An explicit formulation of the Green's operator for general one-dimensional structures

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Abstract

A general procedure for the determination of the explicit expression of the Green's function of an ordinary differential operator is described in the paper, that requires only the knowledge of the null spaces of the operator and its adjoint. Although the main results concerning the structure of the Green's function are known in the literature, the proposed approach is suitable for symbolic mathematics systems and has been addressed in particular for frame structures including various beam models. The general expression of all the influence functions is given and some analytical results are presented.

Keywords Green function; Influence function; Frame; Beam

1. Introduction

As it is well known, the Green operator of a linear elliptic differential problem is the inverse operator that yields the solution as a linear map of the data (Courant and Hilbert, 1953). Of course, it gives a complete knowledge of the problem. In structural mechanics it has the immediate physical meaning of the response function of a system when a unit excitation is applied at an arbitrary point.

The use of Green's function in mechanics arises in many applications, as soil-structure and liquid-structure interaction, elastic–plastic problems involving the application of Colonnetti's principle (Kønig, 1967), elasticity and steady-state heat conduction problems in multiple connected layered bodies (Melnikov, 1997), computation of response functions (e.g., the response function appearing in the Duhamel integral), etc.

Noteworthy is Fichera's theory of orthogonal invariants of the Green's operator (Fichera, 1965, 1969, 1973) that yields the error estimate in the evaluation of eigenvalues (natural frequencies and critical loads) of an elastic system. This method has been applied to the case of frame structures by M. Romano (Romano, 1975, 1987) and others (Cuomo et al., 1987, 1989).

Explicit expressions of the Green function are known in very few cases, and in general approximate forms have to be evaluated. Exact expressions can always be obtained, at least in principle, for problems governed by an operator defined on subsets of $\mathcal{R}$. In this case, in fact, ordinary differential equations arise, and the dimension of the null space $\mathcal{N}$ of the differential operator L is finite, so that the algebra of finite dimensional vector spaces can be applied.

In structural mechanics one-dimensional problems are related to frames and trusses, but many other problems can be reformulated as sets of ODEs, like plane structures with axial symmetry, circular plates subjected to particular boundary conditions, infinite strips, etc.

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In the paper is derived an explicit expression for the Green's function of this class of structures, that requires only the knowledge of a basis for the null space of the equilibrium operator. First, some properties of the Green's function are recalled, then it is proved that the Green's function is given by a linear combination of $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$, L^* being the adjoint operator of L. This result is known in the literature (e.g. (Chitwood, 1973; Locker, 1977)), where attention is mainly focused on existence and representation results in the compatible case (i.e. when the solution to the homogeneous problem, if exists, is not unique), and for nth-order operators. Here some of the formalism used in (Chitwood, 1973; Locker, 1977) is adopted, but the focus of the paper is to give a constructive proof of the structure of the Green's operator for incompatible problems, where the solution exists and is unique. The proof given in the paper allows for a direct evaluation of the Green's operator given a basis of the kernel of the differential problem for any assigned two-points boundary conditions. The case of self-adjoint operators is then addressed, and it is recognized that a suitable base system is given by the fundamental shape functions of the elements of the structure, for clamped end (homogeneous) boundary conditions, and a simplified expression for the Green's function is derived for this case. Subsequently, the case of general frame structures is considered. The Green's operator is obtained splitting the problem in two parts, the first accounting for the nodal boundary conditions, the other related to the single element with homogeneous boundary conditions.

In the closing part of the paper the Green's operator is determined for several beam elements, and some graphs of influence functions computed by the proposed approach are reported as well.

2. Formulation of the problem for an operator on $\mathcal{R}$

2.1. General properties of the Green's function

Let L be a linear differential operator of order $2n$ in the variable v, $\mathcal{V}$-elliptic in a closed subspace $\mathcal{V}$ of $\mathcal{H}^n(\Omega)$:

$$L\big(v(x)\big) : \mathcal{D}(L) \subset \mathcal{V} \to \mathcal{L}^2(\Omega), \tag{1}$$

where $v(x) : \Omega \to \mathcal{R}^m$, $\Omega = [x_0, x_1] \subset \mathcal{R}$. Let suppose that the differential problem

$$L(v) = f, \quad f \in \mathcal{L}^2,\ v \in \mathcal{D}(L) \tag{2}$$

admits a solution.

Since v is a function defined over a scalar interval, the condition $v \in \mathcal{D}(L)$ states that the function v has to satisfy a set of $2n$ (linear) boundary conditions. If ρ is the number of linearly independent boundary conditions then the index of compatibility of problem (2) is $\mu = 2n - \rho$. In the following it will be assumed that $\mu = 0$, i.e. the problem admits a unique solution representable in the form (Chitwood, 1973)

$$v(x) = \int_{x_0}^{x_1} G(x, \xi)\, f(\xi)\, \mathrm{d}\xi, \tag{3}$$

$G(x, \xi) : \Omega \times \Omega \to \mathcal{D}(L)$ being the Green's function of the problem (2) with the prescribed boundary conditions.

The linear differential operator L may always be written in normal form; i.e. a matrix differential operator $\mathbf{L}$ of the first order in the vector variable $\mathbf{u} : \Omega \to \mathcal{R}^{2n}$ exists, such that $\mathbf{Lu}$ is the system of differential equations in normal form associated with L. The problem (2) is therefore formulated as

$$\mathbf{Lu} = \mathbf{f}, \quad \mathbf{f} \in \mathcal{L}^2\big(\mathcal{R}^{2n}\big), \tag{4}$$

$\mathbf{L}$ having always the form

$$\mathbf{Lu} = \mathbf{P}(x)\mathbf{u}_{,x} - \mathbf{Q}(x)\mathbf{u} \tag{5}$$

in which $\__{,x}$ is the derivative with respect to the independent variable x while $\mathbf{P}$ and $\mathbf{Q}$ are matrices of coefficients that, in the more general case, are functions of x. Furthermore $\mathbf{P}$ is nonsingular on the whole domain Ω.

The matrix $\mathbf{P}$ in (5) can always be put, without loss of generality, in the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{0} & \mathbf{P}_{10} \\ \mathbf{P}_{01} & \mathbf{0} \end{bmatrix} \tag{6}$$

with the elements of $\mathbf{P}_{10}$ and $\mathbf{P}_{01}$ equal to 0 or ± 1.

In the following it will be determined the Green's function associated with the operator $\mathbf{L}$ together with homogeneous boundary conditions of the form

$$\mathbf{B}_0\mathbf{u}(x_0) + \mathbf{B}_1\mathbf{u}(x_1) = \mathbf{0}, \tag{7}$$

where $\mathbf{B}_0$ and $\mathbf{B}_1$ are $2n \times 2n$ matrices of constants. The compatibility index of the problem (4) with the boundary conditions (7) is

$$\mu = 2n - \operatorname{rank}\big[\mathbf{B}_0\mathbf{u}(x_0) + \mathbf{B}_1\mathbf{u}(x_1)\big] \tag{8}$$

and therefore the condition that has to be verified in order that an unique solution exists is that $\operatorname{rank}[\mathbf{B}_0\mathbf{u}(x_0) + \mathbf{B}_1\mathbf{u}(x_1)] = 2n$. The null space of $\mathbf{L}$ is given by all and only the solutions of the homogeneous problem

$$\mathbf{Lu} = \mathbf{0}. \tag{9}$$

Before proceeding to the construction of the Green's function for the differential problem

$$\begin{cases} \mathbf{Lu} = \mathbf{f}(x), \\ \mathbf{B}_0\mathbf{u}(x_0) + \mathbf{B}_1\mathbf{u}(x_1) = \mathbf{0}, \end{cases} \tag{10}$$

it is useful to restate some properties of the Green's function in the case of a linear vector differential operator defined on a subset Ω of $\mathcal{R}$.

Theorem 1 (Properties of the Green's function). *Let $\mathcal{C}^0_{2n}(\Omega)$ be the space of the $2n \times 2n$ matrices whose coefficients are continuous functions of Ω. The Green's function $\mathbf{G}(x,\xi)$ of the problem* (10) *with zero compatibility index has the following properties*:

(1) $\mathbf{G}(x,\xi)$ *is defined on $\Omega \times \Omega$ and has values on $\mathcal{R}^{2n}$;*
(2) $\mathbf{G}(x,\xi)$ *considered as a function of x, satisfies the boundary conditions* (7);
(3) $\mathbf{G}(x,\xi)$ *considered as a function of x, belongs to $\mathcal{C}^0_{2n}(\Omega)$;*
(4) $\mathbf{G}(x,\xi)$ *considered as a function of x, satisfies* (4) *in $\Omega - \{\xi\}$;*
(5) $\lim_{h\to 0} \mathbf{G}(x, x+h) - \mathbf{G}(x, x-h) = -\mathbf{P}^{-1}$; $\mathbf{P}$ *being defined in* (5).

Proof. If $\mathbf{u}(x)$ is a solution to problem (10) then, because of the definition of Green's function, it can be represented as

$$\mathbf{u}(x) = \int_{x_0}^{x_1} \mathbf{G}(x, y)\,\mathbf{f}(\xi)\,\mathrm{d}\xi. \tag{11}$$

Let us prove then that if we consider a function $\widetilde{\mathbf{G}}$ for which the properties (1)–(5) hold then the function

$$\mathbf{u}(x) = \int_{x_0}^{x_1} \widetilde{\mathbf{G}}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi \tag{12}$$

is a solution of the differential boundary problem (10). Furthermore because of the uniqueness of the solution it results $\mathbf{G} = \widetilde{\mathbf{G}}$.

In fact the boundary conditions (7) are satisfied because of property (2) and, differentiating (12) with respect to x one has

$$\begin{aligned} \mathbf{u}_{,x} &= \int_{x_0}^{x} \widetilde{\mathbf{G}}_{,x}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi + \widetilde{\mathbf{G}}_{,x}(x,x^-)\mathbf{f}(x) + \int_{x}^{x_1} \widetilde{\mathbf{G}}_{,x}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi - \widetilde{\mathbf{G}}_{,x}(x,x^+)\mathbf{f}(x) \\ &= \int_{x_0}^{x_1} \widetilde{\mathbf{G}}_{,x}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi + \mathbf{P}^{-1}\mathbf{f}(x). \end{aligned} \tag{13}$$

Then the evaluation of (5) yields

$$\mathbf{Lu} = \mathbf{P}\int_{x_0}^{x_1} \widetilde{\mathbf{G}}_{,x}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi + \mathbf{P}\mathbf{P}^{-1}\mathbf{f}(x) - \mathbf{Q}\int_{x_0}^{x_1} \widetilde{\mathbf{G}}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi, \tag{14}$$

that is,

$$\mathbf{Lu} = \int_{x_0}^{x_1} \big(\mathbf{P}\widetilde{\mathbf{G}}_{,x}(x,\xi) - \mathbf{Q}\widetilde{\mathbf{G}}\big)\mathbf{f}(\xi)\,\mathrm{d}\xi + \mathbf{f}(x). \tag{15}$$

Since $\widetilde{\mathbf{G}}$ by property (4) satisfies the homogeneous equation $\mathbf{L}\widetilde{\mathbf{G}}(x,\xi)=0,\ \forall x\in\Omega-\{\xi\}$, it is readily obtained the result

$$\mathbf{Lu}=\mathbf{f}(x). \tag{16}$$

The function (12) is therefore the solution of the problem (10).

