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UNIQUENESS AND LONG TIME ASYMPTOTIC FOR THE KELLER-SEGEL EQUATION: THE PARABOLIC-ELLIPTIC CASE

G. EGAÑA, S. MISCHLER

ABSTRACT. The present paper deals with the parabolic-elliptic Keller-Segel equation in the plane in the general framework of weak (or “free energy”) solutions associated to initial datum with finite mass M , finite second moment and finite entropy. The aim of the paper is threefold:

(1) We prove the uniqueness of the “free energy” solution on the maximal interval of existence $[0, T^*)$ with $T^* = \infty$ in the case when $M \leq 8\pi$ and $T^* < \infty$ in the case when $M > 8\pi$. The proof uses a DiPerna-Lions renormalizing argument which makes possible to get the “optimal regularity” as well as an estimate of the difference of two possible solutions in the critical $L^{4/3}$ Lebesgue norm similarly as for the $2d$ vorticity Navier-Stokes equation.

(2) We prove immediate smoothing effect and, in the case $M < 8\pi$, we prove Sobolev norm bound uniformly in time for the rescaled solution (corresponding to the self-similar variables).

(3) In the case $M < 8\pi$, we also prove weighted $L^{4/3}$ linearized stability of the self-similar profile and then universal optimal rate of convergence of the solution to the self-similar profile. The proof is mainly based on an argument of enlargement of the functional space for semigroup spectral gap.

Keywords: Keller-Segel model; chemotaxis; weak solutions; free energy; entropy method; logarithmic Hardy-Littlewood-Sobolev inequality; Hardy-Littlewood-Sobolev inequality; subcritical mass; uniqueness; large time behavior; self-similar variables.

AMS Subject Classification: 35B45, 35B30, 35D05, 35K15, 35B40, 35D10, 35K60

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1. INTRODUCTION

The aim of the paper is to prove uniqueness of weak “free energy” solutions to the the so-called parabolic-elliptic Keller-Segel equation in the plane associated to initial datum with finite mass $M \geq 0$, finite polynomial moment and finite entropy, and in the subcritical case $M < 8\pi$, to prove optimal rate of convergence to self-similarity of these solutions. In [19] our analysis will be extended to the parabolic-parabolic Keller-Segel equation in a similar context.

The Keller-Segel (KS) system for chemotaxis describes the collective motion of cells that are attracted by a chemical substance that they are able to emit ([34, 27]). We refer to [8] and the references quoted therein for biological motivation and mathematical introduction. In this paper we are concerned with the parabolic-elliptic KS model in the plane which takes the form

$$(1.1) \quad \begin{aligned} \partial_t f &= \Delta f - \nabla(f \nabla c) & \text{in } (0, \infty) \times \mathbb{R}^2, \\ c &:= -\bar{\kappa} = -\kappa * f & \text{in } (0, \infty) \times \mathbb{R}^2, \end{aligned}$$

with $\kappa := \frac{1}{2\pi} \log |z|$, so that in particular

$$-\nabla c = \bar{\mathcal{K}} := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa = \frac{1}{2\pi} \frac{z}{|z|^2}.$$

Here $t \geq 0$ is the time variable, $x \in \mathbb{R}^2$ is the space variable, $f = f(t, x) \geq 0$ stands for the *mass density of cells* while $c = c(t, x) \in \mathbb{R}$ is the *chemo-attractant concentration* which solves the (elliptic) Poisson equation $-\Delta c = f$ in $(0, \infty) \times \mathbb{R}^2$.

The evolution equation (1.1) is complemented with an initial condition

$$(1.2) \quad f(0, \cdot) = f_0 \quad \text{in } \mathbb{R}^2,$$

where throughout this paper, we shall assume that

$$(1.3) \quad 0 \leq f_0 \in L^1_{loc}(\mathbb{R}^2), \quad f_0 \log f_0 \in L^1(\mathbb{R}^2).$$

Here and below for any weight function $\varpi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ we define the weighted Lebesgue space $L^p(\varpi)$ for $1 \leq p \leq \infty$ by

$$L^p(\varpi) := \{f \in L^1_{loc}(\mathbb{R}^2); \|f\|_{L^p(\varpi)} := \|f \varpi\|_{L^p} < \infty\},$$

as well as $L^1_+(\mathbb{R}^2)$ the cone of nonnegative functions of $L^1(\mathbb{R}^2)$. We also use the shorthand L^p_k , $k \geq 0$, for the weighted Lebesgue space associated to the polynomial growth weight function $\varpi(x) := \langle x \rangle^k$, $\langle x \rangle := (1 + |x|^2)^{1/2}$.

The fundamental identities are that any solution to the Keller-Segel equation (1.1) satisfies at least formally the conservation of mass

$$(1.4) \quad M(t) := \int_{\mathbb{R}^2} f(t, x) dx = \int_{\mathbb{R}^2} f_0(x) dx =: M,$$

the second moment equation

$$(1.5) \quad M_2(t) := \int_{\mathbb{R}^2} f(t, x) |x|^2 dx = C_1(M) t + M_{2,0}, \quad M_{2,0} := \int_{\mathbb{R}^2} f_0(x) |x|^2 dx,$$

$C_1(M) := 4M(1 - \frac{M}{8\pi})$, and the free energy-dissipation of the free energy identity

$$(1.6) \quad \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s) ds = \mathcal{F}_0,$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0)$ is defined by

$$\mathcal{F} = \mathcal{F}(f) := \int_{\mathbb{R}^2} f \log f dx + \frac{1}{2} \int_{\mathbb{R}^2} f \bar{\kappa} dx,$$

and the dissipation of free energy is defined by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f) := \int_{\mathbb{R}^2} f |\nabla(\log f) + \nabla \bar{\kappa}|^2 dx.$$

It is worth emphasizing that the critical mass $M_* := 8\pi$ is a threshold because one sees from (1.5) that there does not exist nonnegative and mass preserving solution when $M > 8\pi$ (the identity (1.5) would imply that the second moment becomes negative in a finite time shorter than $T^{**} := 2\pi M_{2,0}/[M(8\pi - M)]$ which is in contradiction with the positivity of the solution).

