \quad m = 0, 1, 2. \quad (10,29)$$

Since the point $(0, \dots, 0, 1)$ is invariant by the group of order $n-1$ this means that the last column and the last row are the components of the unit vectors, with $2^m n-1$ parameters, of points on the unit sphere $S^{m, n-1}$ of the Euclidian space $E_n(K)$, $K = R, C, H = Q$.

b- The second relation:

$$N_m(n) = 2N_m(n-1) - N_m(n-2) + 2^m, \quad m = 0, 1, 2. \quad (10,30)$$

It is quite evident that the left and right parameters of (10,26) are different

Therefore

$$\begin{aligned} A_n^m &= A_{n-2}^m B_{n-1}^m A_{n-2}^m B_n^m A_{n-2}^m B_{n-1}^m A_{n-2}^m \\ &= A_{n-2}^m B_{n-1}^m [A_{n-2}^m B_n^m A_{n-2}^m] B_{n-1}^m A_{n-2}^m \end{aligned} \quad (10,31)$$

If

$$[B_n^m, A_{n-2}^m] = 0 \quad (10,32)$$

The number of parameters of A_n^m becomes $2N_m(n-1) - N_m(n-2) + 2^m$ and the number of parameters of B_n^m is 2^m therefore we find the same result of the above recurrences relations (10,29-30).

To find A_n^m we must choose the parameters such that $[B_n^m, A_{n-2}^m] = 0$ and the last line, or the last column, are the components of the unit vector $\vec{r} = (x_1, \dots, x_n)$ of $E_n(K)$ and from this property we can deduce simply the measure of integration of SO(n) and SU(n) [25].

5.2 Parameterization of SO(n)

In this case $A_n^m = A_{n-1}^m B_n^m A_{n-1}^m$, $m = 0$ the matrix B_n^0 is function of one variable and The expression $[B_n^0, A_{n-2}^0] = 0$ means that B_n^0 leave invariant A_{n-2}^0 .

If we choose in E_n the spherical coordinates $\theta_1, \theta_2, \dots, \theta_{n-1}$ we find the Vilenkin's parameterization [44] for $SO(n)$. and any rotation g of the group $SO(n)$ can be set as follows:

$$g = g^{(n-1)} \dots g^{(1)}$$

$$g^{(k)} = g_l(\theta_l^k) \dots g_k(\theta_k^k)$$
(10,33)

Where

And $g_k(\theta)$ is the rotation matrix in the plane (x_k, x_{k+1}) .

5.3 Parameterization of $SU(n)$

In the case of $m = 1$ the matrix B_n^l is function of two variables and $\det(B_n^l) = 1$. The expression $[B_n^l, A_{n-2}^l] = 0$ means that B_n^l leave invariant A_{n-2}^l and the solution is not unique for $n > 2$.

We can consider the useful options (22), for example

$$u = u^{(n-1)} \dots u^{(1)}$$

$$u^{(k)} = u_0^k(\psi_1^k) u_1^k(\theta_1^k, \psi_2^k) \dots u_k^k(\theta_k^k, \psi_{k+1}^k)$$

$$u_i^k(\theta_i^k, \psi_i^k) = B_n^2 = g_i(\theta_i^k) d_i(\psi_i^k)$$
(10,34)

Where

$$u_i^k = \begin{pmatrix} e^{-i\psi_1^k} & 0 \\ 0 & e^{+i\psi_1^k} \end{pmatrix}, d_i(\psi_i^k) = \begin{pmatrix} e^{-i\psi_i^k} I_{n-1} & 0 \\ 0 & e^{-i(n-1)\psi_i^k} \end{pmatrix}$$
(10,35)

With

We have for $SU(3)$

$$U_3 = A_3^2 = A_2^2 [B_3^2] A_2^2$$

$$\begin{pmatrix} a_1 & a_2 & 0 \\ -\bar{a}_2 & \bar{a}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\nu_3}{2} & \sin \frac{\nu_3}{2} \\ 0 & -\sin \frac{\nu_3}{2} & \cos \frac{\nu_3}{2} \end{pmatrix} \begin{pmatrix} d_3 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & \bar{d}_3^2 \end{pmatrix} \right] \begin{pmatrix} b_1 & b_2 & 0 \\ -\bar{b}_2 & \bar{b}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
(10,36)

with $0 \leq \nu_3 \leq \pi$, $d_3 = e^{i\beta_3}$, $0 \leq \beta_3 \leq \pi$.

$$\text{And } b_1 = \exp(\varphi_1') \cos\left(\frac{\theta'}{2}\right), \quad b_2 = \exp(\varphi_2') \sin\left(\frac{\theta'}{2}\right)$$
(10,37)

Note: Symmetry and prime numbers

The right and left symmetry is fundamental in physics this incite us to ask whether this symmetry is in primes:

Find the numbers n with $a + \bar{a} = n$.

a, \bar{a} are primes and $\{a\}$ are all the primes with: $n/2 \leq a < n$, and $n = 2, 3, \dots$

Calculations for $n \leq 100$ million we find six numbers $n = 5, 7, 10, 16, 36, 210$.

XI -The representation of SU (3) group

The group SU (3) plays a fundamental role in nuclear physics and elementary particle and we will determine first the generators of SU (3).

The expression of the vectors of the basis was found by several methods but we will find it simply using the generating function and the Schwinger approach [40].

1. Generators and Casimir operators of SU (3)

1.1 Gell-Mann represented the generators of SU (3) by the following matrices:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \quad (11,1)$$

Put $F_i = \lambda_i / 2$

$$\begin{aligned} \text{We write : } T_{\pm} &= F_1 \pm iF_2, \quad U_{\pm} = F_6 \pm iF_7, \quad V_{\pm} = F_4 \pm iF_5 \\ T_z &= F_3, \quad Y = \frac{2}{\sqrt{3}} F_8, \quad U_3 = \frac{1}{2} \left(\frac{3Y}{2} - T_z \right), \quad V_3 = \frac{1}{2} \left(\frac{3Y}{2} + T_z \right) \end{aligned} \quad (11,2)$$

After a simple calculation we find

$$[Y, T_{\pm}] = 0, \quad [T_+, T_-] = 2T_z, \quad [U_+, U_-] = 2U_3, \quad [V_+, V_-] = 2V_3, \quad (11,3)$$

SU (3) has rank two then by Racah theorem the number of Casimir operators are two:

$$C_1 = \sum_{i=1}^8 F_i^2, \quad C_2 = \sum_{i,j,k} d_{i,j,k} F_i F_j F_k, \quad d_{i,j,k} = \frac{1}{4} \text{Trace} ([\lambda_i, \lambda_j]_+ \lambda_k) \quad (11,4)$$

1.2 The basis of the group SU (2) \subset SU (3)

Let be the space $D_{[\lambda, \mu]}$ of homogeneous polynomials and $V_{(t, \varepsilon, y)}^{\lambda \mu}(z^1, z^2)$ is the orthogonal basis. In the case of SU (3) we write:

$$\begin{aligned} z^1 &= (\xi_1, \eta_1, \sigma_1) = (x^1, \sigma_1), \quad z^2 = (\xi_2, \eta_2, \sigma_2) = (x^2, \sigma_2) \\ \text{And } T_+ &= \xi \frac{\partial}{\partial \eta}, \quad T_- = \eta \frac{\partial}{\partial \xi}, \quad T_z = \frac{1}{2} \left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) \end{aligned} \quad (11,5)$$

$$Y = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} - 2\sigma \frac{\partial}{\partial \sigma}$$

With

$$T_{ij} = \xi_i \frac{\partial}{\partial \xi_j} + \eta_i \frac{\partial}{\partial \eta_j} + \sigma_i \frac{\partial}{\partial \sigma_j},$$

The vectors $V_{(t,t_z,y)}^{\lambda\mu}(z^1, z^2)$ are eigenfunctions of the Casimir operator and the projection of \vec{T} on the z-axis and y is the triple of the hypercharge Y ($y = 3Y$).

The numbers I, I_z or t, t_z are isospin and its projection on the z-axis, $p=\lambda+\mu$ and $q=\mu$.