The sufficiency of conditions (1)–(5) for a function $\mathbf{u}$ given by (11) to be the solution of problem (10) can be obtained going through the same steps in inverse order. □

The formal adjoint operator $\mathbf{L}^*$ to $\mathbf{L}$ is given by

$$\mathbf{L}^*=-\mathbf{P}^{\mathrm{T}}\frac{\partial}{\partial x}-\mathbf{Q}^{\mathrm{T}}, \tag{17}$$

where the superscript $^{\mathrm{T}}$ denotes the transpose. The null spaces of $\mathbf{L}$, $\mathbf{L}^*$ are related by the property stated in the following theorem:

Theorem 2 (Property of the elements of the null spaces $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$). *Let $\mathbf{u}\in\mathcal{N}(\mathbf{L})$ be an arbitrary element of the null space of the differential operator $\mathbf{L}$ and $\mathbf{w}\in\mathcal{N}(\mathbf{L}^*)$ be an arbitrary element of the null space of the adjoint differential operator $\mathbf{L}^*$. It results*

$$\mathbf{Pu}\cdot\mathbf{w}=\mathbf{const},\quad \forall x\in\Omega. \tag{18}$$

Proof. Let $\mathbf{u}$ and $\mathbf{w}$ be two $\mathcal{R}^{2n}$-valued derivable functions defined on Ω. The standard dot product in $\mathcal{R}^{2n}$ of the vectors $\mathbf{Lu}$ and $\mathbf{w}$ yields

$$\begin{aligned}\mathbf{Lu}\cdot\mathbf{w} &= (\mathbf{Pu}_{,x}-\mathbf{Qu})\cdot\mathbf{w}=\mathbf{Pu}_{,x}\cdot\mathbf{w}-\mathbf{Qu}\cdot\mathbf{w}=(\mathbf{Pu}\cdot\mathbf{w})_{,x}-\mathbf{Pu}\cdot\mathbf{w}_{,x}-\mathbf{Qu}\cdot\mathbf{w}\\ &= (\mathbf{Pu}\cdot\mathbf{w})_{,x}-\mathbf{P}^{\mathrm{T}}\mathbf{w}_{,x}\cdot\mathbf{u}-\mathbf{Q}^{\mathrm{T}}\mathbf{w}\cdot\mathbf{u}=(\mathbf{Pu}\cdot\mathbf{w})_{,x}+\mathbf{L}^*\mathbf{w}\cdot\mathbf{u},\end{aligned} \tag{19}$$

therefore,

$$\mathbf{Lu}\cdot\mathbf{w}-\mathbf{L}^*\mathbf{w}\cdot\mathbf{u}=(\mathbf{Pu}\cdot\mathbf{w})_{,x}\,. \tag{20}$$

If $\mathbf{u}$ and $\mathbf{w}$ are respectively two arbitrary elements of $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$, the first member of (20) is identically equal to zero and it is

$$(\mathbf{Pu}\cdot\mathbf{w})_{,x}=0 \quad \text{from which} \tag{21}$$

$$\mathbf{Pu}\cdot\mathbf{w}=\mathbf{const},\quad \forall x\in\Omega, \tag{22}$$

which completes the proof. □

Corollary 1 (Property of the bases of $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$). *Let $\boldsymbol{\Phi}$ be a $2n\times 2n$ matrix whose columns $\mathbf{u}^{(i)}$ are a basis of $\mathcal{N}(\mathbf{L})$ and $\boldsymbol{\Phi}^*$ be a $2n\times 2n$ matrix whose columns $\mathbf{w}^{(j)}$ are a basis of $\mathcal{N}(\mathbf{L}^*)$. The product*

$$\boldsymbol{\Lambda}=-\boldsymbol{\Phi}^*(x)^{\mathrm{T}}\mathbf{P}\boldsymbol{\Phi}(x)\quad \left(\boldsymbol{\Lambda}_{ij}=\mathbf{Pu}^{(i)}\cdot\mathbf{w}^{(j)}\right),\quad \forall x\in\Omega, \tag{23}$$

is a nonsingular matrix of constants.

Proof. The matrix $\boldsymbol{\Lambda}$ is nonsingular since it is the product of nonsingular matrices. The result is readily obtained from the property expressed in (22) which applies to any element of the matrix. □

2.2. Construction of the Green's function

It will now be presented a constructive proof of the property that the Green's function can always be expressed as a bilinear combination of a basis for the null spaces of the direct operator and its adjoint.

Because of properties (3)–(5) the Green's function of the differential operator $\mathbf{L}$ with the boundary conditions (7) may be written in the following form:

$$\mathbf{G}(x,\xi)=\begin{cases}\mathbf{G}_1(x,\xi), & x\leqslant\xi\\ \mathbf{G}_2(x,\xi), & x\geqslant\xi\end{cases}=\begin{cases}\boldsymbol{\Phi}(x)\boldsymbol{\Gamma}_1(\xi), & x\leqslant\xi,\\ \boldsymbol{\Phi}(x)\boldsymbol{\Gamma}_2(\xi), & x\geqslant\xi,\end{cases} \tag{24}$$

where $\boldsymbol{\Gamma}_1(\xi)$ and $\boldsymbol{\Gamma}_2(\xi)$ are matrices of unknown functions. In fact the function $\mathbf{G}$ defined in (24) belongs to $C^0_{2n}(\Omega - \{\xi\})$ (property 3) as $\boldsymbol{\Phi}(x) \in C^1_{2n}(\Omega)$ and the matrices $\boldsymbol{\Gamma}_1(\xi)$ and $\boldsymbol{\Gamma}_2(\xi)$ are, for fixed ξ, constant matrices; moreover, as the columns of $\mathbf{G}_1$ and $\mathbf{G}_2$ are linear combinations of the columns of $\boldsymbol{\Phi}$, it results $\mathbf{LG} = \mathbf{0}$ as requested by property (4).

The matrices $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ can be determined enforcing properties (2) and (5). From property (5) at the point $x = \xi$ it must be

$$\mathbf{G}_1(\xi,\xi) - \mathbf{G}_2(\xi,\xi) = -\mathbf{P}^{-1} \tag{25}$$

and, by substitution of definition (24),

$$\boldsymbol{\Phi}(\xi)\big[\boldsymbol{\Gamma}_1(\xi) - \boldsymbol{\Gamma}_2(\xi)\big] = -\mathbf{P}^{-1} \tag{26}$$

letting $\boldsymbol{\Gamma}(\xi) = \boldsymbol{\Gamma}_2(\xi) - \boldsymbol{\Gamma}_1(\xi)$,

$$\boldsymbol{\Phi}(\xi)\boldsymbol{\Gamma}(\xi) = \mathbf{P}^{-1}, \tag{27}$$

$$\big(\mathbf{P}\boldsymbol{\Phi}(\xi)\big)\boldsymbol{\Gamma}(\xi) = \mathbf{I}, \tag{28}$$

and, taking the dot product of each side by $\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}$,

$$\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}\big(\mathbf{P}\boldsymbol{\Phi}(\xi)\big)\boldsymbol{\Gamma}(\xi) = \boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}, \tag{29}$$

it is finally obtained, using (23):

$$\boldsymbol{\Gamma}(\xi) = -\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}. \tag{30}$$

So requiring that the function defined by (24) satisfy property (5), the difference $\boldsymbol{\Gamma}(\xi) = \boldsymbol{\Gamma}_2(\xi) - \boldsymbol{\Gamma}_1(\xi)$ has been uniquely determined. Moreover, $\mathbf{G}$, because of property (2), must fulfil the boundary conditions (7):

$$\mathbf{B}_0\mathbf{G}_1(x_0,\xi) + \mathbf{B}_1\mathbf{G}_2(x_1,\xi) = \mathbf{0}, \quad \forall \xi \in \Omega. \tag{31}$$

Substituting definition (24)

$$\mathbf{B}_0\boldsymbol{\Phi}(x_0)\boldsymbol{\Gamma}_1(\xi) + \mathbf{B}_1\boldsymbol{\Phi}(x_1)\boldsymbol{\Gamma}_2(\xi) = \mathbf{0}, \quad \forall \xi \in \Omega. \tag{32}$$

Expressing $\boldsymbol{\Gamma}_2(\xi)$ as a function of $\boldsymbol{\Gamma}(\xi)$ and $\boldsymbol{\Gamma}_1(\xi)$, the following linear system of equations is obtained:

$$\big[\mathbf{B}_0\boldsymbol{\Phi}(x_0) + \mathbf{B}_1\boldsymbol{\Phi}(x_1)\big]\boldsymbol{\Gamma}_1(\xi) = -\mathbf{B}_1\boldsymbol{\Phi}(x_1)\boldsymbol{\Gamma}(\xi). \tag{33}$$

Since $\mu = 0$, the matrix

$$\mathbf{H} = \mathbf{B}_0\boldsymbol{\Phi}(x_0) + \mathbf{B}_1\boldsymbol{\Phi}(x_1) \tag{34}$$

is nonsingular, so (33) admits the unique solution

$$\boldsymbol{\Gamma}_1(\xi) = -\mathbf{H}^{-1}\mathbf{B}_1\boldsymbol{\Phi}(x_1)\boldsymbol{\Gamma}(\xi) \tag{35}$$

which may be written, introducing (30),

$$\boldsymbol{\Gamma}_1(\xi) = \mathbf{H}^{-1}\mathbf{B}_1\boldsymbol{\Phi}(x_1)\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}. \tag{36}$$

A similar expression holds for $\boldsymbol{\Gamma}_2(\xi)$:

$$\boldsymbol{\Gamma}_2(\xi) = -\mathbf{H}^{-1}\mathbf{B}_0\boldsymbol{\Phi}(x_0)\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}. \tag{37}$$

Introducing the constant matrices

$$\mathbf{C} = \mathbf{H}^{-1}\mathbf{B}_1\boldsymbol{\Phi}(x_1)\boldsymbol{\Lambda}^{-1}, \tag{38}$$

$$\mathbf{D} = -\mathbf{H}^{-1}\mathbf{B}_0\boldsymbol{\Phi}(x_0)\boldsymbol{\Lambda}^{-1}, \tag{39}$$

it is possible to write the Green's function (24) in the form

$$\mathbf{G}(x,\xi) = \begin{cases} \boldsymbol{\Phi}(x)\mathbf{C}\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}, & x \leqslant \xi, \\ \boldsymbol{\Phi}(x)\mathbf{D}\boldsymbol{\Phi}^*(\xi)^{\mathrm{T}}, & x \geqslant \xi, \end{cases} \tag{40}$$

which is a bilinear form of a basis of $\mathcal{N}(\mathbf{L})$ and a basis of $\mathcal{N}(\mathbf{L}^*)$.

It is observed that the evaluation of the matrix $\mathbf{C}$ (as well as $\mathbf{D}$) may be carried out with respect to any arbitrary basis of $\mathcal{N}(\mathbf{L})$ and then transformed to the basis chosen to represent the Green's function. In fact let $\boldsymbol{\Phi}(x)$ and $\boldsymbol{\Theta}(x)$ be two bases and let $\mathbf{Z}$ be the linear transformation such that

$$\boldsymbol{\Phi}^{\mathrm{T}}(x) = \mathbf{Z}\boldsymbol{\Theta}^{\mathrm{T}}(x) \tag{41}$$

and suppose that $\mathbf{C}_\Phi$ is the expression of $\mathbf{C}$ with respect to the basis $\boldsymbol{\Phi}$. It is readily found, by substitution, that

$$\mathbf{C}_\Theta = \mathbf{Z}^{\mathrm{T}}\mathbf{C}_\Phi\mathbf{Z}. \tag{42}$$

In the previous section it has been treated the general case of a differential operator, and the role of its adjoint has resulted evident. In the following the result will be particularized to the case of a self-adjoint operator. In this case, being $\mathbf{L} = \mathbf{L}^*$, from (17) it is obtained

$$\mathbf{P}\frac{\partial}{\partial x}\mathbf{u} - \mathbf{Q}\mathbf{u} = -\mathbf{P}^{\mathrm{T}}\frac{\partial}{\partial x}\mathbf{u} - \mathbf{Q}^{\mathrm{T}}\mathbf{u}, \quad \forall \mathbf{u}, \tag{43}$$

$$\left(\mathbf{P} + \mathbf{P}^{\mathrm{T}}\right)\frac{\partial}{\partial x}\mathbf{u} - \left(\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\right)\mathbf{u} = \mathbf{0}, \quad \forall \mathbf{u}; \tag{44}$$

these imply

$$\mathbf{P} = -\mathbf{P}^{\mathrm{T}}, \qquad \mathbf{Q} = \mathbf{Q}^{\mathrm{T}}, \tag{45}$$

that is, $\mathbf{P}$ is skew-symmetric and $\mathbf{Q}$ is symmetric. In particular, (6) and (45) imply $\mathbf{P}_{01} = -\mathbf{P}_{10}^{\mathrm{T}}$. Without loss of generality it is always possible to write the system in normal form so that $\mathbf{P}_{10} = -\mathbf{P}_{01} = \mathbf{I}$ can be obtained.

