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# EXISTENCE AND STABILITY FOR A NON-LOCAL ISOPERIMETRIC MODEL OF CHARGED LIQUID DROPS

MICHAEL GOLDMAN, MATTEO NOVAGA, AND BERARDO RUFFINI

ABSTRACT. We consider a variational problem related to the shape of charged liquid drops at equilibrium. We show that this problem never admits global minimizers with respect to  $L^1$  perturbations preserving the volume. This leads us to study it in more regular classes of competitors, for which we show existence of minimizers. We then prove that the ball is the unique solution for sufficiently small charges.

## 1. INTRODUCTION

In this paper we investigate a variational model describing the shape of a charged liquid drop at equilibrium. It is known since the pioneering paper of Lord Rayleigh [26], that such a drop maintains a spherical shape as long as the excess charge  $Q$  is not too large. When the charge overcomes a critical threshold  $Q_c$ , which depends on the volume of the drop and on the characteristic constants of the liquid (surface tension and dielectric constant), there is a symmetry breaking. Typically, the drops tends to deform into a very eccentric prolate spheroid and ejects a very thin jet from its poles (see [2, 10]). These jets carry about 1 % of the mass but up to 30 % of the charge. The unstable regime is still only poorly understood both experimentally and mathematically (see [25, 11] and the references therein).

In order to introduce the model, we denote by  $\mathcal{I}_\alpha$  the Riesz potential energy

$$\mathcal{I}_\alpha(E) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} : \mu(E) = 1 \right\},$$

where  $\alpha \in (0, d)$  and  $E$  is a compact subset of  $\mathbb{R}^d$ . We refer to the monograph [20] for a detailed analysis of this family of functionals and we recall that the Coulomb interaction corresponds to  $\alpha = d - 2$ . A charged liquid drop in equilibrium is then a local minimizer among sets of prescribed volume of the energy

$$P(E) + Q^2 \mathcal{I}_{d-2}(E),$$

where  $P(E)$  denotes the perimeter of  $E$  (which is equal to  $\mathcal{H}^{d-1}(\partial E)$  when  $E$  is an open set) and  $Q$  measures the excess charge of the drop. Up to renormalization constants, we are thus led to consider the (more general) functional

$$\mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E)$$

where  $Q > 0$  and  $\alpha \in (0, d)$ . Quite surprisingly, when  $\alpha \in (0, d - 1)$  it turns out that, for every given charge and volume, the functional  $\mathcal{F}_{\alpha,Q}$  has no minimizer among subsets of  $\mathbb{R}^d$  of this given volume. Indeed, it is more convenient to spread the excess charge into little drops far away from each other (see Theorem 3.2). This is somewhat reminiscent of the creation of the liquid jets, experimentally observed in [2, 10].

As a by-product of our analysis we also get that  $\mathcal{F}_{\alpha,Q}$  does not even have local minimizers in the  $L^1$  topology. This comes from the fact that the perimeter is defined up to sets of Lebesgue measure zero while the Riesz potential energy is defined up to sets of zero capacity. This phenomenon is further illustrated when considering the problem among sets which are contained in a fixed bounded domain  $\Omega$ . In this case we prove that the isoperimetric problem and the *charge minimizing* problem completely decouple (see Theorem 3.3). This negative result strongly supports the idea that global (or even local)  $L^1$  minimizers are not the physically relevant objects to consider. One should instead look for stable configurations which are typically local minimizers for a stronger topology. It is then reasonable to look for minimizers of  $\mathcal{F}_{\alpha,Q}$  in some smaller class of sets with some extra regularity conditions. In particular, in Theorem 4.2 we prove that, for all  $\delta > 0$  there exists  $Q_\delta > 0$  such that, if  $Q < Q_\delta$ , one can find a minimizer of  $\mathcal{F}_{\alpha,Q}$  in the class of closed sets satisfying the  $\delta$ -ball condition (see Definition 2.18). If we further assume that the sets are connected, then a minimizer exists for all  $Q > 0$  (see Theorem 4.3). Then, we show that when  $\alpha = d - 2$  and possibly reducing the threshold  $Q_\delta$ , the ball is the unique minimizer in the class of sets satisfying the  $\delta$ -ball condition (see Corollary 5.6). This shows that for small charges, the ball is stable under small  $C^{1,1}$  perturbations. This extends a previous result of M.A. Fontelos and A. Friedman [11], which asserts the stability with respect to  $C^{2,\alpha}$  perturbations. These authors also gave a detailed analysis of the linear stability. We remark that our proof of the stability of the ball is quite different from the one in [11], and is inspired by the proofs in [19, 7]. In particular, it makes use of the quantitative isoperimetric inequality with optimal exponent, which has been recently established in [13] (see also [1]).

It is interesting to compare our results with the analysis in [18, 19] (see also [7, 22, 6]) of the non-local isoperimetric problem, known as the sharp interface Ohta-Kawasaki model,

$$\min_{|E|=m} P(E) + \int_{E \times E} \frac{dx dy}{|x - y|^\alpha},$$

which is motivated by the theory of diblock copolymers and the stability of atomic nuclei. The authors show that there exist two (possibly equal) critical volumes  $0 < m_1(\alpha) \leq m_2(\alpha)$  such that minimizers exist if  $m \leq m_1$ , while there are no minimizers if  $m > m_2$ . Moreover, the minimizers are balls when  $\alpha < d - 1$  and the volume is sufficiently small. A crucial difference between our model and the Ohta-Kawasaki model is that in the latter, the non-local term is Lipschitz with respect to the measure of the symmetric difference between sets (see for instance [7, Prop. 2.1]). Hence, on small scales, the perimeter dominates the non-local part of the energy. This implies in particular that minimizers enjoy the same regularity properties as minimal surfaces. In our case, it is quite the contrary since on small scales, the functional  $\mathcal{I}_\alpha$  dominates the perimeter. This prevents *a priori* the hope to get any regularity result for stable configurations. Let us notice that the same type of existence/non-existence issues in variational models where the perimeter competes against a non-local energy has been recently addressed in other models. For instance, in [5] the authors study a model related to epitaxial growth where the non-local part forces compactness whereas the perimeter part favors spreading.

The paper is organized as follows. In Section 2, we recall some properties of the Riesz potentials  $\mathcal{I}_\alpha$ . In Section 3, we prove the non-existence of minimizers for the functional  $\mathcal{F}_{\alpha,Q}$ . In Section 4, we study this existence issue in some smaller class, before proving in Section 5 the stability of the ball. Finally, in Section 6, we extend our results to the logarithmic potential

energy

$$I_{\log}(E) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \log \left( \frac{1}{|x - y|} \right) d\mu(x) d\mu(y) : \mu(E) = 1 \right\}.$$

## 2. THE RIESZ POTENTIAL ENERGY

In this section we recall some results regarding the Riesz potential energy (see Definition 2.1 below). Most of the material presented here comes from [20].

In the following, given an open set  $\Omega \subset \mathbb{R}^d$ , we denote by  $\mathcal{M}(\Omega)$  the set of all Borel measures with support in  $\Omega$ . For  $x \in \mathbb{R}^d$  and  $r > 0$  we denote by  $B_r(x)$  the open ball of radius  $r$  centered in  $x$  and simply by  $B$  the unit ball and by  $\omega_d = |B|$  its Lebesgue measure. For  $k \in [0, d]$ , we will denote by  $H^k$  the  $k$ -dimensional Hausdorff measure.

**Definition 2.1.** Let  $d \geq 2$  and  $\alpha > 0$ . Given  $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ , we define the interaction energy (or potential energy) between  $\mu$  and  $\nu$  by

$$\mathcal{I}_\alpha(\mu, \nu) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x) d\nu(y)}{|x - y|^\alpha} \in [0, +\infty].$$

When  $\mu = \nu$ , we simply write  $\mathcal{I}_\alpha(\mu) := \mathcal{I}_\alpha(\mu, \mu)$ . When the measures are absolutely continuous with respect to the Lebesgue measure, that is  $\mu = f\mathcal{H}^d \llcorner E$  and  $\nu = g\mathcal{H}^d \llcorner E$  for some set  $E$  and functions  $f$  and  $g$ , we denote  $\mathcal{I}_\alpha(\mu, \nu) = \mathcal{I}_\alpha^E(f, g)$  (and when  $f = g$  we denote it by  $\mathcal{I}_\alpha^E(f)$ ). Similarly, when  $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$  and  $\nu = g\mathcal{H}^{d-1} \llcorner \partial E$  we write  $\mathcal{I}_\alpha(\mu, \nu) = \mathcal{I}_\alpha^{\partial E}(f, g)$  (and when  $f = g$  we denote it by  $\mathcal{I}_\alpha^{\partial E}(f)$ ).

The following proposition can be found in [20, (1.4.5)].

**Proposition 2.2.** The functional  $\mathcal{I}_\alpha$  is lower semicontinuous for the weak\* convergence of measures.

**Definition 2.3.** Let  $d \geq 2$  and  $\alpha > 0$  then for every Borel set  $A$  we define the Riesz potential energy of  $A$  by

$$\mathcal{I}_\alpha(A) := \inf \left\{ \mathcal{I}_\alpha(\mu) : \mu \in \mathcal{M}(\mathbb{R}^d), \mu(A) = 1 \right\}. \quad (2.1)$$

**Remark 2.4.** Notice that, if we change  $\mu$  in  $Q\mu$  for a given charge  $Q > 0$ , then for any Borel set  $A \subset \mathbb{R}^d$ , it holds

$$Q^2 \mathcal{I}_\alpha(A) := \inf \left\{ \mathcal{I}_\alpha(\mu) : \mu \in \mathcal{M}(\mathbb{R}^d), \mu(A) = Q \right\}.$$

Notice also that, for all  $\lambda > 0$ , there holds

$$\mathcal{I}_\alpha(\lambda A) = \lambda^{-\alpha} \mathcal{I}_\alpha(A). \quad (2.2)$$

**Remark 2.5.** An important notion related to  $\mathcal{I}_\alpha(A)$  is the so-called  $\alpha$ -capacity [21, 20, 23]

$$C_\alpha(A) := \frac{1}{\mathcal{I}_\alpha(A)}.$$

For  $\alpha = d - 2$  and  $K$  compact, we have the following representation of the capacity [21]:

$$C_{d-2}(K) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla f|^2 : f \in C_c^1(\mathbb{R}^d), f \geq 0, f \geq 1 \text{ on } K \right\}.$$

We stress however, for the sake of clearness, that this is not the only definition of capacity one may find in the literature, see for instance the discussion in [21, Section 11.15]. The proof of the following result is given in [20, pages 131 and 132].

**Lemma 2.6.** *If  $A$  is a compact set, the infimum in (2.1) is achieved.*

**Remark 2.7.** When the set  $A$  is unbounded, there does not always exist an optimal measure  $\mu$ , i.e. in (2.1) is not achieved. Indeed, it is possible to construct a set  $E$  of finite volume with  $\mathcal{I}_\alpha(E) = 0$ . To this aim, consider  $\alpha \in (0, d-1)$ ,  $\gamma \in (\frac{1}{d-1}, +\infty)$  and the set  $E = \{(x, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : |x'| \leq 1 \text{ and } |x| \leq \frac{1}{|x'|^\gamma}\}$ . The set  $E$  has finite volume and taking  $N$  balls of radius  $r = N^{-\beta}$  inside  $E$ , at mutual distance  $\ell = N^{\frac{\beta}{\gamma}-1}$ , with charge  $1/N$  distributed uniformly on each ball, we have

$$\mathcal{I}_\alpha(E) \leq C \left( N^{\alpha\beta-1} + N^{(1-\frac{\beta}{\gamma})\alpha} \right)$$

for some  $C > 0$ , so that  $\mathcal{I}_\alpha(E) = 0$  if  $\frac{1}{d-1} < \gamma < \beta < \frac{1}{\alpha}$ . Similarly, if  $d > 2$  and  $\alpha < d-2$ , taking  $\gamma > \frac{1}{d-2}$  one can even construct a set with finite perimeter for which the same property holds.