We have:

$$\begin{aligned} Y V_{(\alpha)}^{\lambda\mu} &= y V_{(\alpha)}^{\lambda\mu}, & T_z V_{(\alpha)}^{\lambda\mu} &= t_z V_{(\alpha)}^{\lambda\mu} \\ \vec{T}^2 V_{(\alpha)}^{\lambda\mu} &= t(t+1) V_{(\alpha)}^{\lambda\mu}. \end{aligned} \quad (11,6)$$

The Young tableau imposes the following condition on $V_{(\alpha)}^{\lambda\mu}$ (see also table 8):

$$T_{12} V_{(\alpha)}^{\lambda\mu}(z^1, z^2) = 0 \quad (11,7)$$

2. Schwinger approach and the generating function of SU (3)

2.1 Schwinger approach

The vectors $V_{(t,t_z,y)}^{\lambda\mu}(z^1, z^2)$ are the elements of the product space $D_{t_1} \otimes D_{t_2} \otimes D_{t_3}$ which has the basis:

$$\varphi_{j_1 m_1}(\xi_1, \eta_1) \varphi_{j_2 m_2}(\xi_2, \eta_2) \varphi_{j_3 m_3}(\sigma_1, \sigma_2).$$

But

$$T_+ = \mathfrak{T}_+ + \xi_3 \frac{\partial}{\partial \eta_3}, \quad T_- = \mathfrak{T}_- + \eta_3 \frac{\partial}{\partial \xi_3}, \quad T_z = \mathfrak{T}_3 + \frac{1}{2} \left(\xi_3 \frac{\partial}{\partial \xi_3} - \eta_3 \frac{\partial}{\partial \eta_3} \right) \quad (11,8)$$

And it is easy to verify that $\mathfrak{T}_+, \mathfrak{T}_-$ et \mathfrak{T}_3 are the generators of SU (2).

So the determination of the basis $\{V_{(\alpha)}^{\lambda\mu}\}$ by Schwinger generating function method.

This method consists of making the following coupling (7,40):

a-The first coupling $(j_1 m_1) \times (j_2 m_2) \rightarrow (j(m'), m)$:

$$\varphi_{j_1 m_1}(\xi_1, \eta_1) \varphi_{j_2 m_2}(\xi_2, \eta_2) \Rightarrow {}_{(m')} F_{(m)}^j(x), \quad m' = j_1 - j_2$$

b-We apply to the result the new coupling $(jm') \times (j_3 m_3)$:

$${}_{(m')} F_{(m)}^j(x) \varphi_{j_3 m_3}(\sigma_1, \sigma_2) \Rightarrow V_{(t,t_z,y)}^{(\lambda\mu)}(z^{(1)}, z^{(2)}, z^{(12)})$$

c-We then apply the Cartan basis (table 8), or the Young condition (11,7), to cancel the parameters $\vec{z}^{(2)}$ so we write:

$$V_{(t,t_z,y)}^{(\lambda\mu)}(z^{(1)}, z^{(12)}) = V_{(t,t_z,y)}^{(\lambda\mu)}(z^{(1)}, z^{(2)}, z^{(12)})$$

We obtain the generating function of SU (3) and we write it in a simple form:

$$G((x, y, u), z) = \exp[\vec{f} \cdot \vec{z}^{(1)} + \vec{g} \cdot \vec{z}^{(2)}] = \sum_{\lambda, \mu, t, z, y} \varphi_{(t, z, y)}^{(\lambda, \mu)}(\vec{f}, \vec{g}) V_{(t, z, y)}^{(\lambda, \mu)}(z^{(1)}, z^{(2)}) \quad (11,9)$$

With $\vec{f} = (x_1 \xi, x_1 \eta, x_2)$, $\vec{g} = (y_1 \eta, -y_1 \xi, y_2)$, $\vec{z}^{(i)} = (z_1^i, z_2^i, z_3^i)$
 z^{ij} are the minors:

$$z^{(ij)} = \vec{z}^{(i)} \times \vec{z}^{(j)} = (\Delta_3^1 \vec{i} + \Delta_3^2 \vec{j} + \Delta_3^3 \vec{k})$$

We also have:

$$\varphi_{(t, z, y)}^{(\lambda, \mu)}(\vec{f}, \vec{g}) = N[(\lambda, \mu), (\alpha)] (x_1^r x_2^{\lambda-r} y_1^{(\mu-s)} y_2^s) (\xi^{(t+t_z)} \eta^{(t-t_z)}) \quad (11,10)$$

$$\text{And } N[(\lambda, \mu), (\alpha)] = (-1)^s \sqrt{\frac{(\mu+r+1)!(\mu+\lambda-s+1)!}{(\lambda+1)!(2t+1)\lambda! [r!(\lambda-r)!s!(\mu-s)!(t+t_3)!(t-t_3)!]}$$

2.2 The basis vectors $V_{(\alpha)}^{\lambda, \mu}(z^1, z^2)$ of $SU(3)$

By the development of the generating function and after comparison with the second term we find the basis:

$$V_{(t, z, y)}^{\lambda, \mu}(z^1, z^2) = N[(\lambda, \mu; \alpha)] (-1)^s \times \sum_k \binom{m}{k} \frac{(\mu-s)! r!}{(\mu-s-k)! [r-(n-k)]!} \\ \times \xi_1^{r-(m-k)} \eta_1^{m-k} \sigma_1^{\lambda-r} (\Delta_3^1)^k (-\Delta_3^2)^{\mu-s-k} (\Delta_3^3)^s \\ N[(\lambda, \mu; \alpha)] = \left\{ \frac{(\lambda+1)!(\mu+r-s+1)!}{r!s!(\mu-s)!(\lambda-r)!(\mu+r+1)!(\lambda+\mu-s+1)!} \times \frac{(2t-m)!}{(2t)!m!} \right\}^{\frac{1}{2}} \quad (11,11)$$

And $y = -(2\lambda + \mu) + 3(r + s)$, $0 \leq r \leq \lambda$,

$$t = \frac{\mu}{2} + \frac{r-s}{2}, \quad 0 \leq s \leq \mu,$$

$$t_z = t - m, \quad m = 0, 1, \dots, 2t. \quad (11,12)$$

In part 6 we follow the notations:

$${}_3\Delta_1^1 = z_1^1, \quad {}_3\Delta_1^2 = z_2^1, \quad {}_3\Delta_1^3 = z_3^1, \quad (11,13)$$

$${}_3\Delta_2^4 = \begin{vmatrix} z_2^1 & z_3^1 \\ z_2^2 & z_3^2 \end{vmatrix}, \quad {}_3\Delta_2^5 = \begin{vmatrix} z_1^1 & z_3^1 \\ z_1^2 & z_3^2 \end{vmatrix}, \quad {}_3\Delta_2^6 = \begin{vmatrix} z_1^1 & z_2^1 \\ z_1^2 & z_2^2 \end{vmatrix} \quad (11,14)$$

And the vectors $V_{(\alpha)}^{\lambda, \mu}(z^1, z^2) = \Gamma_3(\Delta(z))$

The dimensions of the subspace of $SU(3)$ are useful for the following:

$h_{13}=\lambda+\mu, h_{23}=\mu$	1,0	1,1	2,0	2,1	3,0	3,1	4,0	4,1	4,2
$d[h_{13}, h_{23}, 0]$	3	3	6	8	10	15	15	24	27

XII- SU(n) generating function of Gel'fand basis and tensor products of SU (3)

To extend the study of angular momentum to the unitary groups it is important to determine the generating functions of these groups. We observed a very important property: the powers of the parameters of the generating functions are the same powers of the ladders operators of the oscillator and SU (2).

We will demonstrate that the generating function of SU (3) that we have built [46] with the help of the Schwinger approach (Chapter 11) verifies the above property.

The generalization to SU (n) of this property give the Gel'fand generating functions and then we deducting the base and the formulas for the conjugate states of SU (n).

We also expose the usual methods which are very useful for the calculation of tensor products of SU (3).

1. Generating functions and ladder operators

1.1 Generating function of the oscillator and the base of Gel'fand

The generating function of the oscillator in terms of Gel'fand indices:

$$|G(z, q)\rangle = \sum_{h_{11}} \frac{(y_1^1)^{h_{11}} (a^+)^{h_{11}}}{(h_{11})!} |0\rangle = \exp[y_1^1 a_1^+] |0\rangle \quad (12,1)$$

2.2 Generating function of SU(2) and the Gel'fand basis

We want to express the generating function of SU(2) in terms of Gel'fand indices.

We have
$$|jm\rangle = \sqrt{\frac{(j \pm m)!}{(2j)!(j \mp m)!}} J_{\mp}^{j \mp m} |j \pm j\rangle, \quad (12,2)$$

In the Gel'fand notation we write the generating function of SU (2) by:

$$G(\xi, z) = \sum_{h_{11}, h_{12}} \frac{(x_2^1)^{h_{12}-h_{11}} (y_2^1)^{h_{11}}}{\sqrt{(h_{12}-h_{11})!(h_{11})!}} \Gamma_2 \left(\begin{matrix} h_{12} \\ h_{11} \end{matrix} \middle| 0 \right) (\Delta(z)) = \exp[y_2^1 \Delta_1^1 + x_2^1 \Delta_2^1] \quad (12,3)$$

With: $h_{12} = 2j, \quad h_{11} = j - m, \quad h_{22} = 0.$

1.3 The generating function of SU(3) and the Gel'fand basis

We observed (12,1-2) that the powers of parameters x and y have the same powers of raising and lowering operators introduced by Nagel and Moshinsky [35].