Moreover, being $\boldsymbol{\Phi}^* = \boldsymbol{\Phi}$, from (23) it follows that the matrix of constants $\boldsymbol{\Lambda}$ is skew symmetric:

$$\boldsymbol{\Lambda}^{\mathrm{T}} = \left[-\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}\boldsymbol{\Phi}\right]^{\mathrm{T}} = -\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{P}\boldsymbol{\Phi} = -\boldsymbol{\Lambda}. \tag{46}$$

In this case the Green's function is symmetric with respect to the variables x and ξ. In fact let $\mathbf{p}$ and $\mathbf{f}$ be two arbitrary vectors of functions generally continuous on the domain Ω. Because of existence and uniqueness of the solution

$$\exists!\mathbf{u}: \quad \mathbf{L}\mathbf{u} = \mathbf{p}, \quad \forall x \in \Omega, \tag{47}$$

$$\exists!\mathbf{v}: \quad \mathbf{L}\mathbf{v} = \mathbf{f}, \quad \forall x \in \Omega, \tag{48}$$

given by

$$\mathbf{u} = \int_{\Omega} \mathbf{G}(x,\xi)\mathbf{p}(\xi)\,\mathrm{d}\xi, \tag{49}$$

$$\mathbf{v} = \int_{\Omega} \mathbf{G}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi. \tag{50}$$

As $\mathbf{L}$ is self-adjoint because of the duality relation

$$\langle \mathbf{L}\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{L}\mathbf{v}, \mathbf{u}\rangle, \quad \forall \mathbf{u}, \mathbf{v}, \tag{51}$$

substituting Eqs. (47), (48) and (49), (50)

$$\int_{\Omega\times\Omega} \mathbf{p}(x)\cdot\mathbf{G}(x,\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_{\Omega\times\Omega} \mathbf{f}(x)\cdot\mathbf{G}(x,\xi)\mathbf{p}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x, \quad \forall \mathbf{p}, \mathbf{f}. \tag{52}$$

Equation (52) implies

$$\mathbf{G}^{\mathrm{T}}(x,\xi) = \mathbf{G}(\xi,x), \quad \forall\{x,\xi\} \in \Omega\times\Omega. \tag{53}$$

Introducing (53) in (40) and being $\boldsymbol{\Phi}^* = \boldsymbol{\Phi}$, it is obtained

$$\mathbf{G}_1^{\mathrm{T}}(x,\xi) = \mathbf{G}_2(\xi,x), \quad \forall\{x,\xi\} \in \Omega\times\Omega, \tag{54}$$

that is,

$$\left[\boldsymbol{\Phi}(x)\mathbf{C}\boldsymbol{\Phi}^{\mathrm{T}}(\xi)\right]^{\mathrm{T}} = \boldsymbol{\Phi}(\xi)\mathbf{D}\boldsymbol{\Phi}^{\mathrm{T}}(x), \quad \forall\{x,\xi\} \in \Omega\times\Omega, \tag{55}$$

$$\boldsymbol{\Phi}(\xi)\mathbf{C}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}(x) = \boldsymbol{\Phi}(\xi)\mathbf{D}\Phi^{\mathrm{T}}(x), \quad \forall\{x,\xi\} \in \Omega\times\Omega. \tag{56}$$

The matrix $\mathbf{D}$ is therefore the transpose of $\mathbf{C}$. In the hypothesis that $\mathbf{L}$ is self-adjoint the Green's function expressed by (40) has then the form

$$\mathbf{G}(x,\xi) = \begin{cases} \boldsymbol{\Phi}(x)\mathbf{C}\boldsymbol{\Phi}^{\mathrm{T}}(\xi), & x \leqslant \xi, \\ \boldsymbol{\Phi}(x)\mathbf{C}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}(\xi), & x \geqslant \xi. \end{cases} \tag{57}$$

The matrix $\mathbf{C}$ can be determined, by virtue of (38), with respect to any basis of $\mathcal{N}(\mathbf{L})$. However, if some assumptions are made in the determination of the basis the final expression of $\mathbf{C}$ can be greatly simplified. In order to reach this result the concept of canonical basis will be introduced in this section. Let

$$\mathbf{B}_0 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{58}$$

A basis $\boldsymbol{\Sigma}(x)$ of $\mathcal{N}(\mathbf{L})$ will be called *canonical* if it satisfies the boundary conditions

$$\mathbf{H} = \mathbf{B}_0 \boldsymbol{\Sigma}(x_0) + \mathbf{B}_1 \boldsymbol{\Sigma}(x_1) = \mathbf{I}. \tag{59}$$

If the basis $\boldsymbol{\Sigma}(x)$ is partitioned into four $n \times n$ matrices as

$$\boldsymbol{\Sigma}(x) = \begin{bmatrix} \boldsymbol{\Sigma}_{00}(x) & \boldsymbol{\Sigma}_{10}(x) \\ \boldsymbol{\Sigma}_{01}(x) & \boldsymbol{\Sigma}_{11}(x) \end{bmatrix} \tag{60}$$

then $\boldsymbol{\Sigma}(x)$ will be canonical if it satisfies the following boundary conditions:

$$\boldsymbol{\Sigma}_{00}(x_0) = \mathbf{I}, \qquad \boldsymbol{\Sigma}_{10}(x_0) = \mathbf{0}, \qquad \boldsymbol{\Sigma}_{00}(x_1) = \mathbf{0}, \qquad \boldsymbol{\Sigma}_{10}(x_1) = \mathbf{I}. \tag{61}$$

In the context of solid mechanics the canonical basis is formed by the extended shape functions of the element on the interval $[x_0, x_1]$, i.e. the integrals $\boldsymbol{\Sigma}_0(x)$ and $\boldsymbol{\Sigma}_1(x)$ of the canonical basis $\boldsymbol{\Sigma}(x)$,

$$\boldsymbol{\Sigma}_0(x) = \begin{bmatrix} \boldsymbol{\Sigma}_{00}(x) \\ \boldsymbol{\Sigma}_{01}(x) \end{bmatrix}, \qquad \boldsymbol{\Sigma}_1(x) = \begin{bmatrix} \boldsymbol{\Sigma}_{10}(x) \\ \boldsymbol{\Sigma}_{11}(x) \end{bmatrix}, \tag{62}$$

satisfy the essential homogeneous boundary conditions at the ends $x = x_1$ and $x = x_0$, respectively, that, after introducing the matrix

$$\mathbf{B} = [\,\mathbf{I} \quad \mathbf{0}\,] \tag{63}$$

can be written in a short form as

$$\mathbf{B}\boldsymbol{\Sigma}_0(x_0) = \mathbf{I}, \qquad \mathbf{B}\boldsymbol{\Sigma}_1(x_1) = \mathbf{I}, \tag{64}$$

$$\mathbf{B}\boldsymbol{\Sigma}_1(x_0) = \mathbf{0}, \qquad \mathbf{B}\boldsymbol{\Sigma}_0(x_1) = \mathbf{0}. \tag{65}$$

If the differential system (5) is written in a normal form it is easy to see that the adjoint differential operator is given by the differential operator (17) with the boundary conditions still given by (58). Therefore, using notation similar to (60) the canonical basis $\boldsymbol{\Sigma}^*(x)$ of the kernel of the adjoint operator $\mathbf{L}^*$ satisfies

$$\boldsymbol{\Sigma}^*_{00}(x_0) = \mathbf{I}, \qquad \boldsymbol{\Sigma}^*_{10}(x_0) = \mathbf{0}, \qquad \boldsymbol{\Sigma}^*_{00}(x_1) = \mathbf{0}, \qquad \boldsymbol{\Sigma}^*_{10}(x_1) = \mathbf{I}. \tag{66}$$

If the determination of the Green's function is made with respect to the canonical bases of $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$, it is possible to derive a simpler expression for the constant matrices $\mathbf{C}$ and $\mathbf{D}$. To this end the expression of the Green's function (40) for $x \leqslant \xi$ is written introducing a suitable partition of the matrix $\mathbf{C}$:

$$\begin{aligned} \mathbf{G}(x,\xi) &= \big[\boldsymbol{\Sigma}_0(x), \boldsymbol{\Sigma}_1(x)\big] \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{10} \\ \mathbf{C}_{01} & \mathbf{C}_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}^*_0(\xi)^{\mathrm{T}} \\ \boldsymbol{\Sigma}^*_1(\xi)^{\mathrm{T}} \end{bmatrix} \\ &= \boldsymbol{\Sigma}_0(x)\big[\mathbf{C}_{00}\boldsymbol{\Sigma}^*_0(\xi)^{\mathrm{T}} + \mathbf{C}_{10}\boldsymbol{\Sigma}^*_1(\xi)^{\mathrm{T}}\big] + \boldsymbol{\Sigma}_1(x)\big[\mathbf{C}_{01}\boldsymbol{\Sigma}^*_0(\xi)^{\mathrm{T}} + \mathbf{C}_{11}\boldsymbol{\Sigma}^*_1(\xi)^{\mathrm{T}}\big]. \end{aligned} \tag{67}$$

As the Green's function satisfies the boundary conditions, in view of the structure (58) of the boundary conditions, it is $\mathbf{BG}(x_0,\xi) = \mathbf{0},\ \forall\xi$, $\mathbf{BG}(x_1,\xi) = \mathbf{0},\ \forall\xi$. The following results can be readily obtained using Eqs. (61) and (66):

$$\mathbf{BG}(x_0,x_0) = \mathbf{0} \implies \mathbf{C}_{00} = \mathbf{0}, \tag{68}$$

$$\mathbf{BG}(x_0,x_1) = \mathbf{0} \implies \mathbf{C}_{10} = \mathbf{0}, \tag{69}$$

$$\mathbf{BG}(x_1,x_1) = \mathbf{0} \implies \mathbf{C}_{11} = \mathbf{0}. \tag{70}$$

Similarly writing the Green's function for $x \geqslant \xi$ and introducing for $\mathbf{D}$ the same partition as in (67) it is obtained

$$\mathbf{BG}(x_0,x_0) = \mathbf{0} \implies \mathbf{D}_{00} = \mathbf{0}, \tag{71}$$

$$\mathbf{BG}(x_1,x_0) = \mathbf{0} \implies \mathbf{D}_{01} = \mathbf{0}, \tag{72}$$

$$\mathbf{BG}(x_1,x_1) = \mathbf{0} \implies \mathbf{D}_{11} = \mathbf{0}. \tag{73}$$

The structures of $\mathbf{C}$ and $\mathbf{D}$ with respect to the canonical bases of $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$ are therefore

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{01} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{10} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{74}$$

and the following final expression of the Green's function can then be given:

$$\mathbf{G}(x,\xi) = \begin{cases} \boldsymbol{\Sigma}_1(x)\mathbf{C}_{01}\boldsymbol{\Sigma}_0^*(\xi)^{\mathrm{T}}, & x \leqslant \xi, \\ \boldsymbol{\Sigma}_0(x)\mathbf{D}_{10}\boldsymbol{\Sigma}_1^*(\xi)^{\mathrm{T}}, & x \geqslant \xi. \end{cases} \tag{75}$$

Moreover, recalling (59), the following equalities hold:

$$\mathbf{H}^{-1}\mathbf{B}_1\boldsymbol{\Sigma}(x_1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{H}^{-1}\mathbf{B}_0\boldsymbol{\Sigma}(x_0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{76}$$

Partitioning the matrix $\boldsymbol{\Lambda}^{-1}$ as

$$\boldsymbol{\Lambda}^{-1} = \begin{bmatrix} (\boldsymbol{\Lambda}^{-1})_{00} & (\boldsymbol{\Lambda}^{-1})_{10} \\ (\boldsymbol{\Lambda}^{-1})_{01} & (\boldsymbol{\Lambda}^{-1})_{11} \end{bmatrix}, \tag{77}$$

evaluating the matrices $\mathbf{C}$ and $\mathbf{D}$ with respect to the canonical bases and using Eqs. (38), (39) and (76), (77), the following expressions are got:

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\boldsymbol{\Lambda}^{-1})_{01} & (\boldsymbol{\Lambda}^{-1})_{11} \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} (\boldsymbol{\Lambda}^{-1})_{00} & (\boldsymbol{\Lambda}^{-1})_{10} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{78}$$

so that by virtue of (74) the matrices $\mathbf{C}_{01}$ and $\mathbf{D}_{10}$ are equal to

$$\mathbf{C}_{01} = (\boldsymbol{\Lambda}^{-1})_{01}, \tag{79}$$

$$\mathbf{D}_{10} = -(\boldsymbol{\Lambda}^{-1})_{10}, \tag{80}$$

while it must be

$$(\boldsymbol{\Lambda}^{-1})_{00} = \mathbf{0}, \tag{81}$$

$$(\boldsymbol{\Lambda}^{-1})_{11} = \mathbf{0}. \tag{82}$$

In the case of self-adjoint operators it is $\mathbf{D}^{\mathrm{T}} = \mathbf{C}$ and, therefore, $\mathbf{D}_{10}^{\mathrm{T}} = \mathbf{C}_{01}$, as follows from (79) and (80), since in this case $\boldsymbol{\Lambda}$ is skew symmetric.