**Definition 2.8.** *Given a non-negative Radon measure  $\mu$  on  $\mathbb{R}^d$  and  $\alpha \in (0, d)$ , we define the potential function*

$$v_\alpha^\mu(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} = \mu * k_\alpha(x)$$

where  $k_\alpha(x) = |x|^{-\alpha}$ . We will sometime drop the dependence of  $\mu$  and  $\alpha$  in the definition of  $v_\alpha^\mu$  and we will refer to it as potential.

**Definition 2.9.** *We say that two functions  $u$  and  $v$  are equal  $\alpha$ -quasi everywhere (briefly  $u = v$   $\alpha$ -q.e.) if they coincide up to a set of  $\alpha$ -capacity 0.*

The Euler-Lagrange equation of  $\mathcal{I}_\alpha(A)$  reads as follows:

**Lemma 2.10.** *Let  $A$  be a compact set and let  $\mu$  be a minimizer for  $\mathcal{I}_\alpha(A)$  then  $v^\mu = \mathcal{I}_\alpha(A)$   $\alpha$ -q.e. on  $\text{spt}(\mu)$ , and  $v^\mu \geq \mathcal{I}_\alpha(A)$   $\alpha$ -q.e. on  $A$ . Moreover, the following equation holds in the distributional sense*

$$(-\Delta)^{\frac{d-\alpha}{2}} v^\mu = c(\alpha, d) \mu, \quad (2.3)$$

where  $(-\Delta)^s$  denotes the fractional Laplacian (see [9]). In particular,

$$(-\Delta)^{\frac{d-\alpha}{2}} v^\mu = 0 \quad \text{on } \mathbb{R}^d \setminus A.$$

*Proof.* The first assertions on  $v^\mu$  follow from [20, Theorem 2.6 and page 137] (see also [14] where these conditions were first derived).

Equation (2.3) can be directly verified by means of the Fourier Transform, namely

$$\widehat{(-\Delta)^{\frac{d-\alpha}{2}} v^\mu}(\xi) = |\xi|^{d-\alpha} \widehat{\mu * k_\alpha}(\xi) = c(\alpha, d) \widehat{\mu}(\xi),$$

where we used the fact [20, Equation (1.1.1)]

$$\widehat{k_\alpha}(\xi) = c(\alpha, d) k_{d-\alpha}(\xi) \quad \text{with} \quad c(\alpha, d) := \pi^{\alpha-\frac{d}{2}} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}.$$

□

We recall another important result which will be exploited in Section 4. We refer to [20, Theorem 1.15] (see also [21, Corollary 5.10]) for its proof.

**Theorem 2.11.** *For any signed measure  $\mu$  and for any  $\alpha \in (0, d)$ , there holds*

$$\mathcal{I}_\alpha(\mu) = \int_{\mathbb{R}^d} \left( v_{\alpha/2}^\mu(x) \right)^2 dx$$

and therefore,

$$\mathcal{I}_\alpha(\mu) \geq 0.$$

Moreover equality holds if and only if  $\mu = 0$ .

**Remark 2.12.** A consequence of Theorem 2.11, is that the functional  $\mathcal{I}_\alpha(\cdot, \cdot)$  is a *positive*, bilinear operator on the product space of Radon measures on  $\mathbb{R}^d$ ,  $\mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$ . In particular it satisfies the Cauchy-Schwarz inequality

$$\mathcal{I}_\alpha(\mu, \nu) \leq \mathcal{I}_\alpha(\mu)^{1/2} \mathcal{I}_\alpha(\nu)^{1/2}. \quad (2.4)$$

The following uniqueness result can be found in [20, page 133].

**Lemma 2.13.** *For every compact set  $A$  the measure minimizing  $\mathcal{I}_\alpha(A)$  is unique.*

**Lemma 2.14.** *Let  $\alpha \in (0, d - 1)$ . For every open bounded set  $E$ , the minimizer  $\mu$  of  $\mathcal{I}_\alpha(E)$  satisfies:*

- i) *If  $\alpha \leq d - 2$  then  $\text{spt}(\mu) \subset \partial E$ . In particular  $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(\partial E)$ .*
- ii) *If  $\alpha > d - 2$  then  $\text{spt}(\mu) = \overline{E}$ .*

Moreover, when  $\alpha \geq d - 2$ ,  $v_\alpha^\mu = \mathcal{I}_\alpha(E)$  on  $\overline{E}$ .

*Proof.* The case  $\alpha \leq d - 2$  can be found in [20, page 162]. If  $\alpha > d - 2$ , by [20, Theorem 2.6 and page 137], we know that  $v_\alpha^\mu = \mathcal{I}_\alpha(E)$   $\alpha$ -q.e. on  $\overline{E}$  and  $v_\alpha^\mu \leq \mathcal{I}_\alpha(E)$  on  $\mathbb{R}^d$ . Moreover, outside of  $\text{spt}(\mu)$ ,  $v_\alpha^\mu$  is smooth and  $\Delta v_\alpha^\mu > 0$ . Assume that there exists  $x \in E \setminus \text{spt}(\mu)$ . Then there exists an open ball  $B_r(x) \subset E \setminus \text{spt}(\mu)$ . But this is impossible since this would imply  $v_\alpha^\mu = \mathcal{I}_\alpha(E)$  in  $B_r(x)$  and hence  $\Delta v_\alpha^\mu = 0$  in  $B_r(x)$  contradicting  $\Delta v_\alpha^\mu > 0$ . The last claim of the lemma follows by the fact that  $v_\alpha^\mu$  is, in this case, a regular function on  $E$  which is  $\alpha$ -q.e. equal to  $\mathcal{I}_\alpha(E)$ .  $\square$

We now prove a density result which is an adaptation of [20, Theorem 1.11 and Lemma 1.2].

**Proposition 2.15.** *Let  $E$  be a smooth connected closed set of  $\mathbb{R}^d$ , then for every  $\alpha \in (0, d)$ ,*

$$\mathcal{I}_\alpha(E) = \inf \left\{ \mathcal{I}_\alpha^E(f) : \mu = f dx, f \in L^\infty(E), \int_E f dx = 1 \right\}.$$

*Proof.* Let  $\mu$  be such that  $\mu(E) = 1$ ,  $\text{spt}(\mu) \subset E$  and  $\mathcal{I}_\alpha(\mu) < +\infty$  then for  $\varepsilon > 0$  consider the measure  $\mu_\varepsilon dx$  defined as

$$\mu_\varepsilon(x) = \frac{1}{|B_\varepsilon(x) \cap E|} \int_{B_\varepsilon(x)} d\mu(y).$$

Since  $\|\mu_\varepsilon\|_{L^\infty(E)} \leq (\min_{x \in E} |B_\varepsilon(x) \cap E|)^{-1} \leq (C\varepsilon^d)^{-1}$ , we only have to prove that  $\mathcal{I}_\alpha^E(\mu_\varepsilon) \rightarrow \mathcal{I}_\alpha(\mu)$ . By Theorem 2.11 we have

$$\mathcal{I}_\alpha^E(\mu_\varepsilon) = \int_{\mathbb{R}^d} \left( v_{\alpha/2}^{\mu_\varepsilon}(x) \right)^2 dx.$$

Let us show that for all  $x \in \mathbb{R}^d$ ,

$$v_{\alpha/2}^{\mu_\varepsilon}(x) \leq C v_{\alpha/2}^\mu(x) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} v_{\alpha/2}^{\mu_\varepsilon}(x) = v_{\alpha/2}^\mu(x)$$

from which we can conclude by means of the Dominated Convergence Theorem. Denoting by  $\chi_A$  the characteristic function of the set  $A$ , we have, for any  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} v_{\alpha/2}^{\mu_\varepsilon}(x) &= \int_E \int_E \frac{1}{|B_\varepsilon(y) \cap E|} \chi_{B_\varepsilon}(y) \frac{d\mu(z)}{|x-y|^{\alpha/2}} dy \\ &= \int_E \left( \int_{B_\varepsilon(z) \cap E} \frac{1}{|B_\varepsilon(y) \cap E|} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} dy \right) \frac{d\mu(z)}{|x-z|^{\alpha/2}} \\ &\leq \int_E \left( \frac{C}{\varepsilon^d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} dy \right) \frac{d\mu(z)}{|x-z|^{\alpha/2}}. \end{aligned}$$

Moreover it is possible to prove that the function  $(x, z, \varepsilon) \mapsto \varepsilon^{-d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\frac{\alpha}{2}}}{|x-y|^{\frac{\alpha}{2}}} dy$  is uniformly bounded in  $(x, z, \varepsilon)$  (see [20, Theorem 1.11]) so that  $v_{\alpha/2}^{\mu_\varepsilon}(x) \leq C v_{\alpha/2}^\mu(x)$  for a suitable constant  $C > 0$ . Consider now a point  $x \in \mathbb{R}^d$  such that  $v_{\alpha/2}^\mu(x) < +\infty$ . Then for every  $\delta > 0$  there is a ball  $B_\eta(x)$  such that  $v_{\alpha/2}^{\mu'} < \delta$  where  $\mu' = \mu \llcorner B_\eta(x)$ . By the previous computations, we know that  $v_{\alpha/2}^{(\mu')^\varepsilon}(x) \leq C\delta$ . Moreover  $\lim_{\varepsilon \rightarrow 0} v_{\alpha/2}^{(\mu-\mu')^\varepsilon}(x) = v_{\alpha/2}^{\mu-\mu'}(x)$  (see again [20, Theorem 1.11]) and  $v_{\alpha/2}^{(\mu-\mu')^\varepsilon} = v_{\alpha/2}^{\mu_\varepsilon} - v_{\alpha/2}^{\mu'}$ . Thus we have

$$\begin{aligned} v_{\alpha/2}^\mu(x) &= v_{\alpha/2}^{\mu'}(x) + v_{\alpha/2}^{\mu-\mu'}(x) \leq \delta + \lim_{\varepsilon \rightarrow 0} v_{\alpha/2}^{(\mu-\mu')^\varepsilon}(x) \\ &\leq (1+C)\delta + \liminf_{\varepsilon \rightarrow 0} v_{\alpha/2}^{\mu_\varepsilon}(x) \leq (1+C)\delta + \overline{\lim}_{\varepsilon \rightarrow 0} v_{\alpha/2}^{\mu_\varepsilon}(x) \\ &\leq (1+C)\delta + \overline{\lim}_{\varepsilon \rightarrow 0} v_{\alpha/2}^{\mu'_\varepsilon}(x) + \overline{\lim}_{\varepsilon \rightarrow 0} v_{\alpha/2}^{(\mu-\mu')^\varepsilon}(x) \\ &\leq 2(1+C)\delta + v_{\alpha/2}^\mu(x) \end{aligned}$$

so that letting  $\delta \rightarrow 0$  we get that  $\lim_{\varepsilon \rightarrow 0} v_{\alpha/2}^{\mu_\varepsilon}(x) = v_{\alpha/2}^\mu(x)$  as claimed.  $\square$

The following result can be proven analogously.

**Proposition 2.16.** *Let  $E$  be a smooth connected closed set of  $\mathbb{R}^d$  then for every  $\alpha \in (0, d)$ ,*

$$\mathcal{I}_\alpha(\partial E) = \inf \left\{ \mathcal{I}_\alpha^{\partial E}(f) : \mu = f d\mathcal{H}^{d-1}, f \in L^\infty(\partial E), \int_{\partial E} f d\mathcal{H}^{d-1} = 1 \right\}.$$

For the unit ball, since the problem is invariant by rotations, it is not hard to compute the exact minimizer of  $\mathcal{I}_\alpha(B)$  or  $\mathcal{I}_\alpha(\partial B)$ , see [20, Chapter II.13].

**Lemma 2.17.** *The uniform measure on the sphere  $\partial B$*

$$d\mathcal{U}_B = \frac{1}{P(B)} d\mathcal{H}^{d-1} \llcorner \partial B$$

is the unique optimizer for  $\mathcal{I}_\alpha(\partial B)$ . For  $d > \alpha > d - 2$ , the measure

$$d\tilde{\mathcal{U}}_B = \frac{C_\alpha}{(1 - |x|^2)^{\frac{\alpha}{2}}} d\mathcal{H}^d \llcorner B$$

is the unique optimizer for  $\mathcal{I}_\alpha(B)$  (where  $C_\alpha$  is a suitable renormalization constant).

