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Paul Lescot, Hung Ngoc Nguyen, Yong Yang

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# ON THE COMMUTING PROBABILITY AND SUPERSOLVABILITY OF FINITE GROUPS

PAUL LESCOT (CORRESPONDING AUTHOR)  
LMRS,CNRS UMR 6085  
UFR DES SCIENCES ET TECHNIQUES, UNIVERSITÉ DE ROUEN  
AVENUE DE L'UNIVERSITÉ BP12  
76801 SAINT-ETIENNE DU ROUVRAY, FRANCE  
PHONE 00 33 (0)2 32 95 52 24  
FAX 00 33 (0)2 32 95 52 86  
E-MAIL PAUL.LESCOT@UNIV-ROUEN.FR

HUNG NGOC NGUYEN  
DEPARTMENT OF MATHEMATICS  
THE UNIVERSITY OF AKRON  
AKRON, OHIO 44325, USA  
E-MAIL HUNGNGUYEN@UAKRON.EDU

YONG YANG  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF WISCONSIN-PARKSIDE  
KENOSHA, WI 53141, USA  
E-MAIL YANGY@UWP.EDU

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ABSTRACT. For a finite group  $G$ , let  $d(G)$  denote the probability that a randomly chosen pair of elements of  $G$  commute. We prove that if  $d(G) > 1/s$  for some integer  $s > 1$  and  $G$  splits over an abelian normal nontrivial subgroup  $N$ , then  $G$  has a nontrivial conjugacy class inside  $N$  of size at most  $s - 1$ . We also extend two results of Barry, MacHale, and Ní Shé on the commuting probability in connection with supersolvability of finite groups. In particular, we prove that if  $d(G) > 5/16$  then either  $G$  is supersolvable, or  $G$  isoclinic to  $A_4$ , or  $G/\mathbf{Z}(G)$  is isoclinic to  $A_4$ .

## 1. INTRODUCTION

For a group  $G$ , let  $d(G)$  denote the probability that a randomly chosen pair of elements of  $G$  commute. That is,

$$d(G) := \frac{1}{|G|^2} |\{(x, y) \in G \times G \mid xy = yx\}|.$$

This quantity is often referred to as the *commuting probability* of  $G$ . The study of the commuting probability of finite groups dates back to work of Gustafson in the seventies. In [6], he showed that

$$d(G) = \frac{k(G)}{|G|},$$

where  $k(G)$  is the number of conjugacy classes of  $G$ .

It is clear that  $d(G) = 1$  if and only if  $G$  is abelian. Therefore, when  $d(G)$  is close to 1, one might expect that  $G$  is close to abelian. For instance, it was proved by Gustafson in the same paper that if  $d(G) > 5/8$  then  $G$  must be abelian. In [12], the first author classified all groups with commuting probability at least  $1/2$  – if  $d(G) \geq 1/2$  then  $G$  is isoclinic to the trivial group, an extraspecial 2-group, or  $S_3$ . As a consequence, if  $d(G) > 1/2$  then  $G$  must be nilpotent. Going further, the first author proved in [10, 11] that  $G$  is solvable whenever  $d(G) > 1/12$ . This was improved by Guralnick and Robinson in [5, Theorem 11] where they showed that if  $d(G) > 3/40$  then either  $G$  is solvable or  $G \cong A_5 \times A$  for some abelian group  $A$ . In [1], Barry, MacHale, and Ní Shé proved that  $G$  must be supersolvable whenever  $d(G) > 1/3$  and pointed out that, since  $d(A_4) = 1/3$ , the bound cannot be improved.

Two finite groups are said to be *isoclinic* if there exists isomorphisms between their inner automorphism groups and commutator subgroups such that these isomorphisms are compatible with the commutator map, see §3 for the detailed definition. This concept is weaker than isomorphism and was introduced by Hall [7] in connection with the enumeration of  $p$ -groups. It was shown by the first author [12] that the commuting probability is invariant under isoclinism. It follows that any group isoclinic to  $A_4$  has commuting probability exactly equal to  $1/3$ . Our first result highlights the special role of  $A_4$  among non-supersolvable groups with commuting probability greater than  $5/16$ . Here and what follows, the center of  $G$ , as usual, is denoted by  $\mathbf{Z}(G)$ .