2.5. Computation of the constant matrices with respect to canonical bases

As shown in the previous section, the matrices $\mathbf{C}_{01}$ and $\mathbf{D}_{10}$ fully define the Green's function of a differential operator with respect to the canonical bases of $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}^*)$. Their computation can be carried out using (79) and (80) either symbolically or numerically, evaluating $\boldsymbol{\Lambda}^{-1}$ at any point $x \in \Omega$ (as $\boldsymbol{\Lambda}$ has been shown to be independent from x). A simpler formula can, however, be determined for $\mathbf{C}_{01}$ and $\mathbf{D}_{10}$ using the properties of the canonical bases.

Recalling the structure of the matrix $\mathbf{P}$, Eq. (6), the constant matrix $\boldsymbol{\Lambda}$ can be written in the form

$$\boldsymbol{\Lambda} = -\boldsymbol{\Sigma}^*(x)^{\mathrm{T}}\mathbf{P}\boldsymbol{\Sigma}(x) = -\begin{bmatrix} \boldsymbol{\Sigma}_{00}^*(x)^{\mathrm{T}} & \boldsymbol{\Sigma}_{01}^*(x)^{\mathrm{T}} \\ \boldsymbol{\Sigma}_{10}^*(x)^{\mathrm{T}} & \boldsymbol{\Sigma}_{11}^*(x)^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{P}_{10} \\ \mathbf{P}_{01} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_{00}(x) & \boldsymbol{\Sigma}_{10}(x) \\ \boldsymbol{\Sigma}_{01}(x) & \boldsymbol{\Sigma}_{11}(x) \end{bmatrix}. \tag{83}$$

Carrying out the products, computing $\boldsymbol{\Lambda}$ for $x = x_0$ and $x = x_1$, accounting for the boundary conditions (61), (66) and equating the results, the following equivalent expressions for $\boldsymbol{\Lambda}$ are found:

$$\boldsymbol{\Lambda} = -\begin{bmatrix} \mathbf{0} & \mathbf{P}_{10}\boldsymbol{\Sigma}_{11}(x_0) \\ \boldsymbol{\Sigma}_{11}^*(x_0)^{\mathrm{T}}\mathbf{P}_{01} & \mathbf{0} \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & \boldsymbol{\Sigma}_{01}^*(x_1)^{\mathrm{T}}\mathbf{P}_{01} \\ \mathbf{P}_{10}\boldsymbol{\Sigma}_{01}(x_1) & \mathbf{0} \end{bmatrix}. \tag{84}$$

Applying the formulas for the computation of the inverse of a partitioned matrix, it is obtained

$$\boldsymbol{\Lambda}^{-1} = -\begin{bmatrix} \mathbf{0} & (\boldsymbol{\Sigma}_{11}^*(x_0)^{\mathrm{T}}\mathbf{P}_{01})^{-1} \\ (\mathbf{P}_{10}\boldsymbol{\Sigma}_{11}(x_0))^{-1} & \mathbf{0} \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & (\mathbf{P}_{10}\boldsymbol{\Sigma}_{01}(x_1))^{-1} \\ (\boldsymbol{\Sigma}_{01}^*(x_1)^{\mathrm{T}}\mathbf{P}_{01})^{-1} & \mathbf{0} \end{bmatrix}; \tag{85}$$

the final expressions for $\mathbf{C}_{01}$ and $\mathbf{D}_{10}$ given by Eqs. (79) and (80) take the form

$$\mathbf{C}_{01} = -\left(\mathbf{P}_{10}\boldsymbol{\Sigma}_{11}(x_0)\right)^{-1} = -\left(\boldsymbol{\Sigma}^*_{01}(x_1)^{\mathrm{T}}\mathbf{P}_{01}\right)^{-1}, \tag{86}$$

$$\mathbf{D}_{10} = \left(\boldsymbol{\Sigma}^*_{11}(x_0)^{\mathrm{T}}\mathbf{P}_{01}\right)^{-1} = \left(\mathbf{P}_{10}\boldsymbol{\Sigma}_{01}(x_1)\right)^{-1}. \tag{87}$$

In the common case of self-adjoint operators, recalling that it is always possible to obtain $\mathbf{P}_{10} = -\mathbf{P}_{01} = \mathbf{I}$, formulas (86) and (87) simplify to

$$\mathbf{C}_{01} = -\left(\boldsymbol{\Sigma}_{11}(x_0)\right)^{-1} = \left(\boldsymbol{\Sigma}_{01}(x_1)\right)^{-\mathrm{T}} = \mathbf{D}^{\mathrm{T}}_{10}. \tag{88}$$

In solid mechanics the elements of the matrices appearing in (88) give the values of the mechanical boundary conditions at the ends of the structural element associated with the ordinary shape functions.

3. The structural problem

The results reported in the previous section can be applied to determine the expression for the Green's function of the equilibrium operator governing the mechanical response of a structural model. In order to match the hypotheses made in the development of the theory, a linear structural model will be considered, described through the following equations:

$$Tu = \varepsilon, \qquad T^*\sigma = f, \qquad E\varepsilon = \sigma, \tag{89}$$

where

- u is the displacement vector, defined on the closed domain Ω occupied by the structure; u is assumed to belong to a subspace $\mathcal{U}$ of the Hilbert space $\mathcal{H}^n$, where the index n depends on the order of the variational operator of the problem;
- f is the applied forces vector, an element of a subspace $\mathcal{F}$ (usually of $\mathcal{L}^2$), dual to $\mathcal{U}$, defined both on the domain Ω and on its boundary;
- ε is the deformation field, whose nature depends on the problem at hand, belonging to a linear vector space $\mathcal{E}$;
- σ is the internal forces (stress) field, defined on the vector space $\mathcal{S} = \mathcal{E}^*$.
- $T : \mathcal{U} \to \mathcal{E}$, $T^* : \mathcal{S} \to \mathcal{F}$ are the adjoint operators of compatibility and equilibrium, assumed to be linear;
- E is the linear symmetric operator describing the constitutive relations.

The hypotheses made, therefore, refer to the case of infinitesimal displacements (T linear) and linear material behavior. The linear elastic case will be addressed, so that E reduces to a constant operator.

The structural analysis according to the standard displacement method leads to the variational problem

$$l(u, v) = (f, v), \quad u \in \mathcal{V},\ \forall v \in \mathcal{V}, \tag{90}$$

$\mathcal{V} \subset \mathcal{U}$ being the subspace of admissible displacements which satisfies the kinematical boundary conditions. In (90) $(\cdot, \cdot)$ indicates the scalar product on $\mathcal{L}^2$, and $l(u, v)$ is the bilinear form on $\mathcal{H}^n$, corresponding to the linear differential operator of equilibrium

$$L = T^*ET : \mathcal{U} \to \mathcal{F} \tag{91}$$

together with both the proper kinematical and mechanical boundary conditions.

Since the operator L is linear, self-adjoint and continuous on $\mathcal{U}$, all the theory developed in the previous sections can be applied to it.

3.1. Frame structures

In this section the problem of the construction of the Green's function for frame structures will be addressed, following the approach introduced in (Romano, 1987).

With the symbols previously introduced, the equilibrium problem for a frame structure can be described by means of the following differential boundary value problem

$$Lu = q, \quad u \in \mathcal{V}, \tag{92}$$

q being the load vector. The variational form of (92) is given by

$$l(u, v) = (q, v), \quad u \in \mathcal{V},\ \forall v \in \mathcal{V}, \tag{93}$$

where l is the bilinear form of the elastic strain energy and $(\cdot\,,\cdot)$ is the scalar product in $\mathcal{V}$, defined by

$$(a,b)=\sum_{i=1,m}\int_{e_i} a\cdot b\,\mathrm{d}x, \tag{94}$$

m being the number of the elements in the structure.

The normal form $\mathbf{L}_{(i)}$ of the restriction of the differential operator of equilibrium (92) to a single beam element is given by the system of differential equations

$$\begin{cases} T^{*}\mathfrak{s}=q+p(u),\\ Tu=E^{-1}\mathfrak{s}+\mathfrak{q}, \end{cases} \tag{95}$$

where u is the vector of the displacement functions, $\mathfrak{s}$ is the vector of the internal forces and $\mathfrak{q}$ are inelastic deformations. In the plane case the elements of u and $\mathfrak{s}$ have, respectively, the physical meaning of axial displacement, transversal displacement and rotation, and axial force, shear force and bending moment. The term $p(u)$ accounts for reactive or inertia forces. The explicit expressions of the differential operators for some beam models will be given in Section 4.

3.1.1. Splitting of the problem

In order to evaluate the Green's function for a frame structure the admissible displacements space $\mathcal{V}$ is split into the direct sum of two orthogonal subspaces with respect to the scalar product defined by the bilinear form of the elastic energy l:

$$\mathcal{V}=\mathcal{V}'\oplus\mathcal{V}''. \tag{96}$$

In this equation $\mathcal{V}'$ is chosen as the space of the displacement functions satisfying homogeneous boundary conditions at the ends, i.e. the displacement functions for the clamped beams. The subspace $\mathcal{V}''$ is then defined as the orthogonal complement to $\mathcal{V}'$ by the relation

$$l\big(u',u''\big)=0,\quad \forall u'\in\mathcal{V},\ \forall u''\in\mathcal{V}, \tag{97}$$

$$\big(Lu',u''\big)=0,\quad \forall u'\in\mathcal{V},\ \forall u''\in\mathcal{V}. \tag{98}$$

The problem (93) is then divided into the following two:

$$l\big(u',v'\big)=\big(q,v'\big),\quad u'\in\mathcal{V}',\ \forall v'\in\mathcal{V}', \tag{P1}$$

$$l\big(u'',v''\big)=\big(q,v''\big),\quad u''\in\mathcal{V}'',\ \forall v''\in\mathcal{V}'', \tag{P2}$$

where, from the definition of the spaces $\mathcal{V}'$ and $\mathcal{V}''$, problem (P1) has homogeneous boundary conditions, while problem (P2) retains the original ones.

If $\mathbb{G}'$ and $\mathbb{G}''$ are the Green's operators for the problems (P1) and (P2), because of the property (96), the following additive decomposition for the Green operator is obtained:

$$u=u'+u''=\mathbb{G}'q+\mathbb{G}''q=\mathbb{G}q; \tag{99}$$

and, because of the linearity of the Green's operator,

$$\mathbb{G}=\mathbb{G}'+\mathbb{G}''. \tag{100}$$

The Green's function can therefore be expressed as the sum of the Green's functions $\mathbf{G}'$ and $\mathbf{G}''$ of the problems (P1) and (P2).