**Definition 2.18.** Given  $\delta > 0$ , we say that  $E$  satisfies the internal  $\delta$ -ball condition if for any  $x \in \partial E$  there is a ball of radius  $\delta$  contained in  $E$  and tangent to  $\partial E$  in  $x$ . Analogously,  $E$  satisfies the external  $\delta$ -ball condition if for any  $x \in \partial E$ , there is a ball of radius  $\delta$  contained in  $E^c$ . Finally, if  $E$  satisfies both the internal and the external  $\delta$ -ball condition we shall say that it satisfies the  $\delta$ -ball condition.

We remark that the sets which satisfies the  $\delta$ -ball condition have  $C^{1,1}$  boundary with principal curvatures bounded from above by  $1/\delta$ , see [8]. We denote by  $\mathcal{K}_\delta$  the class of all the closed sets which satisfy the  $\delta$ -ball condition and by  $\mathcal{K}_\delta^{co}$  the subset of  $\mathcal{K}_\delta$  composed of connected sets.

**Remark 2.19.** An equivalent formulation of Definition 2.18 is requiring that  $d_E \in C^{1,1}(\{|d_E| < \delta\})$ , where

$$d_E(x) = \begin{cases} \text{dist}(x, \partial E) & \text{if } x \notin E \\ -\text{dist}(x, \partial E) & \text{if } x \in E \end{cases}$$

is the signed distance function from  $\partial E$ .

**Lemma 2.20.** Let  $\delta > 0$ , then every set  $E \in \mathcal{K}_\delta^{co}$  with  $|E| = m$  satisfies

$$\text{diam}(E) \leq \sqrt{d} 2^{d+2} \frac{m}{\omega_d} \delta^{1-d}.$$

*Proof.* Consider the tiling of  $\mathbb{R}^d$  given by  $[0, 2\delta)^d + 2\delta\mathbb{Z}^d$  and for  $k \in \mathbb{Z}^d$  let  $C_k = [0, 2\delta)^d + 2\delta k$ . For every  $k \in \mathbb{Z}^d$  such that  $C_k \cap E \neq \emptyset$ , let  $B_\delta(x_k)$  be a ball of radius  $\delta$  such that  $B_\delta(x_k) \subset E$  and  $B_\delta(x_k) \cap C_k \neq \emptyset$ . the existence of such a ball is guaranteed by the  $\delta$ -ball condition. Any such ball can intersect at most  $2^d$  cubes  $C_j$  so that

$$\#\{k \in \mathbb{Z}^d : E \cap C_k \neq \emptyset\} = \frac{1}{|B_\delta|} \sum_{k: C_k \cap E \neq \emptyset} |B_\delta(x_k)| \leq \frac{2^d}{|B_\delta|} |E|,$$

where  $\#A$  is the cardinality of the set  $A$ . The fact that  $E$  is connected implies that, up to translation,  $E \subset [0, 4\delta \frac{2^d}{|B_\delta|} m]^d$ . Thus we can conclude that

$$\text{diam}(E) \leq \text{diam} \left( \left[ 0, 4\delta \frac{2^d}{|B_\delta|} m \right]^d \right) = \sqrt{d} 2^{d+2} \frac{m}{\omega_d} \delta^{1-d}.$$

□

**Remark 2.21.** In some sense the  $\delta$ -ball condition is the analog of the famous density estimates for problem in which the perimeter term is dominant see [16]. Since in the problems we are going to consider, both the perimeter and the Riesz potential energy are of the same order, there is *a priori* no hope to get such density estimates from the minimality. It is a classical feature that for connected sets, these density estimates provide a bound on the diameter [17].

**Proposition 2.22.** *Let  $d \geq 3$ ,  $\alpha = d - 2$ ,  $\delta > 0$  and  $E \subset \mathbb{R}^d$  be a compact set which satisfies the  $\delta$ -ball condition. Then the optimal measure  $\mu$  for  $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(\partial E)$  can be written as  $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$  with  $\|f\|_{L^\infty(\partial E)} \leq \mathcal{I}_\alpha(E)(d-2)\delta^{-1}$ .*

*Proof.* By Lemma 2.14 we know that the optimizer  $\mu$  is concentrated on  $\partial E$ . Denote by  $v = v_{d-2}^\mu$  the potential related to  $\mu$  on  $E$ . By Lemma 2.14, we know that  $v = \mathcal{I}_\alpha(E)$  on  $E$ , and that  $-\Delta v = \mu$ . By classical elliptic regularity (see for instance [15, Cor. 8.36]),  $v$  is regular in  $\mathbb{R}^d \setminus E$ , and  $C^{1,\beta}$  up to the boundary of  $E$ . Consider now a point  $x \in \partial E$  and let  $y \in E$  such that the ball  $B_\delta(y)$  is contained in  $E$  and is tangent to  $\partial E$  in  $x$ . The existence of such a  $y$  is guaranteed by the  $\delta$ -ball condition satisfied by  $E$ . Let  $u$  be a solution of

$$\Delta u = 0 \quad \text{in } B_\delta^c(y); \quad u = v(x) = \mathcal{I}_\alpha(E) \quad \text{on } \partial B_\delta(y).$$

Notice that  $u(z) = \frac{\mathcal{I}_\alpha(E)\delta^{d-2}}{|z-y|^{d-2}}$  out of  $B_\delta(y)$ . By the maximum principle for harmonic functions,  $u \leq \mathcal{I}_\alpha(E)$  on  $\partial E$ . Thus, again by the maximum principle, applied to  $u - v$ , we get that  $v \geq u$  on  $\mathbb{R}^d \setminus E$ . Since  $u(x) = v(x)$ ,

$$|\nabla v(x)| \leq |\nabla u(x)| = \mathcal{I}_\alpha(E)(d-2)\delta^{-1}. \quad (2.5)$$

Let us prove that  $\mu = |\nabla v|\mathcal{H}^{d-1} \llcorner \partial E$ . For this, let  $x \in \partial E$  and  $r > 0$  and consider a test function  $\varphi \in C_c^\infty(\mathbb{R}^d)$ . Then we have

$$\begin{aligned} \int_{\partial E} \varphi d\mu &= - \int_{\mathbb{R}^d} \varphi \Delta v = \int_{\mathbb{R}^d} \langle \nabla \varphi, \nabla v \rangle dy \\ &= \int_{E^c} \langle \nabla \varphi, \nabla v \rangle dy = \int_{\partial E} \varphi \langle \nabla v, \nu^E \rangle d\mathcal{H}^{d-1} \end{aligned} \quad (2.6)$$

where  $\nu^E$  is the external normal to  $E$ . Since  $v$  is constant on  $\partial E$ , its tangential derivative is zero. Thus, since  $v < \mathcal{I}_\alpha(E)$  on  $\mathbb{R}^d \setminus \overline{E}$  we have that  $\langle \nabla v, \nu^E \rangle \geq 0$ . Therefore,  $\langle \nabla v, \nu^E \rangle = |\nabla v|$  on  $\partial E$ . Hence, by (2.6) we conclude that for every test function  $\varphi$ ,

$$\int_{\partial E} \varphi d\mu = \int_{\partial E} \varphi |\nabla v| d\mathcal{H}^{d-1},$$

which is equivalent to the claim  $\mu = |\nabla v|\mathcal{H}^{d-1} \llcorner \partial E$ . □

### 3. NON-EXISTENCE OF MINIMIZERS

**Definition 3.1.** *Let  $d \geq 2$  and  $\alpha > 0$ . For every  $Q > 0$  and every open set  $E \subset \mathbb{R}^d$  we define the functionals,*

$$\mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E), \quad (3.1)$$

and

$$\mathcal{G}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(\partial E). \quad (3.2)$$

Notice that by Lemma 2.14, for  $\alpha \in (0, d-2]$  the functionals  $\mathcal{F}_{\alpha,Q}$  and  $\mathcal{G}_{\alpha,Q}$  coincide. Notice also that  $\mathcal{F}_{\alpha,Q}(E) \equiv +\infty$  if  $\alpha \geq d$ , and  $\mathcal{G}_{\alpha,Q}(E) \equiv +\infty$  if  $\alpha \geq d-1$ .

In this section we consider a closed, connected, regular set  $\Omega \subset \mathbb{R}^d$  (not necessarily bounded) of measure  $|\Omega| > m$  and address the following problems:

$$\inf_{|E|=m, E \subset \Omega} \mathcal{F}_{\alpha,Q}(E), \quad (3.3)$$

and

$$\inf_{|E|=m, E \subset \Omega} \mathcal{G}_{\alpha, Q}(E), \quad (3.4)$$

where the (implicit) parameter  $\alpha$  belongs to  $(0, d)$ .

**Theorem 3.2.** *For every  $\alpha \in (0, d-1)$ , there holds*

$$\inf_{|E|=m} \mathcal{F}_{\alpha, Q}(E) = \inf_{|E|=m} \mathcal{G}_{\alpha, Q}(E) = \min_{|E|=m} P(E) = \left( \frac{m}{\omega_d} \right)^{\frac{d-1}{d}} P(B).$$

*In particular, problems (3.3) and (3.4) do not admit minimizers when  $\Omega = \mathbb{R}^d$ .*

*Proof.* Let  $N \in \mathbb{N}$  and consider a number  $\beta$  which will be fixed later on. Consider  $N$  balls of radius  $r_N = N^{-\beta}$  which we can consider mutually infinitely far away (since sending them away leaves unchanged the perimeter and decrease the potential interaction energy), and put on each of these balls a charge  $\frac{1}{N}$ . Let  $V_N = Nr_N^d \omega_d$  be their total volume and consider the set  $E$  given by the union of these balls with a (non-charged) ball of volume  $m - V_N$ . If we choose  $\beta \in (1/(d-1), 1/\alpha)$ , then we get

$$\lim_{N \rightarrow +\infty} Nr_N^{d-1} = 0 \quad \text{and} \quad \lim_{N \rightarrow +\infty} \frac{1}{N} \frac{1}{r_N^\alpha} = 0. \quad (3.5)$$

which implies that  $V_N \rightarrow 0$  and

$$\left( \frac{m}{\omega_d} \right)^{\frac{d-1}{d}} P(B) \leq P(E) + Q^2 \mathcal{I}_\alpha(E) \leq \left( \frac{m - V_N}{\omega_d} \right)^{\frac{d-1}{d}} P(B) + C \left( Nr_N^{d-1} + \frac{Q^2}{N} \frac{1}{r_N^\alpha} \right).$$

Since the right-hand side converges to  $\left( \frac{m}{\omega_d} \right)^{\frac{d-1}{d}} P(B)$ , as  $N$  tends to  $+\infty$ , the claim follows.  $\square$

We now consider the case of bounded  $\Omega$  where the situation is more involved.

**Theorem 3.3.** *Let  $\Omega$  be a compact subset of  $\mathbb{R}^d$  with smooth boundary, and let  $0 < m < |\Omega|$ . Let  $E_0$  be a solution of the constrained isoperimetric problem*

$$\min \{P(E) : E \subset \Omega, |E| = m\}. \quad (3.6)$$

*Then, for  $\alpha \in (0, d-1)$  and  $Q > 0$  we have*

$$\inf_{|E|=m, E \subset \Omega} \mathcal{F}_{\alpha, Q}(E) = \inf_{|E|=m, E \subset \Omega} \mathcal{G}_{\alpha, Q}(E) = P(E_0) + Q^2 \mathcal{I}_\alpha(\Omega). \quad (3.7)$$

*Proof.* We divide the proof into three steps.