On the one hand, in the subcritical case $M < 8\pi$, thanks to the logarithmic Hardy-Littlewood Sobolev inequality (see e.g. [3, 18])

$$(1.7) \quad \forall f \geq 0, \quad \int_{\mathbb{R}^2} f(x) \log f(x) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| dx dy \geq C_2(M),$$

with $C_2(M) := M(1 + \log \pi - \log M)$, one can easily check (see [8, Lemma 7]) that

$$(1.8) \quad \mathcal{H} := \mathcal{H}(f) = \int_{\mathbb{R}^2} f \log f dx \leq C_3(M) \mathcal{F} + C_4(M),$$

with $C_3(M) := 1/(1 - \frac{M}{8\pi})$, $C_4(M) := C_3(M)C_2(M)M/(8\pi)$. Then from (1.8) and the very classical functional inequality (see for instance [8, Lemma 8])

$$(1.9) \quad \mathcal{H}^+ := \mathcal{H}^+(f) = \int_{\mathbb{R}^2} f(\log f)_+ dx \leq \mathcal{H} + \frac{1}{4}M_2 + C_5(M),$$

with $C_5(M) := 2M \log(2\pi) + 2/e$, one concludes that (1.4), (1.5) and (1.6) provide a convenient family of a priori estimates in order to define weak solutions. More precisely, we get

$$(1.10) \quad \begin{aligned} \mathcal{H}^+(f(t)) + M_2(f(t)) + C_3(M) \int_0^t \mathcal{D}_{\mathcal{F}}(f(s)) ds &\leq \\ &\leq C_3(M) \mathcal{F}_0 + \frac{5}{4}M_{2,0} + 2C_1(M)t + C_4(M) + C_5(M), \end{aligned}$$

where the RHS term is finite under assumption (1.3) on f_0 , since

$$(1.11) \quad \begin{aligned} \mathcal{F}_0 &\leq \mathcal{H}_0 + \frac{1}{4\pi} \iint f_0(x) f_0(y) (\log|x-y|)_+ dx dy \\ &\leq \mathcal{H}_0 + \frac{1}{4\pi} \iint f_0(x) f_0(y) |x-y|^2 dx dy \leq \mathcal{H}_0 + \frac{1}{\pi} M M_{2,0}, \end{aligned}$$

with $\mathcal{H}_0 := \mathcal{H}(f_0)$. In other words, we have

$$(1.12) \quad \mathcal{A}_T(f) := \sup_{t \in [0, T]} \{ \mathcal{H}^+(f(t)) + M_2(f(t)) \} + \int_0^T \mathcal{D}_{\mathcal{F}}(f(s)) ds \leq C(T) \quad \forall T \in (0, T^*)$$

with $T^* = +\infty$ and a constant $C(T)$ which depends on M , $M_{2,0}$, \mathcal{H}_0 and the final time T .

On the other hand, in the critical case $M = 8\pi$ and the supercritical case $M > 8\pi$, the above argument using the logarithmic Hardy-Littlewood Sobolev inequality (1.7) fails, but one can however prove that (1.12) holds with $T^* = +\infty$ when $M = 8\pi$ and that (1.12) holds with some final time $T^* \in (0, T^{**}]$ when $M > 8\pi$ (see [6] for details as well as Remark 2.3 below).

Definition 1.1. For any initial datum f_0 satisfying (1.3) and any final time $T^* > 0$, we say that

$$(1.13) \quad 0 \leq f \in L^\infty(0, T; L^1(\mathbb{R}^2)) \cap C([0, T]; \mathcal{D}'(\mathbb{R}^2)), \quad \forall T \in (0, T^*),$$

is a weak solution to the Keller-Segel equation in the time interval $(0, T^*)$ associated to the initial condition f_0 whenever f satisfies (1.4), (1.5) and

$$(1.14) \quad \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s) ds \leq \mathcal{F}_0 \quad \forall t \in (0, T^*),$$

as well as the Keller-Segel equation (1.1)-(1.2) in the distributional sense, namely

$$(1.15) \quad \int_{\mathbb{R}^2} f_0(x) \varphi(0, x) dx = \int_0^{T^*} \int_{\mathbb{R}^2} f(t, x) \left\{ (\nabla_x(\log f) + \bar{\mathcal{K}}) \cdot \nabla_x \varphi - \partial_t \varphi \right\} dx dt$$

for any $\varphi \in C_c^2([0, T] \times \mathbb{R}^2)$.

It is worth emphasizing that thanks to the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}^2} f |\nabla_x(\log f) + \bar{\mathcal{K}}| dx \leq M^{1/2} \mathcal{D}_{\mathcal{F}}^{1/2},$$

and the RHS of (1.15) is then well defined thanks to (1.10).

This framework is well adapted for the existence theory.

Theorem 1.2. For any initial datum f_0 satisfying (1.3) there exists at least one weak solution on the time interval $(0, T^*)$ in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2) with $T^* = +\infty$ when $M \leq 8\pi$ and $T^* < +\infty$ when $M > 8\pi$.

We refer to [8, Theorem 1] for the subcritical case $M \in (0, 8\pi)$ and to [6] for the critical and supercritical cases $M \geq 8\pi$.

Our first main result establishes that this framework is also well adapted for the well-posedness issue.