3.1.2. Evaluation of the Green's function $\mathbf{G}'$

The results of Section 2 can be directly applied to problem (P1), that reduces to m differential equations with homogeneous boundary conditions, one for every element of the structure. If $\mathbf{G}'_{(i)}$ is the Green's function for the ith element of the structure considered as clamped at both ends (homogeneous boundary conditions), then the Green's function for the whole structure is given by

$$\mathbf{G}'=\mathrm{diag}\big[\mathbf{G}'_{(i)}\big]. \tag{101}$$

Because of the particular boundary conditions there is in fact no interaction between the elements of the structure. The evaluation of the Green's function $\mathbf{G}'_{(i)}$ for each single element is obtained determining a basis for the kernel of $\mathbf{L}_{(i)}$. In order to simplify the expression of the Green's function $\mathbf{G}''$ reported in the next paragraph, this evaluation can be conveniently carried out with respect to the canonical basis $\boldsymbol{\Sigma}_{(i)}$ made by the *extended shape functions* of the element, that is, a basis made by the ordinary

shape functions for the displacements plus those for the internal forces. Accordingly, the basis $\boldsymbol{\Sigma}_{(i)}$ can be partitioned in two groups of rows as in (60):

$$\boldsymbol{\Sigma}_{(i)} = \begin{bmatrix} \boldsymbol{\Sigma}_u \\ \boldsymbol{\Sigma}_\mathfrak{s} \end{bmatrix}_{(i)}. \tag{102}$$

The extended shape functions $\boldsymbol{\Sigma}_{(i)}$ satisfy boundary conditions of the kind (61) and the differential system is self-adjoint. Therefore, the Green's function is given by expression (75) and the matrix $\mathbf{C}_{01}$ is given by the formula (88).

3.1.3. Evaluation of the Green's function $\mathbf{G}''$

The subspace $\mathcal{V}''$, as defined by the orthogonality relation (98), is generated by a basis of the homogeneous differential system $Lu'' = 0$, with the boundary conditions of the original problem (92). It follows that a suitable basis of this space can be given by the set $\boldsymbol{\Sigma}$ of the shape functions of the elements.

If $\mathbf{d}$ is the vector of the nodal displacements in the global structure coordinate system and $\mathbf{s}$ is the vector of the end displacements for all the elements in the structure in their local coordinate systems, it is possible to define a linear operator $\mathbf{R}$ accounting for the kinematical constraints such that $\mathbf{s} = \mathbf{R}\mathbf{d}$ and then write u'' and the corresponding internal forces $\mathfrak{s}''$ as

$$\begin{bmatrix} u''(x) \\ \mathfrak{s}''(x) \end{bmatrix} = \boldsymbol{\Sigma}(x)\mathbf{s} = \boldsymbol{\Sigma}(x)\mathbf{R}\mathbf{d}. \tag{103}$$

The vector of the equivalent nodal loads for the structure $\mathbf{f}$ is given by

$$\mathbf{f} = \mathbf{R}^{\mathrm{T}} \int \boldsymbol{\Sigma}(\xi)^{\mathrm{T}} \begin{bmatrix} q(\xi) \\ \mathfrak{q}(\xi) \end{bmatrix} \mathrm{d}\xi, \tag{104}$$

where $\mathfrak{q}$ are the inelastic distortions dual to $\mathfrak{s}''$. The nodal displacements $\mathbf{d}$ can be expressed as a linear function of $\mathbf{f}$ through the flexibility matrix $\mathbf{F}$ of the system in the form $\mathbf{d} = \mathbf{F}\mathbf{f}$. The flexibility matrix $\mathbf{F}$ is readily obtained from the variational formulation (93) as $\mathbf{F} = (\int_{0,l_i} \mathbf{R}^{\mathrm{T}} \boldsymbol{\Sigma}_\mathfrak{s}^{\mathrm{T}} \mathbf{E} \Sigma_\mathfrak{s} \mathbf{R}\, \mathrm{d}\xi)^{-1}$, once the shape functions are known.

From (103) and (104) the following expression is obtained:

$$\begin{bmatrix} u''(x) \\ \mathfrak{s}''(x) \end{bmatrix} = \int \left[\boldsymbol{\Sigma}(x)\mathbf{R}\mathbf{F}\mathbf{R}^{\mathrm{T}} \boldsymbol{\Sigma}^{\mathrm{T}}(\xi) \right] \begin{bmatrix} q(\xi) \\ \mathfrak{q}(\xi) \end{bmatrix} \mathrm{d}\xi. \tag{105}$$

The Green's function $\mathbf{G}''$ is then, by definition, equal to the term in square brackets:

$$\mathbf{G}''(x,\xi) = \boldsymbol{\Sigma}(x)\mathbf{R}\mathbf{F}\mathbf{R}^{\mathrm{T}} \boldsymbol{\Sigma}^{\mathrm{T}}(\xi). \tag{106}$$

It is immediately observed that the matrix $\mathbf{F}_e = \mathbf{R}\mathbf{F}\mathbf{R}^{\mathrm{T}}$ is a "disassembled flexibility matrix" for the structure. If the structure is made by m elements and n_d is the number of the degrees of freedom at one end of the generic element, then $\mathbf{F}_e$ is a $2m\,n_d \times 2m\,n_d$ matrix formed by m^2 submatrices $2n_d \times 2n_d$. Each submatrix $\mathbf{F}_{e(ij)}$ is the linear operator that gives the end displacements of the element i (in its local coordinate system) as a function of the end forces applied to the element j (expressed in the local coordinate system of the element j). The following restriction of the Green's function $\mathbf{G}''$ can therefore be considered:

$$\mathbf{G}''_{(ij)}(x,\xi) = \boldsymbol{\Sigma}_{(i)}(x)\mathbf{F}_{e(ij)} \boldsymbol{\Sigma}^{\mathrm{T}}_{(j)}(\xi). \tag{107}$$

The Green's matrix $\mathbf{G}''_{(ij)}(x,\xi)$ has dimensions $2n_d \times 2n_d$ and its elements have the physical meaning of effect (i.e. displacement or rotation or internal force) at the abscissa x of the ith beam due to an action (i.e. force or couple or distortion) applied at the abscissa ξ of the jth beam.

3.1.4. Evaluation of the Green's function $\mathbf{G}$

Using the results obtained in the previous sections, the Green's function for two elements i and j of a frame structure is given by

$$\mathbf{G}_{(ij)}(x,\xi) = \begin{cases} \boldsymbol{\Sigma}_{(i)}(x)\left[\mathbf{F}_{e(ij)} + \delta_{ij}\mathbf{C}_{(i)}\right] \boldsymbol{\Sigma}^{\mathrm{T}}_{(j)}(\xi), & x \leqslant \xi, \\ \boldsymbol{\Sigma}_{(i)}(x)\left[\mathbf{F}_{e(ij)} + \delta_{ij}\mathbf{C}^{\mathrm{T}}_{(i)}\right] \boldsymbol{\Sigma}^{\mathrm{T}}_{(j)}(\xi), & x \geqslant \xi, \end{cases} \tag{108}$$

where δ_{ij} is the Kronecker's function ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise) and $\mathbf{C}_{(i)}$ is trivially obtained from Eq. (74) with $\mathbf{C}_{(i)01}$ given by (88).

It is noted in (108) that the term $\mathbf{G}'$ is added only when $i = j$ as the "clamped beam effects" are of course present only if the action and the effect are in the same element, as it was pointed out in the determination of $\mathbf{G}'$. In the plane case, if

w, v, φ, N, T, M are respectively the influence functions of the axial displacement, transversal displacement, rotation, axial force, shear and bending moment due to an axial force a, a transversal force t, a couple m, an inelastic axial distortion λ, an inelastic shear distortion θ and an inelastic curvature μ, then the elements of $\mathbf{G}$ are

$$\mathbf{G}(x,\xi) = \begin{bmatrix} w_a & w_t & w_m & w_\lambda & w_\theta & w_\mu \\ v_a & v_t & v_m & v_\lambda & v_\theta & v_\mu \\ \varphi_a & \varphi_t & \varphi_m & \varphi_\lambda & \varphi_\theta & \varphi_\mu \\ N_a & N_t & N_m & N_\lambda & N_\theta & N_\mu \\ T_a & T_t & T_m & T_\lambda & T_\theta & T_\mu \\ M_a & M_t & M_m & M_\lambda & M_\theta & M_\mu \end{bmatrix}, \tag{109}$$

where the indexes represent the moving actions.

4. Evaluation of the Green's function for some beam models

In the previous sections it has been shown that the computation of the Green's function for a frame structure requires only the knowledge of a basis for the kernel of the equilibrium operator. This allows either to evaluate the flexibility matrix of the system (and hence $\mathbf{G}''$) and to compute the matrix $\mathbf{C}$ necessary to define $\mathbf{G}'$ through one of the previously given formulas (38) or (88), depending on the basis being used.

As the computation of $\mathbf{G}''$ is trivial, here some results regarding the clamped beam are given. It is also explicitly calculated the value of the linear invariant of the Green's operator. It represents the sum of the eigenvalues of the operator $\mathbf{L}_{(i)}$ in $\mathcal{V}'$, and allows to bound the approximation error in the estimate of eigenperiods for the whole structure through the formula (Romano, 1975)

$$\omega_k^+ = \omega_k^- + \operatorname{tr} \Xi \mathbf{G}', \tag{110}$$

where ω_k^-, ω_k^+ are the lower and upper bound to the ith vibration period and Ξ is the matrix of the mass distribution of the system. If $\rho_i(x)$ is the mass per unit length pertaining to the ith member and $J_{p_i}(x)$ the corresponding polar inertia moment, it is (in the plane case) $\Xi(x) = \operatorname{diag}[\rho_i(x), \rho_i(x), J_{p_i}(x), 0, 0, 0]$. The trace appearing in (110) is equal to

$$\operatorname{tr} \Xi \mathbf{G}' = \sum_{i=1}^{m} \int_0^{l_i} \operatorname{tr} \Xi_i(x) \mathbf{G}_i'(x,x)\, \mathrm{d}x. \tag{111}$$

Introducing the explicit expression for $\mathbf{G}_i'$, the following expression for the linear invariant is obtained:

$$\operatorname{tr} \Xi \mathbf{G}' = \sum_{i=1}^{m} \operatorname{tr} \mathbf{C}_i \int_0^{l_i} \boldsymbol{\Sigma}_{u(i)}^{\mathrm{T}}(x) \widetilde{\Xi}_i(x) \boldsymbol{\Sigma}_{u_{(i)}}(x)\, \mathrm{d}x = \sum_{i=1}^{m} \operatorname{tr} \mathbf{C}_i \mathbf{M}_i = \sum_{i=1}^{m} \operatorname{tr} \mathbf{C}_{01_i} \mathbf{M}_{10_i}, \tag{112}$$

having indicated by $\widetilde{\Xi}$ the matrix formed by the first three rows and columns of Ξ. In this formula $\mathbf{M}_i$ has the meaning of the consistent mass matrix for the element, and a partition similar to (60) has been introduced for it.

For a plane straight beam the flexural part of the homogeneous differential problem, written in the normal form (5), is (disregarding inertia forces and Eulerian instability effects)

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{,x} \\ \varphi_{,x} \\ T_{,x} \\ M_{,x} \end{bmatrix} + \begin{bmatrix} k(x) & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -\chi(x)/(GA) & 0 \\ 0 & 0 & 0 & -1/(EJ) \end{bmatrix} \begin{bmatrix} v \\ \varphi \\ T \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{113}$$

so that

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{Q} = -\begin{bmatrix} k(x) & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -\chi(x)/(GA) & 0 \\ 0 & 0 & 0 & -1/(EJ) \end{bmatrix}, \tag{114}$$

where v and φ are the transversal displacement and the rotation of the cross section, T and M are the shear force and the bending moment. In the matrix $\mathbf{Q}$ the bending and shear deformabilities appear as well as the Winkler constant k and they can be functions of x. From this equation various particular cases can be obtained.