The extension of this observation to the expression of the generating function of SU(3) give:

$$\sum_{\lambda, \mu} \prod_{\lambda=2}^3 \prod_{\mu=1}^{\lambda-1} A_{\lambda, \mu} ((x_{\lambda}^{\mu})^{h_{\mu\lambda}-h_{\mu\lambda-1}} (y_{\lambda}^{\mu})^{h_{\mu\lambda-1}-h_{\mu+1\lambda}}) \Gamma_3 \left(\begin{matrix} [h]_3 \\ (h)_3 \end{matrix} \right) (\Delta(z)) = \exp[_3\Delta_{12}^{(12)} y_3^2 + z_3^2 (_3\Delta_{23}^{(12)} x_2^1 + _3\Delta_{13}^{(12)} y_2^1) + y_3^1 (_3\Delta_1^1 y_2^1 + _3\Delta_2^1 x_2^1) + _3\Delta_3^1 x_3^1] \quad (12,4)$$

The comparison of the expressions (8.3) and (10.28) yields:

$$\begin{pmatrix} [h]_3 \\ (h)_2 \end{pmatrix} = \begin{pmatrix} \lambda + \mu & \mu & 0 \\ t + \frac{Y}{2} + \frac{(\lambda + 2\mu)}{3} & -t + \frac{Y}{2} + \frac{(\lambda + 2\mu)}{3} & \\ & t_z + \frac{Y}{2} + \frac{(\lambda + 2\mu)}{3} & \end{pmatrix} \quad (12,5)$$

Thus we find the well known result [36]:

$$\begin{array}{ll} \text{a-} & h_{13} = p, h_{23} = q \\ \text{c-} & Y = h_{12} + h_{22} - 2(h_{13} + h_{23}) / 3 \end{array} \quad \begin{array}{ll} \text{b-} & I_z = t_z = h_{11} - (h_{12} + h_{22}) / 2 \\ \text{d-} & I = t = (h_{12} - h_{22}) / 2 \end{array} \quad (12,6)$$

We give only an expression of the basis of SU (3) in terms of the hypergeometric function ${}_2F_1$:

$$\left| \begin{pmatrix} h_{13} & h_{23} & 0 \\ h_{12} & h_{22} & \\ & h_{12} & \end{pmatrix} \right\rangle = M^{-1/2} ({}_3\Delta_2^6)^{h_{22}} ({}_3\Delta_2^5)^{h_{23} - h_{22}} ({}_3\Delta_1^2)^{h_{12} - h_{11}} ({}_3\Delta_1^3)^{h_{13} - h_{12}} \\ \times {}_2F_1(h_{22} - h_{23}, h_{11} - h_{12}; h_{11} - h_{23} + 1; ({}_3\Delta_1^1 {}_3\Delta_1^4) / ({}_3\Delta_1^2 {}_3\Delta_2^5)) \quad (12,7)$$

1.4 The fundamental representations of quarks and antiquarks

SU(3) i =	1	2	3	4	5	6
Binary basis	100	010	001	011	101	100
Gel'fand basis	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 \\ 1 \end{pmatrix}$
Young basis	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 3 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline 3 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline 3 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}$
Quarks (L, Y)	(1/2, 1/3)	(-1/2, 1/3)	(0, -2/3)	(-1/2, -1/3)	(1/2, -1/3)	(0, 2/3)

Table 10 Baryon $J^P=1/2^+$

2. The generating function of SU(n)

Generalizing the generating functions of the oscillator, SU (2) and SU (3) we write:

$$\sum_{\lambda, \mu} \prod_{\lambda=2}^n \prod_{\mu=1}^{\lambda-1} A_{\lambda, \mu} ((x_\lambda^\mu)^{h_{\mu, \lambda} - h_{\mu, \lambda-1}} (y_\lambda^\mu)^{h_{\mu, \lambda-1} - h_{\mu+1, \lambda}}) \Gamma_n(\Delta(z)) \\ = \exp[\sum_{m, i} \varphi_{n, i}^m(x, y)_n \Delta_j^m(z)], \quad (12,8)$$

With $e(\mu, \lambda) = h_{\mu, \lambda} - h_{\mu, \lambda-1}$, and $f(\mu, \lambda) = h_{\mu, \lambda-1} - h_{\mu+1, \lambda}$.

2.1 Calculus of the coefficients $\varphi_{n,i}^m(x, y)$

The coefficients $\varphi_{n,i}^m(x, y)$ may be written as product of parameters $y_\lambda^\mu = y(\lambda, \mu)$ and

$x_\lambda^\mu = x(\lambda, \mu)$. We determine the indices of these parameters by using the following rules:

a- We associate to each "one" which appeared after the first zero a parameter x_λ^μ whose index λ is the number of boxes and μ the number of "one" before it, plus one.

b- We associate to each zero after the first "one" a parameter y_λ^μ whose index λ is the number of boxes and μ is the number of "one" before him.

2.2 The symmetry and the conjugate states of $SU(n)$

We know that each binary number has a complement then we deduce that $\Delta_{n,[i]}^k$ has a complement $\overline{\Delta}_{n,[i]}^k$, Therefore the generating function is invariant by the transformation:

$${}_n\Delta_i^k \rightarrow {}_n\overline{\Delta}_i^k \quad (12,9)$$

2.2.1 The conjugate state $\Gamma_n((h^c)_n)$ of Gel'fand basis

The generating function of the conjugate state $\Gamma_n((h^c)_n)$ becomes:

$$\begin{aligned} & \sum_{\lambda\mu} \prod_{\lambda=2}^n \prod_{\mu=1}^{\lambda-1} A_{\lambda\mu} ((x_\lambda^\mu)^{h_{\mu,\lambda} - h_{\mu,\lambda-1}} (y_\lambda^\mu)^{h_{\mu,\lambda-1} - h_{\mu+1,\lambda}}) \Gamma_n((h)_n)_c \\ & = \exp \left[\sum_{m,i} \varphi_{n,i}^m(x, y) {}_n\overline{\Delta}_j^m \right] \end{aligned} \quad (12,10)$$

We change x_λ^μ and y_λ^μ by $y_\lambda^{\lambda-\mu}$ and $x_\lambda^{\lambda-\mu}$ in $\varphi_{n,i}^m(x, y)$ to get $\varphi_{n,i}^m(x, y)$.

Put $v=\lambda-\mu$ so we write:

$$\begin{aligned} & \sum_{\lambda\mu} \prod_{\lambda=2}^n \prod_{\mu=1}^{\lambda-1} A_{\lambda,v} ((x_\lambda^v)^{h_{\mu,\lambda}^c - h_{\mu,\lambda-1}^c} (y_\lambda^v)^{h_{\mu,\lambda-1}^c - h_{\mu+1,\lambda}^c}) \Gamma_n((h^c)_n) \\ & = \exp \left[\sum_{m,i} \varphi_{n,i}^m(x, y) \times {}_n\overline{\Delta}_j^m \right], \end{aligned} \quad (12,11)$$

Comparing (5,2) and (5,3) we find:

$$\boxed{h_{\mu,\lambda-1}^c - h_{\mu+1,\lambda}^c = h_{\lambda-\mu,\lambda} - h_{\lambda-\mu,\lambda-1}}$$

et
$$\boxed{h_{\mu,\lambda}^c - h_{\mu,\lambda-1}^c = h_{\lambda-\mu,\lambda-1} - h_{\lambda-\mu+1,\lambda}},$$

avec
$$\boxed{h_{n,n}^c = h_{n,n} = 0} \quad (12,12)$$

We will determine the conjugate state $\left| (h^c)_n \right\rangle$ with the help of (5,4).

2.2.2 Expressions of the indices of the conjugate states:

We proceed by induction to determine the indices of the conjugate states:

After the calculations for $\lambda = n, n-1, \dots, 1$ and $\mu = \lambda-1, \dots, 1$ we find:

$$\boxed{h_{i,j}^c = h_{1,n} - h_{j-i+1,j}} \quad (12,13)$$

We also get a known but found by other methods result [36].

2.2.3 The phase factor

By extension of phase factor φ of SU (2) [6], we write:

$$\boxed{\varphi = \sum_{i,j} h_{i,j} - h_{1n}} \quad (12,14)$$

a- The conjugate state of SU (2) is:

$$h_{12}^c = 2j, \quad h_{22}^c = 0, \quad h_{11}^c = j + m \quad (12,15)$$

b - The conjugate state of SU (3) is the transformation : $(t, t_z, Y) \rightarrow (t, -t_z, -Y)$

So we get the conjugate basis and the R-Conjugation of Gell-Mann of SU (3) [36].

It is important to note that obtaining results (12,4) and (12,12) shows also the validity of our observation (12,1-2) that the powers of the parameters of the generating function have the same powers of ladders operators.