*Step 1.* For  $\varepsilon > 0$  and  $f \in L^\infty(\Omega)$ , with  $f \geq 0$  and  $\int_\Omega f dx = 1$ , we shall construct a measure  $\tilde{\mu}_\varepsilon$  with  $\text{spt}(\tilde{\mu}_\varepsilon) \subset \Omega$ ,  $\tilde{\mu}_\varepsilon(\Omega) = 1$ , satisfying

$$P(\text{spt}(\tilde{\mu}_\varepsilon)) \leq \varepsilon \quad (3.8)$$

and

$$\mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) \leq \mathcal{I}_\alpha^\Omega(f) + \varepsilon. \quad (3.9)$$

Let  $\delta > \lambda > 0$  be small parameters to be fixed later and consider the tiling of the space given by  $[0, \lambda)^d + \lambda \mathbb{Z}^d$ . For every  $k \in \mathbb{Z}^d$  such that  $(\lambda k + [0, \lambda)^d) \cap \Omega \neq \emptyset$ , we let  $C_k = \lambda k + [0, \lambda)^d$  and

denote by  $x_k$  be the centre of  $C_k$ . Notice that the number  $N$  of such squares  $C_k$  is bounded by  $C(\Omega)\lambda^{-d}$ . Letting  $f_k := \int_{C_k} f \, dx$ , it holds

$$\begin{aligned} \sum_{|x_k - x_j| \geq 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} &= \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x - y|^\alpha} \frac{|x - y|^\alpha}{|x_k - x_j|^\alpha} dx \, dy \\ &\leq \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x - y|^\alpha} \frac{(|x_k - x_j| + 2\lambda)^\alpha}{|x_k - x_j|^\alpha} dx \, dy \\ &\leq \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x - y|^\alpha} \left(1 + C(\alpha) \frac{\lambda}{\delta}\right) dx \, dy \end{aligned} \quad (3.10)$$

where we used the fact that

$$\sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x - y|^\alpha} dx dy \leq \int_{\Omega \times \Omega} \frac{f(x)f(y)}{|x - y|^\alpha} dx dy = \mathcal{I}_\alpha^\Omega(f) < \infty.$$

Let now  $r = (\lambda/2)^\beta$ , with  $\beta > 1$ . If  $\text{dist}(x_k, \mathbb{R}^d \setminus \Omega) \leq r$ , we replace the point  $x_k$  with a point  $\tilde{x}_k \in C_{j(k)}$ , with  $|\tilde{x}_k - x_{j(k)}| \geq \lambda/4$ , where  $C_{j(k)} \subset \Omega$  is a cube adjacent to  $C_k$ . For simplicity of notation, we still denote  $\tilde{x}_k$  by  $x_k$ . We consider  $N$  balls of radius  $r$  centered at the points  $x_k$ , and we set

$$\tilde{\mu}_\varepsilon := \sum_k \frac{f_k}{|B_r|} \chi_{B_r(x_k)}.$$

Notice that, by construction, it holds  $\text{spt}(\tilde{\mu}_\varepsilon) \subset \Omega$  and  $\tilde{\mu}_\varepsilon(\Omega) = \int_\Omega f \, dx = 1$ . We have

$$\begin{aligned} \mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) &= \sum_{j,k} \frac{f_k f_j}{|B_r|^2} \int_{B_r(x_j) \times B_r(x_k)} \frac{dx dy}{|x - y|^\alpha} \\ &= \sum_k \frac{f_k^2}{|B_r|^2} \int_{B_r(x_k) \times B_r(x_k)} \frac{dx dy}{|x - y|^\alpha} \\ &\quad + \sum_{|x_j - x_k| < 2\delta, k \neq j} \frac{f_k f_j}{|B_r|^2} \int_{B_r(x_j) \times B_r(x_k)} \frac{dx dy}{|x - y|^\alpha} \\ &\quad + \sum_{|x_j - x_k| \geq 2\delta} \frac{f_k f_j}{|B_r|^2} \int_{B_r(x_j) \times B_r(x_k)} \frac{dx dy}{|x - y|^\alpha} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Moreover we have that

$$I_1 \leq CN \|f\|_{L^\infty(\Omega)}^2 |C_k|^2 \frac{1}{r^\alpha} \leq C \|f\|_{L^\infty(\Omega)}^2 \lambda^{d-\alpha\beta}, \quad (3.11)$$

and

$$I_2 \leq C \delta^d N^2 \|f\|_{L^\infty(\Omega)}^2 |C_k|^2 \frac{1}{\lambda^\alpha} \leq C \|f\|_{L^\infty(\Omega)}^2 \frac{\delta^d}{\lambda^\alpha}. \quad (3.12)$$

Eventually, from (3.10) it follows

$$\begin{aligned}
I_3 &= \sum_{|x_j - x_k| \geq 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} \frac{1}{|B_r|^2} \int_{B_r(x_j) \times B_r(x_k)} \frac{|x_k - x_j|^\alpha}{|x - y|^\alpha} dx dy \\
&\leq \sum_{|x_k - x_j| \geq 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} \left(1 + C(\alpha) \frac{r}{\delta}\right) \\
&\leq \mathcal{I}_\alpha^\Omega(f) \left(1 + C(\alpha) \frac{\lambda}{\delta}\right) \left(1 + C(\alpha) \frac{r}{\delta}\right) \\
&\leq \mathcal{I}_\alpha^\Omega(f) + C(\alpha) \mathcal{I}_\alpha^\Omega(f) \frac{\lambda}{\delta}.
\end{aligned} \tag{3.13}$$

Letting  $\lambda = \delta^\gamma$ , from (3.11), (3.12), (3.13) we then get

$$\mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) = I_1 + I_2 + I_3 \leq \mathcal{I}_\alpha^\Omega(f) + C(\alpha) \mathcal{I}_\alpha^\Omega(f) \delta^{\gamma-1} + C \|f\|_{L^\infty(\Omega)}^2 \left( \delta^{\gamma(d-\alpha\beta)} + \delta^{d-\alpha\gamma} \right).$$

Choosing  $1 < \beta < d/\alpha$  and  $1 < \gamma < d/\alpha$ , for  $\delta$  small enough we obtain (3.9).

We now show that (3.8) also holds. To this aim, we notice that

$$P(\text{spt}(\tilde{\mu}_\varepsilon)) \leq C N r^{d-1} = C N \lambda^{\beta(d-1)} = C \lambda^{\beta(d-1)-d} \tag{3.14}$$

so that, for  $\lambda$  small enough, (3.8) follows from (3.14) by letting  $d/\alpha > \beta > d/(d-1)$ , choice which is allowed since  $\alpha < d-1$ .

*Step 2.* Let now  $E_0$  be a solution of the constrained isoperimetric problem (3.6), and let

$$E_\varepsilon := \left( E_0 \cup \bigcup_k B_r(x_k) \right) \setminus B_\eta, \quad \mu_\varepsilon := \frac{\tilde{\mu}_\varepsilon \llcorner E_\varepsilon}{1 - \tilde{\mu}_\varepsilon(B_\eta)},$$

where  $B_\eta \subset E_0$  is a ball such that  $|E_\varepsilon| = m$ . Notice that  $\text{spt}(\mu_\varepsilon) \subset E_\varepsilon$  and  $\mu_\varepsilon(E_\varepsilon) = 1$ . By (3.14) we have

$$|B_\eta|^{\frac{d-1}{d}} \leq \left| \bigcup_k B_r(x_k) \right|^{\frac{d-1}{d}} \leq C P \left( \bigcup_k B_r(x_k) \right) \leq C \lambda^{\beta(d-1)-d},$$

so that  $\eta \leq C \lambda^{\beta - \frac{d}{d-1}}$ . In particular, recalling (3.9), for  $\lambda$  sufficiently small the measure  $\mu_\varepsilon$  satisfies

$$\mathcal{I}_\alpha(\mu_\varepsilon) \leq \mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) + \varepsilon \leq \mathcal{I}_\alpha^\Omega(f) + 2\varepsilon. \tag{3.15}$$

From (3.15) we then get

$$\overline{\lim}_{\varepsilon \rightarrow 0} P(E_\varepsilon) + Q^2 \mathcal{I}_\alpha(\mu_\varepsilon) = P(E_0) + Q^2 \mathcal{I}_\alpha^\Omega(f). \tag{3.16}$$

*Step 3.* By Proposition 2.15 we can find a function  $f \in L^\infty(\Omega)$  such that  $\int_\Omega f dx = 1$  and  $\mathcal{I}_\alpha^\Omega(f) \leq \mathcal{I}_\alpha(\Omega) + \varepsilon$ . Thus (3.7) follows by (3.16) and a diagonal argument.  $\square$

**Remark 3.4.** Notice that when  $\alpha \in (d-2, d-1)$ , Problem (3.4) relaxes to its “natural” domain, in the sense that the infimum is  $P(E_0) + Q^2\mathcal{I}_\alpha(\Omega)$  and not  $P(E_0) + Q^2\mathcal{I}_\alpha(\partial\Omega)$  as one might expect.

**Remark 3.5.** An interpretation of Theorem 3.3 is that Problem (3.7) decouples into the isoperimetric problem (3.6) and the *charge-minimizing* problem (2.1), which are minimized separately. This is essentially due to the fact that the perimeter is defined up to a set of zero Lebesgue measure, while the Riesz potential energy is defined up to a set of zero capacity [20, Chapter 2]. A consequence of this is that the minimum problem

$$\min \{ \mathcal{F}_{\alpha,Q}(E) : |E| = m, E \subset A \}$$

has in general no solution.

**Remark 3.6.** When considering a bounded domain  $A$  it is also interesting to study the Riesz potential associated to the Green kernel  $G_A$ , with Dirichlet or Neumann boundary conditions. Since

$$G_A(x, y) = k_{d-2}(|x - y|) + h(x, y)$$

with  $h$  harmonic in  $A$  (see [20, Chapter 1.3], [7]), Theorem 3.3 can be easily extended to that case.

**Remark 3.7.** For  $\alpha \in [d-1, d)$ , it seems difficult to construct a sequence of open sets with vanishing perimeter but of positive capacity. This is due to the fact that sets of positive  $\alpha$ -capacity have Hausdorff measure at least  $\alpha$  (see [23]). As a consequence, the infimum of (3.7) should be strictly larger than  $P(E_0)$ . In order to study the question of existence or non-existence of minimizers, one would need to extend the definition of  $\mathcal{F}_{\alpha,Q}$  to sets which are not open. There are mainly two possibilities to do it. The first is to let for every Borel set  $E$

$$\mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2\mathcal{I}_\alpha(E)$$

where now  $P(E)$  denotes the total variation of  $\chi_E$  (see [3]). It is easy to see that the problem is still ill posed in this class. Indeed, for every set  $E$ , it is possible to consider a set  $F$  of positive  $\alpha$ -capacity but of Lebesgue measure zero so that  $\mathcal{F}_{\alpha,Q}(E \cup F) < \mathcal{F}_{\alpha,Q}(E)$ . The second possibility would be to consider the relaxation of the functional  $\mathcal{F}_{\alpha,Q}$  defined on open sets for a suitable topology. Because of the previous discussion, we see that the  $L^1$  topology, for which the perimeter has good compactness and lower semicontinuity properties, is not the right one. The Hausdorff topology might be more adapted to this situation. Unfortunately, the resulting functional seems hard to identify.

#### 4. EXISTENCE OF MINIMIZERS UNDER SOME REGULARITY CONDITIONS

In the previous section we have seen that we cannot hope to get existence for Problem (3.3) without some further assumptions on the class of minimization. In this section we investigate the existence question in the classes  $K_\delta$  and  $K_\delta^{co}$ . More precisely, we consider the following problems:

$$\min \{ \mathcal{F}_{\alpha,Q}(E) : |E| = m, E \in \mathcal{K}_\delta^{co} \}, \quad (4.1)$$

$$\min \{ \mathcal{G}_{\alpha,Q}(E) : |E| = m, E \in \mathcal{K}_\delta^{co} \}, \quad (4.2)$$

$$\min \{ \mathcal{F}_{\alpha,Q}(E) : |E| = m, E \in \mathcal{K}_\delta \}, \quad (4.3)$$

$$\min \{ \mathcal{G}_{\alpha, Q}(E) : |E| = m, E \in \mathcal{K}_\delta \}. \quad (4.4)$$

Notice that, up to rescaling, we can always assume that  $|E| = \omega_d$ . Indeed, if we let  $\tilde{E} := (\frac{\omega_d}{m})^{1/d} E$ , so that  $|\tilde{E}| = \omega_d$ , from (2.2) we get

$$\mathcal{F}_{\alpha, Q}(E) = \mathcal{F}_{\alpha, Q} \left( \left( \frac{m}{\omega_d} \right)^{1/d} \tilde{E} \right) = \left( \frac{m}{\omega_d} \right)^{\frac{d-1}{d}} \mathcal{F}_{\alpha, (\frac{\omega_d}{m})^{\frac{d-1+\alpha}{2d}}} Q(\tilde{E}) \quad (4.5)$$

$$\mathcal{G}_{\alpha, Q}(E) = \mathcal{G}_{\alpha, Q} \left( \left( \frac{m}{\omega_d} \right)^{1/d} \tilde{E} \right) = \left( \frac{m}{\omega_d} \right)^{\frac{d-1}{d}} \mathcal{G}_{\alpha, (\frac{\omega_d}{m})^{\frac{d-1+\alpha}{2d}}} Q(\tilde{E}). \quad (4.6)$$

**Definition 4.1.** For any set  $E$  with  $|E| = \omega_d$ , we let  $\delta P(E) := P(E) - P(B) \geq 0$  be the isoperimetric deficit of  $E$ .

