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Stochastic isentropic Euler equations

F. Berthelin* and J. Vovelle†

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Abstract

We study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ -law for the pressure. When $1 < \gamma \leq 2$, we show the existence of martingale solutions; we also discuss the existence of invariant measure in the concluding section.

Keywords: Stochastic partial differential equations, isentropic Euler equations, kinetic formulation, entropy solutions.

MSC: 60H15, 35R60, 35L65, 76N15

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1 Introduction

In this paper, we study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ -law for the pressure.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis, let \mathbb{T}^1 be the one-dimensional torus, let $T > 0$ and set $Q_T := \mathbb{T}^1 \times (0, T)$. We study the system

$$d\rho + (\rho u)_x dt = 0, \quad \text{in } Q_T, \quad (1.1a)$$

$$d(\rho u) + (\rho u^2 + p(\rho))_x dt = \Phi(\rho, u) dW(t), \quad \text{in } Q_T, \quad (1.1b)$$

$$\rho = \rho_0, \quad \rho u = \rho_0 u_0, \quad \text{in } \mathbb{T}^1 \times \{0\}, \quad (1.1c)$$

where p follows the γ -law

$$p(\rho) = \kappa \rho^\gamma, \quad \kappa = \frac{\theta^2}{\gamma}, \quad \theta = \frac{\gamma - 1}{2}, \quad (1.2)$$

for $\gamma > 1$, W is a cylindrical Wiener process and $\Phi(0, u) = 0$. Therefore the noise affects the momentum equation only and vanishes in vacuum regions. Our aim is to prove the existence of solutions to (1.1) for general initial data (including vacuum). When $\gamma \leq 2$, we will manage to show the existence of martingale solutions, *cf.* Theorem 2.7 below.

There are to our knowledge no existing results on stochastically forced systems of first-order conservation laws¹. The question of existence of solutions for (1.1) is one of the first problem to be solved, and what we will do here for

¹see however the works by Tornare and Fujita [TFY97] and by Feireisl, Maslowski, Novotny [FMN13] on stochastically forced flows with viscosity

$1 < \gamma \leq 2$, but there are some other ones to be considered (see the conclusion part, Section 6 on this subject). In the deterministic case, the existence of (entropy) solutions has been proved by Lions, Perthame, Souganidis in [LPS96] (let us mention also the anterior papers by Di Perna [DiP83a], Ding, Chen, Luo [DCL85], Chen [Che86], Lions, Perthame, Tadmor [LPT94a]). The uniqueness of entropy solutions is still an open question nowadays.

For *scalar* non-linear hyperbolic equations with a stochastic forcing term, the theory has recently known a lot of developments. Well-posedness has been proved in different contexts and under different hypotheses and also with different techniques: by Lax-Oleinik formula (E, Khanin, Mazel, Sinai [EKMS00]), Kruzhkov doubling of variables for entropy solutions (Kim [Kim03], Feng, Nualart [FN08], Vallet, Wittbold [VW09], Chen, Ding, Karlsen [CDK12], Bauzet, Vallet, Wittbold [BVW12], kinetic formulation (Debussche, Vovelle [DV10, DV], the more general results being the resolution in L^1 given in Debussche, Vovelle [DV13]. Let us also mention the works of Hofmanová in this fields (extension to second-order scalar degenerate equations, convergence of the BGK approximation [Hof13b, DHV13, Hof13a]) and the recent works by Lions, Perthame, Souganidis [LPS12, LPS13] on scalar conservation laws with quasilinear stochastic terms.

We will show existence of martingale solutions to (1.1), see Theorem 2.7 below, under the restriction $\gamma \leq 2$. The procedure is standard: we prove the convergence of (subsequence of) solutions to the parabolic approximation to (1.1). For this purpose we have to adapt the concentration compactness technique (*cf.* [DiP83a, LPS96]) of the deterministic case to the stochastic case. Such an extension has already been done for scalar conservation laws by Feng and Nualart [FN08] and what we do is quite similar. The mode of convergence for which there is compactness, if we restrict ourselves to the alea, is the convergence in law. That is why we obtain martingale solutions. There is a usual trick (the Gyöngy-Krylov characterization of convergence in probability) that allow to recover pathwise solutions once pathwise uniqueness of solutions is known (*cf.* [Hof13b, Section4.5]). However for the stochastic problem (1.1) (as it is already the case for the deterministic one), no such results of uniqueness are known and we will say nothing else about pathwise solutions.

Besides the proof of convergence of the parabolic approximation to Problem (1.1) (*cf.* Problem (3.1)) which needs serious adaptation due to the loss of L^∞ bounds with respect to the deterministic case, a large part of our analysis is devoted to the proof of existence of martingale solutions to the parabolic approximation (3.1) which is not a classical point here. What is challenging, and differ from the deterministic case, is the issue of positivity of the density. We solve this problem by using the uniformization effects of parabolic equations with drifts and a bound given by the entropy, in the spirit of Mellet, Vasseur [MV09], *cf.* Theorem A.1. We will give more details

about the main problematic of the paper in Section 2.4, after our framework has been introduced more precisely.

2 Notations and main result

2.1 Stochastic forcing

Our hypotheses on the stochastic forcing term $\Phi(\rho, u)W(t)$ are the following ones. We assume that $W = \sum_{k \geq 1} \beta_k e_k$ where the β_k are independent brownian processes and $(e_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space \mathfrak{U} . For each $\rho \geq 0, u \in \mathbb{R}$, $\Phi(\rho, u): \mathfrak{U} \rightarrow L^2(\mathbb{T}^1)$ is defined by

$$\Phi(\rho, u)e_k = \sigma_k(\cdot, \rho, u) = \rho \sigma_k^*(\cdot, \rho, u), \quad (2.1)$$

where $\sigma_k^*(\cdot, \rho, u)$ is a 1-periodic continuous function on \mathbb{R} . More precisely, we assume $\sigma_k^* \in C(\mathbb{T}_x^1 \times \mathbb{R}_+ \times \mathbb{R})$, with the bound

$$\mathbf{G}^*(x, \rho, u) := \left(\sum_{k \geq 1} |\sigma_k^*(x, \rho, u)|^2 \right)^{1/2} \leq D_0(1 + |u| + \rho^{\frac{\gamma-1}{2}}), \quad (2.2)$$

and

$$\mathbf{G}_j^*(x, \rho, u) := \left(\sum_{k \geq 1} |\nabla_{\rho, u}^j \sigma_k^*(x, \rho, u)|^2 \right)^{1/2} \leq D_1(1 + \rho^{\frac{\gamma-1}{2}-j}), \quad (2.3)$$

where $x \in \mathbb{T}^1, \rho \geq 0, u \in \mathbb{R}$ and $j \in \{1, 2\}$.

We define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 = \left\{ v = \sum_{k \geq 1} \alpha_k e_k; \sum_{k \geq 1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}, \quad v = \sum_{k \geq 1} \alpha_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of W are \mathbb{P} -a.s. in $C([0, T]; \mathfrak{U}_0)$ (see Da Prato, Zabczyk [DPZ92]).

2.2 Notations

We denote by

$$\mathbf{U} = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \mathbf{A}(\mathbf{U}) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}, \quad q = \rho u, \quad (2.4)$$

the 2-dimensional unknown and flux of the conservative part of the problem. We also set

$$\Phi^*(\mathbf{U}) = \rho^{-1}\Phi(\mathbf{U}), \quad \psi_k(\mathbf{U}) = \begin{pmatrix} 0 \\ \sigma_k(\mathbf{U}) \end{pmatrix}, \quad \Psi(\mathbf{U}) = \begin{pmatrix} 0 \\ \Phi(\mathbf{U}) \end{pmatrix}.$$

Note that Φ^* is well defined by (2.1) and that, with the notation above, (1.1) rewrites more concisely as

$$d\mathbf{U} + \partial_x \mathbf{A}(\mathbf{U})dt = \Psi(\mathbf{U})dW(t). \quad (2.5)$$

We will also set, for $\rho \geq 0$,

$$\mathbf{G}(\mathbf{U}) = \rho \mathbf{G}^*(\mathbf{U}). \quad (2.6)$$

If E is a space of real-valued functions on \mathbb{T}^1 , we will denote $\mathbf{U}(t) \in E$ instead of $\mathbf{U}(t) \in E \times E$ when this occur (with $E = W^{1,\infty}(\mathbb{T}^1)$ or $E = L^\infty(\mathbb{T}^1)$ for example in Definition 3.1, Theorem 3.2, etc.)

We denote by \mathcal{P} the predictable σ -algebra on $\Omega \times [0, T]$ generated by (\mathcal{F}_t) and we denote by \mathcal{P}_2 the completion of $\mathcal{P} \otimes \mathcal{B}(\mathbb{T}^1)$, where $\mathcal{B}(\mathbb{T}^1)$ is the Borel σ -algebra on \mathbb{T}^1 .

If $\eta \in C(\mathbb{R}^2)$ is an entropy (cf. (2.8) below) and $\mathbf{U}: \mathbb{T}^1 \rightarrow \mathbb{R}^2$, we let

$$\Gamma_\eta(\mathbf{U}) := \int_{\mathbb{T}^1} \eta(\mathbf{U}(x))dx \quad (2.7)$$

denote the total entropy of \mathbf{U} .

2.3 Entropy Solution

In relation with the kinetic formulation for (1.1) in [LPT94b], there is a family of entropies

$$\eta(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)\chi(\rho, \xi - u)d\xi, \quad \text{with } q = \rho u, \quad (2.8)$$

for (1.1), where

$$\chi(\mathbf{U}) = c_\lambda(\rho^{2\theta} - u^2)_+^\lambda, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \quad c_\lambda = \left(\int_{-1}^1 (1 - z^2)_+^\lambda dz \right)^{-1},$$

$s_+^\lambda := s^\lambda \mathbf{1}_{s>0}$ and the parameter $g \in C^2(\mathbb{R})$ is a convex function. In what follows we will consider only such sub-polynomial g , with the following

Definition 2.1 (Sub-polynomial function). *We say that $g \in C(\mathbb{R})$ is sub-polynomial if there exists $m \in \mathbb{N}$ and $C \geq 0$ such that*

$$|g(\xi)| \leq C(1 + |\xi|^m),$$

for all $\xi \in \mathbb{R}$.

Formally, by the Itô Formula, solutions to (1.1) satisfy

$$d\mathbb{E}\eta(\mathbf{U}) + \mathbb{E}H(\mathbf{U})_x dt = \frac{1}{2}\mathbb{E}\partial_{qq}^2\eta(\mathbf{U})\mathbf{G}^2(\mathbf{U})dt, \quad (2.9)$$

where the entropy flux H is given by

$$H(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)[\theta\xi + (1-\theta)u]\chi(\rho, \xi - u)d\xi, \quad \text{with } q = \rho u. \quad (2.10)$$

Note that, by a change of variable, we also have

$$\eta(\mathbf{U}) = \rho c_\lambda \int_{-1}^1 g\left(\frac{q}{\rho} + z\rho^{(\gamma-1)/2}\right)(1-z^2)_+^\lambda dz \quad (2.11)$$

and

$$H(\mathbf{U}) = \rho c_\lambda \int_{-1}^1 g\left(\frac{q}{\rho} + z\rho^{(\gamma-1)/2}\right)\left(\frac{q}{\rho} + \frac{\gamma-1}{2}z\rho^{(\gamma-1)/2}\right)(1-z^2)_+^\lambda dz. \quad (2.12)$$

In particular, for $g(\xi) = 1$ we obtain the density $\eta(\mathbf{U}) = \rho$. To $g(\xi) = \xi$ corresponds the impulsion $\eta(\mathbf{U}) = q$ and to $g(\xi) = \frac{1}{2}\xi^2$ corresponds the energy

$$\eta_E(\mathbf{U}) = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma.$$

If (2.9) is satisfied with an inequality \leq , then formally (2.2) and the Gronwall Lemma give a bound on $\mathbb{E}\eta(\mathbf{U})(t)$ in terms of $\mathbb{E}\eta(\mathbf{U})(0)$. Indeed, we have

$$\eta_E(\mathbf{U}) = \rho c_\lambda \int \frac{1}{2} \left[\frac{q}{\rho} + z\rho^{\frac{\gamma-1}{2}} \right]^2 (1-z^2)_+^\lambda dz,$$

and $\partial_{qq}^2\eta_E(\mathbf{U}) = \frac{1}{\rho}$, which gives

$$\mathbb{E}\partial_{qq}^2\eta_E(\mathbf{U})\mathbf{G}^2(\mathbf{U}) \leq C\mathbb{E}\eta_E(\mathbf{U}),$$

where C is a constant. Similarly, by taking $g(\xi) = \xi^{2m}$, $m \geq 2$, we shall see (cf. Section 3.1.2) that if (2.9) is satisfied with an inequality \leq , then (2.2) and the Gronwall Lemma give a bound on $\mathbb{E}\eta(\mathbf{U})(t)$ in terms of $\mathbb{E}\eta(\mathbf{U})(0)$. Note however that, in contrast with the deterministic case $\Phi \equiv 0$, and unless Φ is compactly supported, it is not possible to obtain an L^∞ -estimate on \mathbf{U} .

We will prove rigorously uniform bounds for approximated parabolic solutions in Section 3.1.2. The above formal computations are however sufficient for the moment to introduce the following definition.

Definition 2.2 (Entropy solution). *Let $\rho_0, u_0 \in L^\infty(\mathbb{T}^1)$ with $\rho_0 \geq 0$ a.e.. A \mathcal{P}_2 -measurable function*

$$\mathbf{U}: \Omega \times (0, T) \times \mathbb{T}^1 \rightarrow \mathbb{R}_+ \times \mathbb{R}$$

is said to be an entropy solution to (1.1) with initial datum \mathbf{U}_0 if

- we have

$$\Phi(\mathbf{U}) \in L^2(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; L_2(\mathfrak{U}; L^2(\mathbb{T}^1))), \quad (2.13)$$

where $L_2(\mathfrak{U}; L^2(\mathbb{T}^1))$ is the space of Hilbert-Schmidt operators from \mathfrak{U} into $L^2(\mathbb{T}^1)$,

- for all $m \in \mathbb{N}^*$,

$$\mathbb{E} \operatorname{ess\,sup}_{0 \leq t \leq T} \Gamma_\eta(\mathbf{U}(t)) \leq C(\mathbb{E}\Gamma_\eta(\mathbf{U}_0), T, m), \quad (2.14)$$

for all entropy η of the form (2.8) with $g(\xi) = \xi^{2m}$,

- for any (η, H) given by (2.8)-(2.10), where $g \in C^2(\mathbb{R})$ is convex and sub-polynomial, for all $t \in (0, T]$, for all nonnegative $\varphi \in C^1(\mathbb{T}^1)$, and nonnegative $\alpha \in C_c^1([0, t])$, we have

$$\begin{aligned} & \int_0^t \langle \eta(\mathbf{U})(s), \varphi \rangle \alpha'(s) + \langle H(\mathbf{U})(s), \partial_x \varphi \rangle \alpha(s) ds \\ & + \int_0^t \langle \mathbf{G}^2(\mathbf{U}) \partial_q^2 \eta(\mathbf{U}), \varphi \rangle \alpha(s) ds + \langle \eta(\mathbf{U}_0), \varphi \rangle \alpha(0) \\ & + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\mathbf{U}) \partial_q \eta(\mathbf{U}), \varphi \rangle \alpha(s) d\beta_k(s) \geq 0, \end{aligned} \quad (2.15)$$

\mathbb{P} -almost surely.

Remark 2.3. An entropy solution \mathbf{U} is not defined as a process. It is indeed difficult to specify a Banach space in which the process $\mathbf{U}(t)$ would evolve. Indeed the natural bounds for q are in $L^p(\nu)$ spaces where the measure ν depends on ρ . Actually, even if we had found such a Banach space X , we would still have to consider entropy solutions \mathbf{U} as equivalence classes in, say, $L^1(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; X)$, for the simple reason that the entropy solutions that we obtain are issued from a limiting process that does not give convergence for every time t . Certainly, if we knew that a

$$\mathbf{U} \in L^1(\Omega \times [0, T], \mathcal{P}, d\mathbb{P} \times dt; X)$$

had a representative with some continuity property in time, this one may be considered as a stochastic process in its own. However, we do not know how to obtain such continuity properties for entropy solutions to (1.1); this is related to the lack of techniques for proving uniqueness.

Remark 2.4. By (2.13), the stochastic integral $t \mapsto \int_0^t \Phi(\mathbf{U})(s) dW(s)$ is a well defined process taking values in $L^2(\mathbb{T}^1)$ (see [DPZ92] for detailed construction). There is a little redundancy here in the definition of entropy solutions since, apart from the predictability, the integrability property (2.13) will follow from (2.2) and the bounds (2.14).

Remark 2.5. *If we take (for $g(\xi) = 1$ and ξ respectively)*

$$(\eta(\mathbf{U}), H(\mathbf{U})) = (\rho, q), \quad (\eta(\mathbf{U}), H(\mathbf{U})) = (q, \frac{q}{\rho} + p(\rho)),$$

we can deduce from (2.15) the weak formulation of (1.1).

Definition 2.6 (Martingale solution). *Let $\rho_0, u_0 \in L^\infty(\mathbb{T}^1)$ with $\rho_0 \geq 0$ a.e. A martingale solution to (1.1) with initial datum \mathbf{U}_0 is a multiplet*

$$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), W, \mathbf{U}),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, with filtration (\mathcal{F}_t) and Wiener process W , and $\mathbf{U}(t)$ defines an entropy solution to (1.1) with initial datum \mathbf{U}_0 .

Theorem 2.7 (Main result). *Assume $1 < \gamma \leq 2$ and the growth hypotheses (2.2), (2.3) on the noise. Let $\rho_0, u_0 \in L^\infty(\mathbb{T}^1)$ with $\rho_0 \geq 0$ a.e. Then there exists a martingale solution to (1.1) with initial datum \mathbf{U}_0 .*

2.4 Organization of the paper and main problematic

The paper is organized as follows. In Section 3, we prove the existence of martingale solutions to the parabolic approximation of Problem (1.1). This is done by a splitting method. One difficulty here is to obtain gradient estimates on high moments of the solution (*cf.* Section 3.1.3) since no L^∞ bounds are satisfied due to the stochastic perturbation. An other difficulty is to obtain some positivity results on the density: we need quantitative estimates, *cf.* Section 3.1.4. To that purpose we use De Giorgi type estimates in a way developed by Mellet and Vasseur in [MV09]: this is the subject of Appendix A. Once the existence of martingale solution to the parabolic approximation of Problem (1.1) has been proved, we want to take the limit on the regularizing parameter to obtain a martingale solution to (1.1). As in the deterministic case [DiP83a, DiP83b, LPS96], we use the concept of measure-valued solution (Young measure) to achieve this. In Section 4 we develop the tools on Young measure (in our stochastic framework) which are required. This is taken in part (but quite different) from Section 4.3 in [FN08]. We also use the probabilistic version of Murat's Lemma from [FN08, Appendix A], to identify the limiting Young measure. This is the content of Section 5, which requires two other fundamental tools: the analysis of the consequences of the div-curl lemma in [LPS96, Section I.5] and an identification result for densely defined martingales from [Hof13b, Appendix A]. We obtain then the existence of a martingale solution to (1.1). In Section 6 we conclude on some open questions about Problem (1.1) (uniqueness inlaw, existence of an invariant measure). As explained above, we need at some point some bounds from below on solutions to (1-dimensional here) parabolic equations, which are developed in Appendix A. We also need some regularity results, with few variations, on the (1-dimensional) heat semi-group, and those are given in Appendix B.