3. The decomposition of tensor products of SU (3)

3.1. Graphical representation of Quarks

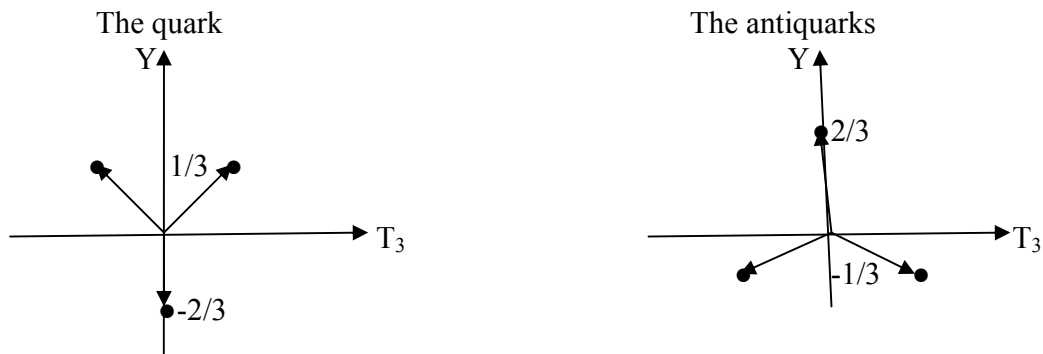
We will present the decomposition of the tensor product by the graphical method because it is a simple and original method [19] and then the Young method tableau.

We represent the fundamental basis of quarks by $||I_z, Y >$ with:

a- The quark [3]: $(I_z, Y) = (1/2, 1/3), (-1/2, 1/3), (0, -2/3)$

b- The antiquarks $[\bar{3}]$: $(I_z, Y) = (-1/2, -1/3), (1/2, -1/3), (0, 2/3)$

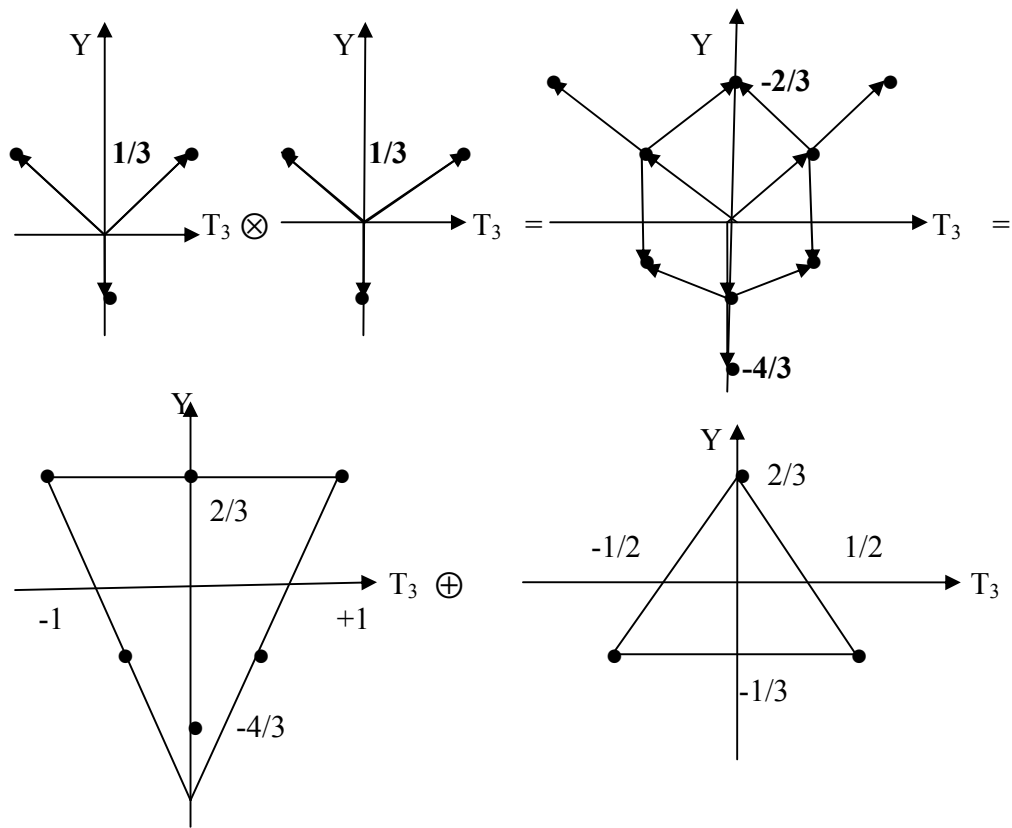
The graphical representation is :



3.2 The decomposition of the tensor products

We want to determine, for example, the tensor product $[3] \otimes [3]$. Let us put the center of the second pattern at the ends of the first and we get the second figure.

But among the ends there are double points imposes second figures for double points.



3.3 The decomposition of the tensor products

We will calculate the tensor products with Young tableaux

a- $\square \otimes \square = \square\square \oplus \begin{matrix} \square \\ \square \end{matrix}$
 $[3] \otimes [3] = [6] \oplus [\bar{3}]$ (12,16)

b- $\square \otimes \begin{matrix} \square \\ \square \end{matrix} = \begin{matrix} \square & \square \\ \square \end{matrix} \oplus \begin{matrix} \square \\ \square \\ \square \end{matrix}$
 $[3] \otimes [\bar{3}] = [8] \oplus [1]$, (12,17)

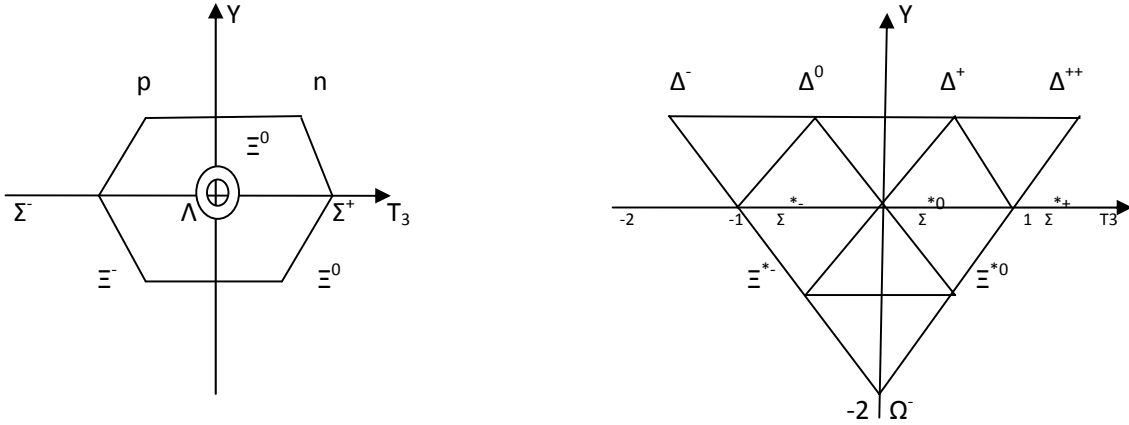
c- $\square \otimes \square\square = \square\square\square \oplus \begin{matrix} \square & \square \\ \square \end{matrix}$
 $[3] \otimes [6] = [10] \oplus [8]$ (12,18)

We can apply this method to other decompositions and are:

$$\begin{aligned} [3] \otimes [3] \otimes [3] &= [1] \oplus [8] \oplus [8] \oplus [10] \\ [8] \otimes [8] &= [27] \oplus [10] \oplus [8] \oplus [8] \oplus [8] \oplus [10] \oplus [1] \end{aligned} \quad (12,19)$$

3.4 The Baryon Octet and The Gel'fand basis

1-The Baryon Octet and the plane Y-T₃



3.5 The Baryon Octet and the Gel'fand basis

particle	proton	neutron	Σ^+	Σ^0	Σ^-
States	$\begin{matrix} 1 & 1 \\ 2 \end{matrix}$	$\begin{matrix} 1 & 2 \\ 2 \end{matrix}$	$\begin{matrix} 1 & 3 \\ 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 3 \end{matrix}$	$\begin{matrix} 1 & 2 \\ 3 \end{matrix}$
Gel'fand basis	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 \\ 0 \end{pmatrix}$
Quarks (I_z, Y)	(1/2,1)	(-1/2,1)	(1,0)	(0,0)	(-1,0)

particle	Λ^0	Ξ^0	Ξ^-
States	$\begin{matrix} 1 & 3 \\ 2 \end{matrix}$	$\begin{matrix} 1 & 3 \\ 3 \end{matrix}$	$\begin{matrix} 2 & 3 \\ 3 \end{matrix}$
Gel'fand basis	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 \\ 0 \end{pmatrix}$
Quarks (I_z, Y)	(0,0)	(1/2,-1)	(-1/2,-1)

Table 11 Baryon $J^P=1/2^+$

3.6 Coupled representations of SU(3)

Several methods have been proposed for the determination of coupled representations and Clebsch-Gordan but calculations by these methods is beyond the undergraduate level [25,36].

XIII-Appendices

Appendix 1: Equations of Klein-Gordon and Dirac

We want to give only the formulas [9].