Theorem 1.3. *For any initial datum f_0 satisfying (1.3) there exists at most one weak solution in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2).*

Theorem 1.3 improves the uniqueness result proved in [20] in the class of solutions $f \in C([0, T]; L^1_2(\mathbb{R}^2)) \cap L^\infty((0, T) \times \mathbb{R}^2)$ which can be built under the additional assumption $f_0 \in L^\infty(\mathbb{R}^2)$ (see also [24] where a uniqueness result is established for a related model). Our proof follows a strategy introduced in [23] for the 2D viscous vortex model. It is based on a DiPerna-Lions renormalization trick (see [21]) which makes possible to get the optimal regularity of solutions for small time and then to follow the uniqueness argument introduced by Ben-Artzi for the 2D viscous vortex model (see [4, 10]). More precisely, we start proving the optimal regularity for short time $t^{1/4} \|f(t)\|_{L^{4/3}} \rightarrow 0$ as $t \rightarrow 0$ for any weak solution f , and next we estimate the $L^{4/3}$ -norm of the difference of two possible solutions written in mild formulation. We emphasize that the $L^{4/3}$ -space is critical for the Hardy-Littlewood-Sobolev inequality (see e.g. [28, Theorem 4.3]) because it writes in that case

$$(1.16) \quad \|f * \mathcal{K}\|_{L^{(4/3)' }(\mathbb{R}^2)} = \|f * \mathcal{K}\|_{L^4(\mathbb{R}^2)} \leq C \|f\|_{L^{4/3}(\mathbb{R}^2)},$$

where $p' \in [1, \infty]$ is the conjugate exponent associated to $p \in [1, \infty]$ defined by $1/p + 1/p' = 1$. That last inequality is the key estimate in order to control the nonlinear term in (1.1). One probably could perform a similar argument with the L^q -norm, $q \geq 4/3$, see [19].

Next we consider the smoothness issue and the long time behaviour of solution for subcritical mass issue. For that last purpose it is convenient to work with self-similar variables. We introduce the rescaled functions g and u defined by

$$(1.17) \quad g(t, x) := R(t)^{-2} f(\log R(t), R(t)^{-1}x), \quad u(t, x) := c(\log R(t), R(t)^{-1}x),$$

with $R(t) := (1 + 2t)^{1/2}$. The rescaled parabolic-elliptic KS system reads

$$(1.18) \quad \begin{aligned} \partial_t g &= \Delta g + \nabla(gx - g \nabla u) && \text{in } (0, \infty) \times \mathbb{R}^2, \\ u &= -\kappa * g && \text{in } (0, \infty) \times \mathbb{R}^2. \end{aligned}$$

Our second main result concerns the regularity of the solutions.

Theorem 1.4. *For any initial datum f_0 satisfying (1.3) the associated solution f is smooth for positive time, namely $f \in C^\infty((0, T^*) \times \mathbb{R}^2)$, and satisfies the identity (1.6) on $(0, T^*)$. Moreover, when $M < 8\pi$, the rescaled solution g defined by (1.17)-(1.1) satisfies the uniform in time moment estimate*

$$(1.19) \quad \sup_{t \geq 0} M_k(g(t)) \leq \max((k-1)^{k/2} M, M_k(f_0)) \quad \forall k \geq 2,$$

with $M_k(g) := \|g\|_{L^1_k}$, as well as the uniform in time regularity estimate for positive time

$$(1.20) \quad \sup_{t \geq \varepsilon} \|g(t, \cdot)\|_{W^{2, \infty}} \leq \mathcal{C} \quad \forall \varepsilon > 0,$$

for some explicit constant \mathcal{C} which depends on ε , M , \mathcal{F}_0 and $M_{2,0}$.

It is worth mentioning that L^p bounds on g for positive time and for $p \in [1, \infty)$ were known, but non uniformly in time and as an a priori bound, while here (1.20) is proved as an a posteriori estimate. Our proof is merely based on the same estimates as those established in [8], on a bootstrap argument (using the DiPerna-Lions renormalization trick) and on the observation that the rescaled free energy provides uniform in time estimates.

From now on in this introduction, we definitively restrict ourself to the subcritical case $M < 8\pi$ and we focus on the long time asymptotic of the solutions. It has been proved in [8, Theorem 1.2] that the solution given by Theorem 1.2 satisfies

$$(1.21) \quad g(t, \cdot) \rightarrow G \quad \text{in } L^1(\mathbb{R}^2) \quad \text{as } t \rightarrow \infty,$$

where G is a solution to the rescaled stationary problem

$$(1.22) \quad \begin{aligned} \Delta G + \nabla(Gx - G \nabla U) &= 0 \quad \text{in } \mathbb{R}^2, \\ 0 \leq G, \quad \int_{\mathbb{R}^2} G dx &= M, \quad U = -\mathcal{K} * G. \end{aligned}$$

Moreover, the uniqueness of the solution G to (1.22) has been proved in [8, 5], see also [15, 16, 17]. From now on, $G = G_M$ stands for the unique self-similar profile with same mass M as f_0 and it is given in implicit form by

$$(1.23) \quad G = M \frac{e^{-G*\kappa-|x|^2/2}}{\int_{\mathbb{R}^2} e^{-G*\kappa-|x|^2/2} dx},$$

or equivalently $U = -G * \kappa$ satisfies

$$(1.24) \quad \Delta U + \frac{M}{\int_{\mathbb{R}^2} e^{U-|x|^2/2} dx} e^{U-|x|^2/2} = 0.$$

Our third main result is about the convergence to self-similarity.

Theorem 1.5. *For any $M \in (0, 8\pi)$, and any finite positive real numbers \mathcal{F}_0^* , $k^* > 3$, $M_{k^*,0}^*$, there exists a (non explicit) constant C such that for any initial datum f_0 satisfying (1.3) with*

$$M_0(f_0) = M, \quad M_{k^*}(f_0) \leq M_{k^*,0}^*, \quad \mathcal{F}(f_0) \leq \mathcal{F}_0^*,$$

the associated solution in self-similar variables g defined by (1.17)-(1.1) satisfies the optimal rate of convergence

$$\|g(t, \cdot) - G\|_{L^{4/3}} \leq C e^{-t} \quad \forall t \geq 1,$$

where G stands for the self-similar profile with same mass M as f_0 .

Let us emphasize that assuming only the second moment bound $M_2(f_0) < \infty$, the same proof leads to a not optimal rate of convergence to the self-similar profile, namely $\|g(t, \cdot) - G\|_{L^{4/3}} \leq C_\eta e^{-\eta t}$ for all $t \geq 1$ and for some $\eta \in (0, 1)$, $C_\eta \in (0, \infty)$. It is likely that stronger moment assumption on the initial datum leads to the same optimal rate of convergence in L^q -norm with larger values of q , but we do not follow that line of research in the present work.