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3 Parabolic Approximation

Let $\varepsilon > 0$. In this section, we prove the existence of a solution to the following problem.

$$d\mathbf{U}^\varepsilon + \partial_x \mathbf{A}(\mathbf{U}^\varepsilon) dt = \varepsilon \partial_x^2 \mathbf{U}^\varepsilon dt + \Psi(\mathbf{U}^\varepsilon) dW(t), \quad (3.1a)$$

$$\mathbf{U}^\varepsilon|_{t=0} = \mathbf{U}_0^\varepsilon. \quad (3.1b)$$

Definition 3.1 (Weak martingale solution to the parabolic approximation). *Let $\mathbf{U}_0^\varepsilon \in L^\infty(\mathbb{T}^1)$ and $T > 0$. A sextuplet $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}}^\varepsilon)$ is said to be a weak martingale solution to (3.1) with initial datum \mathbf{U}_0 if $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is a probability space, $(\tilde{\mathcal{F}}_t)$ a filtration, \tilde{W} a cylindrical Wiener process and $\tilde{\mathbf{U}}^\varepsilon$ is a predictable process with values in $C([0, T]; W^{1,2}(\mathbb{T}^1))$ satisfying: for all test function $\varphi \in C^\infty(\mathbb{T}^1; \mathbb{R}^2)$,*

$$\begin{aligned} \langle \tilde{\mathbf{U}}^\varepsilon(t), \varphi \rangle &= \langle \tilde{\mathbf{U}}_0, \varphi \rangle - \int_0^t \langle \partial_x \mathbf{A}(\tilde{\mathbf{U}}^\varepsilon) - \partial_x^2 \tilde{\mathbf{U}}^\varepsilon, \varphi \rangle ds \\ &\quad + \int_0^t \langle \Psi(\tilde{\mathbf{U}}^\varepsilon) d\tilde{W}(s), \varphi \rangle, \quad t \in [0, T], \quad \tilde{\mathbb{P}}\text{-a.s.} \end{aligned}$$

Recall that Γ_η is defined by (2.7). We will prove the following result

Theorem 3.2 (Martingale solution to (3.1)). *Assume $\mathbf{U}_0^\varepsilon \in W^{1,\infty}(\mathbb{T}^1)$. Then (3.1) admits a martingale solution $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), W, \mathbf{U}^\varepsilon)$ which satisfies the following estimates (uniform in ε)*

1. *some moment estimates: for all $m \in \mathbb{N}$,*

$$\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{T}^1} \left(|u^\varepsilon|^{2m} + |\rho^\varepsilon|^{m(\gamma-1)} \right) \rho^\varepsilon dx = \mathcal{O}(1), \quad (3.2)$$

where $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2), on m and on $\mathbb{E}\Gamma_\eta(\mathbf{U}_0^\varepsilon)$ for the entropy η associated to $|\xi|^{2m}$,

2. *some gradient estimates: for all $\alpha, \beta \geq 0$,*

$$\varepsilon \mathbb{E} \iint_{Q_T} \left(|u^\varepsilon|^\alpha |\rho^\varepsilon|^{\gamma-2} + |\rho^\varepsilon|^{\gamma-2+\beta} \right) |\partial_x \rho^\varepsilon|^2 dx dt = \mathcal{O}(1), \quad (3.3)$$

and

$$\varepsilon \mathbb{E} \iint_{Q_T} \left(|u^\varepsilon|^\alpha \rho^\varepsilon + |\rho^\varepsilon|^{1+\beta} \right) |\partial_x u^\varepsilon|^2 dx dt = \mathcal{O}(1), \quad (3.4)$$

where $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2) and on a finite number of quantities $\mathbb{E}\Gamma_{\eta_j}(\mathbf{U}_0^\varepsilon)$, for entropies η_j associated to various power-like functions $|\xi|^{r_j}$, for given r_j 's depending on α, β .

Besides, \mathbf{U}^ε satisfies the Itô formula

$$\begin{aligned} d\eta(\mathbf{U}^\varepsilon) + \partial_x H(\mathbf{U}^\varepsilon) dt \\ = \varepsilon \partial_x^2 \eta(\mathbf{U}^\varepsilon) dt + \partial_q \eta(\mathbf{U}^\varepsilon) \Phi(\mathbf{U}^\varepsilon) dW(t) - \varepsilon \eta''(\mathbf{U}^\varepsilon) \cdot (\mathbf{U}_x^\varepsilon, \mathbf{U}_x^\varepsilon) dt \\ + \frac{1}{2} |\mathbf{G}(\mathbf{U}^\varepsilon)|^2 \partial_{qq}^2 \eta(\mathbf{U}^\varepsilon) dt. \end{aligned} \quad (3.5)$$

for all entropy - entropy flux couple (η, H) where η is of the form (2.8) with a subpolynomial function g .

Note that (3.5) has the following meaning: for all test-function $\varphi \in C^\infty(\mathbb{T}^1)$, we have

$$\begin{aligned} \langle \eta(\mathbf{U}^\varepsilon), \varphi \rangle(t) &= \langle \eta(\mathbf{U}_0^\varepsilon), \varphi \rangle(t) + \int_0^t \langle H(\mathbf{U}^\varepsilon(s)), \partial_x \varphi \rangle - \varepsilon \langle \eta(\mathbf{U}^\varepsilon(s)), \partial_x^2 \varphi \rangle ds \\ &\quad - \int_0^t \varepsilon \langle \eta''(\mathbf{U}^\varepsilon) \cdot (\mathbf{U}_x^\varepsilon, \mathbf{U}_x^\varepsilon), x \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \langle \partial_q \eta(\mathbf{U}^\varepsilon) \sigma_k(\mathbf{U}^\varepsilon), \varphi \rangle dW(s) \\ &\quad + \frac{1}{2} \int_0^t \langle |\mathbf{G}(\mathbf{U}^\varepsilon)|^2 \partial_{qq}^2 \eta(\mathbf{U}^\varepsilon), \varphi \rangle ds, \end{aligned}$$

for all $t \in [0, T]$, \mathbb{P} -almost surely.

3.1 Solution to the parabolic problem

3.1.1 Time splitting

Assume, without loss of generality, that $\varepsilon = 1$. To prove the existence of a martingale solution to (3.1), we perform a splitting in time. Let $\tau > 0$. Set $t_k = k\tau$, $k \in \mathbb{N}$. We solve alternatively the deterministic, parabolic part of (3.1) on time intervals $[t_{2k}, t_{2k+1})$ and the stochastic part of (3.1) on time intervals $[t_{2k+1}, t_{2k+2})$, *i.e.*

- for $t_{2k} \leq t < t_{2k+1}$,

$$\partial_t \mathbf{U}^\tau + 2\partial_x \mathbf{A}(\mathbf{U}^\tau) = 2\partial_x^2 \mathbf{U}^\tau \quad \text{in } Q_T, \quad (3.6a)$$

$$\mathbf{U}^\tau(t_{2k}) = \mathbf{U}^\tau(t_{2k}-) \quad \text{in } \mathbb{T}^1, \quad (3.6b)$$

- for $t_{2k+1} \leq t < t_{2k+2}$,

$$d\mathbf{U}^\tau = \sqrt{2} \Psi^\tau(\mathbf{U}^\tau) dW(t) \quad \text{in } Q_T, \quad (3.7a)$$

$$\mathbf{U}^\tau(t_{2k+1}) = \mathbf{U}^\tau(t_{2k+1}-) \quad \text{in } \mathbb{T}^1. \quad (3.7b)$$

Note that we took care to speed up the deterministic equation (3.6a) by a factor 2 and the stochastic equation (3.7a) by a factor $\sqrt{2}$, this rescaling procedure should yield a solution \mathbf{U}^τ consistent with the solution $\mathbf{U}^{\varepsilon=1}$ to (3.1) when $\tau \rightarrow 0$. In (3.7) we have also regularized the coefficient Φ into a coefficient Φ_τ which, together with its derivatives, is globally Lipschitz continuous, satisfies (2.2), (2.3) uniformly in τ and converges punctually to Φ when $\tau \rightarrow 0$.

The deterministic problem (3.6) is solved in [LPS96]: for Lipschitz continuous initial data (ρ_0, q_0) with an initial density ρ_0 uniformly positive, say $\rho_0 \geq c_0 > 0$ on \mathbb{T}^1 , the Problem (3.6) admits a unique solution \mathbf{U} in the class of functions

$$\mathbf{U} \in L^\infty(0, \tau, W^{1,\infty}(\mathbb{T}^1))^2 \cap C([0, \tau]; L^2(\mathbb{T}^1))^2; \quad \rho \geq \mu c_0 \text{ on } \mathbb{T}^1 \times [0, \tau].$$

Here $\mu > 0$ is a constant depending on γ, τ . Besides, \mathbf{U} is smooth on $\mathbb{T}^1 \times (0, \tau]$. The stochastic problem (3.7) is an ordinary stochastic differential equation and does not act on the density. It follows that, starting from an initial datum (ρ_0, q_0) Lipschitz continuous, with an initial density ρ_0 uniformly positive on \mathbb{T}^1 , the splitting method defines a solution \mathbf{U}^τ on $[0, t_K]$, $K \in \mathbb{N}$ fixed, such that, a.s., \mathbf{U}^τ is smooth on $\mathbb{T}^1 \times (0, t_K]$, \mathbf{U}^τ is continuous on $[0, t_K]$ with values in $L^2(\mathbb{T}^1)^2$ and $\rho^\tau(t) > 0$ on \mathbb{T}^1 for all $t \in [0, t_K]$. In the sections that follow we prove some bounds uniform in τ on \mathbf{U}^τ ; we then pass to the limit [$\tau \rightarrow 0$] in Section 3.1.6 and Section 3.1.7. Note that, unless the forcing coefficient Φ is compactly supported, we can not expect some bounds in $L^\infty(\mathbb{T}^1)$ on the solutions, such as the one satisfied when there is no force term in the equations (in particular in the deterministic case [LPS96]).

3.1.2 Entropy bounds

If $\eta \in C(\mathbb{R}^2)$ is an entropy and $\mathbf{U}: \mathbb{T}^1 \rightarrow \mathbb{R}^2$, let

$$\Gamma_\eta(\mathbf{U}) := \int_{\mathbb{T}^1} \eta(\mathbf{U}(x)) dx$$

denote the total entropy of \mathbf{U} . We also define the following indicator functions

$$h_{\text{det}} = \sum_{k \geq 0} \mathbf{1}_{[t_{2k}, t_{2k+1})}, \quad h_{\text{sto}} = 1 - h_{\text{det}}, \quad (3.8)$$

which will be used to localize various estimates below.

Proposition 3.3 (Entropy bounds). *Let $m \in \mathbb{N}$. The solution \mathbf{U}^τ to (3.6)-(3.7) satisfies the estimate*

$$\mathbb{E} \sup_{t \in [0, T]} \Gamma_\eta(\mathbf{U}^\tau(t)) + 2\mathbb{E} \iint_{Q_T} h_{\text{det}} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx dt = \mathcal{O}(1), \quad (3.9)$$

where (η, H) is the entropy-entropy flux couple given by (2.8)-(2.10) corresponding to $g(\xi) = \xi^{2m}$, and $\mathcal{O}(1)$ depends on T, γ , on the constant D_0 in (2.2), on m and $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$.

Proof. Using an entropy identity for (3.6) and Itô Formula for the evolution by (3.7), we obtain

$$\mathbb{E}\Gamma_\eta(\mathbf{U}^\tau(t)) + 2\mathbb{E} \iint_{Q_t} h_{\det} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx ds = \mathbb{E}\Gamma_\eta(\mathbf{U}_0^\tau) + \mathbb{E}R_\eta(t),$$

where

$$R_\eta(t) := \iint_{Q_t} h_{\text{sto}} |\rho^\tau|^2 (\mathbf{G}^*)^2 (\mathbf{U}^\tau) \partial_{qq}^2 \eta(\mathbf{U}^\tau) dx ds$$

is the Itô correction. If $m = 0$, then $\partial_{qq}^2 \eta = 0$ and we obtain (3.9). To estimate R_η in the case $m \geq 1$, we compute, by (2.11),

$$\partial_{qq}^2 \eta(\mathbf{U}) = \frac{1}{\rho} c_\lambda \int_{-1}^1 g'' \left(\frac{q}{\rho} + z \rho^{(\gamma-1)/2} \right) (1 - z^2)_+^\lambda dz. \quad (3.10)$$

Since $g(\xi) = \xi^{2m}$, and $(a + b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha)$, $a, b \in \mathbb{R}$, $\alpha = 2m - 2$, we obtain, using Young's inequality, the bound

$$|\partial_{qq}^2 \eta(\mathbf{U})| [\rho^2 + q^2 + \rho^{\gamma+1}] = \mathcal{O}(1) \left[\rho + \frac{q^{2m}}{\rho^{2m-1}} + \rho^{m(\gamma-1)+1} \right]. \quad (3.11)$$

On the other hand, $\eta(\mathbf{U})$ can be estimated from below: by developing the power,

$$\begin{aligned} \eta(\mathbf{U}) &= \rho c_\lambda \int_{-1}^1 \left(u + z \rho^{(\gamma-1)/2} \right)^{2m} (1 - z^2)_+^\lambda dz \\ &= \rho c_\lambda \int_{-1}^1 \sum_{k=0}^{2m} \binom{2m}{k} u^{2m-k} z^k \rho^{k(\gamma-1)/2} (1 - z^2)_+^\lambda dz \\ &= \rho c_\lambda \int_{-1}^1 \sum_{l=0}^m \binom{2l}{2m} u^{2(m-l)} z^{2l} \rho^{2l(\gamma-1)/2} (1 - z^2)_+^\lambda dz \end{aligned}$$

since the odd terms are zero. Retaining only the extremal terms ($l = 0, m$), we obtain

$$\frac{q^{2m}}{\rho^{2m-1}} + \rho^{m(\gamma-1)+1} = \mathcal{O}(1) \eta(\mathbf{U}). \quad (3.12)$$

By (2.2), (3.11) and the case $m = 0$, we deduce that

$$\mathbb{E}R_\eta(t) = \mathcal{O}(1) \int_0^t \mathbb{E}\Gamma_\eta(\mathbf{U}^\tau(s)) ds \quad (3.13)$$

and, by Gronwall's Lemma,

$$\sup_{t \in [0, T]} \mathbb{E} \Gamma_\eta(\mathbf{U}^\tau(t)) + 2\mathbb{E} \iint_{Q_T} h_{\det} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx dt = \mathcal{O}(1). \quad (3.14)$$

To obtain (3.9), we have to take into account the noise term: we have

$$0 \leq \Gamma_\eta(\mathbf{U}^\tau(t)) = \Gamma_\eta(\mathbf{U}_0^\tau) + M_\eta(t) + R_\eta(t) - D_\eta(t) \quad (3.15)$$

where

$$M_\eta(t) = \sqrt{2} \sum_{k \geq 1} \int_0^t h_{\text{sto}}(s) \langle \sigma_k(\mathbf{U}^\tau(s)), \partial_q \eta(\mathbf{U}^\tau(s)) \rangle_{L^2(\mathbb{T}^1)} d\beta_k(s)$$

and

$$D_\eta(t) = 2 \iint_{Q_t} h_{\det} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx ds.$$

Since $D_\eta \geq 0$, we have

$$0 \leq \Gamma_\eta(\mathbf{U}^\tau(t)) \leq \Gamma_\eta(\mathbf{U}_0^\tau) + M_\eta(t) + R_\eta(t).$$

Similarly as for (3.13), we have $\mathbb{E} \sup_{t \in [0, T]} |R_\eta(t)| = \mathcal{O}(1) \int_0^T \mathbb{E} \Gamma_\eta(\mathbf{U}^\tau(s)) ds$,

then by (3.14), the last term R_η satisfies the bound

$$\mathbb{E} \sup_{t \in [0, T]} |R_\eta(t)| = \mathcal{O}(1).$$

By the Burkholder-Davis-Gundy inequality, we also have

$$\mathbb{E} \sup_{t \in [0, T]} |M_\eta(t)| \leq C \mathbb{E} \left(\int_0^T \sum_{k \geq 1} \langle \sigma_k(\mathbf{U}^\tau(s)), \partial_q \eta(\mathbf{U}^\tau(s)) \rangle_{L^2(\mathbb{T}^1)}^2 ds \right)^{1/2},$$

for a given constant C . Now $\sigma_k(U) = \rho \sigma_k^*(U)$ and $\partial_q \eta(U) = (|u|^{2m-1} + \rho^{(2m-1)(\gamma-1)/2}) \mathcal{O}(1)$ leads for $\mathbb{E} \sup_{t \in [0, T]} |M_\eta(t)|$ to control $\rho(1 + u^{2m} + \rho^{(\gamma-1)m})$

using (2.2). Therefore (3.12) and (3.14) give

$$\mathbb{E} \sup_{t \in [0, T]} |M_\eta(t)| = \mathcal{O}(1).$$

This concludes the proof of the proposition. ■

Corollary 3.4 (Entropy bounds). *Let $m \in \mathbb{N}$. Let I be a Borel subset of $[0, T]$. The solution \mathbf{U}^τ to (3.6)-(3.7) satisfies the estimate*

$$\mathbb{E} \left| \int_I \int_{\mathbb{T}^1} h_{\det} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx dt \right|^2 = \mathcal{O}(1)|I|, \quad (3.16)$$

where $|I|$ denote the Lebesgue measure of I , where η is the entropy given by (2.8) corresponding to $g(\xi) = \xi^{2m}$, and $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2), on m and $\mathbb{E}\Gamma_{\eta^2}(\mathbf{U}_0)$.

Proof. We use the proof of Proposition 3.3: the identity (3.15) gives

$$D_\eta^I(t) = \Gamma_\eta^I(\mathbf{U}_0^\tau) + M_\eta^I(t) + R_\eta^I(t) - \Gamma_\eta^I(\mathbf{U}^\tau(t)),$$

where the superscript I indicates that we integrate on $\mathbb{T}^1 \times I \cap (0, t)$ now, instead of $Q_t = \mathbb{T}^1 \times (0, t)$. Taking square, then expectancy we easily obtain

$$\mathbb{E}|D_\eta(T)|^2 = \mathcal{O}(1)|I| + \mathbb{E}|M_\eta^I(T)|^2.$$

By the Itô isometry, we also have $\mathbb{E}|M_\eta(T)|^2 = \mathcal{O}(1)|I|$. ■

To convert the entropy estimates of Proposition 3.3 into moment estimates, it is practical to work with the (ρ, u) -variables. If η is as in Proposition 3.3, then $\eta(\mathbf{U})$ satisfies the converse inequality from (3.12):

$$\eta(\mathbf{U}) = \mathcal{O}(1) \left(\frac{q^{2m}}{\rho^{2m-1}} + \rho^{m(\gamma-1)+1} \right). \quad (3.17)$$

In particular, we have the following corollary.

Corollary 3.5 (Bounds on the moments). *The solution \mathbf{U}^τ to (3.6)-(3.7) satisfies:*

$$\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{T}^1} \left(|u^\tau|^{2m} + |\rho^\tau|^{m(\gamma-1)} \right) \rho^\tau dx = \mathcal{O}(1), \quad (3.18)$$

for all $m \in \mathbb{N}$, where $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2), on m and on $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ for the entropy η associated to $|\xi|^{2m}$.

If we introduce the measure

$$\mu_t^\tau : \varphi \mapsto \int_{\mathbb{T}^1} \varphi(x) \rho^\tau(x, t) dx,$$

(which depend on the solution), we thus obtain estimates on

$$\mathbb{E} \int_{\mathbb{T}^1} |u^\tau(x, t)|^\alpha d\mu_t^\tau(x) \quad \text{and} \quad \mathbb{E} \int_{\mathbb{T}^1} |\rho^\tau(x, t)|^\beta d\mu_t^\tau(x),$$

for all $\alpha, \beta \geq 0$. It is then straightforward, using Young's and Hölder's inequality to obtain some bounds on the moments of the entropy and entropy flux:

Proposition 3.6 (Bounds on the moments of the entropy). *The solution \mathbf{U}^τ to (3.6)-(3.7) satisfies the estimate*

$$\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{T}^1} [|\eta(\mathbf{U}^\tau(t))|^p + |H(\mathbf{U}^\tau(t))|^p] dx = \mathcal{O}(1), \quad (3.19)$$

for all $m \in \mathbb{N}$, $p \geq 1$, where (η, H) is the entropy-entropy flux couple given by (2.8)-(2.10) corresponding to $g(\xi) = \xi^{2m}$, and $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2), on m , p and $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$.