**Theorem 1.** *Let  $G$  be a finite group. If  $d(G) > 5/16$ , then*

- (i)  $G$  is supersolvable, or
- (ii)  $G$  is isoclinic to  $A_4$ , or
- (iii)  $G/\mathbf{Z}(G)$  is isoclinic to  $A_4$ .

Theorem 1 has two consequences. The obvious one is the aforementioned result of Barry, MacHale, and Ní Shé. We would like to note that their proof is somewhat more complicated and requires a large amount of computations with GAP [4]. The second one is less obvious and shows that the groups isoclinic to  $A_4$  are the only non-supersolvable groups of commuting probability at least  $1/3$ .

**Corollary 2.** *Let  $G$  be a finite group with  $d(G) \geq 1/3$ , then  $G$  is either supersolvable or isoclinic to  $A_4$ .*

We remark that, the average size of a conjugacy class of  $G$ , denoted by  $\text{acs}(G)$ , is exactly the reciprocal of  $d(G)$ . Therefore, Theorem 1 is equivalent to: that if  $\text{acs}(G) < 16/5$  then either  $G$  is supersolvable or  $G$  is isoclinic to  $A_4$  or  $G/\mathbf{Z}(G)$  is isoclinic to  $A_4$ . Recently, Isaacs, Loukaki, and Moretó [9] have obtained some dual results on solvability and nilpotency in connection with average character degree of finite groups. For instance, they showed that a finite group is supersolvable whenever its average character degree is less than  $3/2$ . It would be interesting if there is a dual result of Theorem 1 for the average character degree.

For groups of odd order, it is possible to obtain better bounds. It was proved in [1] that if  $G$  is a group of odd order with  $d(G) > 11/75$ , then  $G$  must be supersolvable. Let  $C_n$  denote the cyclic group of order  $n$ . We notice that  $(C_5 \times C_5) \rtimes C_3$  is the smallest non-supersolvable group of odd order. Here we can show that the groups isoclinic to  $(C_5 \times C_5) \rtimes C_3$  have commuting probability ‘substantially’ larger than that of other non-supersolvable groups of odd order.

**Theorem 3.** *Let  $G$  be a finite group of odd order. If  $d(G) > 35/243 < 11/75$ , then  $G$  is either supersolvable or isoclinic to  $(C_5 \times C_5) \rtimes C_3$ .*

Our last result provides a characteristic of certain groups with ‘large’ commuting probability and therefore can be applied to obtain the inside structure of these groups. For an example, see §4.

**Theorem 4.** *Let  $s \geq 2$  be an integer and  $G$  a finite group with  $d(G) > 1/s$ . Let  $N$  be an abelian normal nontrivial subgroup of  $G$  and suppose that  $G$  splits over  $N$ . Then there exists a nontrivial conjugacy class of  $G$  inside  $N$  of size at most  $s - 1$ . In particular, we have either  $\mathbf{Z}(G) \neq 1$  or  $G$  has a proper subgroup of index at most  $s - 1$ .*

Theorems 1, 3 and 4 are respectively proved in Sections 2, 3, and 4. Corollary 2 is proved at the end of Section 2.

## 2. GROUPS WITH COMMUTING PROBABILITY GREATER THAN $5/16$

We will prove Theorem 1 and Corollary 2 in this section. We first recall a well-known result of Gallagher [3].

**Lemma 5.** *If  $N$  is a normal subgroup of  $G$ , then*

$$k(G) \leq k(G/N)k(N)$$

and the equality is equivalent to

$$C_{G/N}(gN) = C_G(g)N/N \text{ for each } g \in G.$$

This gives an immediate consequence.

**Lemma 6.** *Let  $N$  be a normal subgroup of  $G$ . Then*

- (i)  $d(G) \leq d(G/N)d(N)$ ,
- (ii)  $d(G) = d(G/N)$  if and only if  $N$  is abelian and  $C_{G/N}(gN) = C_G(g)N/N$  for each  $g \in G$ , and
- (iii) if  $N \subseteq \mathbf{Z}(G)$  and  $d(G) = d(G/N)$ , then  $\mathbf{Z}(G/N) = \mathbf{Z}(G)/N$ .