Theorem 1.5 drastically improves some anterior results which establish the same exponential rate of convergence for some particular class of initial data. On the one hand, for a radially symmetric initial datum with finite second moment it has been proved in [15, Theorem 1.2] the same convergence in Wasserstein distance W_2 by a direct and nice entropy method. On the other hand, the same convergence in $L^2(e^{\nu|x|^2})$ norm, $\nu \in (0, 1/4)$, has been recently proven to hold in [17, Theorem 1] (see also [7, 16] for previous results in that direction) for any initial datum f_0 with mass $M \in (0, 8\pi)$ and which satisfies (roughly speaking) the strong confinement condition $f_0 \leq \tilde{G}$ for some self-similar profile \tilde{G} associated to some mass $\tilde{M} \in [M, 8\pi)$. In that last work [17], the uniform exponential stability (with optimal rate) of the linearized rescaled equation is established in $L^2(G^{-1/2})$ by the mean of the analysis of the associated linear operator in a well chosen (equivalent) Hilbert norm. The nonlinear exponential stability is then deduced from that linear stability together with an uniform in time estimate deduced from the strong confinement assumption made on the initial datum.

Our proof follows a strategy of “enlarging the functional space of semigroup spectral gap” initiated in [32] for studying long time convergence to the equilibrium for the homogeneous Boltzmann equation, and then developed in [30, 25, 12, 11, 29] (see also [31]) in the framework of kinetic equations and growth-fragmentation equations. More precisely, taking advantage of the uniform exponential stability of the linearized rescaled equation established in [17] in the small (strongly

confining) space $L^2(G^{-1/2})$ (observe that $\log G(x) \sim -|x|^2/2$ in the large position asymptotic) we prove that the same uniform exponential stability (with the same optimal rate) result holds in the larger space $L_\ell^{4/3}$, $\ell > 3/2$. It is worth emphasizing that the choice of the exponent $4/3$ is made in order to handle the singularity of the force field (thanks to the critical Hardy-Littlewood-Sobolev inequality (1.16)) while the choice of the moment exponent $\ell > 3/2$ is made in order to have enough confinement and then to get the optimal rate. We probably can perform a similar semigroup spectral gap analysis in a different space L_ℓ^q , $q \geq 4/3$, $\ell \geq 3/2$, but also probably at the cost of a stronger confinement (higher moment bound) assumption of the initial datum since at some point in the proof we use the embedding $L_\ell^q \subset L^\infty \cap L_{q\ell}^1$. Anyway, we do not follow that line of research in the present work. Next, gathering the long time convergence (without rate) to self-similarity (1.21) with the estimates of Theorem 1.4, we obtain that any solution reaches a small $L_\ell^{4/3}$ -neighborhood of G in finite time and we conclude to Theorem 1.5 by nonlinear stability in $L_\ell^{4/3}$. It is worth emphasizing that it is only in that last nonlinear step that we use the a bit stronger initial (and then uniform in time) moment estimate (1.19) with $k = k^* > 3$.

Let us end the introduction by describing the plan of the paper. In Section 2 we present some functional inequalities which will be useful in the sequel of the paper, we establish several a posteriori bounds satisfied by any weak solution, and we prove Theorem 1.4. Section 3 is dedicated to the proof of the uniqueness result stated in Theorem 1.3. In Section 4 we prove the long time behaviour result as stated in Theorem 1.5.

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2. A POSTERIORI ESTIMATES - PROOF OF THEOREM 1.4

We start by presenting some elementary functional inequalities which will be of main importance in the sequel. The two first estimates are picked up from [23, Lemma 3.2] but are probably classical and the third one is a variant of the Gagliardo-Nirenberg-Sobolev inequality.

Lemma 2.1. *For any $0 \leq f \in L^1(\mathbb{R}^2)$ with finite mass M and finite Fisher information*

$$I = I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f},$$

there hold

$$(2.1) \quad \forall p \in [1, \infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \leq C_p M^{1/p} I(f)^{1-1/p},$$

$$(2.2) \quad \forall q \in [1, 2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \leq C_q M^{1/q-1/2} I(f)^{3/2-1/q}.$$

For any $0 \leq f \in L^1(\mathbb{R}^2)$ with finite mass M , there holds

$$(2.3) \quad \forall p \in [2, \infty) \quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \leq C_p M^{1/(p+1)} \|\nabla(f^{p/2})\|_{L^2}^{2/(p+1)}.$$

For the sake of completeness we give the proof below.

Proof of Lemma 2.1. We start with (2.2). Let $q \in [1, 2)$ and use the Hölder inequality:

$$\|\nabla f\|_{L^q}^q = \int \left| \frac{\nabla f}{\sqrt{f}} \right|^q f^{q/2} \leq \left(\int \frac{|\nabla f|^2}{f} \right)^{q/2} \left(\int f^{q/(2-q)} \right)^{(2-q)/2} = I(f)^{q/2} \|f\|_{L^{q/(2-q)}}^{q/2}.$$

Denoting by $q^* = 2q/(2-q) \in [2, \infty)$ the Sobolev exponent associated to q in dimension 2, we have, thanks to a standard interpolation inequality and to the Sobolev inequality,

$$(2.4) \quad \begin{aligned} \|f\|_{L^{q/(2-q)}} &= \|f\|_{L^{q^*/2}} \leq \|f\|_{L^1}^{1/(q^*-1)} \|f\|_{L^{q^*}}^{(q^*-2)/(q^*-1)} \\ &\leq C_q \|f\|_{L^1}^{1/(q^*-1)} \|\nabla f\|_{L^q}^{(q^*-2)/(q^*-1)}. \end{aligned}$$

Gathering these two inequalities, it comes

$$\|\nabla f\|_{L^q} \leq C_q I(f)^{1/2} \|f\|_{L^1}^{1/(2(q^*-1))} \|\nabla f\|_{L^q}^{(q^*-2)/(2(q^*-1))},$$

from which we deduce (2.2).

We now establish (2.1). For $p \in (1, \infty)$, write $p = q^*/2 = q/(2-q)$ with $q := 2p/(1+p) \in [1, 2)$ and use (2.4) and (2.2) to get

$$\|f\|_{L^p} \leq C_p \|f\|_{L^1}^{\frac{1}{q^*-1} + \frac{q^*-2}{q^*-1}(\frac{1}{q} - \frac{1}{2})} I(f)^{\frac{q^*-2}{q^*-1}(\frac{3}{2} - \frac{1}{q})},$$

from which one easily concludes.