3.1.3 Gradient estimates

In Proposition 3.3 above, we have obtained an estimate on \mathbf{U}_x^τ . In the case where η is the energy (this corresponds to $g(\xi) = \frac{1}{2}\xi^2$), some computations show that

$$\eta''(\mathbf{U}) \cdot (\mathbf{U}_x, \mathbf{U}_x) = \kappa\gamma|\rho|^{\gamma-2}|\rho_x|^2 + \rho|u_x|^2. \quad (3.20)$$

More generally, we have the following weighted estimates.

Proposition 3.7 (Gradient bounds). *Let $g \in C^2(\mathbb{R})$ be a convex, sub-polynomial function and let η be given by (2.8). Then, the solution \mathbf{U}^τ to (3.6)-(3.7) satisfies the estimate*

$$\begin{aligned} & \mathbb{E} \iint_{Q_T} h_{\det}(t) G^{[2]}(\rho^\tau, u^\tau) [\theta^2 |\rho^\tau|^{\gamma-2} |\partial_x \rho^\tau|^2 + \rho^\tau |\partial_x u^\tau|^2] dx dt \\ & + \mathbb{E} \iint_{Q_T} h_{\det}(t) G^{[1]}(\rho^\tau, u^\tau) 2\theta |\rho^\tau|^{\frac{\gamma-2}{2}} |\partial_x \rho^\tau| \cdot |\rho^\tau|^{1/2} \partial_x u^\tau dx dt = \mathcal{O}(1), \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} G^{[2]}(\rho, u) &= c_\lambda \int_{-1}^1 g''(u + z\rho^{\frac{\gamma-1}{2}}) (1 - z^2)_+^\lambda dz, \\ G^{[1]}(\rho, u) &= c_\lambda \int_{-1}^1 z g''(u + z\rho^{\frac{\gamma-1}{2}}) (1 - z^2)_+^\lambda dz, \end{aligned}$$

and $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2) and $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$.

Proof. We introduce the probability measure

$$dm_\lambda(z) = c_\lambda (1 - z^2)_+^\lambda dz$$

and the 2×2 matrix

$$S = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

which satisfies

$$\partial_x \mathbf{U} = S \mathbf{W}, \quad \mathbf{W} := \begin{pmatrix} \partial_x \rho \\ \rho \partial_x u \end{pmatrix}. \quad (3.22)$$

By (3.9), we then have

$$\int_0^T \mathbb{E} \int_{\mathbb{T}^1} h_{\det}(t) \langle S^* \eta''(\mathbf{U}^\tau) S \mathbf{W}, \mathbf{W} \rangle dx dt = \mathcal{O}(1), \quad (3.23)$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on \mathbb{R}^2 and S^* is the adjoint of S . We compute (recall that $\theta = \frac{\gamma-1}{2}$)

$$\eta''(\mathbf{U}) = \frac{1}{\rho} \int_{\mathbb{R}} \left[A(z) g' \left(u + z \rho^\theta \right) + B(z) g'' \left(u + z \rho^\theta \right) \right] dm_\lambda(z),$$

where

$$A(z) = \begin{pmatrix} \frac{\gamma^2-1}{4} z \rho^\theta & 0 \\ 0 & 0 \end{pmatrix}, \quad B(z) = \begin{pmatrix} (-u + \theta z \rho^\theta)^2 & -u + \theta z \rho^\theta \\ -u + \theta z \rho^\theta & 1 \end{pmatrix}.$$

In particular

$$S^* A S(z) = \begin{pmatrix} \frac{\gamma^2-1}{4} z \rho^\theta & 0 \\ 0 & 0 \end{pmatrix}, \quad S^* B S(z) = \begin{pmatrix} \theta^2 z^2 \rho^{2\theta} & \theta z \rho^\theta \\ \theta z \rho^\theta & 1 \end{pmatrix},$$

and (3.22)-(3.23) give

$$\mathbb{E} \iint_{Q_T} h_{\det}(t) \left(\mathbf{I} |\partial_x \rho^\tau|^2 + \mathbf{J} \partial_x \rho^\tau \cdot |\rho^\tau|^{1/2} \partial_x u^\tau + \mathbf{K} \rho^\tau |\partial_x u^\tau|^2 \right) dx dt = \mathcal{O}(1), \quad (3.24)$$

where

$$\begin{aligned} \mathbf{I} &= |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \theta^2 z^2 g'' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z) \\ &\quad + |\rho^\tau|^{\theta-1} \int_{\mathbb{R}} \frac{\gamma^2-1}{4} z g' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z), \end{aligned}$$

and

$$\begin{aligned} \mathbf{J} &= 2 |\rho^\tau|^{\theta-\frac{1}{2}} \int_{\mathbb{R}} \theta z g'' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z), \\ \mathbf{K} &= \int_{\mathbb{R}} g'' \left(|u^\tau| + z |\rho^\tau|^\theta \right) dm_\lambda(z). \end{aligned}$$

We observe that $2z dm_\lambda(z) = -\frac{c_\lambda}{\lambda+1} d(1-z^2)_+^{\lambda+1}$. By integration by parts, the second term in \mathbf{I} can therefore be written

$$\frac{1}{\lambda+1} |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \frac{\gamma^2-1}{8} (1-z^2) g'' \left(u + z |\rho^\tau|^{\frac{\gamma-1}{2}} \right) dm_\lambda(z).$$

Since $\frac{\gamma^2-1}{8} \frac{1}{\lambda+1} = \theta^2$, we have

$$\mathbf{I} = |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \theta^2 g'' \left(u^\tau + z |\rho^\tau|^{\frac{\gamma-1}{2}} \right) dm_\lambda(z).$$

This gives (3.21). ■

We apply (3.21) with $g(\xi) := |\xi|^{2m+2}$ and η given by (2.8). Then

$$|u|^{2m} + \rho^{m(\gamma-1)} = \mathcal{O}_m(1)G^{[2]}(\rho, u)$$

and since $G^{[1]}$ is dominated by $G^{[2]}$, our estimate (3.21) gives

$$\begin{aligned} & \mathbb{E} \iint_{Q_T} h_{\det}(t) \left[|u^\tau|^{2m} + |\rho^\tau|^{m(\gamma-1)} \right] \left[|\rho^\tau|^{\gamma-2} |\partial_x \rho^\tau|^2 + \rho^\tau |\partial_x u^\tau|^2 \right] dx dt \\ & = \mathcal{O}(1). \end{aligned}$$

By letting m vary, we obtain the following

Corollary 3.8. *The solution \mathbf{U}^τ to (3.6)-(3.7) satisfies the estimates*

$$\mathbb{E} \iint_{Q_T} h_{\det}(t) \left(|u^\tau|^\alpha |\rho^\tau|^{\gamma-2} + |\rho^\tau|^{\gamma-2+\beta} \right) |\partial_x \rho^\tau|^2 dx dt = \mathcal{O}(1), \quad (3.25)$$

and

$$\mathbb{E} \iint_{Q_T} h_{\det}(t) \left(|u^\tau|^\alpha \rho^\tau + |\rho^\tau|^{1+\beta} \right) |\partial_x u^\tau|^2 dx dt = \mathcal{O}(1), \quad (3.26)$$

for all $\alpha, \beta \geq 0$, where $\mathcal{O}(1)$ depends on T, γ , on the constant D_0 in (2.2) and on a finite number of quantities $\mathbb{E}\Gamma_{\eta_j}(\mathbf{U}_0)$, for entropies η_j associated to various power-like functions $|\xi|^{r_j}$, for given r_j 's depending on α, β .

3.1.4 Positivity of the density

Proposition 3.9 (Positivity). *Let \mathbf{U}^τ be the solution to (3.6)-(3.7) with initial datum $\mathbf{U}_0 = (\rho_0, q_0)$ and assume that ρ_0 is uniformly positive: there exists $c_0 > 0$ such that $\rho_0 \geq c_0$ a.e. on \mathbb{T}^1 . Then, a.s., there exists $c > 0$ depending on c_0, T and*

$$\iint_{Q_T} h_{\det}(t) \rho^\tau |\partial_x u^\tau|^2 dx dt$$

only, such that $\rho^\tau \geq c$ a.e. in $\mathbb{T}^1 \times [0, T]$.

Proof. We apply the Corollary A.3 to Theorem A.1 from Appendix A by considering specifically the equation satisfied by ρ^τ . Note that we have then to modify slightly the argument of the proof to Theorem A.1 since the equation satisfied by ρ^τ is

$$\rho_t^\tau + 2h_{\det}(t) [(\rho^\tau u^\tau)_x - \rho_{xx}^\tau] = 0,$$

while Equation (A.2) has no such factor $h_{\det}(t)$. This requires however only minor adaptations (it is clear that, when h_{\det} vanishes, the uniformization

effect due to the evolution by the heat semi-group is but preserved). Specifically, replace U_k in (A.4) by

$$\tilde{U}_k := \sup_{\tau_k \leq t \leq T} \int_{\mathbb{T}^1} |w_k|^2 dx + 4 \iint_{Q_{\tau_k, T}} h_{\det}(t) |\partial_x w_k|^2 dx dt,$$

also replace the estimate (A.10) by the estimate

$$\|h_{\det} w_k\|_{L^6(Q_{\tau_{k-1}, T})}^2 \leq \tilde{U}_k,$$

and so on; we leave the details to the reader. ■

3.1.5 Bounds on higher derivatives of U^τ

Assume, by adjusting τ if necessary, that $T = t_{2K}$, $K \in \mathbb{N}^*$. Let $R > 0$ and let us introduce the stopping time T_R defined as the infimum of the times t_{2k} such that

$$\sup_{t \in [0, t_{2k}]} \Gamma_\eta(\mathbf{U}^\tau(t)) + 2 \iint_{Q_{t_{2k}}} h_{\det} \eta''(\mathbf{U}^\tau) \cdot (\mathbf{U}_x^\tau, \mathbf{U}_x^\tau) dx dt \leq R, \quad (3.27)$$

where the infimum over the empty set is taken to be $t_{2K} = T$ by definition. Here we take for η the energy, *i.e.* the entropy associated to $g(\xi) = \frac{1}{2}|\xi|^2$. In that case, we have by (3.20) the control

$$\iint_{Q_{T_R}} h_{\det}(t) \rho^\tau |\partial_x u^\tau|^2 dx dt \leq R.$$

Let then \mathbf{U}_R^τ denote the process deduced from \mathbf{U}^τ by following a deterministic evolution after the time T_R : \mathbf{U}_R^τ is defined like \mathbf{U}^τ by (3.6) where Φ_τ is replaced with $\Phi_\tau \mathbf{1}_{t < T_R}$. Since $\Gamma_\eta(\mathbf{U}^\tau(T_R)) \leq R$, we can adapt the proof of Proposition 3.3 (it is now purely deterministic) to obtain

$$\int_{T_R}^T \int_{\mathbb{T}^1} h_{\det}(t) \rho_R^\tau |\partial_x u_R^\tau|^2 dx dt \leq M(R),$$

where the deterministic constant $M(R)$ depends on T , γ , D_0 and R only. In particular, we have

$$\iint_{Q_T} h_{\det}(t) \rho_R^\tau |\partial_x u_R^\tau|^2 dx dt \leq M(R) + R.$$

Proposition 3.9 in the previous section then asserts that there exists $c(R) > 0$ depending on c_0 , T and R such that

$$\rho_R^\tau \geq c(R) \text{ a.e. on } \mathbb{T}^1 \times [0, T]. \quad (3.28)$$

In this section we will use this uniform estimate (3.28) and the entropy estimates from Section 3.1.2 to obtain some bounds independent on τ on the higher derivatives of \mathbf{U}_R^τ . Note in particular that the proof of Corollary 3.5 and Proposition 3.7 apply readily to \mathbf{U}_R^τ as well so that we have the following

Remark 3.10. The estimates (3.18), (3.25), (3.26) are satisfied by \mathbf{U}_R^τ .

We will prove the following result.

Proposition 3.11 (Bounds on higher derivatives). *Let $\mathbf{U}_0 = (\rho_0, q_0) \in W^{1,\infty}(\mathbb{T}^1)^2$, \mathbf{U}_0 deterministic. Let \mathbf{U}^τ be the solution to (3.6)-(3.7) with initial datum $\mathbf{U}_0 = (\rho_0, q_0)$. Assume that $\rho_0 \geq c_0$ a.e. on \mathbb{T}^1 where $c_0 > 0$. Let $p > 2$ and $\alpha \in (0, 1/2)$. Then for all $R > 0$, we have*

$$\mathbb{E}\|\rho_R^\tau\|_{C^\alpha([0,T];L^p(\mathbb{T}^1))}, \mathbb{E}\|\rho_R^\tau\|_{L^p(0,T;W^{3/2,p}(\mathbb{T}^1))} = \mathcal{O}(1), \quad (3.29)$$

and

$$\mathbb{E}\|u_R^\tau\|_{C^\alpha([0,T];L^p(\mathbb{T}^1))}, \mathbb{E}\|u_R^\tau\|_{L^p(0,T;W^{3/2,p}(\mathbb{T}^1))} = \mathcal{O}(1), \quad (3.30)$$

where $\mathcal{O}(1)$ depends on R , on c_0 , on p , α , γ , T , on the constants D_0 and D_1 in (2.2), (2.3) and on the moments of \mathbf{U}_0 and $\partial_x \mathbf{U}_0$ up to a given order depending on p and α .

Proof. For $0 \leq s < t \leq T$, let $Q_{s,t}$ denote the cylinder $\mathbb{T}^1 \times (s, t)$. We work on the variables ρ , $u = q/\rho$ and rewrite (3.6) as a system of two heat equations with source terms in the domain $Q_{t_{2k}, t_{2k+1}}$:

$$\left(\frac{1}{2}\partial_t - \partial_x^2\right)\rho^\tau = -u^\tau \partial_x \rho^\tau - \rho^\tau \partial_x u^\tau =: f_1^\tau, \quad (3.31)$$

and

$$\left(\frac{1}{2}\partial_t - \partial_x^2\right)u^\tau = -u^\tau \partial_x u^\tau - \kappa\gamma|\rho^\tau|^{\gamma-2}\partial_x \rho^\tau + 2\frac{\partial_x \rho^\tau}{\rho^\tau} \partial_x u^\tau =: f_2^\tau. \quad (3.32)$$

Let us extend f_1^τ and f_2^τ by 0 in the strips $Q_{t_{2k+1}, t_{2k+2}}$, and set

$$\mathbf{F}^\tau = \begin{pmatrix} f_1^\tau \\ f_2^\tau \end{pmatrix},$$

and let $S(t)$ denote the semi-group associated to $-\partial_x^2$ on $L^p(\mathbb{T}^1)$. For $f \in L^1(Q_T)$, $z_0 \in L^1(\mathbb{T}^1)$, the solution to the equation

$$\left(\frac{1}{2}\partial_t - \partial_x^2\right)z(x, t) = f(x, t), \quad x \in \mathbb{T}^1, t > 0,$$

with initial datum z_0 is

$$z(t) = S(2t)z_0 + 2 \int_0^t S(2(t-s))f(s)ds.$$

By iteration, it follows that \mathbf{U}^τ is given by

$$\begin{aligned} \mathbf{U}^\tau(t) &= S(t_\sharp)\mathbf{U}^\tau(0) + \int_0^{t_\sharp} S(t_\sharp - s)\mathbf{F}^\tau(s_b)ds \\ &\quad + \sum_{k \geq 1} \sqrt{2} \int_0^{t_\sharp} S(t_\sharp - s_\sharp)h_{\text{sto}}(s)\psi_k(\mathbf{U}^\tau(s))d\beta_k(s), \end{aligned}$$

where we have defined

$$t_{\sharp} = \min(2t - t_{2n}, t_{2n+2}), \quad t_{\flat} = \frac{t + t_{2n}}{2}, \quad t_{2n} \leq t < t_{2n+2}.$$

For the “stopped” process \mathbf{U}_R^τ , we have, similarly, a decomposition

$$\mathbf{U}_R^\tau = \mathbf{U}_{0,R}^\tau + \mathbf{U}_{\text{det},R}^\tau + \mathbf{U}_{\text{sto},R}^\tau, \quad (3.33)$$

where

$$\begin{aligned} \mathbf{U}_{0,R}^\tau(t) &= S(t_{\sharp})\mathbf{U}^\tau(0), \\ \mathbf{U}_{\text{det},R}^\tau(t) &= \int_0^{t_{\sharp}} S(t_{\sharp} - s)\mathbf{F}_R^\tau(s)ds, \\ \mathbf{U}_{\text{sto},R}^\tau(t) &= \sum_{k \geq 1} \sqrt{2} \int_0^{t_{\sharp} \wedge T_R} S(t_{\sharp} - s_{\sharp})h_{\text{sto}}(s)\psi_k(\mathbf{U}^\tau(s))d\beta_k(s), \end{aligned}$$

and where \mathbf{F}_R^τ is defined by similarity with \mathbf{F}^τ , *i.e.*

$$\mathbf{F}_R^\tau = \begin{pmatrix} f_{1,R}^\tau \\ f_{2,R}^\tau \end{pmatrix} = \begin{pmatrix} -u_R^\tau \partial_x \rho_R^\tau - \rho_R^\tau \partial_x u_R^\tau \\ -u_R^\tau \partial_x u_R^\tau - \kappa \gamma |\rho_R^\tau|^{\gamma-2} \partial_x \rho_R^\tau + 2 \frac{\partial_x \rho_R^\tau}{\rho_R^\tau} \partial_x u_R^\tau \end{pmatrix}.$$

Higher derivatives in x : to obtain some estimates on the higher derivatives in x of ρ_R^τ, u_R^τ , we need to use a bootstrap method. Actually we will use improved estimates in Equation (3.33) three times. For a better readability of this part of the proof, we will adopt the following convention of notation: a constant C_R , which may vary from lines to lines, is a constant depending on T, γ , on the constant D_0 in (2.2) and on R . A number $\varepsilon > 0$ being given, we use the notation ε for quantities small with ε but possibly different from ε . Therefore in what follows, $L^{r-\varepsilon}$ may well be actually $L^{r-\frac{\varepsilon}{10}}$. Also, L^{m_ε} is L^m for a m (quite high) depending on ε , which we do not specify and again, the value of m_ε may vary from line to line. Eventually, we denote by $M_\varepsilon(U_0)$ a constant depending only on the moments

$$\int_{\mathbb{T}^1} (u_0^{m_\varepsilon} + \rho_0^{m_\varepsilon(\gamma-1)})\rho_0 dx, \quad \int_{\mathbb{T}^1} |\partial_x u_0|^{m_\varepsilon} + |\partial_x \rho_0|^{m_\varepsilon} dx,$$

where the exponent m_ε depends on ε .

We begin by showing that

$$\|\partial_x \rho_R^\tau\|_{L^{6-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0). \quad (3.34)$$

Rewriting

$$f_{1,R}^\tau = -|\rho_R^\tau|^{\frac{2-\gamma}{2}} \cdot |\rho_R^\tau|^{\frac{\gamma-2}{2}} u_R^\tau \partial_x \rho_R^\tau - \rho_R^\tau \partial_x u_R^\tau,$$

we deduce from Remark 3.10, from the moment estimate (3.18), and from the gradient estimates (3.25), (3.26) and from the Hölder inequality that

$$\|f_{1,R}^\tau\|_{L^{2-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0).$$

By (B.7) this implies $\|\partial_x \rho_{\text{det},R}^\tau\|_{L^{6-\varepsilon}(Q_T)} \leq M_\varepsilon(U_0)$. We have besides $\rho_{\text{sto},R}^\tau = 0$ and thus (3.34).