*Proof.* (i) and (ii) are consequences of Lemma 5. We now prove (iii). Assume that  $N \subseteq \mathbf{Z}(G)$  and  $d(G) = d(G/N)$ . We have  $C_{G/N}(gN) = C_G(g)N/N$  for every  $g \in G$  and therefore

$$\begin{aligned} gN \in \mathbf{Z}(G/N) &\Leftrightarrow C_{G/N}(gN) = G/N \\ &\Leftrightarrow C_G(g)N = G \\ &\Leftrightarrow C_G(g) = G \\ &\Leftrightarrow g \in \mathbf{Z}(G) \end{aligned}$$

Therefore,  $\mathbf{Z}(G/N) = \mathbf{Z}(G)/N$ , as desired. □

Two groups  $G$  and  $H$  are said to be *isoclinic* if there are isomorphisms  $\varphi : G/\mathbf{Z}(G) \rightarrow H/\mathbf{Z}(H)$  and  $\phi : G' \rightarrow H'$  such that

$$\begin{aligned} &\text{if } \varphi(g_1\mathbf{Z}(G)) = h_1\mathbf{Z}(H) \\ &\text{and } \varphi(g_2\mathbf{Z}(G)) = h_2\mathbf{Z}(H), \\ &\text{then } \phi([g_1, g_2]) = [h_1, h_2]. \end{aligned}$$

This concept is weaker than isomorphism and was introduced by Hall in [7] as a structurally motivated classification for finite groups, particularly for  $p$ -groups. It is well-known that several characteristics of finite groups are invariant under isoclinism and in particular supersolvability is one of those, see [2]. Furthermore, it is proved in [12] that the commuting probability is also invariant under isoclinism.

A stem group is defined as a group whose center is contained inside its derived subgroup. It is known that every group is isoclinic to a stem group and if we restrict to finite groups, a stem group has the minimum order among all groups isoclinic to

it, see [7] for more details. The following lemma plays an important role in the proof of Theorems 1 and 3.

**Lemma 7.** *For every finite group  $G$ , there is a finite group  $H$  isoclinic to  $G$  such that  $|H| \leq |G|$  and  $\mathbf{Z}(H) \subseteq H'$ .*

The next lemma will narrow down the possibilities for the commutator subgroup of a finite group with commuting probability greater than  $5/16$ .

**Lemma 8.** *Let  $G$  be a finite group with  $d(G) > 5/16$ . Then  $|G'| < 12$ .*

*Proof.* Let  $\text{Irr}_2(G)$  denote the set of nonlinear irreducible complex characters of  $G$ . Then, as  $G$  has exactly  $[G : G']$  linear characters, we have  $|\text{Irr}_2(G)| = k(G) - [G : G']$  where  $k(G)$  is the number of conjugacy classes of  $G$ . We obtain

$$|G| = [G : G'] + \sum_{\chi \in \text{Irr}_2(G)} \chi(1)^2 \geq [G : G'] + 4(k(G) - [G : G']).$$

As  $d(G) = k(G)/|G|$ , it follows that

$$\frac{1}{|G'|} + 4(d(G) - \frac{1}{|G'|}) \leq 1.$$

Using the hypothesis  $d(G) > 5/16$ , we deduce that  $|G'| < 12$ . □

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* Assume that  $G$  is a finite group with  $d(G) > 5/16$  and  $G$  is not supersolvable. We aim to show that either  $G$  is not isoclinic to  $A_4$  or  $G/\mathbf{Z}(G)$  is isoclinic to  $A_4$ . Since commuting probability and supersolvability are both invariant under isoclinism, using Lemma 7, we can assume that  $\mathbf{Z}(G) \subseteq G'$ . Indeed, if  $\mathbf{Z}(G) = G'$ , then  $G$  is nilpotent which violates our assumption. So we assume furthermore that  $\mathbf{Z}(G) \subsetneq G'$ . Recall that  $d(G) > 5/16$  and hence  $|G'| \leq 11$  by Lemma 8. We note that  $G'$  is noncyclic as  $G$  is not supersolvable.

First we remark that  $S_3, D_8$  as well as  $D_{10}$  have a cyclic, characteristic, non-central subgroup and hence it is well-known that they cannot arise as commutator subgroups, see [13] for instance. Thus we are left with the following possibilities of  $G'$ .