**Theorem 4.2.** For all  $Q \geq 0$  problem (4.1) and (4.2) have a solution.

*Proof.* Let us focus on (4.1) since the proof of the existence for (4.2) is very similar. Let  $E_n \in \mathcal{K}_\delta^{co}$  be a minimizing sequence, with  $|E_n| = \omega_d$ . And let  $\mu_n$  be the corresponding optimal measures for  $\mathcal{I}_\alpha(E_n)$ . We can then assume that

$$\delta P(E_n) \leq Q^2 \mathcal{I}_\alpha(B),$$

therefore  $P(E_n)$  is uniformly bounded. By Lemma 2.20, the sets  $E_n$  are also uniformly bounded so that by the compactness criterion for functions of bounded variation (see for instance [3]), there exists a subsequence converging in  $L^1$  to some set  $E$  with  $|E| = m$ . Similarly, up to subsequence,  $\mu_n$  is weakly\* converging to some probability measure  $\mu$ .

Let us prove that  $E_n$  converges to  $E$  also in the Kuratowski convergence, or equivalently, in the Hausdorff metric (see for instance [4]). Namely we have to check the following two conditions:

- (i)  $x_n \rightarrow x, \quad x_n \in E_n \Rightarrow x \in E$ ;
- (ii)  $x \in E \Rightarrow \exists x_n \in E_n$  such that  $x_n \rightarrow x$ .

The second condition is an easy consequence of the  $L^1$ -convergence. To prove the first one, we notice that by the internal  $\delta$ -ball condition, up to choose a radius  $r$  small enough there exists a constant  $c = c(d, \delta) > 0$  such that  $|B(x_n, r) \cap E_n| \geq cr^d$  which implies, together with the  $L^1$ -convergence, that a limit point  $x$  must be in  $\overline{E}$ . Similarly one can also prove the Hausdorff convergence of  $\partial E_n$  to  $\partial E$ . Since the family  $\mathcal{K}_\delta^{co}$  is stable under Hausdorff convergence, we get  $E \in \mathcal{K}_\delta^{co}$ .

Recalling that  $P$  is lower semicontinuous under  $L^1$  convergence, and  $\mathcal{I}_\alpha(\mu)$  is lower semicontinuous under weak\*-convergence (for the kernel is a positive function, and thus  $\mathcal{I}_\alpha(\cdot)$  is the supremum of continuous functional over  $\mathcal{M}$ ), we have

$$\liminf_{n \rightarrow +\infty} P(E_n) + Q^2 \mathcal{I}_\alpha(\mu_n) \geq P(E) + Q^2 \mathcal{I}_\alpha(\mu).$$

By the Hausdorff convergence of  $E_n$ , there also holds  $\text{spt}(\mu) \subset E$ , which concludes the proof.  $\square$

Thanks to the quantitative isoperimetric inequality [13], we can also prove existence for small charges of minimizers even without assuming *a priori* the connectedness. This is reminiscent of [18, 19, 7].

**Theorem 4.3.** *There exists a constant  $Q_0 = Q_0(\alpha, d)$  such that, for every  $\delta > 0$ ,  $m \geq \omega_d \delta^d$  and*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq Q_0 \frac{\delta^d}{m},$$

*problems (4.3) and (4.4) have a solution.*

*Proof.* We only consider (4.3), since the proof of (4.4) is identical. Assume first that  $m = \omega_d$ .

As noticed in Theorem 4.2, for every minimizing sequence  $E_n \in \mathcal{K}_\delta$ , with  $|E_n| = \omega_d$ , we can assume that there holds

$$\delta P(E_n) \leq Q^2 \mathcal{I}_\alpha(B).$$

Thus, up to translating the sets  $E_n$ , by the quantitative isoperimetric inequality [13] we can assume that

$$|B \Delta E_n|^2 \leq C(d) \delta P(E_n) \leq C(d) Q^2 \mathcal{I}_\alpha(B)$$

so that  $|E_n \cap B^c| \leq CQ$ . Since every connected component of  $E_n \in \mathcal{K}_\delta$  has volume at least  $|B_\delta| = \omega_d \delta^d$ , for  $Q \leq c(\alpha, d) \delta^d$  the set  $E_n$  must be connected. The existence of minimizers then follows as in Theorem 4.2.

The case of a general volume  $m$  can be obtained by rescaling from (4.5).  $\square$

**Remark 4.4.** We point out that there are other interesting classes where existence can be obtained. This is for instance the case in  $\mathbb{R}^2$  in the class of connected sets of area  $m$ . Indeed, a minimizing sequence is compact for the Hausdorff topology by the Blaschke Theorem [4, Theorem 4.4.15] and semicontinuous thanks to the Golab Theorem [4, Theorem 4.4.17]. Another remarkable class for which the existence issue can be easily solved, is that of *convex sets*. The proof can be made as in Theorem 4.2, simply by observing that, by the quantitative isoperimetric inequality, if  $E_n$  is a minimizing sequence then

$$|E_n \Delta B|^2 + \delta P(E_n) \leq C(\alpha, d),$$

which gives, together with the convexity of the sets  $E_n$ , the compactness of the sequence in the Hausdorff metric.

It is natural to expect that, for a charge  $Q$  large enough, it is more favorable to have two connected components rather than one, which would lead to non-existence of minimizers in  $\mathcal{K}_\delta$ . Let us prove that it is indeed the case, at least for small enough  $\alpha$ . We start with the following lemma.

**Lemma 4.5.** *Let  $\alpha > 0$  and let  $E$  be a compact set then*

$$\mathcal{I}_\alpha(E) \geq \frac{1}{\text{diam}(E)^\alpha}.$$

*In particular,*

$$\inf_{|E|=\omega_d, E \in \mathcal{K}_\delta^{co}} \mathcal{F}_{\alpha, Q}(E) \geq \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B) + \left(\sqrt{d} 2^{d+2}\right)^{-\alpha} Q^2 \delta^{(d-1)\alpha}, \quad (4.7)$$

*and*

$$\inf_{|E|=\omega_d, E \in \mathcal{K}_\delta^{co}} \mathcal{F}_{\alpha, Q}(E) \geq \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B) + \left(\sqrt{d} 2^{d+2}\right)^{-\alpha} Q^2 \delta^{(d-1)\alpha}. \quad (4.8)$$

*Proof.* Let  $\mu$  be any positive measure with support in  $\overline{E}$  such that  $\mu(E) = 1$  then

$$\mathcal{I}_\alpha(E) \geq \int_{E \times E} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \geq \int_{E \times E} \frac{d\mu(x)d\mu(y)}{\text{diam}(E)^\alpha} = \frac{1}{\text{diam}(E)^\alpha}.$$

By Lemma 2.20 and of isoperimetric inequality, we get (4.7) and (4.8).  $\square$

We can now prove a non-existence result in  $\mathcal{K}_\delta$ .

**Theorem 4.6.** *For all  $\alpha < 1$  there exist  $c_0 = c_0(\alpha) > 0$  and  $Q_0 = Q_0(\alpha) > 0$  such that, for every  $\delta > 0$ ,  $m \geq c_0\delta^d$ , and*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} > Q_0 \left( \frac{m}{\delta^d} \right)^{\frac{d\alpha+1-\alpha}{2d}}$$

*problems (4.3) and (4.4) do not have a solution.*

*Proof.* We only discuss problem (4.3), since the non-existence result for problem (4.4) follows analogously.

As in Theorem 4.3 we first consider the case  $m = \omega_d$ , so that  $\delta \leq 1$ . If there exists a minimizer then the optimal measure  $\mu$  is necessarily contained in a connected component of the minimizer. From (4.8) it then follows that the energy of the minimizer is greater than

$$P(B) + \left( \sqrt{d} 2^{d+2} \right)^{-\alpha} \delta^{(d-1)\alpha} Q^2, \quad (4.9)$$

which bounds from below the energy of any set in  $\mathcal{K}_\delta^{co}$  with volume  $\omega_d$ . Hence, in order to prove the non-existence, it is enough to construct a competitor  $E \in \mathcal{K}_\delta$  with energy less than (4.9).

Consider the set  $E$  given by  $N$  (which we suppose to be an integer) balls of radius  $\delta$ , equally charged. Up to increasing their mutual distances, we can suppose that the Riesz potential energy of  $E$  is made only of the self interaction of each ball with itself. We then have

$$P(E) + Q^2 \mathcal{I}_\alpha(E) = N \delta^{d-1} P(B) + \frac{Q^2}{N} \mathcal{I}_\alpha(B_\delta) = \frac{1}{\delta} P(B) + \mathcal{I}_\alpha(B) \delta^{d-\alpha} Q^2. \quad (4.10)$$

Notice that, if  $d - \alpha > (d - 1)\alpha$ , i.e. if  $\alpha < 1$ , there exists  $\delta_0 = \delta_0(\alpha)$  such that for all  $\delta \leq \delta_0$  there holds

$$\mathcal{I}_\alpha(B) \delta^{d-\alpha} \leq \frac{1}{2} \left( \sqrt{d} 2^{d+2} \right)^{-\alpha} \delta^{(d-1)\alpha}.$$

With this condition in force, from (4.10) we get

$$P(E) + Q^2 \mathcal{I}_\alpha(E) < P(B) + \left( \sqrt{d} 2^{d+2} \right)^{-\alpha} Q^2 \delta^{(d-1)\alpha},$$

for

$$Q > \sqrt{2P(B)} \left( \sqrt{d} 2^{d+2} \right)^{\frac{\alpha}{2}} \frac{1}{\delta^{\frac{d\alpha+1-\alpha}{2}}}.$$

The general case can be obtain by rescaling from (4.5).  $\square$

**Remark 4.7.** If  $\alpha < \frac{d-1}{d}$ , we can improve the previous estimate on  $Q$  by considering a construction similar to the one of Theorem 3.2. Indeed, for  $\beta \in (d\alpha, d-1)$ , taking  $N := \delta^{-\beta}$  charged balls of radius  $\delta$  and a non charged ball of volume  $m - \omega_d N \delta^d$ , we find a contradiction if

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} > \tilde{Q}_0(\alpha) \left( \frac{m}{\delta^d} \right)^{\frac{\beta-(1-\alpha)(d-1)}{2d}}.$$

Notice that, if  $\alpha < \frac{d-1}{2d-1}$ , we can choose  $\beta$  such that the exponent  $\frac{\beta-(1-\alpha)(d-1)}{2d}$  is negative.

## 5. MINIMALITY OF THE BALL

In this section we prove that in the harmonic case  $\alpha = d - 2$ , the ball is a minimizer for Problem (3.3) among sets in the family of the *nearly spherical sets* belonging to  $\mathcal{K}_\delta^{\text{co}}$  introduced in Definition 2.18, that is, the sets which are a small  $W^{1,\infty}$  perturbation of the ball and that satisfy the  $\delta$ -ball condition.