Then, in the equation satisfied by u_R^τ , we use the expression

$$\begin{aligned} f_{2,R}^\tau &= -\frac{1}{|\rho_R^\tau|^{1/2}} \cdot |\rho_R^\tau|^{1/2} u_R^\tau \partial_x u_R^\tau - \kappa \gamma \frac{1}{|\rho_R^\tau|^{\frac{2-\gamma}{2}}} \cdot |\rho_R^\tau|^{\frac{\gamma-2}{2}} \partial_x \rho_R^\tau \\ &\quad + 2 \frac{1}{|\rho_R^\tau|^2} \cdot \partial_x \rho_R^\tau \cdot \rho_R^\tau \partial_x u_R^\tau. \end{aligned}$$

We deduce from Remark 3.10, from (3.18), (3.25), (3.26), (3.34), from (3.28) and from the Hölder inequality that $\|f_{2,R}^\tau\|_{L^{3/2-\varepsilon}(Q_T)} \leq C_R M_\varepsilon(U_0)$, and thus, by (B.7),

$$\|\partial_x u_{\text{det},R}^\tau\|_{L^{3-\varepsilon}(Q_T)} \leq C_R M_\varepsilon(U_0). \quad (3.35)$$

To get a bound on $u_{\text{sto},R}^\tau$, we use the following lemma.

Lemma 3.12. *Let $q \geq 2$. For $k \in \mathbb{N}^*$, let $h_k \in L^2(\Omega \times (0, T); L^q(\mathbb{T}^1))$ be some predictable processes such that*

$$\mathbf{H} := \left(\sum_{k \geq 1} |h_k|^2 \right)^{1/2} \quad (3.36)$$

satisfies $\mathbf{H} \in L^q(\Omega \times (0, T) \times \mathbb{T}^1)$. Then we have the estimate

$$\mathbb{E} \left\| \sum_{k \geq 1} \int_0^{t_\sharp} S(t_\sharp - s_\sharp) h_k(s) d\beta_k(s) \right\|_{L^q(\mathbb{T}^1)}^q \leq C_{\text{BDG}}(q) t_\sharp^{\frac{q-1}{2}} \mathbb{E} \|\mathbf{H}\|_{L^q(Q_{t_\sharp})}^q, \quad (3.37)$$

for a given constant $C_{\text{BDG}}(q)$ depending on q only which is bounded for bounded values of $q \in [2, +\infty)$.

Proof of Lemma 3.12: we apply the Burkholder-Davis-Gundy inequality in the 2-smooth Banach space $L^q(\mathbb{T}^1)$ to obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq \sigma \leq t_\sharp} \left\| \sum_{k \geq 1} \int_0^\sigma S(t_\sharp - s_\sharp) h_k(s) d\beta_k(s) \right\|_{L^q(\mathbb{T}^1)}^q \\ \leq C_{\text{BDG}}(q) \mathbb{E} \left(\int_0^{t_\sharp} \|S(t_\sharp - s_\sharp) \mathbf{H}(s)\|_{L^2(\mathbb{T}^1)}^2 ds \right)^{q/2}. \end{aligned}$$

Since $S(t)$ is a contraction from $L^2(\mathbb{T}^1)$ into $L^2(\mathbb{T}^1)$, we obtain (3.37) by Jensen's inequality. ■

We apply Lemma 3.12 with

$$h_k(t) = \partial_x (h_{\text{sto}}(t) \sigma_k(\mathbf{U}_R^\tau(t))) \mathbf{1}_{t < T_R}$$

and $q = 3 - \varepsilon$. By the growth hypothesis (2.3) and (3.28) we obtain, for $T \geq 1$,

$$\mathbb{E} \|\partial_x u_{\text{sto},R}^\tau(t)\|_{L^{3-\varepsilon}(\mathbb{T}^1)}^{3-\varepsilon} \leq \bar{C} T C_R \left(\mathbb{E} \|\partial_x u_R^\tau\|_{L^{3-\varepsilon}(Q_{t_\sharp})}^{3-\varepsilon} + \mathbb{E} \|\partial_x \rho_R^\tau\|_{L^{3-\varepsilon}(Q_{t_\sharp})}^{3-\varepsilon} \right). \quad (3.38)$$

Here $\bar{C} := \sup_{2 \leq q \leq 3} C_{\text{BDG}}(q)$ is an absolute constant.

From the identity (3.33), from (3.34) and the estimates (3.35), (3.38) and the Gronwall Lemma, we obtain

$$\mathbb{E} \|\partial_x u_R^\tau\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_\varepsilon(U_0). \quad (3.39)$$

Now, Remark 3.10, the estimates (3.18), (3.28), (3.34), (3.39) and the identity

$$f_{1,R}^\tau = -|\rho_R^\tau|^{-\frac{1}{6}} \cdot |\rho_R^\tau|^{\frac{1}{6}} u_R^\tau \cdot \partial_x \rho_R^\tau - \rho_R^\tau \cdot \partial_x u_R^\tau,$$

give

$$\mathbb{E} \|f_{1,R}^\tau\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_\varepsilon(U_0).$$

Notice that for example, a bound for the term $\rho_R^\tau \cdot \partial_x u_R^\tau$ in L^p is given by $\|\rho_R^\tau\|_{L^m} \|\partial_x u_R^\tau\|_{L^q}$ with $q = 3 - \varepsilon$, $m = (3 - \varepsilon)(3 - 2\varepsilon)/\varepsilon$, and the obtained space is L^p with $p = 3 - 2\varepsilon$ which is also denoting by $L^{3-\varepsilon}$. By (B.7) and (B.4b), it follows that

$$\mathbb{E} \|\partial_x \rho_R^\tau\|_{L^{m_\varepsilon}(Q_T)}^{m_\varepsilon} \leq C_R M_\varepsilon(U_0). \quad (3.40)$$

Eventually, it follows from the decomposition

$$\begin{aligned} f_{2,R}^\tau &= -\frac{1}{|\rho_R^\tau|^{1/(2m)}} \cdot |\rho_R^\tau|^{1/(2m)} u_R^\tau \cdot \partial_x u_R^\tau - \kappa \gamma |\rho_R^\tau|^{\gamma-2} \cdot \partial_x \rho_R^\tau \\ &\quad + 2 \frac{1}{|\rho_R^\tau|} \cdot \partial_x \rho_R^\tau \cdot \partial_x u_R^\tau, \end{aligned}$$

from Remark 3.10 and from estimates (3.18), (3.28), (3.39) and (3.40) that

$$\mathbb{E} \|f_{2,R}^\tau\|_{L^{3-\varepsilon}(Q_T)}^{3-\varepsilon} \leq C_R M_\varepsilon(U_0),$$

which gives by (B.4a) the estimate

$$\mathbb{E} \|\partial_x u_{\text{det},R}^\tau\|_{L^{m_\varepsilon}(Q_T)}^{m_\varepsilon} \leq C_R M_\varepsilon(U_0). \quad (3.41)$$

We also apply Lemma 3.12 to the martingale part in (3.33). Let $\bar{m} > 3$ be fixed (quite large). If $2 \leq m_\varepsilon \leq \bar{m}$, we have

$$\mathbb{E} \|\partial_x u_{\text{sto},R}^\tau(t)\|_{L^{m_\varepsilon}(\mathbb{T}^1)}^{m_\varepsilon} \leq \tilde{C} T \mathbb{E} \|\partial_x u_R^\tau\|_{L^{m_\varepsilon}(Q_{t_\sharp})}^{m_\varepsilon}, \quad (3.42)$$

where $\tilde{C} = \sup_{2 \leq q \leq \bar{m}} C_{\text{BDG}}(q)$. By Gronwall Lemma it follows that

$$\mathbb{E} \|\partial_x u_R^\tau\|_{L^{m_\varepsilon}(Q_T)}^{m_\varepsilon} \leq C_R M_\varepsilon(U_0). \quad (3.43)$$

Now the ‘‘near L^∞ ’’ estimates (3.40), (3.43) together with the positivity estimate (3.28) give $f_{2,R}^\tau \in L^p(Q_T)$ where p is arbitrary. We use the estimate

$$\left\| |D|^{3/2} \int_0^{t_\sharp} S(t_\sharp - s) f(s) ds \right\|_{L^q(Q_T)} \leq C \|f\|_{L^p(Q_T)} \quad \text{if } \frac{1}{q} > \frac{1}{p} - \frac{1}{6}, \quad (3.44)$$

which can be derived similarly to (B.7) by using the estimate

$$\| |D|^{3/2} S(t) \|_{L_x^p \rightarrow L_x^q} \leq C t^{-\frac{1}{2} \left(\frac{1}{p} - \frac{1}{q} \right) - \frac{3}{4}}. \quad (3.45)$$

Here, $|D|$ denotes the square-root of the Laplacian: $D = (-\partial_x^2)^{1/2}$. We also deduce from (3.45) that

$$\| |D|^{3/2} S(t_\sharp) u_0 \|_{L^p(Q_T)} \leq C \|u_0\|_{L^p(Q_T)}.$$

Therefore (3.33) and (3.44) give

$$\mathbb{E} \| |D|^{3/2} (u_{0,R}^\tau + u_{\text{det},R}^\tau) \|_{L^p(Q_T)}^p = \mathcal{O}(1). \quad (3.46)$$

By (2.3) and by interpolation, we have

$$\| \mathbf{H} \|_{L^p(Q_T)} \leq C_R (1 + \| |D|^{3/2} u_R^\tau \|_{L^p(Q_T)}),$$

where \mathbf{H} is defined by (3.36) with $h_k = |D|^{3/2} (h_{\text{sto}}(t) \sigma_k(\mathbf{U}_R^\tau(t))) \mathbf{1}_{t < T_R}$. By Lemma 3.12, we deduce that

$$\mathbb{E} \| |D|^{3/2} u_{\text{sto},R}^\tau \|_{L^p(\mathbb{T}^1)}^p \leq C (1 + \mathbb{E} \| |D|^{3/2} u_R^\tau \|_{L^p(Q_T)}^p).$$

By the Gronwall Lemma and (3.46) it follows that $\mathbb{E} \| u_R^\tau \|_{L^p(0,T;W^{3/2,p}(\mathbb{T}^1))} = \mathcal{O}(1)$. To obtain time estimates on u_R^τ now, we use the Kolmogorov criterion and the a priori bounds of Section 3.1.2. Let $p > 2$. Set

$$h_k(s) = h_{\text{sto}}(s) \rho_R^\tau(s) \sigma_k(\mathbf{U}_R^\tau(s)).$$

By the Burkholder-Davis-Gundy inequality in the 2-smooth Banach space $L^p(\mathbb{T}^1)$ we have, for $0 \leq t \leq t' \leq T$,

$$\begin{aligned} & \mathbb{E} \left\| \sum_{k \geq 1} \int_{t_\sharp}^{t'_\sharp} S(t'_\sharp - s_\sharp) h_k(s) d\beta_k(s) \right\|_{L^p(\mathbb{T}^1)}^p \\ & \leq C_{\text{BDG}}(p) \mathbb{E} \left(\int_{t_\sharp}^{t'_\sharp} \| S(t'_\sharp - s_\sharp) \mathbf{H}(s) \|_{L^2(\mathbb{T}^1)}^2 dx ds \right)^{p/2}, \end{aligned}$$

where \mathbf{H} is defined as in (3.36). Since $S(t)$ is a contraction from $L^2(\mathbb{T}^1)$ into $L^2(\mathbb{T}^1)$, (2.2), Remark 3.10 and the moment estimate (3.18) show that

$$\begin{aligned} E \left\| \sum_{k \geq 1} \int_{t_{\#}}^{t'_{\#}} S(t'_{\#} - s_{\#}) h_k(s) d\beta_k(s) \right\|_{L^p(\mathbb{T}^1)}^p &\leq C_R M_{\varepsilon}(\mathbf{U}_0) |t'_{\#} - t_{\#}|^{p/2} \\ &\leq C_R M_{\varepsilon}(\mathbf{U}_0) |t' - t|^{p/2}. \end{aligned}$$

The Kolmogorov continuity Theorem ([DPZ92], Theorem 3.3) therefore gives

$$\mathbb{E} \|u_{\text{sto}, R}^{\tau}\|_{C^{\alpha}([0, T]; L^p(\mathbb{T}^1))} = \mathcal{O}(1),$$

where $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. We can use a similar argument to deal with the term $u_{\text{det}, R}^{\tau}$: we have

$$\mathbb{E} \left\| \int_{t_{\#}}^{t'_{\#}} S(t'_{\#} - s_{\#}) f_{2, R}^{\tau}(s) d\beta_k(s) \right\|_{L^p(\mathbb{T}^1)}^p \leq C_R M_{\varepsilon}(\mathbf{U}_0) |t'_{\#} - t_{\#}|^{p-1},$$

and thus

$$\mathbb{E} \|u_{\text{det}, R}^{\tau}\|_{C^{\alpha}([0, T]; L^p(\mathbb{T}^1))} = \mathcal{O}(1),$$

where $0 < \alpha < 1 - \frac{2}{p}$. By the identity

$$S(t'_{\#})u_0 - S(t_{\#})u_0 = \int_{t_{\#}}^{t'_{\#}} \partial_x S(t) \partial_x u_0 dt$$

and (B.3), we obtain also $\|u_{0, R}^{\tau}\|_{C^{1/2}([0, T]; L^p(\mathbb{T}^1))} = \mathcal{O}(1)$. This concludes the proof of (3.30). The proof of (3.29) is similar and simpler since $\rho_{\text{sto}, R}^{\tau} = 0$.

■

3.1.6 Compactness argument

For $k \in \mathbb{N}^*$, we introduce the independent processes

$$X_k(t) = \sqrt{2} \int_0^t h_{\text{sto}}(s) d\beta_k(s)$$

and set $W^{\tau}(t) = \sum_{k \geq 1} X_k(t) e_k$. Then $X_k(t) - \beta_k(t)$ is a centred Gaussian process with variance

$$\int_0^t |\sqrt{2} h_{\text{sto}}(s) - 1|^2 ds = \mathcal{O}(\tau),$$

and therefore, if $\eta > 0$

$$\mathbb{P}(\|W^{\tau}(t) - W(t)\|_{\mathfrak{U}_0} > \eta) \leq \sup_{k \geq 1} \mathbb{P}(|X_k(t) - \beta_k(t)| > \frac{\sqrt{6}}{\pi} \eta) \leq C \frac{\tau}{\eta}. \quad (3.47)$$

Let us fix $p > 2$. Define the path space $\mathcal{X} = \mathcal{X}_{\mathbf{U}} \times \mathcal{X}_W$, where

$$\mathcal{X}_{\mathbf{U}} = [C([0, T]; W^{1,p}(\mathbb{T}^1))]^2, \quad \mathcal{X}_W = C([0, T]; \mathfrak{U}_0).$$

Let us denote by $\mu_{\mathbf{U}}^\tau$ the law of \mathbf{U}^τ on $\mathcal{X}_{\mathbf{U}}$ and by μ_{W^τ} the law of W^τ on \mathcal{X}_W . Their joint law on \mathcal{X} is then denoted by μ^τ .

Proposition 3.13. *The set $\{\mu^\tau; \tau \in (0, 1)\}$ is tight and therefore relatively weakly compact in \mathcal{X} .*

Proof. First, we prove tightness of $\{\mu_{\mathbf{U}}^\tau; \tau \in (0, 1)\}$ in $\mathcal{X}_{\mathbf{U}}$. Let $\alpha \in (0, 1/2)$ and $M > 0$. Then

$$K_M := \{\mathbf{U} \in \mathcal{X}_{\mathbf{U}}; \|\mathbf{U}\|_{C^\alpha([0, T]; L^p(\mathbb{T}^1))} + \|\mathbf{U}\|_{L^p([0, T]; W^{3/2, p}(\mathbb{T}^1))} \leq M\}$$

is compact in $\mathcal{X}_{\mathbf{U}}$. Recall that the stopping time T_R is defined by (3.27). By the Markov inequality and the entropy estimate (3.9), we have

$$\mathbb{P}(T_R < T) \leq \frac{C}{R},$$

where the constant C depends on γ, T, D_0 and on the moments of \mathbf{U}_0 and $\partial_x \mathbf{U}_0$ up to a given order. We have therefore, by Proposition 3.11 and the Markov inequality,

$$\mathbb{P}(\mathbf{U}^\tau \notin K_M) \leq \frac{C}{R} + \mathbb{P}(\mathbf{U}_R^\tau \notin K_M) \leq \frac{C}{R} + \frac{C_R}{M},$$

by possibly augmenting the constant C . Therefore, given $\eta > 0$ there exists $R, M > 0$ such that

$$\mu_{\mathbf{U}}^\tau(K_M) \geq 1 - \eta.$$

Besides, the law μ_{W^τ} is tight by (3.47). Consequently the set of the joint laws $\{\mu^\tau; \tau \in (0, 1)\}$ is tight. By Prokhorov's theorem, it is relatively weakly compact. ■

Let now (τ_n) be a sequence decreasing to 0. For simplicity we will keep the notation τ for (τ_n) and eventual subsequences. Let μ be an adherence value, for the weak convergence, of μ^τ . By the Skorokhod Theorem we can assume a.e. convergence of the random variables by changing the probability space.

Proposition 3.14. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with a sequence of \mathcal{X} -valued random variables $(\tilde{\mathbf{U}}^\tau, \tilde{W}^\tau)$, $\tau \in (0, 1)$, and $(\tilde{\mathbf{U}}, \tilde{W})$ such that*

1. *the laws of $(\tilde{\mathbf{U}}^\tau, \tilde{W}^\tau)$ and $(\tilde{\mathbf{U}}, \tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with μ^τ and μ respectively,*
2. *$(\tilde{\mathbf{U}}^\tau, \tilde{W}^\tau)$ converges $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{\mathbf{U}}, \tilde{W})$ in the topology of \mathcal{X} .*

3.1.7 Identification of the limit

Our aim in this section is to identify the limit $(\tilde{\mathbf{U}}, \tilde{W})$ given by Proposition 3.14. Let $(\tilde{\mathcal{F}}_t)$ be the $\tilde{\mathbb{P}}$ -augmented canonical filtration of the process $(\tilde{\mathbf{U}}, \tilde{W})$, i.e.

$$\tilde{\mathcal{F}}_t = \sigma(\sigma(\varrho_t \tilde{\mathbf{U}}, \varrho_t \tilde{W}) \cup \{N \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(N) = 0\}), \quad t \in [0, T],$$

where ϱ_t is the operator of restriction to the interval $[0, t]$ defined as follows: if E is a Banach space and $t \in [0, T]$, then

$$\begin{aligned} \varrho_t : C([0, T]; E) &\longrightarrow C([0, t]; E) \\ k &\longmapsto k|_{[0, t]}. \end{aligned} \tag{3.48}$$

Clearly, ϱ_t is a continuous mapping. In this section we will prove the following result, where $\tilde{\mathbf{U}}^\varepsilon$ denotes the solution with the scaling ε for which we have the same construction of the objects.

Proposition 3.15 (Martingale solution to (3.1)). *The sextuplet*

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}}^\varepsilon)$$

is a weak martingale solution to (3.1).

The proof of Proposition 3.15 uses a method of construction of martingale solutions of SPDEs that avoids in part the use of representation Theorem. This technique has been developed in Ondreját [Ond10], Brzeźniak, Ondreját [BO11] and used in particular in Hofmanová, Seidler [HS12] and in [Hof13b, DHV13].