**Case  $G' \cong C_2 \times C_2$ :** If  $\mathbf{Z}(G) \cong C_2$  then the normal series  $1 < \mathbf{Z}(G) < G' < G$  implies that  $G$  is supersolvable. So we assume that  $\mathbf{Z}(G) = 1$ . Thus  $G'$  is a minimal normal subgroup of  $G$ . Now, since  $G$  is not supersolvable,  $G' \not\subseteq \Phi(G)$ , see [8] for instance. Therefore,  $G'$  is not contained in a maximal subgroup  $M$ , say  $M$ . We have  $G = G'M$ . Also, as  $G'$  is abelian, we see that  $G' \cap M \triangleleft G$ . Now the minimality of  $G'$  and the fact that  $G'$  is not contained in  $M$  imply that  $G' \cap M = 1$ . This means  $G$  splits over  $G'$  or equivalently  $G \cong G' \rtimes M$ . Thus, as  $M \cong G/G'$  is abelian and  $\mathbf{Z}(G) = 1$ , we deduce that  $\mathbf{C}_M(G') = 1$ . It follows that

$$M \leq \text{Aut}(G') \cong \text{Aut}(C_2 \times C_2) \cong S_3,$$

and hence

$$M \cong C_2 \text{ or } M \cong C_3.$$

In the former case,  $|G| = 8$  and  $G$  would be nilpotent, a contradiction. In the latter case,  $G \cong A_4$  and we are done.

**Case  $G' \cong C_3 \times C_3$ :** As in the previous case, we can assume  $\mathbf{Z}(G) = 1$ ,  $G'$  is a minimal normal subgroup of  $G$ , and  $G \cong G' \rtimes M$ . Here  $M$  is an abelian subgroup of  $\text{Aut}(G') \cong \text{GL}_2(3)$ . Consulting the list of subgroups of  $\text{GL}_2(3)$  reveals:  $M \cong C_2, C_3, C_4, C_2 \times C_2, C_6$ , or  $C_8$ . However, it is routine to check that all these possibilities result in either  $G$  is supersolvable or  $d(G) \leq 5/16$ .

In the remaining cases, we let  $N$  be a minimal normal subgroup of  $G$  with  $N \subseteq G'$ . Recall that  $\mathbf{Z}(G) \subsetneq G'$  and so in the case  $\mathbf{Z}(G) \neq 1$ , we can even take  $N \subseteq \mathbf{Z}(G)$ .

**Case  $G' \cong Q_8, C_4 \times C_2$ , or  $C_2 \times C_2 \times C_2$  and  $N \cong C_2$ :** If  $N < \mathbf{Z}(G)$  then the normal series  $1 < N < \mathbf{Z}(G) < G' < G$  would imply that  $G$  is supersolvable, a contradiction. Thus  $\mathbf{Z}(G) = N \cong C_2$ . Also, as  $G$  is not supersolvable,  $G/\mathbf{Z}(G)$  is not supersolvable as well. It follows that  $(G/\mathbf{Z}(G))'$  is not cyclic so that

$$\left(\frac{G}{\mathbf{Z}(G)}\right)' \cong C_2 \times C_2.$$

By Lemma 6, we know that  $d(G/N) \geq d(G) > 5/16$ . Now we are in the first case with  $G/\mathbf{Z}(G)$  replacing  $G$ . Therefore, we conclude that either  $G/\mathbf{Z}(G)$  is supersolvable, a contradiction, or  $G/\mathbf{Z}(G)$  is isoclinic to  $A_4$ , as desired.

**Case  $G' \cong C_4 \times C_2$ , or  $C_2 \times C_2 \times C_2$  and  $N \cong C_2 \times C_2$ :** Then  $G'/N \cong C_2$  is a normal subgroup of  $G/N$ . In particular,  $G'/N \subseteq \mathbf{Z}(G/N)$  and as  $G/G'$  is abelian, we deduce that  $G/N$  is supersolvable. As in the previous case, by using the non-supersolvability of  $G$ , we deduce that  $G \cong N \rtimes M$  where  $M$  is a maximal subgroup of  $G$ . It then follows that  $C_2 \cong G' \cap M \triangleleft M$  and hence  $G' \cap M$  centralizes  $M$ , whence  $G' \cap M$  centralizes  $G$ . This implies that  $\mathbf{Z}(G) \neq 1$ , which in turn implies that  $N = \mathbf{Z}(G)$  since  $N \subseteq \mathbf{Z}(G) \subsetneq G'$ . This violates the minimality of  $N$ .