We verify (2.3). From the Sobolev inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|w^{2(1+1/p)}\|_{L^1(\mathbb{R}^2)} &= \|w^{1+1/p}\|_{L^2(\mathbb{R}^2)}^2 \leq \|\nabla(w^{1+1/p})\|_{L^1(\mathbb{R}^2)}^2 \\ (2.5) \qquad \qquad \qquad &\leq (1+1/p)^2 \|w^{1/p}\|_{L^2}^2 \|\nabla w\|_{L^2(\mathbb{R}^2)}^2 \end{aligned}$$

and we conclude to (2.3) by taking $w := f^{p/2}$. \square

The proof of (1.20) in Theorem 1.4 is split into several steps that we present as some intermediate autonomous a posteriori bounds.

Lemma 2.2. *For any weak solution f and any final time $T \in (0, T^*)$ there exists a constant $C := C(M, \mathcal{A}_T(f))$ such that*

$$(2.6) \qquad \qquad \qquad \frac{1}{2} \int_0^T I(f(t)) dt \leq C.$$

In particular, in the subcritical case $M < 8\pi$ the constant C only depends on M , \mathcal{H}_0 , $M_{2,0}$ and $T \in (0, \infty)$.

Proof of Lemma 2.2. On the one hand, we write

$$\begin{aligned} \mathcal{D}_{\mathcal{F}}(f) &= \int f |\nabla(\log f + \bar{\kappa})|^2 \\ &\geq \int f |\nabla \log f|^2 + 2 \int \nabla f \cdot \nabla \bar{\kappa} = I(f) - 2 \int f^2. \end{aligned}$$

On the other hand, for any $A > 1$, using the Cauchy-Schwarz inequality and the inequality (2.1) for $p = 3$, we have

$$\begin{aligned} \int f^2 \mathbf{1}_{f \geq A} &\leq \left(\int f \mathbf{1}_{f \geq A} \right)^{1/2} \left(\int f^3 \right)^{1/2} \\ &\leq \left(\int f \frac{(\log f)_+}{\log A} \right)^{1/2} \left(C_3^3 M I(f)^2 \right)^{1/2}, \end{aligned}$$

from what we deduce for $A = A(M, \mathcal{H}^+(f))$ large enough, and more precisely taking A such that $\log A = 16 \mathcal{H}^+(f) C_3^3 M$,

$$(2.7) \qquad \int f^2 \mathbf{1}_{f \geq A} \leq C_3^{3/2} M^{1/2} \frac{\mathcal{H}^+(f)^{1/2}}{(\log A)^{1/2}} I(f) \leq \frac{1}{4} I(f).$$

Together with the first estimate, we find

$$\begin{aligned} \frac{1}{2} I(f) &\leq \mathcal{D}_{\mathcal{F}}(f) + 2 \int f^2 \mathbf{1}_{f \leq A} \\ &\leq \mathcal{D}_{\mathcal{F}}(f) + 2 M \exp(16 \mathcal{H}^+(f) C_3^3 M), \end{aligned}$$

and we conclude thanks to (1.12) in the general case and thanks to (1.4)–(1.11) in the subcritical case $M < 8\pi$. \square

Remark 2.3. As we have already mentioned we are not able to use the logarithmic Hardy-Littlewood-Sobolev inequality (1.7) in the critical and supercritical cases. However, introducing the Maxwell function $\mathcal{M} := M(2\pi)^{-1} \exp(-|x|^2/2)$ of mass M and the relative entropy

$$H(h|\mathcal{M}) := \int_{\mathbb{R}^2} (h \log(h/\mathcal{M}) - h + \mathcal{M}) dx,$$

one classically shows that any solution f to the Keller-Segel equation (1.1) formally satisfies

$$\begin{aligned} \frac{d}{dt} H(f(t)|\mathcal{M}) &= -I(f(t)) + \int f(t)^2 + C_1/2 \\ &\leq -I(f(t)) + MA + C_3^{3/2} M^{1/2} \frac{\mathcal{H}^+(f(t))^{1/2}}{(\log A)^{1/2}} I(f(t)) + C_1/2 \quad (\forall A > 0) \\ &= -I(f(t)) + M \exp(4C_3^3 M \mathcal{H}^+(f(t))) + C_1/2 \\ &= -I(f(t)) + M \exp\{C_6 H(f(t)|\mathcal{M})\} + C_1/2, \end{aligned}$$

for a constant $C_6 = C_6(M)$ and where $C_1 = C_1(M)$ is defined in (1.5). In the above estimates, we have used (2.7), we have made the choice $\log A := 4C_3^3 M \mathcal{H}^+(f(t))$ and we have used a variant of inequality (1.9). This differential inequality provides a local a priori estimate on the relative entropy which is the key estimate in order to prove local existence result for supercritical mass as well as global existence result for critical mass in [6].

As an immediate consequence of Lemmas 2.1 and 2.2, we have

Lemma 2.4. For any $T \in (0, T^*)$, any weak solution f satisfies

$$(2.8) \quad f \in L^{p/(p-1)}(0, T; L^p(\mathbb{R}^2)), \quad \forall p \in (1, \infty),$$

$$(2.9) \quad \bar{\mathcal{K}} \in L^{p/(p-1)}(0, T; L^{2p/(2-p)}(\mathbb{R}^2)), \quad \forall p \in (1, 2),$$

$$(2.10) \quad \nabla_x \bar{\mathcal{K}} \in L^{p/(p-1)}(0, T; L^p(\mathbb{R}^2)), \quad \forall p \in (2, \infty).$$