1. Klein-Gordon equation

Starting from the relativistic energy

$$\vec{p}^2 + m^2 = E^2 \quad (\text{A1,1})$$

And the correspondence principle: $E \rightarrow i \frac{\partial}{\partial t}$, $\vec{P} \rightarrow \frac{1}{i} \vec{\nabla}$, $\hbar = 1$

We find the Klein-Gordon equation

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta - m^2\right)\psi(\vec{r}, t) = 0 \quad (\text{A1,2})$$

This equation is an equation of relativistic spinless particles.

A particle of mass m and charge e and is located in the electromagnetic potential (φ, \vec{A}) has the equation:

$$E = e\varphi \pm \sqrt{(\vec{p} - e\vec{A})^2 + m^2} \quad (\text{A1,3})$$

We deduce in this case the Klein-Gordon equation:

$$\left[\left(i \frac{\partial}{\partial t} - e\varphi \right)^2 - \left(\frac{1}{i} \vec{\nabla} - e\vec{A} \right)^2 \right] \Psi = m^2 \Psi \quad (\text{A1,4})$$

2. Dirac equation of the electron

Following Dirac we are looking an equation of the form:

$$i \partial \psi / \partial t = H_D \psi \quad (\text{A1,5})$$

The theory of relativity imposes a symmetry between the coordinates and the time so we are looking for a first-order equation and a hermetic operator, H_D , of the form:

$$H_D = \vec{\alpha} \cdot \vec{p} + \alpha_4 m \quad (\text{A1,6})$$

It is required that the energy of the electron System verify the equation:

$$H^2 = m^2 c^4 + c^2 \vec{p}^2 \quad (\text{A1,7})$$

Replacing H_D and making identification we find:

$$(\alpha_k)^2 = 1 \quad \alpha^k \alpha^l + \alpha^l \alpha^k = 2 \delta_{k,l} \quad (\text{A1,8})$$

We can write the matrix $(\vec{\alpha})$ in a form similar to the Pauli matrices by putting:

$$\gamma^0 = \alpha_4, \quad \gamma^i = \alpha_4 \alpha_i$$

$$\gamma^0 = \begin{pmatrix} \text{I} & 0 \\ 0 & -\text{I} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \text{I} \\ \text{I} & 0 \end{pmatrix} \quad (\text{A1,9})$$

a-The equation is then written :

$$H_D = \vec{\alpha} \cdot \vec{p} + \alpha_4 m \quad (\text{A1,10})$$

3. The equation for the electron in an electromagnetic field is:

$$\left[(E - e\phi) - \vec{\alpha} \cdot (\vec{p} - e\vec{A}) - \alpha_4 m \right] \Psi = 0 \quad (\text{A1,11})$$

Appendix 2: Electronic configuration of the chemical elements

Electronic configuration of neutral atoms in the ground state:
sub-shells abbreviated by detailed sub-shells and the number of electrons in each shell.

1 H Hydrogen : $1s^1$	1
2 He Helium : $1s^2$	2
3 Li Lithium : $1s^2 2s^1$	2, 1
4 Be Beryllium : $1s^2 2s^2$	2, 2
5 B Bore : $1s^2 2s^2 2p^1$	2, 3
6 C Carbone : $1s^2, 2s^2 2p^2$	2, 4
7 N Azotes : $1s^2, 2s^2 2p^3$	2, 5
8 O Oxygen : $1s^2, 2s^2 2p^4$	2, 6
9 F Fluor : $1s^2, 2s^2 2p^5$	2, 7
10 Ne Neon : $1s^2 2s^2 2p^6$	2, 8
11 Na Sodium : [Ne] $3s^1$	2, 8, 1
12 Mg Magnesium : [Ne] $3s^2$	2, 8, 2
13 Al Aluminum : [Ne] $3s^2 3p^1$	2, 8, 3
14 Si Siliceous : [Ne] $3s^2 3p^2$	2, 8, 4
15 P Phosphor : [Ne] $3s^2 3p^3$	2, 8, 5
16 S Sulfur : [Ne] $3s^2 3p^4$	2, 8, 6
17 Cl Chlorine : [Ne] $3s^2 3p^5$	2, 8, 7
18 Ar Argon : [Ne] $3s^2 3p^6$	2, 8, 8
19 K Potassium : [Ar] $4s^1$	2, 8, 8, 1
20 Ca Calcium : [Ar] $4s^2$	2, 8, 8, 2
21 Sc Scandium : [Ar] $3d^1 4s^2$	2, 8, 9, 2
22 Ti Titan : [Ar] $3d^2 4s^2$	2, 8, 10, 2
23 V Vanadium : [Ar] $3d^3 4s^2$	2, 8, 11, 2
24 Cr Chrome : [Ar] $3d^5 4s^1$	2, 8, 13, 1
25 Mn Manganese : [Ar] $3d^5 4s^2$	2, 8, 13, 2
26 Fe Iron : [Ar] $3d^6 4s^2$	2, 8, 14, 2

Table 12

Appendix 3: hypercomplex numbers, quadratic transformations of octonions and Pauli and Dirac matrices

3.1 Octonion Quadratic transformations

From the generalization of the transformation (6, 28) we obtain the antisymmetric matrices H_2 (2,2) "complex", H_4 (4,4) "Quaternion" and H_8 (8,8) "Octonions" (A3,3).

$$\text{by posing: } V_4 = (u_3, u_4, u_1, u_2) \quad \text{et } V_8 = (u_5, u_6, u_7, u_8, u_1, u_2, u_3, u_4) \quad (\text{A3,1})$$

We get the transformations $R^4 \rightarrow R^3$ et $R^8 \rightarrow R^5$:

$$V_4 H_4 = (x_1, x_2, x_3, 0) \quad V_8 H_8 = (z_5, -z_3, z_2, -z_1, z_0, 0, 0, 0) \quad (\text{A3,2})$$

3.2 Hypercomplex numbers: complex, quaternion and octonion.

The matrix of octonions H_8 is given by:

$$\text{And } H_8 = \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 & -u_5 & -u_6 & -u_7 & -u_8 \\ u_2 & u_1 & -u_4 & u_3 & -u_6 & u_5 & u_8 & -u_7 \\ u_3 & u_4 & u_1 & -u_2 & -u_7 & -u_8 & u_5 & u_6 \\ u_4 & -u_3 & u_2 & u_1 & -u_8 & u_7 & -u_6 & u_5 \\ u_5 & u_6 & u_7 & u_8 & u_1 & -u_2 & -u_3 & -u_4 \\ u_6 & -u_5 & u_8 & -u_7 & u_2 & u_1 & u_4 & -u_3 \\ u_7 & -u_8 & -u_5 & u_6 & u_3 & -u_4 & u_1 & u_2 \\ u_8 & u_7 & -u_6 & -u_5 & u_4 & u_3 & -u_2 & u_1 \end{pmatrix} \quad (\text{A3,3})$$

$$\text{With } H_8 = u_1 I + \sum_{i=2}^8 u_i e_i, \quad e_i \cdot e_i = -1, \quad (\text{A3,4})$$

$$\text{And } {}^t H_8 = u_1 I - \sum_{i=2}^8 u_i e_i, \quad {}^t H_8 \times H_8 = \sum_{i=1}^8 u_i^2 \quad (\text{A3,5})$$

3.3 Quadratic transformations and Pauli, Dirac matrices

$$\begin{aligned} \text{3.3.1 Put : } & {}^t(u) = (u_1 + iu_2, u_3 + iu_4) \\ \text{and } & {}^t(v) = (u_1 + iu_2, u_3 + iu_4, u_5 + iu_6, u_7 + u_8) \end{aligned} \quad (\text{A3,6})$$

The Pauli and Dirac matrices are \vec{s} and γ^i and It's simple to shows the relations:

$$x_1 = {}^t(\bar{u})(s_x)(u), \quad x_2 = {}^t(\bar{u})(s_y)(u), \quad x_3 = {}^t(\bar{u})(s_z)(u), \quad (\text{A3,7})$$

$$z_4 = {}^t(\bar{v})\gamma^5(v), \quad z_i = i {}^t(\bar{v})\gamma^i(v), \quad i = 1, 2, 3, \quad z_0 = {}^t(\bar{v})\gamma^0(v) \quad (\text{A3,8})$$

3.3.2 Identifying ${}^t(u) \lambda_i$ with the columns of $V_8 H_8$ we deduce the matrices λ_i which are the Dirac representations matrices of γ^i in R^8 with ${}^t(u) = (u_1, u_2, \dots, u_8)$. In addition we find the three matrices $\lambda_i, z_i = 0, i=5, 6, 7$, which commute with the $\lambda_i, i = 0, 1, \dots, 4$.