**Case  $N = G' \cong C_2 \times C_2 \times C_2$ :** Since  $\mathbf{Z}(G) \subsetneq G'$ , we obtain  $\mathbf{Z}(G) = 1$ . As before, we can show that  $G \cong G' \rtimes M$  where  $M$  is an abelian subgroup of  $\text{Aut}(G') \cong \text{GL}_3(2)$ . This implies that  $M \cong C_2, C_3, C_4, C_2 \times C_2$ , or  $C_7$ . The cases  $M \cong C_2, C_4$  or  $C_2 \times C_2$  would imply that  $G$  is a 2-group. The case  $M \cong C_7$  would imply that  $d(G) = 1/7$ . Finally, the case  $M \cong C_3$  implies that  $G \cong C_2 \times A_4$  and hence  $G' \cong C_2 \times C_2$ , which is the final contradiction.  $\square$

Now we prove Corollary 2.

*Proof of Corollary 2.* Assume, to the contrary, that the statement is false and let  $G$  be a minimal counterexample. Again we know that  $\mathbf{Z}(G) \subsetneq G'$  and as in the proof of Lemma 8, we also have  $|G'| \leq 9$ . Using Theorem 1, we deduce that  $G/\mathbf{Z}(G)$  is

isoclinic to  $A_4$  and  $\mathbf{Z}(G)$  is nontrivial. In particular,  $G'/\mathbf{Z}(G) \cong A'_4 \cong C_2 \times C_2$  and hence  $\mathbf{Z}(G) \cong C_2$ . Using Lemma 6(i), we have

$$\frac{1}{3} = d\left(\frac{G}{\mathbf{Z}(G)}\right) \geq d(G) \geq \frac{1}{3}.$$

Therefore  $d(G) = d(G/\mathbf{Z}(G))$ , which implies that  $\mathbf{Z}(G/\mathbf{Z}(G)) = \mathbf{Z}(G)/\mathbf{Z}(G) = 1$  by Lemma 6(iii). It follows that  $G/\mathbf{Z}(G) \cong A_4$  and as  $|\mathbf{Z}(G)| = 2$ , we have  $G = C_2 \times A_4$ . This violates the assumption that  $\mathbf{Z}(G) \not\subseteq G'$ .  $\square$

### 3. GROUPS OF ODD ORDER

We will prove Theorem 3 in this section. As in Section 2, we narrow down the possibility for the order of the commutator subgroup of a group in consideration.

**Lemma 9.** *Let  $G$  be an odd order finite group with  $d(G) > 35/243$ . Then  $|G'| < 27$ .*

*Proof.* We repeat some of the arguments in the proof of Lemma 8. Recall that  $\text{Irr}_2(G)$  denotes the set of nonlinear irreducible complex characters of  $G$  and we have  $|\text{Irr}_2(G)| = k(G) - [G : G']$ . Since  $|G|$  is odd, every character in  $\text{Irr}_2(G)$  has degree at least 3. We obtain

$$|G| = [G : G'] + \sum_{\chi \in \text{Irr}_2(G)} \chi(1)^2 \geq [G : G'] + 9(k(G) - [G : G']),$$

and therefore

$$\frac{1}{|G'|} + 9(d(G) - \frac{1}{|G'|}) \leq 1.$$

Since  $d(G) > 35/243$ , it follows that  $|G'| < 27$ , as wanted.  $\square$

*Proof of Theorem 3.* We argue by contradiction and let  $G$  be a minimal counterexample. Since commuting probability and supersolvability are both invariant under isoclinism, using Lemma 7, we can assume that  $\mathbf{Z}(G) \subseteq G'$ . Now  $G$  is a non-supersolvable group of odd order with  $d(G) > 35/243$ . Applying Lemma 9, we have  $|G'| < 27$  so that  $G'$  is a noncyclic odd order group of order at most 25.

We choose a minimal normal subgroup  $N$  of  $G$  with  $N \subseteq G'$ . and note that  $N$  is elementary abelian. By Lemma 6, we have  $d(G/N) \geq d(G) > 35/243$  so that  $G/N$  is supersolvable or isoclinic to  $(C_5 \times C_5) \rtimes C_3$  by the minimality of  $G$ .

First we show that the case where  $G/N$  is isoclinic to  $(C_5 \times C_5) \rtimes C_3$  cannot happen. Assume so. Then

$$G'/N = (G/N)' \cong ((C_5 \times C_5) \rtimes C_3)' = C_5 \times C_5,$$

which implies that  $|G'|$  is at least 50, a contradiction. We conclude that  $G/N$  is supersolvable. In particular, if  $N$  is cyclic then  $G$  is supersolvable and we have a contradiction. Note that, when  $G' \not\cong C_3 \times C_3$  and  $C_5 \times C_5$ , a routine check on groups of odd order at most 25 shows that  $N$  must be cyclic. Thus, it remains to consider



the cases where  $G' \cong C_3 \times C_3$  or  $G' \cong C_5 \times C_5$  and  $G'$  is a minimal normal subgroup of  $G$ .