Recall that \mathbf{A} (the flux of Equation (1.1)) is defined by (2.4). Let us define for all $t \in [0, T]$ and a test function $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathbb{T}^1; \mathbb{R}^2)$,

$$\begin{aligned} M^\tau(t) &= \langle \mathbf{U}^\tau(t), \varphi \rangle - \langle \mathbf{U}_0, \varphi \rangle + 2 \int_0^t h_{\det}(s) \langle \partial_x \mathbf{A}(\mathbf{U}^\tau) - \partial_x^2 \mathbf{U}^\tau, \varphi \rangle ds, \\ \tilde{M}^\tau(t) &= \langle \tilde{\mathbf{U}}^\tau(t), \varphi \rangle - \langle \tilde{\mathbf{U}}_0, \varphi \rangle + 2 \int_0^t h_{\det}(s) \langle \partial_x \mathbf{A}(\tilde{\mathbf{U}}^\tau) - \partial_x^2 \tilde{\mathbf{U}}^\tau, \varphi \rangle ds, \\ \tilde{M}(t) &= \langle \tilde{\mathbf{U}}(t), \varphi \rangle - \langle \tilde{\mathbf{U}}_0, \varphi \rangle + \int_0^t \langle \partial_x \mathbf{A}(\tilde{\mathbf{U}}) - \partial_x^2 \tilde{\mathbf{U}}, \varphi \rangle ds. \end{aligned}$$

In what follows, we fix some times $s, t \in [0, T]$, $s \leq t$, and a continuous function

$$\gamma : C([0, s]; L^p(\mathbb{T}^1)) \times C([0, s]; \mathfrak{U}_0) \longrightarrow [0, 1].$$

The proof of Proposition 3.15 will be a consequence of the following two lemmas.

Lemma 3.16. *The process \tilde{W} is a $(\tilde{\mathcal{F}}_t)$ -cylindrical Wiener process, i.e. there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t)$ -Wiener processes $\{\tilde{\beta}_k\}_{k \geq 1}$ such that $\tilde{W} = \sum_{k \geq 1} \tilde{\beta}_k e_k$.*

Proof. By (3.47), \tilde{W} is a \mathfrak{U}_0 -valued cylindrical Wiener process and is $(\tilde{\mathcal{F}}_t)$ -adapted. According to the Lévy martingale characterization theorem, it remains to show that it is also a $(\tilde{\mathcal{F}}_t)$ -martingale. We have

$$\tilde{\mathcal{F}} \gamma(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau) [\tilde{W}^\tau(t) - \tilde{W}^\tau(s)] = \mathcal{F} \gamma(\varrho_s \mathbf{U}^\tau, \varrho_s W^\tau) [W^\tau(t) - W^\tau(s)] = 0$$

since W^τ is a martingale and the laws of $(\tilde{\mathbf{U}}^\tau, \tilde{W}^\tau)$ and $(\mathbf{U}^\tau, W^\tau)$ coincide. Next, the uniform estimate

$$\sup_{\tau \in (0,1)} \tilde{\mathcal{F}} \|\tilde{W}^\tau(t)\|_{\mathfrak{U}_0}^2 = \sup_{\tau \in (0,1)} \mathcal{F} \|W^\tau(t)\|_{\mathfrak{U}_0}^2 < \infty$$

and the Vitali convergence theorem yield

$$\tilde{\mathcal{F}} \gamma(\varrho_s \tilde{\mathbf{U}}, \varrho_s \tilde{W}) [\tilde{W}(t) - \tilde{W}(s)] = 0$$

which finishes the proof. ■

Lemma 3.17. *The processes*

$$\tilde{M}, \quad \tilde{M}^2 - \sum_{k \geq 1} \int_0^\cdot \langle \psi_k(\tilde{\mathbf{U}}), \varphi \rangle^2 dr, \quad \tilde{M} \tilde{\beta}_k - \int_0^\cdot \langle \psi_k(\tilde{\mathbf{U}}), \varphi \rangle dr$$

are $(\tilde{\mathcal{F}}_t)$ -martingales.

Proof. Here, we use the same approach as in the previous lemma. Let us denote by \tilde{X}_k^τ , $k \geq 1$ the real-valued Wiener processes corresponding to \tilde{W}^τ , that is $\tilde{W}^\tau = \sum_{k \geq 1} \tilde{X}_k^\tau e_k$. For all $\tau \in (0, 1)$, the process

$$M^\tau = \sum_{k \geq 1} \int_0^\cdot \langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \rangle dX_k^\tau(r)$$

is a square integrable (\mathcal{F}_t) -martingale and therefore

$$(M^\tau)^2 - \sum_{k \geq 1} \int_0^\cdot \langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \rangle^2 d\langle\langle X^\tau \rangle\rangle(r),$$

and

$$M^\tau \beta_k - \int_0^\cdot \langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \rangle d\langle\langle X^\tau \rangle\rangle(r)$$

are (\mathcal{F}_t) -martingales, where we have denoted by

$$\langle\langle X^\tau \rangle\rangle(t) = 2 \int_0^t h_{\text{sto}}(r) dr$$

the quadratic variation of X_k^τ . Besides, it follows from the equality of laws that

$$\begin{aligned} & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau) [\tilde{M}^\tau(t) - \tilde{M}^\tau(s)] \\ &= \mathbb{E} \gamma(\varrho_s \mathbf{U}^\tau, \varrho_s W^\tau) [M^\tau(t) - M^\tau(s)] = 0, \end{aligned} \quad (3.49)$$

$$\begin{aligned} & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau) \left[(\tilde{M}^\tau)^2(t) - (\tilde{M}^\tau)^2(s) - \sum_{k \geq 1} \int_s^t \langle \sigma_k^\tau(\tilde{\mathbf{U}}^\tau), \varphi_2 \rangle^2 d\langle\langle \tilde{X}^\tau \rangle\rangle(r) \right] \\ &= \mathbb{E} \gamma(\varrho_s \mathbf{U}^\tau, \varrho_s W^\tau) \left[(M^\tau)^2(t) - (M^\tau)^2(s) \right. \\ & \quad \left. - \sum_{k \geq 1} \int_s^t \langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \rangle^2 d\langle\langle X^\tau \rangle\rangle(r) \right] \\ &= 0, \end{aligned} \quad (3.50)$$

$$\begin{aligned} & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau) \left[\tilde{M}^\tau(t) \tilde{X}_k^\tau(t) - \tilde{M}^\tau(s) \tilde{X}_k^\tau(s) - \int_s^t \langle \sigma_k^\tau(\tilde{\mathbf{U}}^\tau), \varphi_2 \rangle \langle\langle \tilde{X}^\tau \rangle\rangle(r) \right] \\ &= \mathbb{E} \gamma(\varrho_s \mathbf{U}^\tau, \varrho_s W^\tau) \left[M^\tau(t) X_k^\tau(t) - M^\tau(s) X_k^\tau(s) \right. \\ & \quad \left. - \int_s^t \langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \rangle d\langle\langle X^\tau \rangle\rangle(r) \right] \\ &= 0. \end{aligned} \quad (3.51)$$

We can pass to the limit in (3.49)-(3.50)-(3.51) due to the moment estimates (3.18) and the Vitali convergence theorem. We obtain

$$\begin{aligned} & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}, \varrho_s \tilde{W}) [\tilde{M}(t) - \tilde{M}(s)] = 0, \\ & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}, \varrho_s \tilde{W}) \left[\tilde{M}^2(t) - \tilde{M}^2(s) - \sum_{k \geq 1} \int_s^t \langle \sigma_k(\tilde{\mathbf{U}}), \varphi_2 \rangle^2 dr \right] = 0, \\ & \tilde{\mathbb{E}} \gamma(\varrho_s \tilde{\mathbf{U}}, \varrho_s \tilde{W}) \left[\tilde{M}(t) \tilde{\beta}_k(t) - \tilde{M}(s) \tilde{\beta}_k(s) - \int_s^t \langle \sigma_k(\tilde{\mathbf{U}}), \varphi_2 \rangle dr \right] = 0, \end{aligned}$$

which gives the $(\tilde{\mathcal{F}}_t)$ -martingale property. ■

Proof of Proposition 3.15. Once the above lemmas are established, we infer that

$$\left\langle\left\langle \tilde{M} - \sum_{k \geq 1} \int_0^\cdot \langle \sigma_k(\tilde{\mathbf{U}}) d\tilde{\beta}_k, \varphi_2 \rangle \right\rangle\right\rangle = 0,$$

where $\langle\langle \cdot \rangle\rangle$ denotes the quadratic variation process. Accordingly, we have

$$\begin{aligned} \langle \tilde{\mathbf{U}}(t), \varphi \rangle &= \langle \tilde{\mathbf{U}}_0, \varphi \rangle - \int_0^t \langle \partial_x \mathbf{A}(\tilde{\mathbf{U}}) - \partial_x^2 \tilde{\mathbf{U}}, \varphi \rangle ds \\ &\quad + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{\mathbf{U}}) d\tilde{\beta}_k, \varphi \rangle, \quad t \in [0, T], \quad \tilde{\mathbb{P}}\text{-a.s.}, \end{aligned}$$

and the proof is complete. ■

Proof of Theorem 3.2. We have just obtained the existence of a martingale solution

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}}^\varepsilon)$$

to (3.1). By passing to the limit on the approximation \mathbf{U}^ε , taking care to the occurrence of the factor ε now, we also obtain the various estimates of Theorem 3.2 and the Itô formula (3.5). ■

Remark 3.18. Let $m \in \mathbb{N}$. Let I be a Borel subset of $[0, T]$. By (3.16) we obtain the following estimate for $\tilde{\mathbf{U}}^\varepsilon$:

$$\mathbb{E} \left| \int_I \int_{\mathbb{T}^1} \eta''(\tilde{\mathbf{U}}^\varepsilon) \cdot (\tilde{\mathbf{U}}_x^\varepsilon, \tilde{\mathbf{U}}_x^\varepsilon) dx dt \right|^2 = \mathcal{O}(1)|I|, \quad (3.52)$$

where $|I|$ denotes the Lebesgue measure of I , where η is the entropy given by (2.8) corresponding to $g(\xi) = \xi^{2m}$, and $\mathcal{O}(1)$ depends on T , γ , on the constant D_0 in (2.2), on m and $\mathbb{E}\Gamma_{\eta^2}(\mathbf{U}_0^\varepsilon)$.

4 Probabilistic Young measures

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, \mathbf{U}^\varepsilon)$ be a martingale solution to (3.1). Our aim is to prove the convergence of \mathbf{U}^ε . The standard tool for this is the notion of measure-valued solution (introduced by Di Perna, [DiP83a]). In this section we give some precisions about it in our context of random solutions. More precisely, we know that, almost surely, (\mathbf{U}^ε) defines a Young measure ν^ε on $\mathbb{R}_+ \times \mathbb{R}$ by the formula

$$\langle \nu_{x,t}^\varepsilon, \varphi \rangle := \langle \delta_{\mathbf{U}^\varepsilon(x,t)}, \varphi \rangle = \varphi(\mathbf{U}^\varepsilon(x,t)), \quad \forall \varphi \in C_b(\mathbb{R}_+ \times \mathbb{R}). \quad (4.1)$$

Our aim is to show that $\nu^\varepsilon \rightharpoonup \nu$ (in the sense to be specified), where ν has some specific properties. To that purpose, we will use the probabilistic compensated compactness method developed in the Appendix of [FN08] and some results on the convergence of probabilistic Young measures that we introduce here.

4.1 Young measures embedded in a space of Probability measures

Let $(Q, \mathcal{A}, \lambda)$ be a finite measure space. Without loss of generality, we will assume $\lambda(Q) = 1$. A Young measure on Q (with state space E) is a measurable map $Q \rightarrow \mathcal{P}_1(E)$, where E is a topological space endowed with the σ -algebra of Borel sets, $\mathcal{P}_1(E)$ is the set of probability measures on E , itself endowed with the σ -algebra of Borel sets corresponding to the topology defined by the weak² convergence of measures, *i.e.* $\mu_n \rightarrow \mu$ in $\mathcal{P}_1(E)$ if

$$\langle \mu_n, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle, \quad \forall \varphi \in C_b(E).$$

As in (4.1), any measurable map $w: Q \rightarrow E$ can be viewed as a Young measure ν defined by

$$\langle \nu_z, \varphi \rangle = \langle \delta_{w(z)}, \varphi \rangle = \varphi(w(z)), \quad \forall \varphi \in C_b(E), \quad \forall z \in Q.$$

A Young measure ν on Q can itself be seen as a probability measure on $Q \times E$ defined by

$$\langle \nu, \psi \rangle = \int_Q \int_E \psi(p, z) d\nu_z(p) d\lambda(z), \quad \forall \psi \in C_b(E \times Q).$$

We then have, for all $\psi \in C_b(Q)$ (ψ independent on $p \in E$), $\langle \nu, \psi \rangle = \langle \lambda, \psi \rangle$, that is to say

$$\pi_* \nu = \lambda, \tag{4.2}$$

where π is the projection $E \times Q \rightarrow Q$ and the push-forward of ν by π is defined by $\pi_* \nu(A) = \nu(\pi^{-1}(A))$, for all Borel subset A of Q . Assume now that Q is a subset of \mathbb{R}^s and E is a closed subset of \mathbb{R}^m , $m, s \in \mathbb{N}^*$, and, conversely, let μ is a probability measure on $E \times Q$ such that $\pi_* \mu = \lambda$: by the Slicing Theorem (*cf.* Attouch, Buttazzo, Michaille [ABM06, Theorem 4.2.4]), we have: for λ -a.e. $z \in Q$, there exists $\mu_z \in \mathcal{P}_1(E)$ such that,

$$z \mapsto \langle \mu_z, \varphi \rangle$$

is measurable from Q to \mathbb{R} for every $\varphi \in C_b(Q)$, and

$$\langle \mu, \psi \rangle = \int_Q \int_E \psi(p, z) d\mu_z(p) d\lambda(z),$$

for all $\psi \in C_b(E \times Q)$, which precisely means that μ is a Young measure on Q . We therefore denote by

$$\mathcal{Y} = \{ \nu \in \mathcal{P}_1(E \times Q); \pi_* \nu = \lambda \}$$

²actually, weak convergence of probability measures, also corresponding to the tight convergence of finite measures

the set of Young measures on Q .

We use now the Prohorov's Theorem, *cf.* Billingsley [Bil99], to give a compactness criteria in \mathcal{Y} . We assume that Q is a compact subset of \mathbb{R}^s and E is a closed subset of \mathbb{R}^m . We also assume that the σ -algebra \mathcal{A} of Q is the σ -algebra of Borel sets of Q .

Proposition 4.1 (Bound against a Lyapunov functional). *Let $\eta: E \rightarrow \mathbb{R}_+$ satisfy the growth condition*

$$\lim_{p \in E, |p| \rightarrow +\infty} \eta(p) = +\infty.$$

Let $C > 0$ be a positive constant. Then the set

$$K_C = \left\{ \nu \in \mathcal{Y}; \int_{Q \times E} \eta(p) d\nu(z, p) \leq C \right\} \quad (4.3)$$

is a compact subset of \mathcal{Y} .

Proof. The condition $\pi_* \nu = \lambda$ being stable by weak convergence, \mathcal{Y} is closed in $\mathcal{P}_1(E \times Q)$. By Prohorov's Theorem, [Bil99], K_C is relatively compact in \mathcal{Y} if, and only if it is tight. Besides, K_C is closed since

$$\int_{Q \times E} \eta(p) d\nu(z, p) \leq \liminf_{n \rightarrow +\infty} \int_{Q \times E} \eta(p) d\nu_n(z, p)$$

if (ν_n) converges weakly to ν . It is therefore sufficient to prove that K_C is tight, which is classical: let $\varepsilon > 0$. For $R \geq 0$, let

$$V(R) = \inf_{|p| > R} \eta(p).$$

Then $V(R) \rightarrow +\infty$ as $R \rightarrow +\infty$ by hypothesis and, setting $M_R = [\overline{B}(0, R) \cap E] \times Q$, we have

$$V(R) \nu(M_R^c) \leq \int_{Q \times E} \eta(p) d\nu(z, p) \leq C,$$

for all $\nu \in K$, whence $\sup_{\nu \in K} \nu(M_R^c) < \varepsilon$ for R large enough. ■

4.2 A compactness criterion for probabilistic Young measures

As above, we assume that Q is a compact subset of \mathbb{R}^s and E is a closed subset of \mathbb{R}^m .

Definition 4.2. *A random Young measure is a \mathcal{Y} -valued random variable.*

Proposition 4.3. *Let $\eta: E \rightarrow \mathbb{R}_+$ satisfy the growth condition*

$$\lim_{p \in E, |p| \rightarrow +\infty} \eta(p) = +\infty.$$

Let $M > 0$ be a positive constant. If (ν_n) is a sequence of random Young measures on Q satisfying the bound

$$\mathbb{E} \int_{Q \times E} \eta(p) d\nu_n(z, p) \leq M,$$

then, up to a subsequence, (ν_n) is converging in law.

Proof. We endow $\mathcal{P}_1(E \times Q)$ with the Prohorov's metric d . Then $\mathcal{P}_1(E \times Q)$ is a complete, separable metric space, weak convergence coincides with d -convergence, and a subset A is relatively compact if, and only if it is tight, [Bil99]. Let $\mathcal{L}(\nu_n) \in \mathcal{P}_1(\mathcal{Y})$ denote the law of ν_n . To prove that it is tight, we use the Prohorov's Theorem. Let $\varepsilon > 0$. For $C > 0$, let K_C be the compact set defined by (4.3). For $\nu \in \mathcal{Y}$, we have

$$\mathbb{P}(\nu \notin K_C) = \mathbb{P}\left(1 < \frac{1}{C} \int_{Q \times E} \eta(p) d\nu(z, p)\right) \leq \frac{1}{C} \mathbb{E} \int_{Q \times E} \eta(p) d\nu(z, p),$$

hence

$$\sup_{n \in \mathbb{N}} \mathcal{L}(\nu_n)(\mathcal{Y} \setminus K_C) = \sup_{n \in \mathbb{N}} \mathbb{P}(\nu_n \notin K_C) \leq \frac{M}{C} < \varepsilon,$$

for C large enough, which proves the result. ■

Let $\mathcal{X}_W = C([0, T]; \mathfrak{U}_0)$ be the path space for the Wiener process W . If (ν_n) is as in Proposition 4.3, then the law of (ν_n, W) is tight in $\mathcal{Y} \times \mathcal{X}_W$ since the law of W is tight in \mathcal{X}_W . We obtain the following extension:

Proposition 4.4. *Under the hypothesis of Proposition 4.3, there exists a subsequence of (ν_n, W) converging in law on $\mathcal{Y} \times \mathcal{X}_W$.*

Remark 4.5 (Almost-sure convergence). *Assume that (ν_n, W) is converging in law on $\mathcal{Y} \times \mathcal{X}_W$. Then we can apply the Skorokhod Theorem [Bil99] to (ν_n, W) : there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, some random Young measures $\tilde{\nu}_n, \tilde{\nu}: \tilde{\Omega} \rightarrow \mathcal{Y}$, some random variable $\tilde{W}: \tilde{\Omega} \rightarrow \mathcal{X}_W$, such that $\text{Law}(\tilde{\nu}_n) = \text{Law}(\nu_n)$, $\text{Law}(\tilde{\nu}) = \text{Law}(\nu)$, $\text{Law}(\tilde{W}) = \text{Law}(W)$ and $\tilde{\nu}_n \rightarrow \tilde{\nu}$ a.s. in $\mathcal{P}_1(E \times Q)$. Let $q > 1$. If ν_n is issued from a sequence of random variables bounded in $L^q(Q; E)$ (as this will be the case in our context), say*

$$\nu_n = \delta_{u_n}, \quad u_n \in L^q(Q; E), \quad (4.4)$$

then $\tilde{\nu}_n$ is itself a Dirac mass.