Proof of Lemma 2.4. The bound (2.8) is a direct consequence of (2.6) and (2.1). The bound (2.9) then follows from the definition of K , the Hardy-Littlewood-Sobolev inequality (see e.g. [28, Theorem 4.3])

$$(2.11) \quad \left\| \frac{1}{|z|} * f \right\|_{L^{2r/(2-r)}(\mathbb{R}^2)} \leq C_r \|f\|_{L^r(\mathbb{R}^2)}, \quad \forall r \in (1, 2),$$

with $r = p$ and (2.8). Finally, from (2.6) and (2.2) we have

$$\nabla f \in L^{\frac{2q}{3q-2}}(0, T; L^q(\mathbb{R}^2)), \quad \forall q \in (1, 2).$$

Applying the Hardy-Littlewood-Sobolev inequality (2.11) to $\nabla_x \bar{\mathcal{K}} = \mathcal{K} * (\nabla_x f)$ with $r = q$, we get

$$\nabla_x \bar{\mathcal{K}} \in L^{\frac{2q}{3q-2}}(0, T; L^{\frac{2q}{2-q}}(\mathbb{R}^2)), \quad \forall q \in (1, 2),$$

which is nothing but (2.10). □

Lemma 2.5. Any weak solution f satisfies

$$(2.12) \quad \begin{aligned} &\int_{\mathbb{R}^2} \beta(f_{t_1}) dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s) |\nabla f_s|^2 dx ds \\ &\leq \int_{\mathbb{R}^2} \beta(f_{t_0}) dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} (\beta'(f_s) f_s^2 - \beta(f_s) f_s)_+ dx ds, \end{aligned}$$

for any times $0 \leq t_0 \leq t_1 < T^*$ and any renormalizing function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(u)| \leq C(1 + u(\log u)_+), \quad (\beta'(u)u^2 - \beta(u)u)_+ \leq C(1 + u^2) \quad \forall u \in \mathbb{R}.$$

Proof of Lemma 2.5. We write

$$\partial_t f - \Delta_x f = \bar{\mathcal{K}} \cdot \nabla_x f + f^2,$$

and we split the proof into three steps.

Step 1. Continuity. Consider a mollifier sequence (ρ_n) on \mathbb{R}^2 , that is $\rho_n(x) := n^2 \rho(nx)$, $0 \leq \rho \in \mathcal{D}(\mathbb{R}^2)$, $\int \rho = 1$, and introduce the mollified function $f_t^n := f_t *_{x} \rho_n$. Clearly, $f^n \in C([0, T], L^1(\mathbb{R}^2))$. Using (2.8) and (2.10), a variant of the commutation Lemma [21, Lemma II.1 and Remark 4] tells us that

$$(2.13) \quad \partial_t f^n - \bar{\mathcal{K}} \cdot \nabla_x f^n - \Delta_x f^n = r^n,$$

with

$$r^n := (f^2) * \rho_n + (\bar{\mathcal{K}} \cdot \nabla_x f) * \rho_n - \bar{\mathcal{K}} \cdot \nabla_x f^n \rightarrow f^2 \quad \text{in } L^1(0, T; L^1_{loc}(\mathbb{R}^2)).$$

The important point here is that $f^2, |\nabla_x \bar{\mathcal{K}}| f \in L^1((0, T) \times \mathbb{R}^2)$, thanks to (2.10) and (2.8).

As a consequence, the chain rule applied to the smooth function f^n reads

$$(2.14) \quad \partial_t \beta(f^n) = \bar{\mathcal{K}} \cdot \nabla_x \beta(f^n) + \Delta_x \beta(f^n) - \beta''(f^n) |\nabla_x f^n|^2 + \beta'(f^n) r^n,$$

for any $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2, \infty}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Because the equation (2.13) with $\bar{\mathcal{K}}$ fixed is linear, the difference $f^{n,k} := f^n - f^k$ satisfies (2.13) with r^n replaced by $r^{n,k} := r^n - r^k \rightarrow 0$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^2))$ and then also (2.14) (with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$). In that last equation, we choose $\beta(s) = \beta_1(s)$ where $\beta_A(s) = s^2/2$ for $|s| \leq A$, $\beta_A(s) = A|s| - A^2/2$ for $|s| \geq A$ and we obtain for any non-negative function $\chi \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned} & \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t, x)) \chi(x) dx \leq \\ & \leq \int_{\mathbb{R}^2} \beta_1(f^{n,k}(0, x)) \chi(x) dx + \int_0^t \int_{\mathbb{R}^2} |r^{n,k}(s, x)| \chi(x) dx ds \\ & + \int_0^t \int_{\mathbb{R}^2} \beta_1(f^{n,k}(s, x)) \left| -f \chi + \Delta \chi(x) - \bar{\mathcal{K}}(s, x) \cdot \nabla \chi(x) \right| dx ds, \end{aligned}$$

where we have used that $\operatorname{div}_x \bar{\mathcal{K}} = f$, that $|\beta'_1| \leq 1$ and that $\beta''_1 \geq 0$. In the last inequality, the RHS term converges to 0 as n, k tend to infinity. More precisely, $\beta_1(f^{n,k}(0)) \rightarrow 0$ in $L^1(\mathbb{R}^2)$ because $f_0 \in L^1(\mathbb{R}^2)$; $r^{n,k} \rightarrow 0$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^2))$ by the DiPerna-Lions commutation Lemma recalled above; $\beta_1(f^{n,k}) \bar{\mathcal{K}} \rightarrow 0$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^2))$ because $\beta_1(s) \leq |s|$, because $f^{n,k} \rightarrow 0$ in $L^3(0, T, L^{3/2}(\mathbb{R}^2))$ by (2.8) with $p = 3/2$ and because $\bar{\mathcal{K}} \in L^6(0, T; L^3(\mathbb{R}^2))$ by (2.9) with $p = 6/5$; $\beta_1(f^{n,k}) f \rightarrow 0$ in $L^1(0, T; L^1(\mathbb{R}^2))$ because again $\beta_1(s) \leq |s|$ and $f \in L^2((0, T) \times \mathbb{R}^2)$ by (2.8) with $p = 2$. All together, we get

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t, x)) \chi(x) dx \xrightarrow{n, k \rightarrow \infty} 0.$$