Some particular examples are reported in the following and have been obtained by means of a symbolic mathematics system as follows. Given the homogeneous differential problem (113) a basis $\boldsymbol{\Phi}$ for the kernel of the differential operator is determined. The basis can always be transformed into a canonical basis by the transformation (41), i.e. solving a system of linear equations. In the following examples it is made reference to canonical bases, and consequently the Green's function is given by expression (75) that requires only the knowledge of the matrix $\mathbf{C}_{01}$ computed using Eq. (88). In this section the expressions of the matrix $\mathbf{C}_{01}$, together with the first invariant of the Green's operator, are reported for some beam models. The matrices of the shape functions are listed in the appendix.

4.1. Beam with shear deformability and constant cross section

$$\mathbf{C}_{01} = \begin{bmatrix} \frac{\chi l}{GA} - \frac{l^3}{6EJ} & \frac{l^2}{2EJ} \\ -\frac{l^2}{2EJ} & \frac{l}{EJ} \end{bmatrix}.$$

If $\rho(x) = \rho$ is constant along the beam, the following expression for the linear invariant is obtained:

$$\operatorname{tr} \Xi \mathbf{G}' = \frac{l^4}{(1+12k)} \frac{1}{420\,EJ} \left[\rho\left(840k^2 + 84k + 1\right) + J_p(840k + 28)\right], \quad \text{where} \quad k = \frac{\chi EJ}{GAl^2}.$$

4.2. Beam with zero shear deformability and variable cross section

Let $J(x) = J_0(1+\theta x/l)^3$ and $\rho(x) = \rho_0(1+\theta x/l)$. These laws of variation are typical of the rectangular cross section beam with linearly variable height. The computation yields

$$\mathbf{C}_{01} = \frac{l^3}{E\,J_0} \begin{bmatrix} \dfrac{\ln(1+\theta)}{\theta^3} - \dfrac{\theta+2}{2\theta^2(\theta+1)} & \dfrac{1}{2l(\theta+1)} \\ -\dfrac{1}{2l(\theta+1)^2} & \dfrac{\theta+2}{2l^2(\theta+1)^2} \end{bmatrix},$$

$$\begin{aligned} \operatorname{tr} \Xi \mathbf{G}' = {} & \frac{\rho_0 l^2}{E\,J_0} \frac{1}{75\theta^5[(\theta+2)\ln(1+\theta) - 2\theta]} \\ & \times \left\{\rho l^2\left[2\theta^2\left(12 + 12\theta - 5\theta^2\right) + 3\theta(2+\theta)\ln(1+\theta)\left(\theta^2 + 2\theta + 2\right) - 36(1+\theta)^2\ln^2(1+\theta)\right]\right. \\ & \left. + J_p\left[288\theta^2 - 180\theta^3(2+\theta)\ln(1+\theta) + 36\theta^2\left(\theta^2 + 2\theta + 2\right)\ln^2(1+\theta)\right]\right\}. \end{aligned}$$

4.3. Beam with constant cross section and zero shear deformability resting on a Winkler soil

Let $\beta^4 = k/(4EJ)$, $c = \cos(\beta l)$, $s = \sin(\beta l)$, $ch = \cosh(\beta l)$, $sh = \sinh(\beta l)$. Then

$$\mathbf{C}_{01} = \frac{1}{4\beta^3 EJ} \begin{bmatrix} c\,sh - s\,ch & 2\beta s\,sh \\ -2\beta s\,sh & 2\beta^2(s\,ch + c\,sh) \end{bmatrix},$$

$$\operatorname{tr} \Xi \mathbf{G}' = \frac{1}{4EJ} \left\{ \frac{\rho}{\beta^4} \left[\frac{2 + \beta l(ch\,sh - cs) - ch^2 - sh^2 - c^2 + s^2}{c^2 - s^2 + ch^2 + sh^2 - 2} \right] + \frac{J_p}{\beta^2} \left[\frac{2\beta l(ch\,sh + cs) - ch^2 - sh^2 + c^2 - s^2}{c^2 - s^2 + ch^2 + sh^2 - 2} \right] \right\}.$$

4.4. Bernoulli's beam with constant cross section

This is the limit case toward which tend all the previous examples. The following results are obtained:

$$\mathbf{C}_{01} = \frac{1}{EJ} \begin{bmatrix} -l^3/6 & l^2/2 \\ -l^2/2 & l \end{bmatrix}, \qquad \operatorname{tr} \Xi \mathbf{G}' = \frac{\rho l^4}{420EJ} + \frac{J_p l^2}{15EJ}.$$

4.5. Circular arch

The homogeneous differential problem for a curved beam with constant cross section is given by

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{s,x} \\ u_{r,x} \\ \alpha_{,x} \\ N_{,x} \\ T_{,x} \\ M_{,x} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & -1/r & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1/r & 0 & -1/(EA) & 0 & 1/(EAr) \\ 1/r & 0 & -1 & 0 & -\chi(x)/(GA) & 0 \\ 0 & 0 & 0 & 1/(EAr) & 0 & -1/(EJ)-1/(EAr^2) \end{bmatrix} \begin{bmatrix} u_s \\ u_r \\ \alpha \\ N \\ T \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where x is the curvilinear abscissa along the beam and u_s, u_r are the tangential (axial) and radial displacements respectively. In the case of constant curvature radius r, total length $l = r\alpha$ and no shear deformability ($\chi(x) = 0$), the matrix $\mathbf{C}_{01}$ w r.t. the shape functions of the element is

$$\mathbf{C}_{01} = \frac{r}{EA}\begin{bmatrix} (1+k)\alpha + \left(2+\frac{k}{2}\right)\alpha c - \left(2+\frac{3k}{2}\right)s & \left(2+\frac{k}{2}\right)\alpha s - (2+k)(1-c) & \frac{(1+k)\alpha-(2+k)s}{r} \\ (2+k)(1-c) - \left(2+\frac{k}{2}\right)\alpha s & \frac{1}{2}(4+k)(\alpha c - s) & \frac{1}{r}(2+k)(1-c) \\ \frac{1}{r}\left[(1+k)\alpha-(2+k)s\right] & -\frac{1}{r}(2+k)(1-c) & \frac{1}{r^2}(1+k)\alpha \end{bmatrix},$$

where $c = \cos(l/r)$, $s = \sin(l/r)$, $k = EAr^2/EJ$. The linear invariant of the Green's function for this particular case is not reported because of the excessive length of the algebraic expression, but it can be easily evaluated numerically from Eq. (112).

5. Influence lines for a circular arch

The method proposed for the evaluation of the Green's operator in frame structures has been implemented into a software for two-dimensional frame analysis. As a nonstandard application an arch clamped at both ends and made by 3 circular beam segments is considered, Fig. 1. The arch segments have a curvature radius equal to 15 and the nodal coordinates, form left to right are $(0, 0)$, $(5, 5)$, $(15, 5)$, $(20, 0)$. The influence lines constructed by the present approach and reported in Figs. 2–8 have been verified through the application of the Betti's theorem.

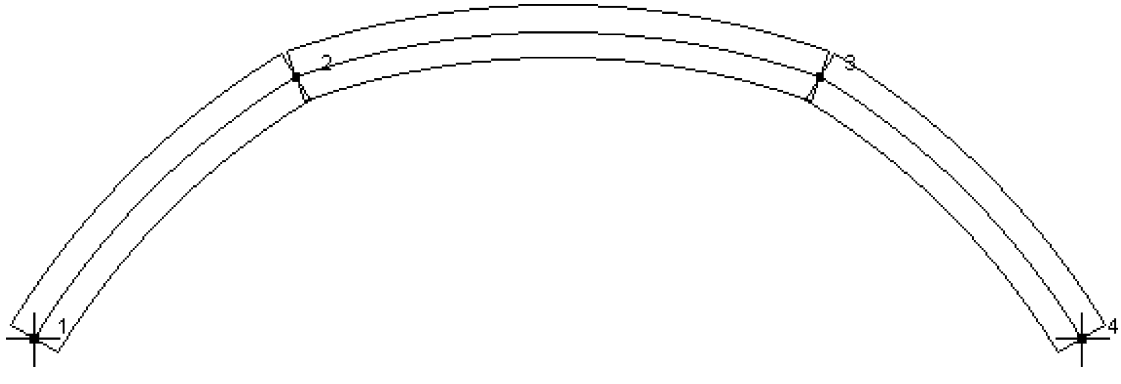

Fig. 1. Geometry of the circular arch.

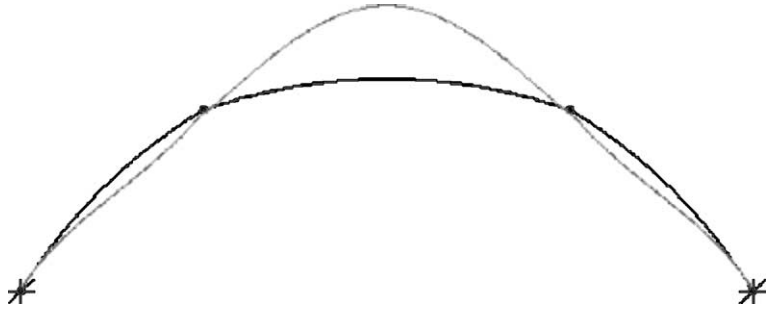

Fig. 2. Influence line of the vertical displacement at the midspan for a vertical moving force.

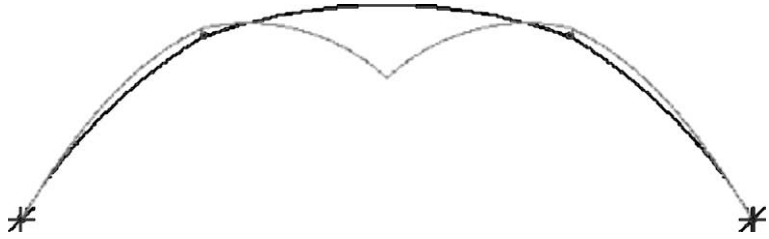

Fig. 3. Influence line of the bending moment at the midspan for a vertical moving force.

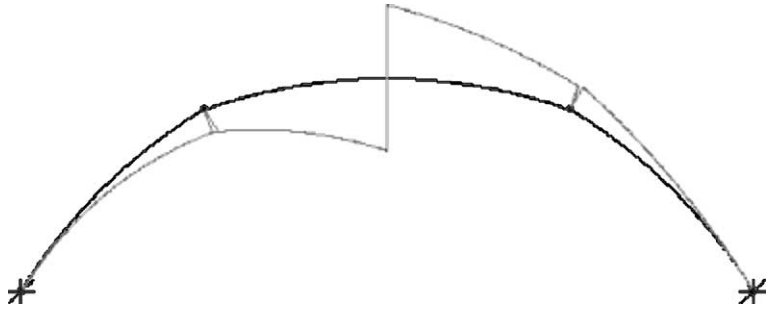

Fig. 4. Influence line of the shear stress at the midspan for a vertical moving force.

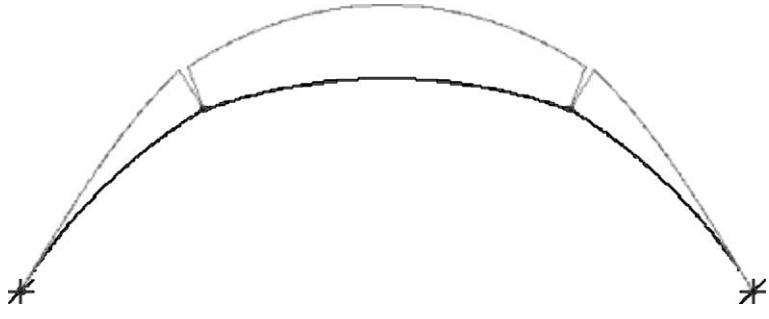

Fig. 5. Influence line of the normal stress at the midspan for a vertical moving force.