The proof of the above statement is as follows: using the Jensen's Formula, (4.4) can be characterized by

$$\mathbb{E} \int_{Q \times E} \psi(p) d\nu_{n,z}(p) d\lambda(z) = \mathbb{E} \int_Q \psi \left(\int_E p d\nu_{n,z}(p) \right) d\lambda(z),$$

where $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is a strictly convex function satisfying the growth condition $|\psi(p)| \leq C(1 + |p|^q)$. In other words,

$$\mathbb{E}\varphi(\nu_n) = \mathbb{E}\theta(\nu_n), \quad (4.5)$$

where

$$\varphi: \mu \mapsto \int_{Q \times E} \psi(p) d\mu_z(p) d\lambda(z), \quad \theta: \mu \mapsto \int_Q \psi \left(\int_E p d\mu_z(p) \right) d\lambda(z).$$

Of course, φ is continuous on \mathcal{Y} and, by the Lebesgue dominated convergence theorem, θ is continuous on the subset

$$\left\{ \mu \in \mathcal{Y}; \int_{Q \times E} |p|^q d\mu_z(p) d\lambda(z) \leq R \right\}.$$

Consequently, by identity of the laws, (4.5) is satisfied by $\tilde{\nu}_n$, *i.e.*

$$\tilde{\nu}_n = \delta_{\tilde{u}_n}, \quad \tilde{u}_n(z) := \int_E p d\tilde{\nu}_{n,z}$$

almost surely.

4.3 Convergence to a random Young measure

We apply the results of paragraphs 4.1-4.2 to the case $\nu^\varepsilon := \delta_{\mathbf{U}^\varepsilon}$, $Q = Q_T$, $E = \mathbb{R}_+ \times \mathbb{R}$. Strictly speaking we will consider a sequence (ε_n) decreasing to 0 and will obtain some limits along some subsequences of (ε_n) . However, for the simplicity of notations we will keep the notation ν^ε instead of ν^{ε_n} in what follows.

Proposition 4.6. *Let $\mathbf{U}_0^\varepsilon \in W^{1,\infty}(\mathbb{T}^1)$ be bounded in $L^\infty(\mathbb{T}^1)$. Let \mathbf{U}^ε be a martingale solution to (3.1) satisfying (3.2) and let $\nu^\varepsilon := \delta_{\mathbf{U}^\varepsilon}$ be the associated random Young measure. Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, some random variables $(\tilde{\nu}^\varepsilon, \tilde{W})$ and $(\tilde{\nu}, \tilde{W})$ with values in $\mathcal{Y} \times \mathcal{X}_W$ such that*

1. *the law of $(\tilde{\nu}^\varepsilon, \tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with the law of (ν^ε, W) ,*
2. *$(\tilde{\nu}^\varepsilon, \tilde{W})$ converges $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{\nu}, \tilde{W})$ in the topology of $\mathcal{Y} \times \mathcal{X}_W$.*

Proof. we apply the Proposition 4.4 to (ν^ε, W) taking for η the energy, *i.e.* the entropy given by (2.8) with $g(\xi) = |\xi|^2$ (which is infinite at infinity by (3.12)). We use the estimate (3.2) for such an η . ■

5 Reduction of the Young measure

Proposition 4.6 above gives the existence of a random young measure ν such that $\nu^\varepsilon := \delta_{\mathbf{U}^\varepsilon}$ converges in law in the sense of Young measures to ν . We will now apply the compensated compactness method to prove that a.s., for a.e. $(x, t) \in Q_T$, either $\nu_{x,t}$ is a Dirac mass or $\nu_{x,t}$ is concentrated on the vacuum region $\{\rho = 0\}$. To do this, we will use the probabilistic compensated compactness method of [FN08] to obtain a set of functional equations satisfied by ν . Then we conclude by adapting the arguments of [LPS96].

5.1 Compensated compactness

In this section, we fix an entropy - entropy flux couple (η, H) of the form (2.8)-(2.10) where $g \in C^2(\mathbb{R})$ is convex, with g sub-quadratic and g' sub-linear.

5.1.1 Murat's Lemma

Proposition 5.1. *Assume $1 < \gamma \leq 2$. Then the sequence of $H^{-1}(Q_T)$ random variables $(\varepsilon\eta(\mathbf{U}^\varepsilon)_{xx})$ is tight.*

Proof. It is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E} \|\eta(\mathbf{U}^\varepsilon)_x\|_{L^2(Q_T)}^2 = 0. \quad (5.1)$$

Indeed, we then have $\varepsilon\eta(\mathbf{U}^\varepsilon)_{xx} \rightarrow 0$ in law on $H^{-1}(Q_T)$ and $(\varepsilon\eta(\mathbf{U}^\varepsilon)_{xx})$ is tight since $H^{-1}(Q_T)$ is separable and complete. The convergence (5.1) follows from the gradient estimates (3.3), (3.4). Indeed, g and g' being sub-polynomial, there exists $m \geq 0$ such that

$$|\partial_\rho \eta(\mathbf{U})|^2 \leq C \left(1 + |u|^m + \rho^{m(\gamma-1)}\right), \quad (5.2)$$

and

$$|\partial_u \eta(\mathbf{U})|^2 \leq C \rho^2 \left(1 + |u|^m + \rho^{m(\gamma-1)}\right),$$

for a given constant $C \geq 0$. The estimates (3.3), (3.4) show that all terms in

$$\varepsilon \mathbb{E} \iint_{Q_T} |\partial_\rho \eta(\mathbf{U}^\varepsilon)|^2 |\partial_x \rho^\varepsilon|^2 + |\partial_u \eta(\mathbf{U}^\varepsilon)|^2 |\partial_x u^\varepsilon|^2 dx dt$$

are bounded, except possibly the contribution

$$\varepsilon \mathbb{E} \iint_{Q_T} (1 + |u^\varepsilon|^m) |\partial_x \rho^\varepsilon|^2 dx dt$$

coming from the first terms in the right-hand side of (5.2). For this term we use the estimate

$$|u|^m \leq |u|^{2m} \rho^{\gamma-2} + \rho^{2-\gamma}.$$

Then, by (3.3), $\varepsilon \mathbb{E} \|\eta(\mathbf{U}^\varepsilon)_x\|_{L^2(Q_T)}^2 = \mathcal{O}(1)$ if $2 - \gamma \geq \gamma - 2$, that is to say $\gamma \leq 2$. ■

The next Proposition is similar to Lemma 4.20 in [FN08].

Proposition 5.2. *Let*

$$M^\varepsilon(t) = \int_0^t \partial_q \eta(\mathbf{U}^\varepsilon)(s) \Phi(\mathbf{U}^\varepsilon)(s) dW(s).$$

Then $\partial_t M^\varepsilon$ is tight in $H^{-1}(Q_T)$.

Proof. The proof is in essential the proof of Lemma 4.19 in [FN08]. However, we will proceed slightly differently (instead of using Marchaud fractional derivative we work directly with fractional Sobolev spaces and an Aubin-Simon compactness lemma). Let $m \geq 2$. For $0 \leq s \leq t \leq T$, we have the following estimates (where the constant C may vary from line to line and is always independent on ε): by a martingale inequality,

$$\mathbb{E} \|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(\mathbb{T}^1)}^m \leq C \int_{\mathbb{T}^1} \mathbb{E} \left| \int_s^t |\partial_q \eta(\mathbf{U}^\varepsilon)|^2 \mathbf{G}^2(\mathbf{U}^\varepsilon) d\sigma \right|^{m/2} dx. \quad (5.3)$$

By (2.2) and since g, g' are sub-polynomial, we have $|\partial_q \eta(\mathbf{U}^\varepsilon)|^2 \mathbf{G}^2(\mathbf{U}^\varepsilon) \leq R(\rho^\varepsilon, |u^\varepsilon|) \rho^\varepsilon$, where R is a polynomial. In particular, by (3.2),

$$\sup_{\sigma \in [s, t]} \int_{\mathbb{T}^1} [|\partial_q \eta(\mathbf{U}^\varepsilon(x, \sigma))|^2 \mathbf{G}^2(\mathbf{U}^\varepsilon(x, \sigma))]^{m/2} dx \leq C.$$

Using Hölder Inequality and (5.3), we obtain

$$\begin{aligned} & \mathbb{E} \|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(\mathbb{T}^1)}^m \\ & \leq C |t - s|^{\frac{m}{2} - 1} \int_{\mathbb{T}^1} \mathbb{E} \int_s^t [|\partial_q \eta(\mathbf{U}^\varepsilon)|^2 \mathbf{G}^2(\mathbf{U}^\varepsilon)]^{m/2} d\sigma dx \leq C |t - s|^{\frac{m}{2}}, \end{aligned}$$

and, by integration with respect to t and s ,

$$\mathbb{E} \int_0^T \int_0^T \frac{\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^m(\mathbb{T}^1)}^m}{|t - s|^{1 + \nu m}} dt ds \leq C, \quad (5.4)$$

as soon as $\nu < 1/2$. The left-hand side in this inequality (5.4) is the norm of M^ε in $L^m(\Omega; W^{\nu, m}(0, T; L^m(\mathbb{T}^1)))$. Since $L^m(\mathbb{T}^1) \hookrightarrow H^{-1}(\mathbb{T}^1)$, it follows that

$$\mathbb{E} \|M^\varepsilon\|_{W^{\nu, m}(0, T; H^{-1}(\mathbb{T}^1))} \leq C. \quad (5.5)$$

Besides, by the growth inequalities (2.2) and the moment estimate (3.2), M^ε also satisfies the bound

$$\mathbb{E} \|M^\varepsilon\|_{L^2(Q_T)} \leq C. \quad (5.6)$$

Let us assume $m > 2$ now. We then have the continuous injection

$$W^{\nu,m}(0, T; H^{-1}(\mathbb{T}^1)) \hookrightarrow C^{0,\mu}([0, T]; H^{-1}(\mathbb{T}^1))$$

for every $0 < \mu < \nu - \frac{1}{m}$. By the Aubin-Simon compactness Lemma (Simon, [Sim87]), the set

$$A_R := \{M \in L^2(Q_T); \|M^\varepsilon\|_{W^{\nu,m}(0,T;H^{-1}(\mathbb{T}^1))} \leq R, \|M\|_{L^2(Q_T)} \leq R\}$$

is compact in $C([0, T]; H^{-1}(\mathbb{T}^1))$, hence compact in $L^2(0, T; H^{-1}(\mathbb{T}^1))$. Consequently (5.5) and (5.6) show that (M^ε) is tight as a $L^2(0, T; H^{-1}(\mathbb{T}^1))$ -random variable, and we conclude that $(\partial_t M^\varepsilon)$ is tight as a $H^{-1}(Q_T)$ -random variable. ■

5.1.2 Functional equation

Let us recall that a sequence of random variables (a_n) with values in a normed space X is said to be stochastically bounded if, for all $\eta > 0$, there exists $M > 0$ such that $\mathbb{P}(\|a_n\|_X \geq M) \leq \eta$ for all n . By the Markov inequality, $\mathbb{E}\|a_n\|_X \leq C$ implies (a_n) stochastically bounded.

We consider the Itô Formula (3.5). Let $p > 2$. By the moment estimates (3.2), which gives bounds on $\eta(\mathbf{U}^\varepsilon)$, $H(\mathbf{U}^\varepsilon)$ in $L^p(Q_T)$, the left-hand side of (3.5) is stochastically bounded in $W^{-1,p}(Q_T)$. By Proposition 5.1 and Proposition 5.2 above, the two terms $\varepsilon\eta(\mathbf{U}^\varepsilon)_{xx}$ and $\partial_t M^\varepsilon$ in the right-hand side of (3.5) are tight in $H^{-1}(Q_T)$. The two remaining terms

$$\varepsilon\eta''(\mathbf{U}^\varepsilon) \cdot (\mathbf{U}_x^\varepsilon, \mathbf{U}_x^\varepsilon) \quad \text{and} \quad \frac{1}{2}|\mathbf{G}(\mathbf{U}^\varepsilon)|^2\partial_{qq}^2\eta(\mathbf{U}^\varepsilon)$$

are stochastically bounded in measure on Q_T by (3.3)-(3.4) and (2.2)-(3.2) respectively. By the stochastic version of the Murat's Lemma, Lemma A.3 in [FN08], we deduce that the sequence of $H^{-1}(Q_T)$ random variables

$$\operatorname{div}_{t,x}(\eta(\mathbf{U}^\varepsilon), H(\mathbf{U}^\varepsilon)) = \eta(\mathbf{U}^\varepsilon)_t + H(\mathbf{U}^\varepsilon)_x$$

is tight. Let now $(\hat{\eta}, \hat{H})$ be a second entropy - entropy flux couple associated by (2.8)-(2.10) to a convex sub-polynomial C^2 function \hat{g} . Similarly, the sequence of random variables

$$\operatorname{curl}_{t,x}(-\hat{H}(\mathbf{U}^\varepsilon), \hat{\eta}(\mathbf{U}^\varepsilon)) = \hat{\eta}(\mathbf{U}^\varepsilon)_t + \hat{H}(\mathbf{U}^\varepsilon)_x$$

is tight. By Theorem A.2 in [FN08] and Proposition 4.6, we obtain, for a.e. $(x, t) \in Q_T$,

$$\langle \hat{\eta} \rangle \langle H \rangle - \langle \eta \rangle \langle \hat{H} \rangle = \langle \hat{\eta}H - \eta\hat{H} \rangle, \quad (5.7)$$

where $\langle \phi \rangle := \langle \phi, \nu_{x,t} \rangle$ and where the identity in (5.7) is the identity between the *laws* of the processes involved. To obtain an almost sure equality, we

apply Skorokhod Theorem: by Remark 4.5, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a new $C([0, T]; W^{1,2}(\mathbb{T}^1))$ random variable $\tilde{\mathbf{U}}^\varepsilon$, $\tilde{\nu}$ a random Young measure, such that $\text{Law}(\delta_{\tilde{\mathbf{U}}^\varepsilon}) = \text{Law}(\delta_{\mathbf{U}^\varepsilon})$, $\text{Law}(\nu) = \text{Law}(\tilde{\nu})$ and, a.s.,

$$\delta_{\tilde{\mathbf{U}}^\varepsilon} \rightarrow \tilde{\nu}.$$

Then, by a slight adaptation of the probabilistic div-curl Lemma, Theorem 2 in [FN08], we obtain (5.7) almost everywhere in $\tilde{\Omega}$, and with $\langle \phi \rangle := \langle \phi, \tilde{\nu}_{x,t} \rangle$.

5.2 Reduction of the Young measure

We now follow [LPS96] to conclude. We switch from the variables (ρ, u) or (ρ, q) to (w, z) , where

$$z = u - \rho^\theta, \quad w = u + \rho^\theta.$$

By (5.7), we have, a.s., for a.e. $(x, t) \in Q_T$,

$$\begin{aligned} & 2\lambda \left(\langle \chi(v)u \rangle \langle \chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v')u \rangle \right) \\ &= (v - v') \left(\langle \chi(v)\chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v') \rangle \right), \end{aligned} \quad (5.8)$$

where

$$\langle \chi(v) \rangle = \int \chi(w, z, v) d\tilde{\nu}_{x,t}(w, z), \quad \chi(w, z, v) := (v - z)_+^\lambda (w - v)_+^\lambda.$$

Let us fix (ω, x, t) such that (5.8) is satisfied. Let \mathcal{C} denote the set

$$\mathcal{C} = \{v \in \mathbb{R}; \langle \chi(v) \rangle > 0\} = \bigcup_{(w,z) \in \text{supp} \tilde{\nu}_{x,t}} \{v; z < v < w\}.$$

Let

$$V = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} | \rho = 0\} = \{(w, z) \in \mathbb{R}^2 | w = z\}$$

denote the vacuum region. If \mathcal{C} is empty, then $\tilde{\nu}_{x,t}$ is concentrated on V . Assume \mathcal{C} not empty. By Lemma I.2 in [LPS96] then, \mathcal{C} is an open interval in \mathbb{R} , say $\mathcal{C} =]a, b[$, where $-\infty \leq a < b \leq +\infty$ (we use here the french notation for open intervals to avoid the confusion with the point (a, b) of \mathbb{R}^2). Furthermore all the computations of [LPS96] apply here, and thus, as in Section I.6 of [LPS96], we obtain

$$\langle \rho^{2\lambda\theta} \langle \chi \circ \pi_i \rangle \phi \circ \pi_i \rangle = 0, \quad (5.9)$$

for any continuous function ϕ with compact support in \mathcal{C} , where $\pi_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the projection on the first coordinate w if $i = 1$, and the projection on the second coordinate z if $i = 2$. If we assume that there exists $Q \in \mathbb{R}^2$ satisfying

$$Q \in \text{supp}(\tilde{\nu}_{x,t}) \setminus V, \quad \pi_i(Q) \in \mathcal{C}, \quad (5.10)$$

for $i \in \{1, 2\}$, then there exists a neighbourhood K of Q such that $K \cap V = \emptyset$, $\nu_{x,t}(K) > 0$, $\pi_i(K) \subset \mathcal{C}$. But then $\langle \chi \circ \pi_i \rangle > 0$ on K , $\rho > 0$ on K and, choosing a continuous function ϕ compactly supported in \mathcal{C} such that $\phi > 0$ on K we obtain a contradiction to (5.9). Consequently (5.10) cannot be satisfied. This implies that there cannot exist *two distinct* points P, Q in $\text{supp}(\tilde{\nu}_{x,t}) \setminus V$. Indeed, if two such points exist, then either $\pi_1(Q) < \pi_1(P)$, and then Q satisfies (5.10) with $i = 1$, or $\pi_1(Q) = \pi_1(P)$ and, say, $\pi_2(P) < \pi_2(Q)$ and then Q also satisfies (5.10). The other cases are similar by symmetry of P and Q .

Therefore if $\mathcal{C} \neq \emptyset$, then the support of the restriction of $\tilde{\nu}_{x,t}$ to \mathcal{C} is reduced to a point. In particular, a and b are finite. Then, by Lemma I.2 in [LPS96], $P := (a, b) \in \text{supp}(\nu_{x,t})$ and $\tilde{\nu}_{x,t} = \tilde{\mu}_{x,t} + \alpha \delta_{\tilde{\mathbf{U}}(x,t)}$, where $\tilde{\mu}_{x,t} = \tilde{\nu}_{x,t}|_V$. Using (5.8), we obtain

$$0 = (v - v')\chi(b, a, v)\chi(b, a, v')(\alpha - \alpha^2),$$

and thus $\alpha = 0$ or 1 . We have therefore proved the following result.

Proposition 5.3 (Reduction of the Young measure). *Either $\tilde{\nu}_{x,t}$ is concentrated on the vacuum region V , or $\tilde{\nu}_{x,t}$ is reduced to a Dirac mass $\delta_{\tilde{\mathbf{U}}(x,t)}$.*

5.3 Martingale solution

By Proposition 5.3, we have

$$\iint_{Q_T} S(\tilde{\mathbf{U}}^\varepsilon(x, t))\varphi(x, t)dxdt \rightarrow \iint_{Q_T} S(\tilde{\mathbf{U}}(x, t))\varphi(x, t)dxdt \quad (5.11)$$

almost surely for every continuous and bounded function S on $\mathbb{R}_+ \times \mathbb{R}$ which vanishes on the vacuum region $\{0\} \times \mathbb{R}$ and every $\varphi \in L^1(Q_T)$. There is also strong convergence of $S(\tilde{\mathbf{U}}^\varepsilon)$ to $S(\tilde{\mathbf{U}})$ in $L^2(Q_T)$, almost-surely. This can be deduced from (5.11), which gives directly the weak convergence in $L^2(Q_T)$ by taking $\varphi \in L^2(Q_T)$, but also the convergence of the norms by taking S^2 instead of S and $\varphi \equiv 1$.