Since χ is arbitrary, we deduce that there exists $\bar{f} \in C([0, \infty); L^1_{loc}(\mathbb{R}^2))$ so that $f^n \rightarrow \bar{f}$ in $C([0, T]; L^1_{loc}(\mathbb{R}^2))$, $\forall T > 0$. Together with the convergence $f^n \rightarrow f$ in $C([0, \infty); \mathcal{D}'(\mathbb{R}^2))$ and the a priori bound (1.10), we deduce that $f = \bar{f}$ and

$$(2.15) \quad f^n \rightarrow f \quad \text{in } C([0, T]; L^1(\mathbb{R}^2)), \quad \forall T > 0.$$

Step 2. Linear estimates. We come back to (2.14), which implies, for all $0 \leq t_0 < t_1$, all $\chi \in C_c^2(\mathbb{R}^2)$,

$$(2.16) \quad \begin{aligned} & \int_{\mathbb{R}^2} \beta(f_{t_1}^n) \chi dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s^n) |\nabla_x f_s^n|^2 \chi dx ds = \int_{\mathbb{R}^2} \beta(f_{t_0}^n) \chi dx \\ & + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \left\{ \beta'(f_s^n) r^n \chi + \beta(f_s^n) \Delta \chi - \beta(f_s^n) \operatorname{div}_x (\bar{\mathcal{K}} \chi) \right\} dx ds. \end{aligned}$$

Choosing $0 \leq \chi \in C_c^2(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W_{loc}^{2,\infty}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and passing to the limit as $n \rightarrow \infty$, we get

$$(2.17) \quad \int_{\mathbb{R}^2} \beta(f_{t_1}) \chi dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s) |\nabla_x f_s|^2 \chi dx ds \leq \int_{\mathbb{R}^2} \beta(f_{t_0}) \chi dx \\ + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \left\{ [\beta'(f) f^2 - \beta(f) f] \chi + \beta(f) [\Delta \chi - \bar{\mathcal{K}} \cdot \nabla \chi] \right\} dx ds.$$

By approximating $\chi \equiv 1$ by the sequence (χ_R) with $\chi_R(x) = \chi(x/R)$, $0 \leq \chi \in \mathcal{D}(\mathbb{R}^2)$, we see that the last term in (2.17) vanishes and we get (2.12) in the limit $R \rightarrow \infty$ for any renormalizing function β with linear growth at infinity.

Step 3. superlinear estimates. Finally, for any β satisfying the growth condition as in the statement of the Lemma, we just approximate β by an increasing sequence of smooth renormalizing functions β_R with linear growth at infinity, and pass to the limit in (2.12) in order to conclude. \square

As a first consequence of Lemma 2.5, we establish an estimate on the quantity

$$(2.18) \quad \mathcal{H}_2(f) := \int_{\mathbb{R}^2} f (\widetilde{\log} f)^2 dx, \quad \widetilde{\log} u := \mathbf{1}_{u \leq e} + (\log u) \mathbf{1}_{u > e}.$$

Lemma 2.6. *For any weak solution f and any time $T \in (0, T^*)$, there exists a constant $C := C(M, T, \mathcal{A}_T)$ such that for any $0 \leq t_0 < t_1 \leq T$*

$$(2.19) \quad \mathcal{H}_2(f(t_1)) \leq \mathcal{H}_2(f(t_0)) + C.$$

Proof of Lemma 2.6. We define the renormalizing function $\beta_K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $K \geq e^2$, by

$$\beta_K(u) := u (\widetilde{\log} u)^2 \text{ if } u \leq K, \quad \beta_K(u) := (2 + \log K) u \log u - 2K \log K \text{ if } u \geq K,$$

so that β_K is convex and piecewise of class C^1 , and moreover there holds

$$\beta_K'(u) u^2 - \beta_K(u) u \leq 2u^2 \widetilde{\log} u \mathbf{1}_{u \leq K} + 4 \log K u^2 \mathbf{1}_{u > K}$$

and

$$\beta_K''(u) \geq 2 \frac{\log u}{u} \mathbf{1}_{e \leq u \leq K} + (2 + \log K) \frac{1}{u} \mathbf{1}_{u > K}.$$

Defining now

$$\widetilde{\log}_K u := \mathbf{1}_{u \leq e} + (\log u) \mathbf{1}_{e < u \leq K} + (\log K) \mathbf{1}_{u > K},$$

we deduce from (2.12) that

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1}) dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} (\widetilde{\log}_K f) \mathbf{1}_{f \geq e} dx ds \\ \leq \int_{\mathbb{R}^2} \beta_K(f_{t_0}) dx + 4 \int_{t_0}^{t_1} \int_{\mathbb{R}^2} f^2 \widetilde{\log}_K f dx ds.$$

Proceeding as in the proof of Lemma 2.2, we have for any $A \in (e, K)$

$$\int_{\mathbb{R}^2} f^2 \widetilde{\log}_K f dx = \int_{\mathbb{R}^2} f^2 \widetilde{\log}_K f \mathbf{1}_{A \leq K} dx + \int_{\mathbb{R}^2} f^2 \widetilde{\log}_K f \mathbf{1}_{A \geq K} dx \\ \leq (A \log A) M + \left(\frac{\mathcal{H}^+(f)}{\log A} \right)^{1/2} \left(\int_{\mathbb{R}^2} (f \widetilde{\log}_K f)^3 dx \right)^{1/2},$$

as well as thanks to inequality (2.1) with $p = 3$

$$\left(\int_{\mathbb{R}^2} (f \widetilde{\log}_K f)^3 dx \right)^{1/2} \leq C_3^{3/2} \left(\int_{\mathbb{R}^2} f \widetilde{\log}_K f dx \right)^{1/2} \int_{\mathbb{R}^2} \frac{|\nabla(f \widetilde{\log}_K f)|^2}{f \widetilde{\log}_K f} dx \\ \leq 4 C_3^{3/2} (M + \mathcal{H}^+(f))^{1/2} \left(\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} (\widetilde{\log}_K f) \mathbf{1}_{f \geq e} dx + I(f) \right).$$

The last three estimates together, we obtain for A large enough and $K > A$

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1}) dx \leq \int_{\mathbb{R}^2} \beta_K(f_{t_0}) dx + 4T(A \log A)M + \int_{t_0}^{t_1} I(f_s) ds,$$

from which (2.19) immediately follows by letting K tends to $+\infty$ and using Lemma 2.2. \square

We now derive some L^p -norm estimate on the solutions to the KS equation.