3.4 Hurwitz theorem:

There are antisymmetric and orthogonal matrices H_n if and only if :

$$n=1, 2, 4 \text{ and } 8 \text{ or " R, C, H=Q and O" } \quad (\text{A3,9})$$

Appendix 4: Inertia tensor and cross product in n-dimensions space

We demonstrated using an elementary method that the tensor of inertia of a material point and the cross product of two vectors were only possible in a three or seven dimensional spaces [14].

1. Inertia Tensor

The kinetic energy of a particle of mass $m=1$ which moves in a system in rotation with angular velocity $(\vec{\omega})$ is:

$$T = \frac{I}{2}(\dot{\vec{x}}) \cdot (\dot{\vec{x}}) = \frac{I}{2}(\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = (V_3)(\omega) \quad (\text{A4,1})$$

Then

$$T = \frac{1}{2} {}^t(\omega) ({}^t(V_3)(V_3)) (\omega) \quad \text{and} \quad (M)_3 = {}^t(V_3)(V_3)$$

We write the inertia matrix as:

$$(M)_3 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} = \begin{pmatrix} \vec{r}^2 - x^2 & -xy & -xz \\ -xy & \vec{r}^2 - y^2 & -yz \\ -xz & -yz & \vec{r}^2 - z^2 \end{pmatrix} \quad (\text{A4,2})$$

2. Inertia tensor and the quaternion

The identification of two sides of the equation (A4,2) may be written as:

$$\begin{aligned} m_{11} + x^2 = \vec{r}^2 & \quad m_{12} + xy = 0 & \quad m_{13} + xz = 0 \\ m_{12} + xy = 0 & \quad m_{22} + y^2 = \vec{r}^2 & \quad m_{23} + yz = 0 \\ m_{13} + xz = 0 & \quad m_{23} + yz = 0 & \quad m_{33} + z^2 = \vec{r}^2 \\ & \quad x^2 + y^2 + z^2 = \vec{r}^2 \end{aligned} \quad (\text{A4,3})$$

We can express these systems in matrix form as ${}^t(V_4)(V_4) = \vec{r}^2 I$

$$\begin{pmatrix} & -x \\ {}^t(V_3) & -y \\ & -z \\ x & y & z & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -x & -y & -z & 0 \end{pmatrix} = \begin{pmatrix} \vec{r}^2 & 0 & 0 & 0 \\ 0 & \vec{r}^2 & 0 & 0 \\ 0 & 0 & \vec{r}^2 & 0 \\ 0 & 0 & 0 & \vec{r}^2 \end{pmatrix} \quad (\text{A4,4})$$

We replace the matrix (V_3) by its expression in (A4,1) then we deduce the orthogonal and anti-symmetric matrix:

$$(V_4) = \begin{pmatrix} 0 & z & -y & x \\ -z & 0 & x & y \\ y & -x & 0 & z \\ -x & -y & -z & 0 \end{pmatrix} \quad (\text{A4,5})$$

With (V_4) is the matrix representation of the quaternion

$$\begin{aligned} V_4 &= xe_1 - ye_2 + ze_3 \\ e_1^2 &= -1, \quad e_2^2 = -1, \quad e_3^2 = -1 \\ e_1e_2 &= -e_3, \quad e_2e_3 = -e_1, \quad e_3e_1 = -e_2 \end{aligned} \quad (\text{A4,6})$$

(V_4) Is an Hurwitz matrix and e_1, e_2 and e_3 are the generators of the quaternion [A4,5].

3. The cross product in n-dimensions

The generalization of the tensor of inertia in an intuitive way is:

$$(M) = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \dots & \dots & \vdots \\ m_{1n} & m_{2n} & \dots & m_{nn} \end{pmatrix} = \begin{pmatrix} \bar{r}^2 - x_1^2 & -x_1x_2 & \dots & -x_1x_n \\ -x_1x_2 & \bar{r}^2 - x_2^2 & \dots & -x_2x_n \\ \vdots & \dots & \dots & \vdots \\ -x_1x_n & -x_2x_n & \dots & \bar{r}^2 - x_n^2 \end{pmatrix} \quad (\text{A4,7})$$

$$\begin{aligned} \text{With } \bar{r} &= \sum_{i=1}^n x_i \bar{e}_i \text{ and} \\ m_{ii} + x_i^2 &= \bar{r}^2, \quad i = 1, \dots, n \\ m_{ij} + x_i x_j &= 0, \quad i \neq j \text{ et } i, j = 1, \dots, n \end{aligned} \quad (\text{A4,8})$$

And the matrix system ${}^t(V_{n+1})(V_{n+1}) = \bar{r}^2 I$ takes the form:

$$\begin{pmatrix} & -x_1 \\ {}^t(V_n) & \vdots \\ & -x_n \\ x_1 & \dots & x_n & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ (V_n) \\ \vdots \\ x_n \\ -x_1 & \dots & -x_n & 0 \end{pmatrix} = \begin{pmatrix} \bar{r}^2 & 0 & \dots & 0 \\ 0 & \bar{r}^2 & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \bar{r}^2 \end{pmatrix} \quad (\text{A4,9})$$

$${}^t(V_{n+1})(V_{n+1}) = \bar{r}^2 I, \text{ and } V = \sum_{i=1}^n x_i e_i \quad (\text{A4,10})$$

The generators of the octonions algebra satisfy:

$$e_i^2 = -I, \quad e_i e_j = -e_j e_i, \quad i, j = 1, \dots, 7$$

Hurwitz showed that the orthogonal and anti-symmetric matrix which lines are a linear combination of components of a vector if only if $n=1, 2, 4$ or 8 .

Consequently the matrix (M) is orthogonal if $n+1=8$, it results that $\dim(R^n)=1, 3$ or 7 . And the restriction of $(V_8) = -u_2 e_2 + u_3 e_3 + u_4 e_4 - u_5 e_5 - u_6 e_6 + u_7 e_7 + u_8 e_8$ to the first seven rows and columns $(7,7)$ we obtain the matrix (V_7) and $(M_7) = {}^t(V_7)(V_7)$.

Appendix 5: The p-representation of the hydrogen

It is well known that the representation $\{\vec{p}\}$ of the hydrogen atom was deduced indirectly by Fock in 1935 by introducing the transformation that bears his name [29].

But in the literature [3-11,28] there is no direct calculation to determine the wave function in the representation $\{\vec{p}\}$ of the hydrogen atom, which is the Fourier transform of the wave function of the representation $\{\vec{r}\}$, except in special cases.

In this section we want to fill this gap by using the method of generating function and the quadratic transformation $R^4 \rightarrow R^3$ and more we find the phase factor.

1. Generating function of hydrogen atom

The wave function of hydrogen atom in momentum representation is

$$\Psi_{nlm}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{p}\cdot\vec{r}} \Psi_{nlm}(\vec{r}) d\vec{r} \quad (\text{A5,1})$$

$$\Psi_{nlm}(\vec{r}) = e^{-(\omega r)/2} R_{nl}(x) Y_{lm}(\theta, \varphi), \vec{r} = (r, \theta, \varphi), \omega = 2/n$$

With $R_{nl}(x)$ is the radial part $R_{nl}(x) = \frac{N_{nl}}{(n+l)!} x^l L_{n-l-1}^{(2l+1)}(x)$ (A5,2)

And $x = \omega r, N_{nl} = \frac{2}{n^2} \sqrt{\frac{(n-l-1)!}{[(n+l)!]}}$, $\omega = 2\delta, \delta = \frac{1}{n}$ (A5,3)

Where $L_n^\alpha(r)$ is the associated Laguerre polynomial.

Atomic unit are used through the text.

1.1 The generating function of Laguerre polynomial $L_{n-l-1}^{2l+1}(r)$

The generating function of Laguerre polynomial $L_n^\alpha(r)$ is:

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)!} L_n^{(\alpha)}(r) = \frac{1}{(1-z)^{\alpha+1}} e^{-\frac{z}{(1-z)}r} \quad (\text{A5,4})$$

From the property $\frac{d}{dr} L_n^{(\alpha)}(r) = -L_{n-1}^{(\alpha+1)}(r)$

We deduce that $\sum_{n=0}^{\infty} \frac{z^n}{(n+1+l)!} L_{n-l-1}^{2l+1}(r) = \frac{(z)^{l+1}}{(1-z)^{2l+2}} e^{-\frac{z}{(1-z)}r}$ (A5,5)

1.2 The generating function of spherical harmonics

$$\frac{(\vec{a} \cdot \vec{r})^l}{2^l l!} = \left[\frac{4\pi}{2l+1} \right]^{\frac{1}{2}} \sum_m \varphi_{lm}(\xi) Y_{lm}(\vec{r}) \quad (\text{A5,6})$$

With \vec{a} is a vector of length zero, $\vec{a} \cdot \vec{a} = 0$ and its components

$$a_1 = -\xi_1^2 + \xi_2^2, \quad a_2 = -i(\xi_1^2 + \xi_2^2), \quad a_3 = 2\xi_1\xi_2,$$

$$\text{With } \varphi_{lm}(\xi) = \frac{\xi_1^{l+m}\xi_2^{l-m}}{\sqrt{(l+m)!(l-m)!}}.$$