Consider a set  $E$  such that  $|E| = \omega_d$ , and such that  $\partial E$  can be written as a graph over  $\partial B$ . In polar coordinates we have

$$E = \{R(x)x : R(x) = 1 + \varphi(x), x \in \partial B\}.$$

The condition  $|E| = \omega_d$  then becomes

$$\int_{\partial B} \left( (1 + \varphi(x))^d - 1 \right) d\mathcal{H}^{d-1}(x) = 0$$

which implies that if  $\|\varphi\|_{L^\infty(\partial B)}$  is small enough, then

$$\int_{\partial B} \varphi d\mathcal{H}^{d-1} = O(\|\varphi\|_{L^2(\partial B)}^2). \quad (5.1)$$

Letting

$$\bar{\varphi} := \frac{1}{|\partial B|} \int_{\partial B} \varphi d\mathcal{H}^{d-1},$$

the Poincaré Inequality gives

$$\begin{aligned} \int_{\partial B} |\nabla \varphi|^2 d\mathcal{H}^{d-1} &\geq C \int_{\partial B} |\varphi - \bar{\varphi}|^2 d\mathcal{H}^{d-1} = C(d) \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} - \frac{C(d)}{d\omega_d} \left( \int_{\partial B} \varphi d\mathcal{H}^{d-1} \right)^2 \\ &= C(d) \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} - \frac{C}{4d\omega_d} \left( \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} \right)^2 \\ &\geq \frac{3}{4} C(d) \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} \end{aligned} \quad (5.2)$$

as soon as

$$\int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} \leq d\omega_d. \quad (5.3)$$

Up to translation, we can also assume that the barycenter of  $E$  is 0. This implies that

$$\left| \int_{\partial B} x\varphi(x) d\mathcal{H}^{d-1}(x) \right| = O\left(\|\varphi\|_{L^2(\partial B)}^2\right). \quad (5.4)$$

**Lemma 5.1.** *Suppose that  $\varphi : \partial B \rightarrow \mathbb{R}^d$  parametrizes  $\partial E$  and  $\|\varphi\|_{L^\infty(\partial B)}$  is small enough so that (5.3) is satisfied. Assume also that the barycenter of  $E$  is in 0. Then,*

$$\delta P(E) \geq c_0 \int_{\partial B} |\nabla \varphi|^2 d\mathcal{H}^{d-1} \geq c_1 \int_{\partial B} |\varphi|^2 d\mathcal{H}^{d-1} = \frac{c_1}{2} \left| \int_{\partial B} \varphi d\mathcal{H}^{d-1} \right|. \quad (5.5)$$

*Proof.* We refer to [12] for the proof of the first inequality. The second inequality is (5.2), while the third one follows from (5.1).  $\square$

A consequence of Lemma 5.1 is the following corollary.

**Corollary 5.2.** *Suppose that  $\partial E$  is parametrized on  $\partial B$  by a function  $\varphi$  which satisfies the hypothesis of Lemma 5.1. Then there exists a positive constant  $C = C(\alpha, d)$  such that*

$$|\mathcal{I}_\alpha^{\partial B}(\varphi)| \leq C \delta P(E), \quad (5.6)$$

and, for any positive constant  $\lambda$ ,

$$|\mathcal{I}_\alpha^{\partial B}(\lambda, \varphi)| \leq C \lambda \delta P(E). \quad (5.7)$$

*Proof.* Inequality (5.7) is an immediate consequence of (5.5). Concerning the first one we have, by the Hölder inequality and the Fubini Theorem,

$$\begin{aligned} \mathcal{I}_\alpha^{\partial B}(\varphi) &= \int_{\partial B \times \partial B} \frac{\varphi(x)\varphi(y)}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\ &\leq \left( \int_{\partial B \times \partial B} \frac{\varphi(x)^2}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \left( \int_{\partial B \times \partial B} \frac{\varphi(y)^2}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \\ &= C \int_{\partial B} \varphi(x)^2 d\mathcal{H}^{d-1}(x). \end{aligned}$$

So (5.6) follows again from (5.5).  $\square$

We will use the following technical lemma.

**Lemma 5.3.** *Let  $E = \{R(x)x : R(x) = 1 + \varphi(x), x \in \partial B\}$  and let  $g \in L^\infty(\partial B)$ , then there exists  $\varepsilon_0(\alpha, d)$  and a constant  $C = C(\alpha, d) > 0$  such that if  $\|\varphi\|_{W^{1,\infty}(\partial B)} \leq \varepsilon_0 \leq 1$ ,*

$$\begin{aligned} &\left| \int_{\partial B \times \partial B} \left( \frac{1}{|R(x) - R(y)|^\alpha} - \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y))}{|x-y|^\alpha} \right) g(x)g(y) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right| \\ &\leq C(\alpha, d)(1 + \varepsilon_0)\|g\|_{L^\infty(\partial B)}^2 \delta P(E). \end{aligned} \quad (5.8)$$

*Proof.* First, notice that since  $|x| = |y| = 1$  we have

$$|R(x)x - R(y)y|^2 = |x - y|^2 (1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y)) \quad (5.9)$$

where  $\psi(x, y) = \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^2}$ . Hence, for any  $x, y \in \partial B$  there holds,

$$|R(x)x - R(y)y|^{-\alpha} = \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y)) + \frac{\alpha(4-\alpha)}{4}\varphi(x)\varphi(y) - \frac{\alpha}{2}(\psi(x, y) + \eta(x, y))}{|x - y|^\alpha} \quad (5.10)$$

where

$$0 \leq \eta(x, y) \leq C (\varphi^2(x) + \varphi^2(y) + \psi^2(x, y)).$$

By (5.10) we get

$$\begin{aligned} &\int_{\partial B \times \partial B} \left( \frac{1}{|R(x) - R(y)|^\alpha} - \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y))}{|x - y|^\alpha} \right) g(x)g(y) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\ &= \frac{\alpha(4-\alpha)}{4} \int_{\partial B \times \partial B} \frac{\varphi(x)\varphi(y)}{|x - y|^\alpha} g(x)g(y) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\ &\quad - \frac{\alpha}{2} \int_{\partial B \times \partial B} \frac{\psi(x, y) + \eta(x, y)}{|x - y|^\alpha} g(x)g(y) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y). \end{aligned} \quad (5.11)$$

By Corollary 5.2 we get

$$\int_{\partial B \times \partial B} \frac{\varphi(x)\varphi(y)}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) = \mathcal{I}_\alpha^{\partial B}(\varphi) \leq C\delta P(E).$$

Furthermore, we have

$$0 \leq \psi(x, y) \leq \|\nabla \varphi\|_{L^\infty(\partial B)}^2 \leq \varepsilon_0,$$

and

$$\int_{\partial B \times \partial B} \frac{\varphi(x)^2 d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} = \int_{\partial B} \frac{d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} \int_{\partial B} \varphi(x)^2 d\mathcal{H}^{d-1}(x) \leq c(\alpha, d)\varepsilon_0^2,$$

for a suitable constant  $c(\alpha, d)$ . Therefore, since  $\eta(x, y) \leq C(\varphi^2(x) + \varphi^2(y) + \psi(x, y))$ , to prove (5.8) we only have to check that

$$\int_{\partial B \times \partial B} \frac{\psi(x, y)}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) \leq C\delta P(E).$$

To this aim, consider  $x, y$  in  $\partial B$  and denote by  $\Gamma_{x,y}$  the geodesic going from  $x$  to  $y$  and by  $\ell(x, y)$  the geodesic distance between  $x$  and  $y$  (that is the length of  $\Gamma_{x,y}$ ). Notice that on  $\partial B$ , the euclidean distance and  $\ell$  are equivalent so that it is enough proving

$$\int_{\partial B \times \partial B} \ell(x, y)^{-(\alpha+2)} (\varphi(x) - \varphi(y))^2 d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) \leq C\delta P(E).$$

We have

$$\begin{aligned} & \int_{\partial B \times \partial B} \ell(x, y)^{-(\alpha+2)} (\varphi(x) - \varphi(y))^2 \\ & \leq c(d) \int_{\partial B \times \partial B} \ell(x, y)^{-(\alpha+1)} \int_{\Gamma_{x,y}} |\nabla \varphi|^2(z) dz d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) \\ & = c(d) \int_{\partial B} \int_0^{2\pi} t^{-(\alpha+1)} t^{d-1} \left( \int_{\{\ell(x,z) \leq t\}} |\nabla \varphi|^2(z) d\mathcal{H}^{d-1}(z) \right) dt d\mathcal{H}^{d-1}(x) \\ & = c(d) \int_0^{2\pi} t^{(d-1)-(\alpha+1)} \left( \int_{\partial B} \int_{\{\ell(x,z) \leq t\}} |\nabla \varphi|^2(z) d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(z) \right) dt \\ & = c(d) \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \int_0^{2\pi} t^{(d-1)-\alpha} \left( \int_{\partial B} |\nabla \varphi|^2(z) d\mathcal{H}^{d-1}(z) \right) dt \\ & = c(d) \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \int_0^{2\pi} t^{(d-1)-\alpha} dt \left( \int_{\partial B} |\nabla \varphi|^2(z) d\mathcal{H}^{d-1}(z) \right) \\ & \leq C\delta P(E) \end{aligned}$$

where  $\mathbb{S}^{d-2}$  is the  $(d-2)$ -dimensional sphere and where we used the fact that  $\alpha < d-1$ .  $\square$

Before we prove our main stability estimates, we recall a classical interpolation inequality.

**Lemma 5.4.** *For every  $0 \leq p < q < r < +\infty$ , there exists a constant  $C(r, p, q)$  such that for every  $\varphi \in H^r(\mathbb{R}^d)$ , there holds*

$$\|\varphi\|_{H^q(\mathbb{R}^d)} \leq C \left( \|\varphi\|_{H^r(\mathbb{R}^d)} \right)^{\frac{r-q}{r-p}} \left( \|\varphi\|_{H^p(\mathbb{R}^d)} \right)^{\frac{q-p}{r-p}}, \quad (5.12)$$

where we adopted the notation  $\|u\|_{H^p(\mathbb{R}^d)} := \|\xi^p \hat{u}\|_{L^2(\mathbb{R}^d)}$  and  $H^p(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) : \|u\|_{H^p} < +\infty\}$ , being  $\hat{u}$  the Fourier transform of the function  $u$ .

*Proof.* Let  $\varphi \in H^r(\mathbb{R}^d)$  and  $\lambda > 0$ , then we have

$$\begin{aligned} \|\varphi\|_{H^q(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |\hat{\varphi}|^2 |\xi|^{2q} d\xi = \int_{|\xi| \leq \lambda} |\hat{\varphi}|^2 |\xi|^{2p} |\xi|^{2(q-p)} d\xi + \int_{|\xi| \geq \lambda} |\hat{\varphi}|^2 |\xi|^{2r} |\xi|^{2(q-r)} d\xi \\ &\leq \lambda^{2(q-p)} \|\varphi\|_{H^p(\mathbb{R}^d)}^2 + \lambda^{-2(r-q)} \|\varphi\|_{H^r(\mathbb{R}^d)}^2. \end{aligned}$$

An optimization in  $\lambda$  yields (5.12). □

**Proposition 5.5.** Let  $\alpha \in [d-2, d-1)$ ,  $f \in L^\infty(\partial E)$  and

$$\partial E = \{R(x)x : R(x) = 1 + \varphi(x), x \in \partial B\}.$$

Then there exist  $\varepsilon_0(\alpha) > 0$  and  $C = C(\alpha) > 0$  such that if  $\|\varphi\|_{W^{1,\infty}(\partial B)} \leq \varepsilon_0$  then

$$\mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\bar{f}) \geq -C\|f\|_{L^\infty(\partial E)}^2 \delta P(E), \quad (5.13)$$

where  $\bar{f} := \frac{1}{P(E)} \int_{\partial E} f d\mathcal{H}^{d-1}$ .