Fig. 6. Influence line of the bending moment at the left clamp for a vertical moving force.

The influence lines have been computed assuming a travelling vertical force and considering the effects at the midspan of the arch and at the left clamp. More precisely they are:

- vertical displacement at the midspan, Fig. 2;
- bending moment at the midspan, Fig. 3;
- shear force at the midspan, Fig. 4;
- normal force at the midspan, Fig. 5;
- bending moment at the left clamp, Fig. 6;

Fig. 7. Influence line of the shear stress at the left clamp for a vertical moving force.

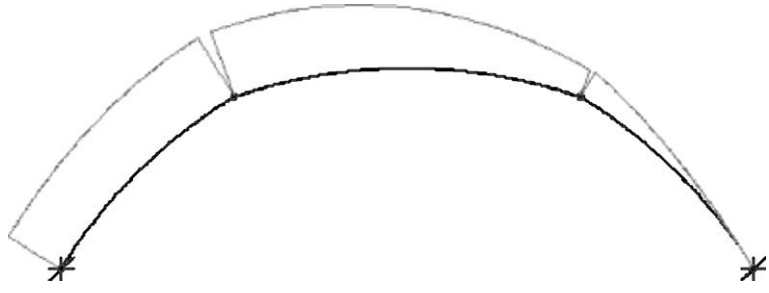

Fig. 8. Influence line of the normal stress at the left clamp for a vertical moving force.

- shear force at the left clamp, Fig. 7;
- normal force at the left clamp, Fig. 8.

The method proposed for the computation of the influence lines appears of very simple introduction in any frame element code. In fact these softwares compute the stiffness matrix of the structure and represent the deformed geometry through the element shape functions. Therefore, according to Eq. (108), only the computation of the constant matrix $\mathbf{C}_{01}$ for a given element is to be added.

6. Conclusions

Starting from the general properties of the Green's function of a differential boundary value problem on $\mathcal{R}$, it has been shown that it can be always expressed as a bilinear combination of bases for the null spaces of the operator and of its adjoint. The proof given is constructive, in the sense that an explicit formula for the evaluation of the Green's function with arbitrary boundary conditions has been given.

The particular case in which the differential operator is self-adjoint has been examined in detail with reference to arbitrary complex frames of one-dimensional structural elements (beams). Here the components of the Green's operator have the immediate physical meaning of influence functions. These functions are expressed as the sum of two parts, one accounting for the nodal flexibility of the frame, and the other accounting for the clamped–clamped beam displacement fields.

The explicit expression of the Green's functions and of the first orthogonal invariant for the Green's operator are reported in the paper for Timoshenko's and variable cross section beams, as well as for beams resting on a Winkler's elastic soil and for circular arches.

Appendix A

In this section the shape functions for the beam models considered in Section 4 are reported. They, together with the matrix $\mathbf{C}$ already given, allow the evaluation of all the influence functions. The physical meaning of the elements of the shape function matrices is briefly recalled. For straight beams, the flexural part of shape function matrix is 4×4 and

- the first row represent the transversal displacement;
- the second row is the rotation;
- the third row is the shear stress;
- the fourth row is the bending moment;

- the first column represents the shape function having unit displacement at the left end, null rotation at both ends, null displacement at the right end;
- the second column represents the shape function having unit rotation at the left end, null displacement at both ends, null rotation at the right end;
- the third column represents the shape function having unit displacement at the right end, null rotation at both ends, null displacement at the left end;
- the fourth column represents the shape function having unit rotation at the left end, null displacement at both ends, null rotation at the right end.

In a similar way in the circular arch, where the axial and flexural behavior are coupled, the shape function matrix is 6×6, the rows having the meaning (from the 1st to the 6th) of tangential displacement, radial displacement, rotation, normal stress, shear stress, bending moment. In the same order the columns are the shape functions for

- unit tangential displacement at the left end, zero being all remaining boundary conditions;
- unit radial displacement at the left end;
- unit rotation at the left end;
- unit tangential displacement at the right end;
- unit radial displacement at the right end;
- unit rotation at the right end.

A.1. Beam with shear deformability

Let $k = \chi EJ/GAl^2$. It is found:

$$\begin{aligned}
\boldsymbol{\Sigma}_{11} &= \frac{(1-\xi)(1+\xi-2\xi^2+12k)}{1+12k}, \qquad \boldsymbol{\Sigma}_{12} = \frac{(1-\xi)\xi l(1-\xi+6k)}{1+12k}, \qquad \boldsymbol{\Sigma}_{13} = \frac{\xi(3\xi-2\xi^2+12k)}{1+12k}, \\
\boldsymbol{\Sigma}_{14} &= -\frac{(1-\xi)\xi l(\xi+6k)}{1+12k}, \qquad \boldsymbol{\Sigma}_{21} = \frac{1}{l}\frac{6(\xi-1)\xi}{1+12k}, \qquad \boldsymbol{\Sigma}_{22} = \frac{(1-\xi)(1-3\xi+12k)}{1+12k}, \qquad \boldsymbol{\Sigma}_{23} = \frac{1}{l}\frac{6(1-\xi)\xi}{1+12k}, \\
\boldsymbol{\Sigma}_{24} &= \frac{\xi(3\xi-2+12k)}{1+12k}, \qquad \boldsymbol{\Sigma}_{31} = -\frac{12EJ}{l^3(1+12k)}, \qquad \boldsymbol{\Sigma}_{32} = -\frac{6EJ}{l^2(1+12k)}, \qquad \boldsymbol{\Sigma}_{33} = \frac{12EJ}{l^3(1+12k)}, \\
\boldsymbol{\Sigma}_{34} &= -\frac{6EJ}{l^2(1+12k)}, \qquad \boldsymbol{\Sigma}_{41} = -\frac{6EJ(1-2\xi)}{l^2(1+12k)}, \qquad \boldsymbol{\Sigma}_{42} = -\frac{2EJ(2-3\xi+6k)}{l(1+12k)}, \qquad \boldsymbol{\Sigma}_{43} = \frac{6EJ(1-2\xi)}{l^2(1+12k)}, \\
\boldsymbol{\Sigma}_{44} &= \frac{2EJ(6k-1+3\xi)}{l(1+12k)}.
\end{aligned}$$

A.2. Bernoulli's beam with variable section

With the notations of Section 4, it is found:

$$\begin{aligned}
\boldsymbol{\Sigma}_{11} &= \frac{(2+\theta)(1+\theta\xi)(\ln(1+\theta)-\ln(1+\theta\xi))-\theta(1-\xi)(2+\theta\xi)}{(1+\theta\xi)((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{12} &= l\frac{(\xi\theta+2\xi-\xi^2)\ln(1+\theta)-(\xi\theta+1)\ln(1+\theta\xi)-\xi\theta(1-\xi)}{(1+\theta\xi)((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{13} &= \frac{(2+\theta)(1+\xi)\ln(1+\theta\xi)-\theta\xi(\theta+\theta\xi+2)}{(1+\theta\xi)((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{14} &= l(1+\theta)\frac{\xi^2(1+\theta)\ln(1+\theta)+\xi\theta(1-\xi)-(1+\xi\theta)\ln(1+\theta\xi)}{(1+\theta\xi)((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{21} &= -\frac{\theta^3(1-\xi)\xi}{l((1+\theta\xi)^2((2+\theta)\ln(1+\theta)-2\theta))}, \qquad \boldsymbol{\Sigma}_{22} = \frac{(\xi-1)((2+\theta\xi)\theta-(2+\theta+\theta\xi)\ln(1+\theta))}{(1+\theta\xi)^2((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{23} &= \frac{\theta^3(1-\xi)\xi}{l((1+\theta\xi)^2((2+\theta)\ln(1+\theta)-2\theta))}, \qquad \boldsymbol{\Sigma}_{24} = \frac{\xi(1+\theta)((1+\theta)(2+\theta\xi)\ln(1+\theta)-\theta(2+\theta+\theta\xi))}{(1+\theta\xi)^2((2+\theta)\ln(1+\theta)-2\theta)}, \\
\boldsymbol{\Sigma}_{31} &= -\frac{EJ_0}{l^3}\frac{(2+\theta)\theta^3}{(2+\theta)\ln(1+\theta)-2\theta}, \qquad \boldsymbol{\Sigma}_{32} = -\frac{EJ_0}{l^2}\frac{\theta^3}{(2+\theta)\ln(1+\theta)-2\theta},
\end{aligned}$$

$$\boldsymbol{\Sigma}_{33} = \frac{EJ_0}{l^3}\frac{(2+\theta)\theta^3}{(2+\theta)\ln(1+\theta)-2\theta}, \qquad \boldsymbol{\Sigma}_{34} = -\frac{EJ_0}{l^2}\frac{(1+\theta)\theta^3}{(2+\theta)\ln(1+\theta)-2\theta},$$

$$\boldsymbol{\Sigma}_{41} = \frac{EJ_0}{l^2}\frac{\theta^3(2\xi+\theta\xi-1)}{(2+\theta)\ln(1+\theta)-2\theta}, \qquad \boldsymbol{\Sigma}_{42} = \frac{EJ_0}{l}\frac{2\theta+\theta^2(3+\theta\xi)-2(1+\theta)^2\ln(1+\theta)}{(2+\theta)\ln(1+\theta)-2\theta},$$

$$\boldsymbol{\Sigma}_{43} = \frac{EJ_0}{l^2}\frac{\theta^3(1-2\xi-\theta\xi)}{(2+\theta)\ln(1+\theta)-2\theta}, \qquad \boldsymbol{\Sigma}_{44} = \frac{EJ_0}{l}(1+\theta)\frac{2(1+\theta)\ln(1+\theta)-2\theta-\theta^2(1-\theta\xi)}{(2+\theta)\ln(1+\theta)-2\theta},$$

A.3. Beam on a Winkler soil

Let us define the following quantities in order to shorten the final algebraic expressions:

$$c = \cos(\beta l), \quad c_x = \cos(\beta x), \quad c_2 = \cos(2\beta l), \quad c_{2x} = \cos(2\beta x), \quad c_{lx} = \cos\big(\beta(l-x)\big),$$
$$c_{2lx} = \cos\big(\beta(2l-x)\big), \quad ch = \cosh(\beta l), \quad ch_x = \cosh(\beta x), \quad ch_2 = \cosh(2\beta l), \quad ch_{2x} = \cosh(2\beta x),$$
$$ch_{lx} = \cosh\big(\beta(l-x)\big), \quad ch_{2lx} = \cosh\big(\beta(2l-x)\big), \quad s = \sin(\beta l), \quad s_x = \sin(\beta x), \quad s_2 = \sin(2\beta l),$$
$$s_{2x} = \sin(2\beta x), \quad s_{lx} = \sin\big(\beta(l-x)\big), \quad s_{2lx} = \sin\big(\beta(2l-x)\big), \quad sh = \sinh(\beta l), \quad sh_x = \sinh(\beta x),$$
$$sh_2 = \sinh(2\beta l), \quad sh_{2x} = \sinh(2\beta x), \quad sh_{lx} = \sinh\big(\beta(l-x)\big), \quad sh_{2lx} = \sinh\big(\beta(2l-x)\big).$$