1.3 Generating function for the basis of the hydrogen atom

We multiply $\psi_{nlm}(\vec{r})$ by $[\frac{4\pi}{2l+1}]^{\frac{1}{2}} z^n \varphi_{lm}(\alpha\xi)$, and summing with respect to n, l, m

$$G(z, \alpha\xi, \vec{r}) = \sum_{nlm} \left[\frac{4\pi}{2l+1} \right]^{\frac{1}{2}} \left[\frac{\pi(n+l)!}{4(n-l-1)!} \right]^{\frac{1}{2}} z^n \varphi_{lm}(\alpha\xi) \psi_{nlm}(\vec{r}) = \sum_{n=0}^{\infty} e^{-(\omega r)^{1/2}} \frac{z^n}{(n+l+1)!} L_{n-l-1}^{2l+1}(\omega r) \sum_m \alpha \varphi_{lm}(\alpha\xi) \psi_{nlm}(\vec{r}) \quad (\text{A5,7})$$

Substituting (A5,5) and (A5,6) in the above expression we obtain:

$$G(z, \alpha\xi, \vec{r}) = \left[\frac{\omega}{\sqrt{\pi}} \frac{z}{(l-z)^2} \right] \exp \left[-\frac{\omega r(l+z)}{2(l-z)} - \frac{\omega \mathbf{z}(\vec{a}, \vec{r})}{(l-z)^2} \right] \quad (\text{A5,8})$$

2. The generating function in momentum representation

2.1 The quadratic transformation $\mathbf{R}^4 \rightarrow \mathbf{R}^3$

Consider the relationship between the well-known Wigner's D matrix spherical harmonics polynomials

$$\rho^{2j} D_{(m,0)}^j(\Omega) = \left(\frac{4\pi}{2l+1} \right)^{1/2} Y_{lm}^*(\theta, \varphi), \quad \mathbf{z} = (z_1, z_2) \quad (\text{A5,9})$$

$$z_1 = u_1 + iu_2, \quad z_2 = u_3 + iu_4,$$

$$\rho = \sqrt{r}, \quad r = z_1 \bar{z}_1 + z_2 \bar{z}_2, \quad r^2 = x^2 + y^2 + z^2$$

We write in terms of Euler's angles or Cayley-Klein parameterization.

$$z_1 = u_1 + iu_2 = \sqrt{r} \cos \frac{\theta}{2} e^{-\frac{i(\phi+\psi)}{2}}, \quad z_2 = u_3 + iu_4 = \sqrt{r} \sin \frac{\theta}{2} e^{\frac{i(\phi-\psi)}{2}} \quad (\text{A5,10})$$

$$\text{And } D_{(m',m)}^j(z_1, \bar{z}_1, z_2, \bar{z}_2) = u^{2j} D_{(m',m)}^j(\psi\theta\varphi), \quad j = 1, 1/2, \dots$$

If we put $l=1$ in (A5,9) we obtain the quadratic transformation

$$\begin{aligned} x &= 2(u_3 u_1 + u_4 u_2) = z_1 \bar{z}_2 + z_2 \bar{z}_1, \quad y = 2(u_4 u_1 - u_3 u_2) = i(z_1 \bar{z}_2 - z_2 \bar{z}_1), \\ z &= u_1^2 + u_2^2 - u_3^2 - u_4^2 = z_1 \bar{z}_1 - z_2 \bar{z}_2, \end{aligned} \quad (\text{A5,11})$$

2.2 The volume element

We consider the transformation $(u_1, u_2, u_3, u_4) \rightarrow (r, \psi\theta\varphi)$

With $0 \leq \theta \leq \pi, 0 \leq \psi, \varphi \leq 2\pi, 0 \leq r \leq \infty, -\infty \leq u_i \leq +\infty, i = 1, \dots, 4$

And $d^4 \bar{u} = |J| dr d\theta d\varphi d\psi$

The calculation of the Jacobian gives

$$|J| = (u^2 / 8) \sin \theta \text{ but } d^3 \bar{r} = r^2 dr d\theta d\varphi d\psi$$

Therefore $8u^2 d\bar{u} = d\bar{r} d\psi,$

And

$$\int f(x, y, z) d^3 \bar{r} = \frac{4}{\pi} \int f(x(u), y(u), z(u)) u^2 d^4 \bar{u} \quad (\text{A5,12})$$

2.3 The generating function in $\{u\}$ representation

We denote the generating function by $G(z, \alpha\xi, \bar{p})$ in the representation $\{u\}$. But to determine the generating function (A5,7) we must multiply by $4 / \pi$ to reflect the change in the measure of integration. We write

$$\Psi_{nlm}(\bar{p}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\bar{p} \cdot \bar{r}} \Psi_{nlm}(\bar{r}) d\bar{r} \quad (\text{A5,13})$$

To calculate this expression we must write (A5,1) in the (u) representation using the formula (A5,11):

$$\Psi_{nlm}(\bar{p}) = \frac{4}{\pi} \frac{1}{(2\pi)^{3/2}} \int e^{-i\bar{p} \cdot \bar{r}} \Psi_{nlm}(\bar{r}) u^2 d^4 u \quad (\text{A5,14})$$

In the expression of $\Psi_{nlm}(\bar{p})$ there is the term u^2 for that we consider a new generating function:

$$G(z, \alpha\xi, \bar{p}, \beta) = \frac{1}{(2\pi)^{3/2}} \frac{4}{\pi} \frac{z}{(1-z)^2} \times \int e^{-i\bar{p} \cdot \bar{r}} \exp\left[-\frac{\omega r(1+z)}{2(1-z)} + \frac{\alpha \omega z(\bar{a} \cdot \bar{r})}{2(1-z)^2}\right] e^{-\beta u^2} d^4 \bar{u} \quad (\text{A5,15})$$

We assume that $\beta \geq 0$ therefore there is no problem of convergence.

We write then: $G(z, \alpha\xi, \bar{p}) = -\partial G(z, \alpha\xi, \bar{p}, \beta) / \partial \beta \Big|_{\beta=0}$ (A5,16)

With $G(z, \alpha\xi, \bar{p}) = \sum_{nlm} \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}} \frac{z^n}{N_{nl}} \alpha^l \varphi_{lm}(\xi) \Psi_{nlm}(p)$ (A5,17)

2.4 The generating function of momentum-space

We can do the integration of (A5,14) by a direct calculation with the variables (u) we can perform the integration using the Gauss formula

$$\left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^n dx_i dy_i \exp(-\bar{z}^t Xz + A^t z + \bar{z}^t \bar{B}) = (\det(X))^{-1} \exp(A^t X^{-1} \bar{B}) \quad (\text{A5,18})$$

With $z = (z_1, z_2, \dots, z_n)$

We have

$$\begin{aligned}
-i\vec{p} \cdot \vec{r} &= -ip_x(z_1\bar{z}_2 + z_2\bar{z}_1) + p_y(z_1\bar{z}_2 - z_2\bar{z}_1) - ip_z(z_1\bar{z}_2 - z_2\bar{z}_1), \\
\vec{a} \cdot \vec{r} &= a_x(z_1\bar{z}_2 + z_2\bar{z}_1) + ia_y(z_1\bar{z}_2 - z_2\bar{z}_1) + a_z(z_1\bar{z}_2 - z_2\bar{z}_1)
\end{aligned} \tag{A5,19}$$

We obtain then

$$X = \begin{pmatrix} \frac{\omega(1+z)}{2(1-z)} + \beta - ip_z + \frac{\omega z}{2(1-z)^2} a_z & -ip_x + \frac{\omega z}{2(1-z)^2} a_x - p_y + \frac{1}{i} \frac{\omega z}{2(1-z)^2} a_y \\ -ip_x + \frac{\omega z}{2(1-z)^2} a_x + p_y - \frac{1}{i} \frac{\omega z}{2(1-z)^2} a_y & \frac{\omega(1+z)}{(1-z)} + \beta + ip_z - \frac{\omega z}{2(1-z)^2} a_z \end{pmatrix}$$

Because $\vec{a}^2=0$ we deduce that:

$$\det(X) = \left[\left(\frac{\delta(1+z)}{(1-z)} + \beta \right)^2 + \bar{p}^2 + i\alpha \frac{2\delta z}{(1-z)^2} \vec{a} \cdot \vec{p} \right], \quad \delta = \omega/2 \tag{A5,20}$$

We find therefore the generating functions

$$G(z, \alpha\xi, \vec{p}, \beta) = \frac{2}{\sqrt{2\pi}} \frac{z}{\left[(\delta(1+z) + \beta(1-z))^2 + (1-z)^2 \bar{p}^2 + 2i\alpha\delta \vec{a} \cdot \vec{p} \right]} \tag{A5,21}$$