If  $G' \subseteq \Phi(G)$  then  $G$  is nilpotent and we are done. Therefore we can assume that  $G' \not\subseteq \Phi(G)$  and, as in the proof of Theorem 1, we see that  $G$  splits over  $G'$  by a maximal subgroup  $M$  of  $G$ :

$$G \cong G' \rtimes M.$$

Since  $\mathbf{Z}(G) \subseteq G'$ , we must have  $\mathbf{Z}(G) = 1$  and, as  $M \cong G/G'$  is abelian, we deduce that  $\mathbf{C}_M(G') = 1$ . It follows that

$$M \leq \text{Aut}(G').$$

First we assume that  $G' \cong C_3 \times C_3$ , then  $M$  is an abelian subgroup of odd order of  $\text{Aut}(C_3 \times C_3) = \text{GL}_2(3)$ . This forces  $M \cong C_3$ , which implies that  $|G| = |G'| |M| = 9 \cdot 3 = 27$  and hence  $G$  is nilpotent, a contradiction. Next we assume that  $G' \cong C_5 \times C_5$ . Arguing similarly, we see that  $M$  is an abelian subgroup of odd order of  $\text{GL}_2(5)$ . This forces  $M \cong C_3, C_5$  or  $C_{15}$ . The case  $M \cong C_5$  would imply that  $G$  is nilpotent whereas the case  $M \cong C_{15}$  would imply that  $d(G) = 23/375$ , which is a contradiction. We conclude that  $M \cong C_3$  so that  $G \cong (C_5 \times C_5) \rtimes C_3$ .  $\square$

#### 4. A CONJUGACY CLASS SIZE THEOREM

In this section, we prove Theorem 4 and then give an example showing how one can obtain some properties of certain groups with ‘large’ commuting probability.

*Proof of Theorem 4.* Assume, to the contrary, that all the nontrivial orbits of the conjugacy action of  $G$  on  $N$  have size at least  $s$ . Since  $G$  splits over  $N$ , let  $G = HN$  where  $H \cap N = 1$ . We denote  $C = \mathbf{C}_H(N)$  and clearly  $C \triangleleft H$ . Every element of  $G$  can be written uniquely as  $ha$  where  $h \in H$  and  $a \in N$ . We now examine the class sizes in  $G$ .

First, let  $g = ha$  with  $a \neq 1$ . Since  $a$  is in an orbit of  $H$  on  $N$  of size greater than or equal to 3, we can find  $s - 1$  other elements  $a_2, a_3, \dots, a_s$  in the orbit of  $a$ . Therefore there exist  $t_2, t_3, \dots, t_s \in H$  such that  $a^{t_i} = a_i$  for  $2 \leq i \leq s$ . Thus

$$g = ha, g^{t_2} = h^{t_2} a_2, g^{t_3} = h^{t_3} a_3, \dots, g^{t_s} = h^{t_s} a_s$$

are  $s$  different elements in the conjugacy class of  $g$ . We now have

$$(4.1) \quad \text{every conjugacy class of an element outside } H \text{ has size at least } s.$$

It remains to consider the conjugacy classes of elements of  $H$ . Let  $g = h_1$  for some  $h_1 \in H$ . If  $h_1 \notin C$ , then there exists some  $a \in N$  which is not fixed by  $h_1$ . Thus

$$ah_1 a^{-1} = h_1 h_1^{-1} a h_1 a^{-1} = h_1 a_1, \text{ where } a_1 = h_1^{-1} a h_1 a^{-1} \neq 1.$$

By the previous paragraph, we know that there are  $t_2, t_3, \dots, t_s \in H$  such that

$$g = h_1, h_1 a_1, h_1^{t_2} a_2, h_1^{t_3} a_3, \dots, h_1^{t_s} a_s$$

are  $s + 1$  different elements in the conjugacy class of  $g$ , where  $a_i$ 's are nontrivial and distinct. Thus  $g$  is in a conjugacy class of size at least  $s + 1$ .