For the moment we have only supposed that $\mathbf{U}_0^\varepsilon \in W^{1,\infty}(\mathbb{T}^1)$ and that $(\mathbf{U}_0^\varepsilon)$ is bounded in $L^\infty(\mathbb{T}^1)$. Assume furthermore

$$\mathbf{U}_0^\varepsilon \rightarrow \mathbf{U}_0 \text{ in } L^1(\mathbb{T}^1). \quad (5.12)$$

The entropy and entropy flux as defined in (2.11), (2.12) are examples of functions S as above: we can therefore pass to the limit in (3.5). We will proceed as follows (along the lines of the proof of Theorem 3.4.12 in [Hof13b]): let $\varphi \in C^1(\mathbb{T}^1)$ be fixed. Since $\eta(\tilde{\mathbf{U}}^\varepsilon) \rightarrow \eta(\tilde{\mathbf{U}})$ in $L^1(\Omega \times Q_T)$ we have, for possibly a further subsequence,

$$\langle \eta(\tilde{\mathbf{U}}^\varepsilon), \varphi \rangle(t) \rightarrow \langle \eta(\tilde{\mathbf{U}}), \varphi \rangle(t), \quad \forall t \in \mathcal{D}, \quad \text{almost surely,} \quad (5.13)$$

where \mathcal{D} is a set of full measure in $[0, T]$ which contains the point $t = 0$. Set

$$e^\varepsilon = \varepsilon \eta''(\mathbf{U}^\varepsilon) \cdot (\mathbf{U}_x^\varepsilon, \mathbf{U}_x^\varepsilon), \quad \tilde{e}^\varepsilon = \varepsilon \eta''(\tilde{\mathbf{U}}^\varepsilon) \cdot (\tilde{\mathbf{U}}_x^\varepsilon, \tilde{\mathbf{U}}_x^\varepsilon).$$

Let $\mathcal{M}_b(Q_T)$ denote the set of bounded Borel measures on Q_T and $\mathcal{M}_b^+(Q_T)$ denote the subset of nonnegative measures. By Remark 3.18, there exists a random variable

$$\tilde{e} \in L_w^2(\tilde{\Omega}; \mathcal{M}_b(Q_T)), \quad \tilde{e} \in \mathcal{M}_b^+(Q_T) \quad \tilde{\mathbb{P}}\text{-almost surely,}$$

such that, for all $\psi \in C(\overline{Q}_T)$, for all $Y \in L^2(\tilde{\Omega})$, and up to a subsequence,

$$\tilde{\mathbb{E}}(\langle \tilde{e}^\varepsilon, \psi \rangle Y) \rightarrow \tilde{\mathbb{E}}(\langle \tilde{e}, \psi \rangle Y). \quad (5.14)$$

In (5.14), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{M}_b(Q_T)$ and $C(\overline{Q}_T)$ (which we extend below in the right-hand side of (5.15) to the pairing between $\mathcal{M}_b(Q_T)$ and $\mathcal{B}_b(Q_T)$, the set of Borel bounded functions over Q_T). The subscript w in $L_w^2(\tilde{\Omega}; \mathcal{M}_b(Q_T))$ indicates that we consider weak-star measurable mappings e from $\tilde{\Omega}$ into $\mathcal{M}_b(Q_T)$, *i.e.* maps e such that $\langle e, \psi \rangle$ is $\tilde{\mathcal{F}}$ -measurable for every $\psi \in C(\overline{Q}_T)$. Then $L_w^2(\tilde{\Omega}; \mathcal{M}_b(Q_T))$ is the dual of the space $L^2(\tilde{\Omega}; C(\overline{Q}_T))$ (Edwards [Edw65, Theorem 8.20.3]), hence (5.14) follows from the Banach-Alaoglu Theorem and from the estimate (3.52) with $I = (0, T)$. Besides (3.52) with arbitrary I implies that, almost surely, \tilde{e} has no atom in time, *i.e.*

$$\mathbb{E} \tilde{e}(\mathbb{T}^1 \times \{t\}) = 0, \quad \forall t \in [0, T].$$

This shows in particular that (5.14) holds true when $\psi = \mathbf{1}_{[0,t)}\varphi$ where $\varphi \in C(\mathbb{T}^1)$: for all $Y \in L^2(\tilde{\Omega})$,

$$\tilde{\mathbb{E}}\left(\int_0^t \langle \tilde{e}^\varepsilon, \varphi \rangle ds Y\right) \rightarrow \tilde{\mathbb{E}}(\langle \tilde{e}, \mathbf{1}_{[0,t)}\varphi \rangle Y). \quad (5.15)$$

Let now $(\tilde{\mathcal{F}}_t)$ be the $\tilde{\mathbb{P}}$ -augmented canonical filtration of the process $(\tilde{\mathbf{U}}, \tilde{W})$, *i.e.*

$$\tilde{\mathcal{F}}_t = \sigma(\varrho_t \tilde{\mathbf{U}}, \varrho_t \tilde{W}) \cup \{N \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(N) = 0\}, \quad t \in [0, T],$$

where the restriction operator ϱ_t is defined in (3.48). Then the sextuplet

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}})$$

is a weak martingale solution to (1.1). To show this, we use a reasoning

analogous to the one followed in Section 3.1.7. Let $\varphi \in C^1(\mathbb{T}^1)$. Define

$$\begin{aligned} M^\varepsilon(t) &= \langle \eta(\mathbf{U}^\varepsilon)(t), \varphi \rangle - \langle \eta(\mathbf{U}_0^\varepsilon), \varphi \rangle + \int_0^t \langle \partial_x H(\mathbf{U}^\varepsilon) - \varepsilon \partial_x^2 \eta(\mathbf{U}^\varepsilon), \varphi \rangle ds \\ &\quad - \int_0^t \langle e^\varepsilon(s), \varphi \rangle ds, \\ \tilde{M}^\varepsilon(t) &= \langle \eta(\tilde{\mathbf{U}}^\varepsilon)(t), \varphi \rangle - \langle \eta(\mathbf{U}_0^\varepsilon), \varphi \rangle + \int_0^t \langle \partial_x H(\tilde{\mathbf{U}}^\varepsilon) - \varepsilon \partial_x^2 \eta(\tilde{\mathbf{U}}^\varepsilon), \varphi \rangle ds \\ &\quad - \int_0^t \langle \tilde{e}^\varepsilon(s), \varphi \rangle ds, \\ \tilde{M}(t) &= \langle \eta(\tilde{\mathbf{U}})(t), \varphi \rangle - \langle \eta(\mathbf{U}_0), \varphi \rangle + \int_0^t \langle \partial_x H(\tilde{\mathbf{U}}), \varphi \rangle ds - \langle \tilde{e}, \mathbf{1}_{[0,t)} \varphi \rangle. \end{aligned}$$

Then M^ε and \tilde{M}^ε have same laws. By the convergence (5.11), (5.13) and (5.15) we show, as in Section 3.1.7 that the processes

$$\tilde{M}, \tilde{M}^2 - \sum_{k \geq 1} \int_0^\cdot \langle \sigma_k(\tilde{\mathbf{U}}) \partial_q \eta(\tilde{\mathbf{U}}), \varphi \rangle^2 dr, \tilde{M} \tilde{\beta}_k - \int_0^\cdot \langle \sigma_k(\tilde{\mathbf{U}}) \partial_q \eta(\tilde{\mathbf{U}}), \varphi \rangle dr \quad (5.16)$$

are $(\tilde{\mathcal{F}}_t)$ -martingales. There is however a notable difference between the result of Lemma 3.17 and the result (5.16) here, in the fact that the martingales in (5.16) are indexed by $\mathcal{D} \subset [0, T]$ since we have used the convergence (5.13). If all the processes in (5.16) were continuous martingales indexed by $[0, T]$, we would infer, as in the proof of Proposition 3.15, that

$$\begin{aligned} \langle \eta(\tilde{\mathbf{U}})(t), \varphi \rangle - \langle \eta(\mathbf{U}_0), \varphi \rangle - \int_0^t \langle H(\tilde{\mathbf{U}}), \partial_x \varphi \rangle ds \\ = -\langle \tilde{e}, \mathbf{1}_{[0,t)} \varphi \rangle + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{\mathbf{U}}) \partial_q \eta(\tilde{\mathbf{U}}), \varphi \rangle d\tilde{\beta}_k(s), \quad (5.17) \end{aligned}$$

for all $t \in [0, T]$, $\tilde{\mathbb{P}}$ -almost surely. Nevertheless, \mathcal{D} contains 0 and is dense in $[0, T]$ since it is of full measure, and it turns out, by the Proposition A.1 in [Hof13b] on densely defined martingales, that this is sufficient to obtain (5.17) for all $t \in \mathcal{D}$, $\tilde{\mathbb{P}}$ -almost surely. Then we conclude as in the *proof of Theorem 4.13* of [Hof13b]: let $N(t)$ denote the continuous semi-martingale defined by

$$N(t) = \int_0^t \langle H(\tilde{\mathbf{U}}), \partial_x \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{\mathbf{U}}) \partial_q \eta(\tilde{\mathbf{U}}), \varphi \rangle d\tilde{\beta}_k(s).$$

Let $t \in (0, T]$ be fixed and let $\alpha \in C_c^1([0, t])$. By the Itô Formula we compute

the stochastic differential of $N(s)\alpha(s)$ to get

$$0 = \int_0^t N(s)\alpha'(s)ds + \int_0^t \langle H(\tilde{\mathbf{U}}), \partial_x \varphi \rangle \alpha(s) ds + \sum_{k \geq 1} \int_0^t \langle \sigma_k(\tilde{\mathbf{U}}) \partial_q \eta(\tilde{\mathbf{U}}), \varphi \rangle \alpha(s) d\tilde{\beta}_k(s). \quad (5.18)$$

By (5.17), we have

$$N(t) = \langle \eta(\tilde{\mathbf{U}})(t), \varphi \rangle - \langle \eta(\mathbf{U}_0), \varphi \rangle + \langle \tilde{e}, \mathbf{1}_{[0,t]} \varphi \rangle,$$

for all $t \in \mathcal{D}$, $\tilde{\mathbb{P}}$ -almost surely. In particular, by the Fubini Theorem,

$$\int_0^t N(s)\alpha'(s)ds = \int_0^t \langle \eta(\tilde{\mathbf{U}})(s), \varphi \rangle \alpha'(s) ds + \langle \eta(\mathbf{U}_0), \varphi \rangle \alpha(0) - \int_0^t \alpha(\sigma) d\tilde{\rho}(\sigma), \quad (5.19)$$

$\tilde{\mathbb{P}}$ -almost surely, where we have defined the measure $\tilde{\rho}$ by $\tilde{\rho}(B) = \langle \tilde{e}, \mathbf{1}_B \varphi \rangle$, for B a Borel subset of $[0, T]$. If $\alpha, \varphi \geq 0$, then

$$\int_0^t \alpha(\sigma) d\tilde{\rho}(\sigma) \geq 0, \quad \tilde{\mathbb{P}} - \text{almost surely}$$

and (2.15) follows from (5.18), (5.19). This concludes the proof of Theorem 2.7.

6 Conclusion

We want to discuss in this concluding section some open questions related to the qualitative behaviour of solutions to (1.1). The first one is the question of uniqueness. Uniqueness of weak entropy solutions is an unsolved question in the deterministic setting. For the stochastic problem (1.1), one may consider the question of uniqueness of martingale solutions, *i.e.* uniqueness in law. Is it a promising question? Would it be easier to answer than the question of uniqueness in the deterministic setting, as, for example, uniqueness in law manifests itself in stochastic differential equations with continuous coefficients, which represent therefore stochastic perturbations of o.d.e's displaying some indeterminacy [SV79]?

An other problem concerns the long-time behaviour of solutions to (1.1). It is known that for *scalar* stochastic conservation laws with additive noise, and for non-degenerate fluxes, there is a unique ergodic invariant measure, *cf.* [EKMS00, DV13]. Since both fields of (1.1) are genuinely non-linear, a form of non-degeneracy condition is clearly satisfied in (1.1). Actually,

in the deterministic case $\Phi \equiv 0$, the solution converges to the constant state determined by the conservation of mass [CF99, Theorem 5.4], which indicates that some kind of dissipation effects (*via* interaction of waves, *cf.* also [GL70]) occur in the Euler system for isentropic gas dynamics. However, in a system there is in a way more room for waves to evolve than in a scalar conservation law, and the long-time behaviour in (1.1) may be different from the one described in [EKMS00, DV13]. The only thing we want to emphasize here is that we observe on numerical simulations that the value of the initial datum get forgotten by the system, which seems to indicate the existence of an invariant measure with ergodic (even mixing) properties.

Specifically, we consider the case $\gamma = 2$. For such a value the system of Euler equations for isentropic gas dynamics is equivalent to the following Saint-Venant system of equations for shallow water:

$$h_t + (hu)_x dt = 0, \quad \text{in } Q_T, \quad (6.1a)$$

$$(hu)_t + (hu^2 + g\frac{h^2}{2})_x + ghZ_x = 0, \quad \text{in } Q_T, \quad (6.1b)$$

with $Z(x, t) = \Phi^*(x)\frac{dW}{dt}$ and $Q_T = \mathbb{T}^1 \times (0, T)$. When $Z = Z(x)$, (6.1) is a model for the one-dimensional flow of a fluid of height h and speed u over a ground described by the curve $z = Z(x)$ ($u(x)$ is the speed of the column of water over the abscissa x)³. For a random Z as in (6.1b), the system (6.1) describes the evolution of the fluid in terms of (h, u) when its behaviour is forced by the moving topography, the question being thus to determine if an equilibrium in law (and which kind of equilibrium) for such a random process can be reached when time goes to $+\infty$. An hint for the existence of an invariant measure and convergence towards it is the “loss of memory in the system” which seems to occur as time goes by, as we illustrate it on the following pictures. Let

$$E := \int_{\mathbb{T}^1} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 dx, \quad E_{\text{tot}} = E + \int_{\mathbb{T}^1} ghZ dx$$

denote respectively the total energy and the full total energy for (6.1). Figure 1 is a drawing⁴ of the evolution in time $t \in [0, 10]$ (for a given path $\{W(t); 0 \leq t \leq 10\}$) of $E(t)$ with the following data:

$$g = 2, \quad \Phi^*(x) = 10 \sum_{k=1}^5 -\frac{1}{2\pi k} \cos(2\pi kx), \quad W(t) = (\beta_k(t))_{k=1, \dots, 5}, \quad (6.2)$$

³the fact that u is independent on the altitude z is admissible as long as h is small compared to the longitudinal length L of the channel, $L = 1$ here, *cf.* [GP01]

⁴the numerical simulations use the kinetic scheme for Saint Venant equations derived in [ABP00]

and for three kind of initial data, respectively called Test 1,2,3, corresponding to the values

$$\begin{pmatrix} h_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - H(x) \end{pmatrix}, \quad \begin{pmatrix} h_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \quad \begin{pmatrix} h_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0.5 + H(x) \\ 0.5 \end{pmatrix}, \quad (6.3)$$

at time $t = 0$, where $H(x)$ is the Heaviside function with origin at $x = 0.5$ ($H = \mathbf{1}_{x>0.5}$). The “loss of memory” effect which we are investigating seems

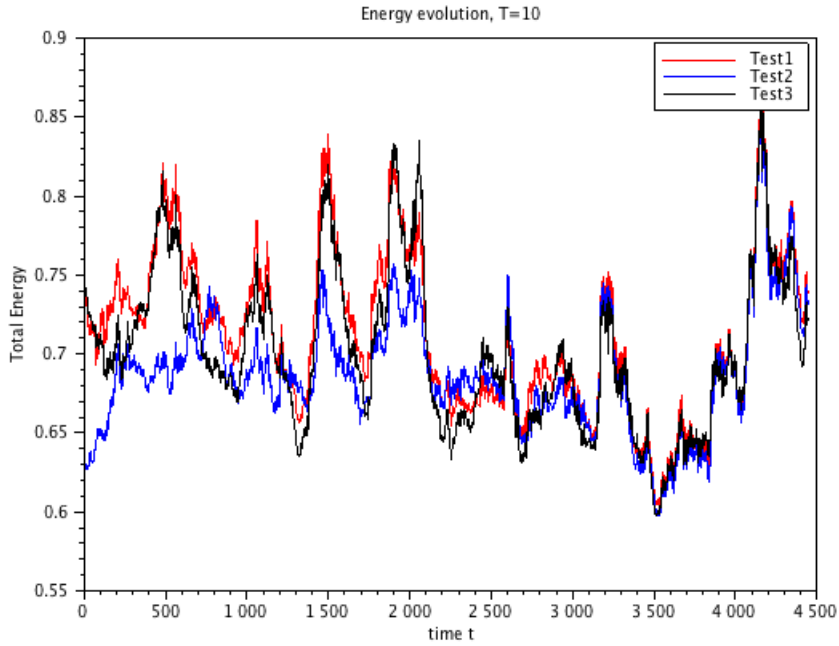


Figure 1: Evolution of the energy

indeed to be displayed on Figure 1 and it seems that there is existence of an invariant measure μ for (6.1). The interpretation of Figure 1 has however to be made with care. Indeed, if an invariant measure shall exist, it would certainly depend on a parameter $\lambda \in \mathbb{R}^2$. Indeed, in the scalar case [EKMS00, DV13], say for the equation

$$dv + (A(v))_x = \partial_x \phi(x) dW(t), \quad x \in \mathbb{T}^1, t > 0,$$

there is a unique invariant measure μ_λ indexed by the constant parameter

$$\lambda = \int_{\mathbb{T}^1} v(x) dx \in \mathbb{R}.$$

For (6.1), the entropy solution is evolving on the manifold

$$\int_{\mathbb{T}^1} h(x) dx = \text{cst.}$$

There should be a second quantity preserved in the evolution. For smooth solutions to (6.1), the quantity

$$\int_{\mathbb{T}^1} u(x) dx \tag{6.4}$$

is preserved, and note that in all three cases considered in (6.3), we have taken the same value for (6.4). However, (6.4) is not the right quantity to be considered. It may be not conserved in the evolution of (6.1) (as soon as shocks occur), and in that case different behaviour for the total energy (as well as for the full total energy actually) are obtained. Below in Figure 2 is such an example: after the first time of relaxation, the evolution of the total energy in Test 2 differs from the one in Tests 1 and 3. Therefore

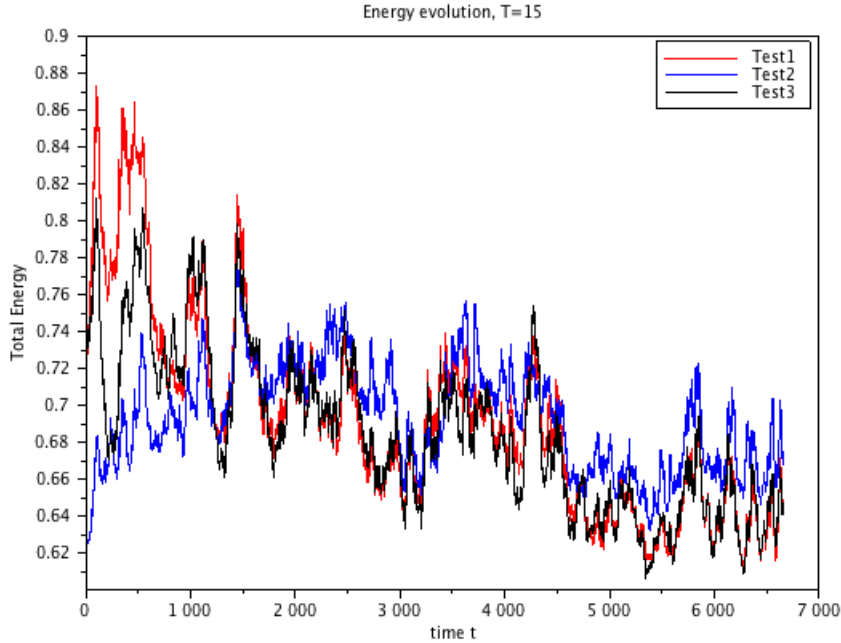


Figure 2: Evolution of the energy - 2

a first interesting problem is to find what quantities do parametrize the invariant measure for (6.1). This question and the other ones described in this concluding section (uniqueness in law, large-time behaviour) will be addressed in a future work.