*Proof.* We have

$$\begin{aligned} \mathcal{I}_\alpha^{\partial E}(f) &= \int_{\partial E \times \partial E} \frac{f(x)f(y)}{|x-y|^\alpha} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\ &= \int_{\partial B \times \partial B} \frac{g(x)g(y)}{|R(x)-R(y)|^\alpha} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \end{aligned} \quad (5.14)$$

where we set

$$g(x) = f(R(x)x)R(x)^{d-2} \sqrt{R(x)^2 + |\nabla R(x)|^2}.$$

Up to choose  $\varepsilon_0$  small enough, we can suppose that

$$\|g\|_{L^\infty(\partial B)} \leq 2\|f\|_{L^\infty(\partial E)}. \quad (5.15)$$

Let  $\bar{g} := \frac{1}{P(B)} \int_{\partial B} g d\mathcal{H}^{d-1} = \frac{P(E)}{P(B)} \bar{f}$ . Then we have

$$\mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\bar{f}) = \mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\bar{g}) + \mathcal{I}_\alpha^{\partial B}(\bar{g}) - \mathcal{I}_\alpha^{\partial B}(\bar{f}).$$

Focusing on the last two terms in the previous equality we have

$$\begin{aligned} \left| \mathcal{I}_\alpha^{\partial B}(\bar{g}) - \mathcal{I}_\alpha^{\partial B}(\bar{f}) \right| &= \mathcal{I}_\alpha^{\partial B}(\bar{f}) \left| 1 - \left( \frac{P(E)}{P(B)} \right)^2 \right| \\ &= C\bar{f}^2 \frac{P(E) + P(B)}{P(B)^2} |P(E) - P(B)| \\ &\leq C(\alpha, d) \|f\|_{L^\infty(\partial E)}^2 \delta P(E). \end{aligned}$$

Therefore, to prove (5.13) we only need to show that

$$\mathcal{I}_\alpha^{\partial E}(f) \geq \mathcal{I}_\alpha^{\partial B}(\bar{g}) - \|g\|_{L^\infty(\partial B)}^2 \delta P(E). \quad (5.16)$$

Formula (5.14) together with Lemma 5.3 imply

$$\mathcal{I}_\alpha^{\partial E}(f) = \mathcal{I}_\alpha^{\partial B} \left( g(1 - \frac{\alpha}{2}\varphi) \right) + \mathcal{R}(g, \varphi)$$

with

$$|\mathcal{R}(g, \varphi)| \leq c \|g\|_{L^\infty(\partial E)}^2 \delta P(E),$$

so that

$$\mathcal{I}_\alpha^{\partial E}(f) \geq \mathcal{I}_\alpha^{\partial B} \left( g(1 - \frac{\alpha}{2}\varphi) \right) - c \|g\|_{L^\infty(\partial E)}^2 \delta P(E). \quad (5.17)$$

We need to estimate  $\mathcal{I}_\alpha^{\partial B}(g(1 - \alpha/2)\varphi)$ . By the bilinearity of  $\mathcal{I}_\alpha^{\partial B}$  we have that

$$\begin{aligned} \mathcal{I}_\alpha^{\partial B}(g(1 - \frac{\alpha}{2}\varphi)) &= \mathcal{I}_\alpha^{\partial B}(g(1 - \frac{\alpha}{2}\varphi), g(1 - \frac{\alpha}{2}\varphi)) \\ &= \mathcal{I}_\alpha^{\partial B}(g, g) - \alpha \mathcal{I}_\alpha^{\partial B}(g, g\varphi) + \frac{\alpha^2}{4} \mathcal{I}_\alpha^{\partial B}(g\varphi, g\varphi) \\ &= \mathcal{I}_\alpha^{\partial B}(\bar{g}, \bar{g}) + \mathcal{I}_\alpha^{\partial B}(g - \bar{g}, g - \bar{g}) - \alpha \mathcal{I}_\alpha^{\partial B}(g - \bar{g}, g\varphi) - \alpha \mathcal{I}_\alpha^{\partial B}(\bar{g}, g\varphi) \\ &\quad + \frac{\alpha^2}{4} \mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, \bar{g}\varphi) + \frac{\alpha^2}{2} \mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, (g - \bar{g})\varphi) + \frac{\alpha^2}{4} \mathcal{I}_\alpha^{\partial B}((g - \bar{g})\varphi, (g - \bar{g})\varphi) \\ &= \mathcal{I}_\alpha^{\partial B}(\bar{g}) + \mathcal{I}_\alpha^{\partial B}(g - \bar{g}) + \frac{\alpha^2}{4} \mathcal{I}_\alpha^{\partial B}((g - \bar{g})\varphi) - \alpha \mathcal{I}_\alpha^{\partial B}(g - \bar{g}, (g - \bar{g})\varphi) \\ &\quad - \alpha \mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi) - \alpha \mathcal{I}_\alpha^{\partial B}(g - \bar{g}, \bar{g}\varphi) + \frac{\alpha^2}{2} \mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, (g - \bar{g})\varphi) \\ &\quad - \alpha \mathcal{I}_\alpha^{\partial B}(\bar{g}, \bar{g}\varphi) + \frac{\alpha^2}{4} \mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi). \end{aligned} \quad (5.18)$$

Thanks to (5.7), the last two terms in the right hand side of (5.18) satisfy:

$$-\mathcal{I}_\alpha^{\partial B}(\bar{g}, \bar{g}\varphi) + \frac{\alpha}{4} \mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi) \geq -c\bar{g}^2 \delta P(E). \quad (5.19)$$

By the Cauchy-Schwarz inequality (2.4) and Young's inequality, we get that for every functions  $h_1$  and  $h_2$  and for any  $\varepsilon > 0$ ,

$$\mathcal{I}_\alpha^{\partial B}(h_1, h_2) \leq \mathcal{I}_\alpha^{\partial B}(h_1)^{\frac{1}{2}} \mathcal{I}_\alpha^{\partial B}(h_2)^{\frac{1}{2}} \leq \varepsilon \mathcal{I}_\alpha^{\partial B}(h_1) + \frac{1}{4\varepsilon} \mathcal{I}_\alpha^{\partial B}(h_2). \quad (5.20)$$

In particular, applying such inequality to the functions  $h_1 = g - \bar{g}$  and  $h_2 = (g - \bar{g})\varphi$  in the fourth term in the right hand side of (5.18), and then to  $h_1 = g - \bar{g}$  and  $h_2 = \bar{g}\varphi$  in the sixth term, and exploiting (5.19), we obtain the existence of a positive constant  $C$  such that

$$\begin{aligned} &\mathcal{I}_\alpha^{\partial B}(g(1 - \frac{\alpha}{2}\varphi)) - \mathcal{I}_\alpha^{\partial B}(\bar{g}) \\ &\geq C \left( \frac{1}{2} \mathcal{I}_\alpha^{\partial B}(g - \bar{g}) - \mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi) - \mathcal{I}_\alpha^{\partial B}((g - \bar{g})\varphi) - \bar{g}^2 \delta P(E) \right). \end{aligned} \quad (5.21)$$

Again, by Lemma 5.1, we have that

$$-\mathcal{I}_\alpha^{\partial B}((g - \bar{g})\varphi) \geq -\|g\|_{L^\infty(\partial B)}^2 \mathcal{I}_\alpha^{\partial B}(\varphi) \geq -C \|g\|_{L^\infty(\partial B)}^2 \delta P(E).$$

Let us show that the term  $\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi)$  can be estimated by the term  $\mathcal{I}_\alpha^{\partial B}(g - \bar{g})$ .

Let  $\tilde{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}$  be a regular extension of  $\varphi$ , and let  $\tilde{g} = (g - \bar{g})d\mathcal{H}^{d-1} \llcorner \partial B$ . By a Fourier transform we get

$$\begin{aligned} \mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi) &= \int_{\partial B} \frac{d\mathcal{H}^{d-1}(x)}{|x - y|^\alpha} \bar{g} \int_{\partial B} (g - \bar{g}) d\mathcal{H}^{d-1}(y) \varphi = c(\alpha, d) \bar{g} \int_{\mathbb{R}^d} \widehat{\tilde{\varphi}} \widehat{\tilde{g}} \\ &\leq \bar{g} \left( \int_{\mathbb{R}^d} \widehat{\tilde{\varphi}}^2 |\xi|^{d-\alpha} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \frac{\widehat{\tilde{g}}^2}{|\xi|^{d-\alpha}} \right)^{\frac{1}{2}} \\ &= \bar{g} \|\tilde{\varphi}\|_{H^{\frac{d-\alpha}{2}}(\mathbb{R}^d)} \mathcal{I}_\alpha^{\partial B}(g - \bar{g}, g - \bar{g})^{\frac{1}{2}} \\ &\leq C(d) \bar{g} \|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g - \bar{g})^{\frac{1}{2}}. \end{aligned}$$

We now observe that, if

$$\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi) \leq \frac{1}{2} \mathcal{I}_\alpha^{\partial B}(g - \bar{g}), \quad (5.22)$$

then we would get

$$\mathcal{I}_\alpha^{\partial B}(g(1 - \frac{\alpha}{2}\varphi)) - \mathcal{I}_\alpha^{\partial B}(\bar{g}) \geq -C\|\bar{g}\|_{L^\infty(\partial B)}^2 \delta P(E),$$

which would imply (5.16) and so the claim of the proposition. On the other hand if (5.22) does not hold, then, up to consider again a regular extension  $\tilde{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}$  of  $\varphi$ , we have

$$\mathcal{I}_\alpha^{\partial B}(g - \bar{g}) < C(d) \bar{g} \|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g - \bar{g})^{\frac{1}{2}},$$

which implies

$$\mathcal{I}_\alpha^{\partial B}(g - \bar{g})^{\frac{1}{2}} < C \bar{g} \|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)},$$

so that

$$\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g - \bar{g})\varphi) \leq C \bar{g} \|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g - \bar{g})^{\frac{1}{2}} \leq C \bar{g}^2 \|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)}^2.$$

If  $\frac{d-\alpha}{2} \leq 1$  then using (5.12) with  $p = 0$ ,  $q = \frac{d-\alpha}{2}$  and  $r = 1$ , up to once again regularly extend  $\varphi$  on  $\mathbb{R}^d$ , we obtain

$$\|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)}^2 \leq c_0 \left( \|\varphi\|_{H^1(\partial B)}^2 \right)^{1-\frac{d-\alpha}{2}} \left( \|\varphi\|_{L^2(\partial B)}^2 \right)^{\frac{d-\alpha}{2}} \leq c_1 \left( \|\varphi\|_{H^1(\partial B)}^2 + \|\varphi\|_{L^2(\partial B)}^2 \right) \leq C \delta P(E),$$

which concludes the proof.  $\square$

**Corollary 5.6.** *Let  $d \geq 3$  and  $\alpha = d - 2$ . Then for any  $\delta > 0$  and  $m \geq \omega_d \delta^d$ , there exists a charge  $\bar{Q} \left( \frac{\delta}{m^{1/d}} \right) > 0$ , such that if*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq \bar{Q} \left( \frac{\delta}{m^{1/d}} \right)$$

*the ball is the unique minimizer of problem (4.3).*

*Proof.* Up to a rescaling we can assume  $m = \omega_d$ . By Theorem 4.3, there exists  $C > 0$  such that problem (4.3) admits a minimizer  $E_Q$  for every  $Q \in (0, C\delta^{\frac{d}{2}})$ . Since  $|E_Q \Delta B|^2 \leq C\delta P(E_Q) \leq Q^2 \mathcal{I}_\alpha(B)$ ,  $E_Q$  converges to  $B$  in  $L^1$  when  $Q \rightarrow 0$ . As in Theorem 4.2, there is also convergence in the Hausdorff sense of  $E_Q$  and  $\partial E_Q$  thanks to the  $\delta$ -ball condition. Again, by the  $\delta$ -ball condition and the Hausdorff convergence of the boundaries, for  $Q$  small enough,  $\partial E_Q$  is a graph over  $\partial B$  of some  $C^{1,1}$  function with  $C^{1,1}$  norm bounded by  $2/\delta$ . From this we see that if  $\partial E_Q = \{(1 + \varphi_Q(x))x : x \in \partial B\}$  then  $\|\varphi_Q\|_{W^{1,\infty}(\partial B)}$  is converging to 0. We can thus assume that  $\varphi_Q$  satisfies the hypotheses of Proposition 5.5.