Using this notation the elements of the shape function matrix are given by

$$\boldsymbol{\Sigma}_{11} = \frac{c_x\, ch_{2lx} + (c_{2lx}-2c_x)\, ch_x + s_x\, sh_{2lx} - s_{2lx}\, sh_x}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{12} = \frac{s_x(ch_{2lx}\, s_x - ch_x) - ss_{lx}\, sh_x}{\beta c_2 + ch_2 - 2},$$

$$\boldsymbol{\Sigma}_{13} = \frac{2\, ch\, s(c_x\, sh_x - ch_x\, s_x) + 2\, sh\, sh_x(cc_x + 2ss_x) - c\, ch_x\, s_x}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{14} = \frac{2ss_x(ch_x\, sh - ch\, sh_x) - 2\, sh\, sh_x\, s_{lx}}{\beta c_2 + ch_2 - 2},$$

$$\boldsymbol{\Sigma}_{21} = 2\beta\frac{s_x(ch_x - ch_{2lx}) - 2s\, sh_x\, s_{lx}}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{22} = \frac{c_x\, ch_{2lx} + ch_x(c_{2lx}-2c_x) - s_x\, sh_{2lx} + s_{2lx}\, sh_x}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{23} = 4\beta\frac{ss_x(ch_x\, sh - ch\, sh_x) + sh\, sh_x\, s_{lx}}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{24} = 2\frac{sh\, c(ch_x\, s_x + sh_x\, c_x) + 2s\, sh\, s_x\, sh_x - ch\, s(ch_x\, s_x + sh_x\, c_x)}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{31} = -8EJ\beta^3\frac{c_{lx}\, ch_x\, s + c_x\, ch_{lx}\, sh}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{32} = -2EJ\beta^2\frac{c_x\, ch_{2lx} - c_{2lx}\, ch_x + 2s_x\, sh_x + s_{2lx}\, sh_x + s_x\, sh_{2lx}}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{33} = 8EJ\beta^3\frac{ch_{lx}\, c_x s + c_{lx}\, ch_x\, sh}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{34} = -4EJ\beta^2\frac{ch\, s(ch_x\, s_x - c_x\, sh_x) + sh\, ch_x(2c_x s - cs_x) + sh\, cc_x\, sh_x}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{41} = 2EJ\beta^2\frac{c_{2lx}\, ch_x - c_x\, ch_{2lx} + s_x\, sh_{2lx} + s_{2lx}\, sh_x + 2s_x\, sh_x}{c_2+ch_2-2}, \qquad \boldsymbol{\Sigma}_{42} = 4EJ\beta\frac{c_{lx}\, ch_x\, s - c_x\, ch_{lx}\, sh}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{43} = 4EJ\beta^2\frac{s_{lx}\, sh_{lx} - ch\, ch_x\, ss_x - cc_x\, sh\, sh_x + c\, ch_x\, s_x\, sh - c_x\, ch\, s\, sh_x}{c_2+ch_2-2},$$

$$\boldsymbol{\Sigma}_{44} = 4EJ\beta\frac{sh\, c_{lx}\, ch_x + c_x s(sh\, sh_x - ch\, ch_x)}{c_2+ch_2-2}.$$

A.4. Bernoulli's beam with constant cross section

The shape functions are obtained from the first case letting k vanish.

A.5. Circular arch

The explicit expressions of the shape functions are very long, so they are expressed in the form $\boldsymbol{\Sigma} = \mathbf{Z}\boldsymbol{\Theta}$, where $\mathbf{Z}$ is the matrix of the constant of integration, and $\boldsymbol{\Theta}$ is a basis for the equilibrium operator of the arch, given in Section 5. Introducing the quantities

$$k = EAr^2/EJ, \qquad \alpha = l/r, \qquad c = \cos(\alpha), \qquad c_1 = \cos(\alpha/2), \qquad c_2 = \cos(2\alpha), \qquad s = \sin(\alpha),$$
$$s_1 = \sin(\alpha/2), \qquad s_2 = \sin(2\alpha),$$

the matrices $\mathbf{Z}, \boldsymbol{\Theta}$ are given by the following expressions:

$$\boldsymbol{\Theta} = \begin{bmatrix} kx & \frac{k-1}{2} r \sin\left(\frac{x}{r}\right) - \frac{3+k}{2} x \cos\left(\frac{x}{r}\right) & \frac{1-k}{2} r \cos\left(\frac{x}{r}\right) - \frac{3+k}{2} x \sin\left(\frac{x}{r}\right) & r & r \sin\left(\frac{x}{r}\right) & -r \cos\left(\frac{x}{r}\right) \\ (1+k)r & \frac{3+k}{2} x \sin\left(\frac{x}{r}\right) & -\frac{3+k}{2} x \cos\left(\frac{x}{r}\right) & 0 & r \cos\left(\frac{x}{r}\right) & r \sin\left(\frac{x}{r}\right) \\ k\frac{x}{r} & (1+k)\sin\left(\frac{x}{r}\right) & -(1+k)\cos\left(\frac{x}{r}\right) & 1 & 0 & 0 \\ 0 & -EA\cos\left(\frac{x}{r}\right) & -EA\sin\left(\frac{x}{r}\right) & 0 & 0 & 0 \\ 0 & EA\sin\left(\frac{x}{r}\right) & -EA\cos\left(\frac{x}{r}\right) & 0 & 0 & 0 \\ EAr & EAr\cos\left(\frac{x}{r}\right) & EAr\sin\left(\frac{x}{r}\right) & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

with

$$\begin{aligned}
\Delta &= \big(\alpha^2-4\big)k^2 + \big(3\alpha^2-8\big)k + (k+3)k\alpha s + 4(k+1)^2 c - 4, \\
C_{11} &= \frac{-2(1+k)s}{r\Delta}, \qquad C_{21} = \frac{(2\alpha^2 - \alpha s_s + 6)k^2 + (12 + 3\alpha s_2 - 6\alpha^2)k + 2(1+k)^2(c_2 - 4c) + 6}{r(k+3)(\alpha - s)\Delta}, \\
C_{31} &= \frac{8c_1 s_1^2 (2(1+k)^2 s_1 - k(k+3)\alpha c_1)}{r(k+3)(\alpha - s)\Delta}, \qquad C_{41} = \frac{8(1+k)c_1 s_1^2(-2(1+k)^2 s_1 6 + k(k+3)\alpha c_1)}{r(k+3)(\alpha - s)\Delta}, \\
C_{51} &= \frac{2(1+k)^2 s}{r\Delta}, \qquad C_{61} = \frac{(4\alpha - \alpha^3 + s_2)k^2 + (2s_2 + 8\alpha - 3\alpha^3)k - 2(1+k)^2(s-2c) + 4\alpha + s_2}{r(\alpha - s)\Delta}, \\
C_{12} &= -\frac{4(1+k)s_1^2}{r\Delta}, \qquad C_{22} = -\frac{2s(2(c-1)(1+k)^2 + k(k+3)\alpha s)}{r(k+3)(\alpha - s)\Delta}, \\
C_{32} &= \frac{2k^2(\alpha^2-1) + k(6\alpha^2 - 4) + k(k+3)\alpha s_2 + 2(1+k)^2 c_2 - 2}{r(k+3)(\alpha - s)\Delta}, \\
C_{42} &= (1+k)\frac{2k^2(\alpha^2-1) + k(6\alpha^2 - 4) + k(k+3)\alpha s_2 + 2(1+k)^2 c_2 - 2}{r(k+3)(\alpha - s)\Delta}, \\
C_{52} &= \frac{(\alpha^2-2)k^2 + (3\alpha^2 - 4)k + (k+3)k\alpha s + 2(k+1)^2 c - 2}{r\Delta}, \\
C_{62} &= \frac{(2\alpha^2-2)k^2 + (6\alpha^2 - 4)k + (k+3)k\alpha s_2 + 2(k+1)^2 c_2 - 2}{2r(\alpha - s)\Delta}, \qquad C_{13} = \frac{(k-1)s - (k+3)\alpha}{\Delta}, \\
C_{23} &= \frac{1}{(k+3)(\alpha - s)\Delta}\big(\big(5 - 2\alpha^2 + 2\alpha s + \alpha s_2\big)k^2 + \big(8 - 6\alpha^2 + 8\alpha s + 3\alpha s_2\big)k \\
&\qquad - 8(1+k)^2 c + \big(3k^2 + 8k + 5\big)c_2 + 3\big), \\
C_{33} &= 4s_1^2 \frac{(2k^2 + 7k + 3)\alpha + k(k+3)\alpha c - (3k^2 + 8k + 5)s}{(k+3)(\alpha - s)\Delta}, \\
C_{43} &= \frac{1}{(k+3)(\alpha - s)\Delta}\big(k^3\big(\alpha^3 - 3/2\alpha + s_2 + \alpha c_2/2\big) + k^2\big(6\alpha^3 - 9\alpha - 2s + s_2 + \alpha c_2\big) \\
&\qquad + k\big(9\alpha^3 - 31/2\alpha - 3/2\alpha c_2 + 2s - s_2\big) - 6\alpha + 2(1+K)^2(3+k)\alpha c + 2s - s_2\big), \\
C_{53} &= (1+k)\frac{(3+k)\alpha - (k-1)s}{\Delta}, \\
C_{63} &= \frac{(k+3)2k\alpha^3 - (6k+8)k\alpha + 2(3k^2 + 4k + 1)\alpha c - 2\alpha + 2s(k^2 - 1) + (1 - k^2)s_2}{2(\alpha - s)\Delta},
\end{aligned}$$

$$
\begin{aligned}
C_{14} &= \frac{2(1+k)s}{r\Delta}, \qquad C_{24} = -2\frac{(3-\alpha s)k^2 + (6-3\alpha s)k + ((\alpha^2-4)k^2 + (3\alpha^2-8)k - 4)c + (1+k)^2c_2 + 3}{r(k+3)(\alpha-s)\Delta},\\
C_{34} &= -2s\frac{(\alpha^2-2)k^2 + (3\alpha^2-4)k + 2(1+k)^2c - 2}{r(k+3)(\alpha-s)\Delta},\\
C_{44} &= -2(1+k)s\frac{(\alpha^2-2)k^2 + (3\alpha^2-4)k + 2(1+k)^2c - 2}{r(k+3)(\alpha-s)\Delta}, \qquad C_{54} = -\frac{2(1+k)^2s}{r\Delta},\\
C_{64} &= -s\frac{(\alpha^2-2)k^2 + (3\alpha^2-4)k + 2(1+k)^2c - 2}{r(\alpha-s)\Delta},\\
C_{15} &= -\frac{4(1+k)s_1^2}{r\Delta}, \qquad C_{25} = 2s\frac{(\alpha^2-2)k^2 + (3\alpha^2-4)k + 2c(1+k)^2 - 2}{r(k+3)(\alpha-s)\Delta},\\
C_{35} &= -2\frac{k(k+3)\alpha(s+\alpha c) - 2(1+k)^2s^2}{r(k+3)(\alpha-s)\Delta}, \qquad C_{45} = -2(1+k)\frac{k(k+3)\alpha(s+\alpha c) - 2(1+k)^2s^2}{r(k+3)(\alpha-s)\Delta},\\
C_{55} &= \frac{4(1+k)^2s_1^2}{r\Delta}, \qquad C_{65} = -\frac{k(k+3)\alpha(s+\alpha c) - 2(1+k)^2s^2}{r(\alpha-s)\Delta},\\
C_{16} &= -\frac{(k-1)s - (k+3)\alpha}{\Delta},\\
C_{26} &= \frac{1}{(k+3)(\alpha-s)\Delta}\big((7-4\alpha s)k^2 + (16-14\alpha s)k + 2c\big(k^2\big(\alpha^2-4\big)k^2 + \big(3\alpha^2-8\big)k - 4\big)\\
&\qquad + \big(k^2-1\big)c_2 - 6\alpha s + 9\big),\\
C_{36} &= \frac{(2\alpha c(k^2+4k+3) + 2s(\alpha^2k^2 - k^2 + 3\alpha^2k + 1) - (1+k)(2k\alpha + 6\alpha - ks_2 + s_2))}{(k+3)(\alpha-s)\Delta},\\
C_{46} &= 2(1+k)s_1\frac{((2\alpha^2-1)k^2 + 6\alpha^2k + 1) + (1+k)((k-1)\cos(3\alpha/2) - 2(3+k)\alpha s_1)}{(k+3)(\alpha-s)\Delta},\\
C_{56} &= -(1+k)\frac{(3+k)\alpha - (k-1)s}{\Delta},\\
C_{66} &= \frac{(\alpha c(k^2+4k+3) + s(\alpha^2k^2 - k^2 + 3\alpha^2k + 1) - (1+k)/2(2k\alpha + 6\alpha - ks_2 + s_2))}{(\alpha-s)\Delta}.
\end{aligned}
$$

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