A A parabolic uniformization effect

The following result, a phenomenon of uniformization from below for a non-negative solution for a parabolic equation with a drift-term with quite low regularity, will be proved with techniques similar to those used in [MV09].

Theorem A.1 (Positivity). *Let $\rho, u \in L^2(0, T; H^1(\mathbb{T}^1))$ satisfy*

$$\rho|\partial_x u|^2 \in L^1(\mathbb{T}^1 \times (0, T)), \quad \partial_t \rho \in L^2(0, T; H^{-1}(\mathbb{T}^1)). \quad (\text{A.1})$$

Assume that ρ is a non-negative solution to the equation

$$\partial_t \rho + \partial_x(\rho u) - \partial_x^2 \rho = 0 \quad (\text{A.2})$$

and that $\rho_0 := \rho(0)$ satisfy $\frac{1}{\rho_0} \in L^2(\mathbb{T}^1)$. Then, for all $\tau \in (0, T]$, there exists a constant $c > 0$ depending on ρ_0, T, τ and

$$\iint_{Q_T} \rho |\partial_x u|^2 dx dt$$

only, such that $\rho \geq c$ a.e. in $\mathbb{T}^1 \times [\tau, T]$.

Proof. Note first that the $L_t^2 H_x^1$ regularity of ρ , together with the $L_t^2 H_x^{-1}$ regularity of $\partial_t \rho$ implies the existence of a representative still denoted ρ which is continuous in time with values in L^2 . This is the value of this representative at time $t = 0$ which is denoted by ρ_0 . Now, to prove the result, we will show a bound from above on $w = \frac{1}{\rho}$. Actually, w is not yet well-defined and we shall more rigorously prove a bound from above on $w_\varepsilon = \frac{1}{\rho + \varepsilon}$ that is uniform with respect to the parameter $\varepsilon > 0$. For simplicity we will work directly on w , the main lines of the proof are easily adapted for w_ε . By a chain-rule formula (cf. Lemma 1.4 in Carrillo, Wittbold [CW99] for example) we derive the following equation for w :

$$\partial_t w - \partial_x^2 w = -2w^{-1} |\partial_x w|^2 + w \partial_x u - u \partial_x w. \quad (\text{A.3})$$

For $M > 0$ to be fixed later and $k \geq 0$, let us set

$$M_k = M(1 - 2^{-k-1}), \quad \tau_k = \tau(1 - 2^{-k}), \quad w_k = (w - M_k)^+,$$

where $s^+ = \max(s, 0)$ denote the positive part of s . Let us also set

$$U_k = \sup_{\tau_k \leq t \leq T} \int_{\mathbb{T}^1} |w_k|^2 dx + 2 \iint_{Q_{\tau_k, T}} |\partial_x w_k|^2 dx dt, \quad (\text{A.4})$$

where $Q_{\tau_k, T} = \mathbb{T}^1 \times (\tau_k, T)$. Multiplying (A.3) by w_k , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^1} |w_k|^2 dx + \int_{\mathbb{T}^1} |\partial_x w_k|^2 dx &\leq \int_{\mathbb{T}^1} w w_k \partial_x u - \frac{u}{2} \partial_x w_k^2 dx \\ &\leq \frac{3}{2} \int_{\mathbb{T}^1} w_k^2 |\partial_x u| dx + M_k \int_{\mathbb{T}^1} w_k |\partial_x u| dx. \end{aligned}$$

Then we sum the result over $[s, \tau']$, with any $s \in (\tau_{k-1}, \tau_k)$ and any $\tau' \in [\tau_k, T]$. This gives

$$\begin{aligned} & \int_{\mathbb{T}^1} |w_k|^2(\tau') dx + 2 \int_s^{\tau'} \int_{\mathbb{T}^1} |\partial_x w_k|^2 dx dt \\ & \leq 3 \int_s^{\tau'} \int_{\mathbb{T}^1} w_k^2 |\partial_x u| dx dt + 2M \int_s^{\tau'} \int_{\mathbb{T}^1} w_k |\partial_x u| dx dt + \int_{\mathbb{T}^1} |w_k|^2(s) dx. \end{aligned}$$

In particular, it implies

$$\begin{aligned} & \int_{\mathbb{T}^1} |w_k|^2(\tau') dx \tag{A.5} \\ & \leq 3 \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k^2 |\partial_x u| dx dt + 2M \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k |\partial_x u| dx dt + \int_{\mathbb{T}^1} |w_k|^2(s) dx \end{aligned}$$

and also

$$\begin{aligned} & 2 \int_{\tau_k}^{\tau'} \int_{\mathbb{T}^1} |\partial_x w_k|^2 dx dt \leq 2 \int_s^{\tau'} \int_{\mathbb{T}^1} |\partial_x w_k|^2 dx dt \\ & \leq 3 \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k^2 |\partial_x u| dx dt + 2M \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k |\partial_x u| dx dt + \int_{\mathbb{T}^1} |w_k|^2(s) dx, \end{aligned}$$

which gives, by taking the sup with respect to $\tau' \in [\tau_k, T]$,

$$\begin{aligned} & 2 \int_{\tau_k}^T \int_{\mathbb{T}^1} |\partial_x w_k|^2 dx dt \tag{A.6} \\ & \leq 3 \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k^2 |\partial_x u| dx dt + 2M \int_{\tau_{k-1}}^T \int_{\mathbb{T}^1} w_k |\partial_x u| dx dt + \int_{\mathbb{T}^1} |w_k|^2(s) dx. \end{aligned}$$

Thus, by (A.5) and (A.6), we obtain

$$U_k \leq 6 \iint_{Q_{\tau_{k-1}, T}} w_k^2 |\partial_x u| dx dt + 4M \iint_{Q_{\tau_{k-1}, T}} w_k |\partial_x u| dx dt + 2 \int_{\mathbb{T}^1} w_k^2(s) dx. \tag{A.7}$$

Eventually, we take the average over $s \in (\tau_{k-1}, \tau_k)$ of the previous inequality to derive the estimate

$$\begin{aligned} U_k & \leq 6 \iint_{Q_{\tau_{k-1}, T}} w_k^2 |\partial_x u| dx dt + 4M \iint_{Q_{\tau_{k-1}, T}} w_k |\partial_x u| dx dt \\ & \quad + \frac{2^{k+1}}{\tau} \iint_{Q_{\tau_{k-1}, T}} w_k^2 dx dt. \tag{A.8} \end{aligned}$$

To obtain an estimate on the right-hand side of (A.8), we set

$$A_k = \{(x, t) \in Q_{\tau_{k-1}, T}; w_k > 0\},$$

and note that, on A_k , $w_{k-1} > M_k - M_{k-1} = 2^{-k-1}M$, hence

$$\mathbf{1}_{A_k} \leq \left(\frac{w_{k-1}}{2^{-k-1}M} \right)^\beta, \quad \forall \beta \geq 0. \quad (\text{A.9})$$

We will also use the estimate

$$\|w_k\|_{L^6(Q_{\tau_{k-1}, T})}^2 \leq U_k, \quad (\text{A.10})$$

which is a direct consequence of the inequality

$$\|w\|_{L^6(Q_{\tau_{k-1}, T})} \leq \left(\sup_{t \in [\tau_{k-1}, T]} \|w(t)\|_{L^2(\mathbb{T}^1)} \right)^{2/3} \|w\|_{L^2(\tau_{k-1}, T; H^1(\mathbb{T}^1))}^{1/3}. \quad (\text{A.11})$$

To prove (A.11), we use the injection $H^\delta(\mathbb{T}^1) \subset L^r(\mathbb{T}^1)$, $\delta \in [0, 1/2)$, $\frac{1}{r} := \frac{1}{2} - \delta$ and an interpolation inequality to obtain

$$\|w(t)\|_{L^r(\mathbb{T}^1)}^r \leq \|w(t)\|_{L^2(\mathbb{T}^1)}^{r(1-\delta)} \|w(t)\|_{H^1(\mathbb{T}^1)}^{r\delta}. \quad (\text{A.12})$$

Then we sum (A.12) over $t \in [\tau_{k-1}, \tau]$. For $r\delta = 2$, an equation which sets the value of (δ, r) to $(1/3, 6)$, we obtain (A.11).

Using (A.9) with $\beta = 4$ and (A.10) (and also the estimate $w_k \leq w_{k-1}$), we obtain

$$\iint_{Q_{\tau_{k-1}, T}} w_k^2 dx dt \leq \frac{2^{4(k+1)}}{M^4} U_{k-1}^3. \quad (\text{A.13})$$

Besides, we have

$$\begin{aligned} \iint_{Q_{\tau_{k-1}, T}} w_k^2 |\partial_x u| dx dt &= \iint_{Q_{\tau_{k-1}, T}} w_k^2 w^{1/2} \rho^{1/2} |\partial_x u| dx dt \\ &\leq \left[\iint_{Q_{\tau_{k-1}, T}} \rho |\partial_x u|^2 dx dt \right]^{1/2} \left[\iint_{Q_{\tau_{k-1}, T}} w_k^4 w dx dt \right]^{1/2}. \end{aligned}$$

Since $w \leq w_k + M$, $\omega_k \leq \omega_{k-1}$, using (A.9) with respectively $\beta = 1, 2$ and also (A.10), we deduce that

$$\begin{aligned} \iint_{Q_{\tau_{k-1}, T}} w_k^2 |\partial_x u| dx dt \\ \leq \left[\iint_{Q_T} \rho |\partial_x u|^2 dx dt \right]^{1/2} \left(\frac{2^{k+1}}{M} + M \frac{2^{2(k+1)}}{M^2} \right)^{1/2} U_{k-1}^{3/2}. \end{aligned} \quad (\text{A.14})$$

Similarly, we have

$$\begin{aligned} \iint_{Q_{\tau_{k-1}, T}} w_k |\partial_x u| dx dt \\ \leq \left[\iint_{Q_T} \rho |\partial_x u|^2 dx dt \right]^{1/2} \left(\frac{2^{3(k+1)}}{M^3} + M \frac{2^{4(k+1)}}{M^4} \right)^{1/2} U_{k-1}^{3/2}. \end{aligned} \quad (\text{A.15})$$

Eventually, assuming with no loss of generality that $M \geq 1$, we conclude from (A.8), (A.14), (A.15) and (A.13) that

$$U_k \leq 2^{5(k+1)} C (U_{k-1}^3 + U_{k-1}^{3/2}), \quad C := \left[\iint_{Q_T} \rho |\partial_x u|^2 dx dt \right]^{1/2} + \frac{1}{\tau}. \quad (\text{A.16})$$

for every $k \geq 1$. It follows from (A.16) that there exists $\varepsilon > 0$ depending on C only such that

$$U_0 < \varepsilon \implies \lim_{k \rightarrow +\infty} U_k = 0.$$

The last statement is equivalent to $\rho \geq M^{-1}$ a.e. on $Q_{\tau, T}$. By an energy estimate (cf. (A.7) with $s = 0$) and also an estimate similar to (A.14), we have

$$\begin{aligned} U_0 &\leq 3 \iint_{Q_T} w_0^2 |\partial_x u| dx dt + \int_{\mathbb{T}^1} w_0^2(0) dx \\ &\leq \eta U_0^{3/2} + \int_{\mathbb{T}^1} |(\rho_0^{-1} - \frac{M}{2})^+|^2 dx, \quad \eta = \frac{3}{M^{1/2}} \left[\iint_{Q_T} \rho |\partial_x u|^2 dx dt \right]^{1/2}. \end{aligned} \quad (\text{A.17})$$

To conclude we choose M_0 such that

$$\int_{\mathbb{T}^1} |(\rho_0^{-1} - \frac{M}{2})^+|^2 dx < \varepsilon_1 := \varepsilon/3,$$

for $M \geq M_0$. For M large enough, i.e. η small enough, (A.17) implies that U_0 is bounded by the first root of the equation

$$X = \eta X^{3/2} + \varepsilon_1, \quad (\text{A.18})$$

(the second root tending to $+\infty$ with M); in particular U_0 is smaller than the root $3\varepsilon_1$ with multiplicity 2 of (A.18), which is obtained for the critical value $\eta = \frac{2}{3\sqrt{3\varepsilon_1}}$. Therefore $U_0 < 3\varepsilon_1 = \varepsilon$. ■

Remark A.2. *The condition on ρ_0 in Theorem A.1 can be relaxed, we may assume for example $\ln(\rho_0) \in L^1(\mathbb{T}^1)$ (and then start the proof with the equation satisfied by $w = \ln(\rho)$ instead of $w = 1/\rho$).*

Corollary A.3 (Positivity). *Let $\rho, u \in L^2(0, T; H^1(\mathbb{T}^1))$ satisfy*

$$\rho |\partial_x u|^2 \in L^1(\mathbb{T}^1 \times (0, T)), \quad \partial_t \rho \in L^2(0, T; H^{-1}(\mathbb{T}^1)). \quad (\text{A.19})$$

Assume that ρ is a non-negative solution to the equation

$$\partial_t \rho + \partial_x(\rho u) - \partial_x^2 \rho = 0 \quad (\text{A.20})$$

and that there exists $c_0 > 0$ such that $\rho_0 := \rho(0) \geq c_0$ a.e. in \mathbb{T}^1 . Then, there exists $c > 0$ depending on c_0, T and

$$\iint_{Q_T} \rho |\partial_x u|^2 dx dt$$

only, such that $\rho \geq c$ a.e. in $\mathbb{T}^1 \times [0, T]$.

Proof. We apply the Theorem A.1 with $\tau = 1$ to the solution $\tilde{\rho}$ of

$$\partial_t \tilde{\rho} + \partial_x(\tilde{\rho} \tilde{u}) - \partial_x^2 \tilde{\rho} = 0 \text{ in } Q_{T+1},$$

with initial datum $\tilde{\rho}(0) = c_0$, where $\tilde{u}(x, t) = u(x, t - 1)$ if $t \in [1, T + 1]$, and 0 otherwise. ■

B Regularizing effects of the one-dimensional heat equation

Let $T > 0$, let $z \in C([0, T]; L^2(\mathbb{T}^1))$ satisfy

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(s)ds, \quad (\text{B.1})$$

for some given data z_0 and f , where $S(t)$ is the semi-group associated to the heat operator $\partial_t - \partial_x^2$ on \mathbb{T}^1 . The function z is a mild solution to the heat equation

$$(\partial_t - \partial_x^2)z = f \text{ in } Q_T,$$

with initial condition $z(0) = z_0$. If $f \in L^p(Q_T)$, $p > \frac{6}{5}$ and $z_0 \in L^2(\mathbb{T}^1)$ then, by the regularizing properties of $S(t)$, (B.1) gives a z which is indeed in $C([0, T]; L^2(\mathbb{T}^1))$. More precisely, we have

$$\|S(t)\|_{L_x^p \rightarrow L_x^q} \leq Ct^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})}, \quad (\text{B.2})$$

for $1 < p \leq q \leq +\infty$, for a given constant C and therefore, for possibly a different constant C ,

$$\|S(t-s)f(s)\|_{L^2(\mathbb{T}^1)} \leq C(t-s)^{-\mu} \|f(s)\|_{L^p(\mathbb{T}^1)}, \quad \mu := \frac{1}{2} \left(\frac{1}{p} - \frac{1}{2} \right).$$

By the Young inequality for the convolutions of functions, $L^{p'} * L^p$ embeds in the space of continuous functions (here p' is the conjugate exponent to p), hence z defined by (B.1) is indeed continuous in time with values in $L^2(\mathbb{T}^1)$ if $t \mapsto t^{-\mu}$ is in $L^{p'}$, which is equivalent to the condition $p > \frac{6}{5}$. Using more generally the regularizing properties

$$\|\partial_x^j S(t)\|_{L_x^p \rightarrow L_x^q} \leq Ct^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{j}{2}} \quad (\text{B.3})$$

for $j \in \mathbb{N}$, we obtain that, for a given constant $C > 0$,

$$\left\| \partial_x \int_0^t S(t-s)f(s)ds \right\|_{L^q(Q_T)} \leq C \|f\|_{L^p(Q_T)} \text{ if } \frac{1}{q} > \frac{1}{p} - \frac{1}{3}, \quad (\text{B.4a})$$

$$\left\| \partial_x S(t)z_0 \right\|_{L^q(Q_T)} \leq C \|z_0\|_{L^p(\mathbb{T}^1)} \text{ if } \frac{1}{q} > \frac{3}{p}. \quad (\text{B.4b})$$

Remark B.1 (Time regularity). *Let $\nu > 0$, $m \geq 1$. If $t \mapsto t^{-\nu}$ is in $L^m(0, T)$ then it is also in $W^{\sigma, m}(0, T)$ for a certain $\sigma > 0$. This shows that we have actually*

$$\left\| \partial_x \int_0^t S(t-s)f(s)ds \right\|_{W^{\sigma, q}(0, T; L^q(\mathbb{T}^1))} \leq C \|f\|_{L^p(Q_T)} \quad \text{if } \frac{1}{q} > \frac{1}{p} - \frac{1}{3}, \quad (\text{B.5})$$

for a given $\sigma > 0$ depending on p and q .

We end this section with a variation over (B.4a) for the solution of a split evolution equation (this is used in Section 3.1.5): let $\tau > 0$, set $t_n = n\tau$, $n \in \mathbb{N}$, set

$$t_{\sharp} := \min(2t - t_{2n}, t_{2n+2}), \quad t_{\flat} := \frac{t + t_{2n}}{2}, \quad t_{2n} \leq t \leq t_{2n+2},$$

and let

$$z(t) = \int_0^{t_{\sharp}} S(t_{\sharp} - s)f(s_b)ds.$$

The function z is the solution to

$$\frac{1}{2} \partial_t z - \partial_x^2 z = f, \quad \text{in } \mathbb{T}^1 \times (t_{2n}, t_{2n+1}), \quad \partial_t z = 0, \quad \text{in } \mathbb{T}^1 \times (t_{2n+1}, t_{2n+2}),$$

for $n = 0, 1, \dots$. If $T = t_{2K}$, $K \in \mathbb{N}^*$, we have, for $h \in L^1(0, T)$,

$$\begin{aligned} \int_0^T h(t_{\sharp})dt &= \frac{1}{2} \int_0^T h(t)dt + \sum_{n=0}^{K-1} \tau h(t_{2n+2}) \\ &= \int_0^T h(t)dt + \frac{1}{2} \sum_{n=0}^{K-1} \int_{t_{2n}}^{t_{2n+2}} (h(t_{2n+2}) - h(t))dt. \end{aligned} \quad (\text{B.6})$$

If there exists a $\sigma \in (0, 1)$ such that $h \in W^{\sigma, 1}(0, T)$, we can estimate the remainder in (B.6) and obtain

$$\left| \int_0^T h(t_{\sharp})dt - \int_0^T h(t)dt \right| \leq C \|h\|_{W^{\sigma, 1}(0, T)} \tau^{\sigma}.$$

By Remark B.1, we deduce in particular for $\tau \leq 1$ that

$$\left\| \partial_x \int_0^{t_{\sharp}} S(t_{\sharp} - s)f(s_b)ds \right\|_{L^q(Q_T)} \leq C \|\tilde{f}\|_{L^p(Q_T)} \quad \text{if } \frac{1}{q} > \frac{1}{p} - \frac{1}{3}, \quad (\text{B.7})$$

where \tilde{f} is the extension by 0 of f to the strips $\mathbb{T}^1 \times (t_{2n+1}, t_{2n+2})$, $n = 0, \dots, K-1$.

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