Let  $\mu = f d\mathcal{H}^{d-1} \llcorner \partial E_Q$  be the minimizer of  $\mathcal{I}_\alpha(E_Q)$ . Since  $\mathcal{I}_\alpha(E_Q) \leq P(B) + Q^2 \mathcal{I}_\alpha(B)$ , by Proposition 2.22,  $\|f\|_{L^\infty(\partial E)} \leq (d-2)\delta^{-1}(P(B) + Q^2 \mathcal{I}_\alpha(B))$ . Let  $\bar{f} := \frac{1}{P(E_Q)} = \frac{1}{P(E_Q)} \int_{\partial E_Q} f d\mathcal{H}^{d-1}$ . By Lemma 2.17 we know that the optimal measure for  $\mathcal{I}_\alpha(B)$  is given by  $\frac{\mathcal{H}^{d-1} \llcorner \partial B}{P(B)}$ . By the minimality of  $E_Q$  we then have

$$\begin{aligned} \delta P(E_Q) &= P(E_Q) - P(B) \leq Q^2 (\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E_Q)) \\ &= Q^2 \left( \mathcal{I}_\alpha^{\partial B}(\bar{f}) - \mathcal{I}_\alpha^{\partial E_Q}(f) + \mathcal{I}_\alpha^{\partial B}(1/P(B)) - \mathcal{I}_\alpha^{\partial B}(1/P(E_Q)) \right). \end{aligned}$$

A simple computation shows that

$$\mathcal{I}_\alpha^{\partial B}(1/P(B)) - \mathcal{I}_\alpha^{\partial B}(1/P(E_Q)) \leq C^2 \delta P(E_Q)$$

for a suitable positive constant  $C = C(\alpha, d)$ . Hence, by Proposition 5.5 we have that

$$\delta P(E_Q) \leq C Q^2 \delta P(E_Q) (1 + \|f\|_{L^\infty(\partial E_Q)}^2) \leq C Q^2 \delta P(E_Q),$$

which implies  $\delta P(E_Q) = 0$  that is  $E_Q = B$ , for  $Q$  small enough.  $\square$

**Remark 5.7.** The previous proof of the stability does not apply to the case  $\alpha > d - 2$ . Indeed, this proof relies on  $L^\infty$  bounds for the optimal measure  $\mu$  for  $\mathcal{I}_\alpha$  which we are not able to obtain in that case. For the very same reason, our approach seems not to work if we replace the class  $\mathcal{K}_\delta$  by the class of convex sets. Indeed, using the maximum principle it is possible to show that if the set has an acute enough corner, then the optimal measure is not in  $L^\infty$ .

## 6. THE LOGARITHMIC POTENTIAL ENERGY

In this section we investigate the same type of questions for the logarithmic potential which is given by  $-\log(|x|)$ . This potential naturally arises in two dimension where it corresponds to the Coulomb interaction. Let then

$$\mathcal{I}_{\log}(E) := \min_{\mu(\bar{E})=1} \int_{\mathbb{R}^d \times \mathbb{R}^d} -\log(|x - y|) d\mu(x) d\mu(y) \quad (6.1)$$

and consider the problem

$$\min_{|E|=m} P(E) + Q^2 \mathcal{I}_{\log}(E). \quad (6.2)$$

In analogy to the notation adopted for the Riesz potential we define, for any Borel functions  $f$  and  $g$ , the following quantity

$$\mathcal{I}_{\log}^{\partial E}(f, g) := \int_{\partial E \times \partial E} -\log(|x - y|) f(x) g(y) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y).$$

We list below some important properties of  $\mathcal{I}_{\log}$  without proof, since they are analogous to those given in Section 2 for the Riesz potential. We refer to [27, 20] for comprehensive guides on the logarithmic potential.

**Proposition 6.1.** *The following properties hold:*

- (i) *for every compact set  $E$ , there exists a unique optimal measure  $\mu$  for  $\mathcal{I}_{\log}(E)$  which is concentrated on the boundary of  $E$ ,*
- (ii) *for every Borel measure  $\mu$  it holds*

$$\mathcal{I}_{\log}(\mu) = \int_{\mathbb{R}^d} \left( v_{d/2}^\mu(x) \right)^2 dx \geq 0$$

where

$$v_{d/2}^\mu(x) = \int_{\mathbb{R}^d} -\log|x-y| d\mu(y),$$

- (iii) *for every smooth set  $E$ , if  $\mu$  is the optimal measure for  $\mathcal{I}_{\log}$ , then the equality  $\int_{\partial E} -\log(|x-y|) d\mu(y) = \mathcal{I}_{\log}(E)$  holds for every  $x \in \partial E$ . Moreover the optimal measure for the ball is the uniform measure,*
- (iv) *if  $d = 2$ , then for every bounded set  $E$  satisfying the  $\delta$ -ball condition, the optimal measure is given by some measure  $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$  with  $\|f\|_{L^\infty(\partial E)} \leq \frac{\mathcal{I}_{\log}(E)}{|\log(\delta)|}$ .*

In this setting, since the potential can be negative, the picture is slightly different from that related to the Riesz energy. Indeed, we have the following Theorem.

**Theorem 6.2.** *The following statements hold true:*

- (i)  $\inf_{|E|=m} P(E) + Q^2 \mathcal{I}_{\log}(E) = -\infty$ .
- (ii) *for any  $\delta > 0$ , if  $m > 2\omega_d \delta^d$  then  $\inf_{|E|=m, E \in \mathcal{K}_\delta} P(E) + Q^2 \mathcal{I}_{\log}(E) = -\infty$ ,*
- (iii) *for every  $Q > 0$  and every  $m > \omega_d \delta^d$ , there exists a minimizer of*

$$\min_{|E|=m, E \in \mathcal{K}_\delta^{\text{co}}} P(E) + Q^2 \mathcal{I}_{\log}(E),$$

- (iv) *for every bounded smooth domain  $\Omega$ ,*

$$\inf_{|E|=m, E \subset \Omega} P(E) + Q^2 \mathcal{I}_{\log}(E) = \min_{|E|=m, E \subset \Omega} P(E) + Q^2 \mathcal{I}_{\log}(\Omega).$$

*Proof.* Statement (ii) implies (i) while (iii) can be proven exactly as in Theorem 4.2 and (iv) as Theorem 3.3. To prove (ii) we set  $E_n = B_\delta(x_1^n) \cup B_\delta(x_2^n)$  and notice that if  $\text{dist}(x_1^n, x_2^n)$  goes to infinity, then  $\mathcal{I}_{\log}(E_n) \rightarrow -\infty$  as  $n \rightarrow +\infty$ .  $\square$

Since  $\mathcal{I}_{\log}(\lambda E) = \mathcal{I}_{\log}(E) - \log(\lambda)$  for every  $\lambda > 0$ , without loss of generality we shall assume that  $m = |B_{1/2}| = \pi/4$  in Problem (6.2). The following result is the counterpart of Proposition 5.5.

**Proposition 6.3.** *Let  $d = 2$ ,  $E = \{R(x)x : R(x) = 1 + \varphi(x), x \in \partial B_{1/2}\}$  and let  $f \in L^\infty(\partial E)$  then there exists  $\varepsilon_0$  and a constant  $C = C(\alpha) > 0$  such that if  $\|\varphi\|_{W^{1,\infty}(\partial B_{1/2})} \leq \varepsilon_0$ . Then*

$$\mathcal{I}_{\log}^{\partial E}(f) - \mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{f}) \geq -C\|f\|_{L^\infty(\partial E)}^2 \delta P(E),$$

where  $\bar{f} := \frac{1}{P(E)} \int_{\partial E} f d\mathcal{H}^1$ .

*Proof.* Notice that since  $E \subset B$ , the logarithmic potential is positive. As in the proof of Proposition 5.5, we have

$$\mathcal{I}_{\log}^{\partial E}(f) = \int_{\partial B_{1/2} \times \partial B_{1/2}} -\log(|R(x) - R(y)|) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y),$$

where  $g(x) = f(R(x)x) \sqrt{R(x)^2 + |\nabla R(x)|^2}$ . Reminding that from (5.9), we have

$$|R(x)x - R(y)y| = |x - y| (1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y))^{1/2},$$

where,  $\psi(x, y) = \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^2}$ , we see that

$$\begin{aligned} \mathcal{I}_{\log}^{\partial E}(f) &= \int_{\partial B_{1/2} \times \partial B_{1/2}} -\log(|x - y|) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) \\ &\quad + \frac{1}{2} \int_{\partial B_{1/2} \times \partial B_{1/2}} -\log(1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y)) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y). \end{aligned}$$

As in Proposition 5.5, letting  $\bar{g} := \frac{1}{P(B_{1/2})} \int_{\partial B_{1/2}} g d\mathcal{H}^1$ , we have

$$\mathcal{I}_{\log}^{\partial B_{1/2}}(g) = \int_{\partial B_{1/2} \times \partial B_{1/2}} -\log(|x - y|) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) = \mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{g}) + \mathcal{I}_{\log}^{\partial B_{1/2}}(g - \bar{g})$$

and

$$\mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{g}) - \mathcal{I}_{\log}^{\partial B}(\bar{f}) \leq C \|f\|_{L^\infty(\partial E)}^2 \delta P(E).$$

Using that for  $|t| \leq 1$ ,  $|\log(1 + t) - t| \leq \frac{t^2}{2}$ , we see that

$$\begin{aligned} &\int_{\partial B_{1/2} \times \partial B_{1/2}} -\log(1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y)) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) \\ &= - \int_{\partial B_{1/2} \times \partial B_{1/2}} (\varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y) + \eta(x, y)) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) \end{aligned}$$

where the function  $\eta(x, y)$  is well controlled. As in Lemma 5.3,

$$\int_{\partial B_{1/2} \times \partial B_{1/2}} \varphi(x)\varphi(y)g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) \leq C \|g\|_{L^\infty(\partial B_{1/2})}^2 \delta P(E)$$

and

$$\int_{\partial B_{1/2} \times \partial B_{1/2}} \psi(x, y) g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) \leq C \left( \int_0^{2\pi} t dt \right) \delta P(E).$$

Since

$$\begin{aligned} \int_{\partial B_{1/2} \times \partial B_{1/2}} \varphi(x)g(x)g(y) d\mathcal{H}^1(x) d\mathcal{H}^1(y) &= \bar{g} \int_{\partial B_{1/2}} \varphi(x) (g(x) - \bar{g}) d\mathcal{H}^1(x) \\ &\quad + \bar{g}^2 P(B_{1/2}) \int_{\partial B_{1/2}} \varphi(x) d\mathcal{H}^1(x) \end{aligned}$$

and since  $\int_{\partial B_{1/2}} \varphi(x) d\mathcal{H}^1(x) \leq C\delta P(E)$ , we are left to prove that

$$\mathcal{I}_{\log}^{\partial B_{1/2}}(g - \bar{g}) - \bar{g} \int_{\partial B_{1/2}} \varphi(x) (g(x) - \bar{g}) d\mathcal{H}^1(x) \geq C\bar{g}^2 \delta P(E). \quad (6.3)$$

As in the proof of Proposition 5.5, we use the Fourier transform to assert that for some regular extension  $\tilde{\varphi}$  of  $\varphi$  and for  $\tilde{g} := (g - \bar{g})\mathcal{H}^1 \llcorner \partial B_{1/2}$ ,

$$\begin{aligned} \int_{\partial B_{1/2}} \varphi(x) (g(x) - \bar{g}) d\mathcal{H}^1(x) &\leq \left( \int_{\mathbb{R}^2} \tilde{\varphi}^2 |\xi|^2 d\xi \right)^{1/2} \left( \int_{\mathbb{R}^2} \tilde{g}^2 |\xi|^{-2} d\xi \right)^{1/2} \\ &\leq C \|\varphi\|_{H^1} \mathcal{I}_{\log}^{\partial B_{1/2}}(g - \bar{g}) \end{aligned}$$

from which (6.3) follows arguing exactly as in the last part of the proof of Proposition 5.5.  $\square$

Arguing as in Corollary 5.6, we get the following result.

**Corollary 6.4.** *Let  $d = 2$  then for any  $\delta > 0$  and  $m > 0$ , there exists a  $\bar{Q} \left( \frac{\delta}{\sqrt{m}} \right) > 0$  such that, if  $\frac{Q}{m^{1/4}} < \bar{Q} \left( \frac{\delta}{\sqrt{m}} \right)$ , the ball is the unique minimizer of problem (6.2) among the sets in  $\mathcal{K}_\delta$  with charge  $Q$ .*

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