In applying the relation (A5,16) we find the generating function $G(z, \alpha\xi, \vec{p})$

$$G(z, \alpha\xi, \vec{p}) = \frac{4\delta}{\sqrt{2\pi}} \frac{z(1-z^2)}{\left[(\delta(1+z) + \beta(1-z))^2 + (1-z)^2 \bar{p}^2 + 2i\alpha\delta \vec{a} \cdot \vec{p} \right]^2} \tag{A5,22}$$

3. The wave functions in momentum space

We drive the basis of momentum-space using the formula (A5,6)

$$\left[\varphi_{jm} (\partial / \partial \xi) \frac{1}{n!} \frac{\partial^n}{\partial z^n} \frac{1}{l!} \frac{\partial^l}{\partial \alpha^l} G(z, \alpha\xi, \vec{p}) \right]_0 = \left(\frac{4\pi}{2l+1} \right)^{l/2} \frac{1}{N_{nl}} \psi_{nlm}(\vec{p}) \tag{A5,23}$$

In this case we must take $\delta=1/n$ and to execute the calculations we proceed by step:

3.1 Derivation with respect to α

$$\left[\frac{1}{l!} \frac{\partial^l}{\partial \alpha^l} G(z, \alpha\xi, \vec{p}) \right]_0 = (i)^l \frac{(l+1)!}{\sqrt{2\pi}} \times (4\delta)^{l+1} \times \frac{(1-z^2)z^{l+1}}{\left[(\delta(1+z))^2 + (1-z)^2 \bar{p}^2 \right]^{l+2}} \frac{(\vec{a} \cdot \vec{p})^l}{2^l l!} \tag{A5,24}$$

We have

$$\begin{aligned}
(\delta(1+z))^2 + (1-z)^2 \bar{p}^2 &= ((\bar{p}^2 + \delta^2) - 2z(\bar{p}^2 - \delta^2) + z^2(\bar{p}^2 + \delta^2)) \\
&= (\bar{p}^2 + \delta^2)[1 - 2zx + z^2], \quad x = \left(\frac{\bar{p}^2 - \delta^2}{\bar{p}^2 + \delta^2} \right)
\end{aligned}$$

We deduce that

$$\left[\frac{1}{l!} \frac{\partial^l}{\partial \alpha^l} G(z, \alpha \xi, \bar{p}) \right]_0 = (i)^l \frac{(l+1)!}{\sqrt{2\pi}} \times \frac{(4\delta)^{l+1}}{(\bar{p}^2 + \delta^2)^{l+2}} \times \frac{(1-z^2)z^{l+1}}{[1-2zx+z^2]^{l+2}} \frac{(\bar{a} \cdot \bar{p})^l}{2^l l!} \quad (\text{A5,25})$$

3.2 Derivation with respect to z

Using the familiar formula for the generating function of Gegenbauer polynomials

$$(1-2zt+z^2)^{-\alpha} = \sum_{m=0}^{\infty} z^m C_m^\alpha(t) = \sum_{n=1}^{\infty} z^n [C_{n-l-1}^{l+2}(x) - C_{n-l-3}^{l+2}(x)] \quad (\text{A5,26})$$

With $m+1+1=n$, $m+1+3=n$ and $\delta=1/n$ therefore

$$\left[\frac{1}{n!} \frac{\partial^n}{\partial z^n} \frac{1}{l!} \frac{\partial^l}{\partial \alpha^l} G(z, \alpha \xi, \bar{p}) \right]_0 = (i)^l \frac{(l+1)!}{\sqrt{2\pi}} \times \frac{(4\delta)^{l+1}}{(\bar{p}^2 + \delta^2)^{l+2}} \times [C_{n-l-1}^{l+2}(x) - C_{n-l-3}^{l+2}(x)] \frac{(\bar{a} \cdot \bar{p})^l}{2^l l!} \quad (\text{A5,27})$$

Put $\vec{y} = (y_1, y_2, y_2, y_3, y_4) = \left(\frac{\delta p_x}{(\bar{p}^2 + \delta^2)}, \frac{\delta p_y}{(\bar{p}^2 + \delta^2)}, \frac{\delta p_y}{(\bar{p}^2 + \delta^2)}, \frac{(\bar{p}^2 - \delta^2)}{(\bar{p}^2 + \delta^2)} \right)$

We obtain $\vec{y} \cdot \vec{y} = 0$. Thus we find the transformation introduced by Fock.

3.3 Derivation with respect to $\varphi_{lm}(\frac{\partial}{\partial \xi})$

By using the formula (A5, 6) we get the following expression

$$\left[\varphi_{jm}(\partial/\partial \xi) \frac{1}{n!} \frac{\partial^n}{\partial z^n} \frac{1}{l!} \frac{\partial^l}{\partial \alpha^l} G(z, \alpha \xi, \bar{p}) \right]_0 = (i)^l \frac{(l+1)!}{\sqrt{2\pi}} \times \frac{(4\delta)^{l+1}}{(\bar{p}^2 + \delta^2)^{l+2}} \times [C_{n-l-1}^{l+2}(x) - C_{n-l-3}^{l+2}(x)] Y_{lm}(\bar{p}) \quad (\text{A5,28})$$

3.4 The wave functions in momentum space

The comparisons of (A5,26) and (A5,27) give us the result:

$$\Psi_{nlm}(\vec{p}) = (i)^l N_{nl} a^{-3/2} \frac{(l+1)!}{\sqrt{2\pi}} \times (4\delta)^{l+1} \frac{[C_{n-l-1}^{l+2}(x) - C_{n-l-3}^{l+2}(x)]}{(\bar{p}^2 + \delta^2)^{l+2}} Y_{lm}(\vec{p}) \quad (\text{A5,29})$$

And with the help of the recurrences formula [30]:

$$(n+\alpha)C_{n+1}^{(\alpha-l)}(x) = (\alpha-l)[C_{n+1}^{(\alpha)}(x) - C_{n-l}^{(\alpha)}(x)] \quad (\text{A5,30})$$

We derive finally the wave functions in momentum space:

$$\Psi_{nlm}(\vec{p}) = (i)^l N_{nl} \frac{(l)!}{\sqrt{2\pi}} \times \frac{n(4\delta)^{l+1}}{(\bar{p}^2 + \delta^2)^{l+2}} C_{n-l-1}^{l+1} \left(\frac{\bar{p}^2 - \delta^2}{\bar{p}^2 + \delta^2} \right) Y_{lm}(\vec{p}) \quad (\text{A5,31})$$

It is clear that we obtain by an elementary method and direct calculus not only the wave function in momentum representation but also the phase factor.

Appendix 6: Generalization of Cramer's rule and classical polynomials

1. Generalization of Cramer's rule

The expression of the determinant formed from $\det(A) = \det(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$ by substituting the components of some vector (\vec{a}_i) by the components of the vectors $\vec{b}_i, 1 \leq i \leq s \leq n$ result from the resolution of the system:

$$\sum_{j=1}^n \vec{a}_j x(k, j) = \vec{b}_k, \quad k = 1, 2, \dots, s, \quad s \leq n \quad (\text{A6,1})$$

The solution of this system is given by the formula (25):

$$\det(\vec{a}_1, \dots, \vec{a}_{i_1-1}, \vec{b}_1, \vec{a}_{i_1+1}, \dots, \vec{a}_{i_s-1}, \vec{b}_s, \vec{a}_{i_s+1}, \dots, \vec{a}_n) = \det(A) \begin{vmatrix} x(1, i_1) & \dots & x(1, i_s) \\ \vdots & \ddots & \vdots \\ x(s, i_1) & \dots & x(s, i_s) \end{vmatrix} \quad (\text{A6,2})$$

In the particular case $s=1$ we find the well known Cramer's rule.

The formula (A6, 2) is very useful for calculating the spectrum of nucleus rotations but it is fundamental for the study of nucleus vibrations (25).

2. Orthogonal polynomials

1.1 The hypergeometric functions

$${}_2F_1(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + \dots \quad (\text{A6,3})$$

1.2 Laguerre polynomials

The differential equation [2,30]:

$$\left[z \frac{d^2}{dz^2} + (k+1-z) \frac{d}{dz} + p \right] L_p^k = 0 \quad (\text{A6,4})$$

Orthonormality relation

$$\int_0^\infty e^{-z} z^k L_p^k L_q^k dz = \frac{[(p+k)!]^2}{p!} \delta_{pq} \quad (\text{A6,5})$$

1.3 Hermite polynomials

The differential equation:

$$\left[\frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2n \right] H_n(z) = 0 \quad (\text{A6,6})$$

1.4 Legendre polynomials

The differential equation:

$$\left[(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + \ell(\ell+1) - \frac{m^2}{1-x^2} \right] P_\ell^m(x) = 0 \quad (\text{A6,7})$$

Orthonormality relation

$$\int_{-1}^{+1} P_k^m P_\ell^m dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{k\ell} \quad (\text{A6,8})$$

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