If  $h_1 \neq h_2$ ,  $h_1, h_2 \in H \setminus C$ , and  $h_1, h_2$  are in the same conjugacy class, then we know that

$$h_1, h_1 a_1, h_1^{t_2} a_2, h_1^{t_3} a_3, \dots, h_1^{t_s} a_s, h_2$$

are  $s + 2$  distinct elements in the same conjugacy class. This implies in general that if  $h_1, h_2, \dots, h_t$  are distinct elements in  $H \setminus C$  and  $h_1, h_2, \dots, h_t$  are in the same conjugacy class, then the size of the conjugacy class is greater than or equals to

$$s + t.$$

Denote  $k = |H|/|C|$ , then there are  $(k - 1)|C|$  elements in  $H \setminus C$ . We consider all the conjugacy classes of  $G$  which contain some elements in  $H$ . Suppose all the elements in  $H \setminus C$  belong to  $n$  different conjugacy classes and each conjugacy class contains  $t_1, \dots, t_n$  elements in  $H \setminus C$  respectively. Then

$$\sum_{i=1}^n t_i = |H \setminus C| = (k - 1)|C|$$

and the sum of sizes of these  $n$  classes is at least

$$\sum_{i=1}^n (s + t_i) = ns + \sum_{i=1}^n t_i.$$

Therefore, the average size of a conjugacy class of an element in  $H$  is at least

$$\frac{ns + \sum_{i=1}^n t_i + |C|}{n + |C|} = \frac{ns + k|C|}{n + |C|}.$$

Since  $C$  acts trivially on  $N$  and  $|H/C| = k$ , the conjugacy action of  $G$  on  $N$  has orbit of size at most  $k$ . Since  $N$  is nontrivial and every nontrivial orbit of  $G$  on  $N$  has size at least  $s$ , we deduce that  $k \geq s$ . It follows that

$$\frac{ns + k|C|}{n + |C|} \geq \frac{ns + s|C|}{n + |C|} = s$$

and hence

(4.2) the average size of a class of an element in  $H$  is at least  $s$ .

Combining (4.1) and (4.2), we conclude that the average class size of  $G$  is at least  $s$ , which violates the hypothesis that  $d(G) > 1/s$ .  $\square$

The following is an application of Theorem 4 to the study of finite groups with commuting probability greater than  $1/3$ .

**Corollary 10.** *Let  $G = (C_2 \times C_2) \rtimes H$  and assume that  $d(G) > 1/3$ . Then there exists a nontrivial element of  $C_2 \times C_2$  that is fixed under the conjugation action of  $H$ . In particular,  $\mathbf{Z}(G) \neq 1$ .*

*Proof.* By Theorem 4, the group  $H$  has a nontrivial orbit of size at most 2 on  $C_2 \times C_2$ . If this orbit has size 1 then we are done. Otherwise, it has size 2 and hence the other orbit must have size 1, as wanted.  $\square$

#### REFERENCES

- [1] F. Barry, D. MacHale, and Á. Ní Shé, Some supersolvability conditions for finite groups, *Math. Proc. R. Ir. Acad.* **106A** (2006), 163–177.
- [2] J.C. Bioch and R.W. van der Waall, Monomiality and isoclinism of groups, *J. Reine Angew. Math.* **298** (1978), 74-88.
- [3] P.X. Gallagher, The number of conjugacy classes in a finite group, *Math. Z.* **118** (1970), 175-179.
- [4] The GAP Group, *GAP - Groups, Algorithms, and Programming, Version 4.6.2*, 2013, <http://www.gap-system.org>.
- [5] R.M. Guralnick and G.R. Robinson, On the commuting probability in finite groups, *J. Algebra* **300** (2006), 509-528.
- [6] W.H. Gustafson, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973), 1031-1034.
- [7] P. Hall, The classification of prime-power groups, *J. Reine Angew. Math.* **182** (1940), 130-141.
- [8] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-New York, 1967.
- [9] I.M. Isaacs, M. Loukaki, and A. Moretó, The average degree of an irreducible character of a finite group, *Israel J. Math.*, to appear.
- [10] P. Lescot, Degré de commutativité et structure d'un groupe fini, *Rev. Math. Spéciales* **8** (1988), 276-279.
- [11] P. Lescot, Degré de commutativité et structure d'un groupe fini (2), *Rev. Math. Spéciales* **4** (1989), 200-202.
- [12] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177** (1995), 847-869.
- [13] D. MacHale and P. Ó Murchú, Commutator subgroups of groups with small central factor groups, *Proc. Roy. Irish Acad.* **93A** (1993), 123–129.