Lemma 2.7. *For any weak solution f , any time $T \in (0, T^*)$ and any $p \in [2, \infty)$ and $t_0 \in [0, T)$ such that $f_{t_0} \in L^p$, there exists a constant $C := C(M, T, \mathcal{A}_T, p, \|f_{t_0}\|_{L^p})$ such that*

$$(2.20) \quad \forall t_1 \in [t_0, T], \quad \|f(t_1)\|_{L^p}^p + \frac{1}{2} \int_{t_0}^{t_1} \|\nabla_x(f^{p/2})\|_{L^2}^2 dt \leq C.$$

Proof of Lemma 2.7. We define the renormalizing function $\beta_K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $K \geq 2$, by

$$\beta_K(u) := \frac{u^p}{p} \text{ if } u \leq K, \quad \beta_K(u) := \frac{K^{p-1}}{\log K}(u \log u - u) - \frac{1}{p'}K^p + \frac{K^p}{\log K} \text{ if } u \geq K,$$

so that β_K is convex and of class C^1 , and moreover there holds

$$\beta'_K(u)u^2 - \beta_K(u)u \leq \frac{1}{p'}u^{p+1} \mathbf{1}_{u \leq K} + 2K^{p-1}u^2 \mathbf{1}_{u > K},$$

as well as

$$\beta''_K(u) = (p-1)u^{p-2} \mathbf{1}_{u \leq K} + \frac{K^{p-1}}{\log K} \frac{1}{u} \mathbf{1}_{u > K}.$$

Thanks to Lemma 2.5, we may write

$$\begin{aligned} & \int_{\mathbb{R}^2} \beta_K(f_{t_1}) dx + \frac{4}{p'p} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx ds + \frac{K^{p-1}}{\log K} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} \mathbf{1}_{f \geq K} dx ds \\ & \leq \int_{\mathbb{R}^2} \beta_K(f_{t_0}) dx + \frac{1}{p'} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} f^{p+1} \mathbf{1}_{f \leq K} dx ds + 2K^{p-1} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} f^2 \mathbf{1}_{f \geq K} dx ds. \end{aligned}$$

On the one hand, using the splitting $f = (f \wedge A) + (f - A)_+$, we have

$$\mathcal{T}_1 := \int_{\mathbb{R}^2} f^{p+1} \mathbf{1}_{f \leq K} dx \leq 2^p A^p M + 2^p \int_{\mathbb{R}^2} f_{A,K}^{p+1} dx,$$

where we have defined $f_{A,K} := \min((f - A)_+, K - A)$, $K > A > 0$. Moreover, thanks to inequality (2.3) and the same trick as in the proof of Lemma 2.2, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f_{A,K}^{p+1} dx & \leq C_p \int_{\mathbb{R}^2} f_{A,K} dx \int_{\mathbb{R}^2} |\nabla(f_{A,K}^{p/2})|^2 dx \\ & \leq C_p \frac{\mathcal{H}^+(f)}{\log A} \int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx. \end{aligned}$$

As a consequence, we obtain

$$\frac{1}{p'} \int_{t_0}^{t_1} \mathcal{T}_1 ds \leq \frac{2^p}{p'} A^p M T + \frac{1}{p'p} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx ds,$$

for $A = A(p, \mathcal{A}_T) > 1$ large enough.

On the other hand, thanks to the Sobolev inequality (line 2) and the Cauchy-Schwarz inequality (line 3), we have

$$\begin{aligned}
\mathcal{T}_2 &:= 2K^{p-1} \int_{\mathbb{R}^2} f^2 \mathbf{1}_{f \geq K} dx \leq 8K^{p-1} \int_{\mathbb{R}^2} (f - K/2)_+^2 dx \\
&\leq 8K^{p-1} \left(\int_{\mathbb{R}^2} |\nabla(f - K/2)_+| dx \right)^2 = 8K^{p-1} \left(\int_{\mathbb{R}^2} |\nabla f| \mathbf{1}_{f \geq K/2} dx \right)^2 \\
&\leq 8K^{p-1} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} \mathbf{1}_{f \geq K/2} dx \int_{\mathbb{R}^2} f \mathbf{1}_{f \geq K/2} dx \\
&\leq 8K^{p-1} \left\{ \frac{4}{p^2} \int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 \left(\frac{2}{K}\right)^{p-1} \mathbf{1}_{f \leq K} + \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} \mathbf{1}_{f \geq K} \right\} \frac{\mathcal{H}_2(f)}{(\log(K/2))^2}.
\end{aligned}$$

Recalling that from Lemma 2.6 we have

$$\sup_{[t_0, T]} \mathcal{H}_2(f) \leq \mathcal{H}_2(f_{t_0}) + C' \leq M + \|f_{t_0}\|_{L^p}^p + C' =: C'',$$

we deduce

$$\begin{aligned}
\int_{t_0}^{t_1} \mathcal{T}_2 ds &\leq \frac{32C''}{(\log K)^2} \left\{ \frac{2^{p+1}}{p^2} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx ds + K^{p-1} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} \mathbf{1}_{f \geq K} dx ds \right\} \\
&\leq \frac{1}{p'p} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx ds + \frac{K^{p-1}}{\log K} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f} \mathbf{1}_{f \geq K} dx ds,
\end{aligned}$$

for any $K \geq K^* = K^*(p, \mathcal{A}_T) > \max(A, 4)$ large enough.

All together, we have proved that for some constant A and K^* only depending on p, T, \mathcal{A}_T and f_{t_0} , and for any $K \geq K^*$ there holds

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1}) dx + \frac{2}{p'} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x(f^{p/2})|^2 \mathbf{1}_{f \leq K} dx ds \leq \int_{\mathbb{R}^2} \beta_K(f_{t_0}) dx + 2^p A^p M T.$$

We conclude to (2.20) by passing to the limit $K \rightarrow \infty$. \square

Lemma 2.8. *Any weak solution f is smooth, that is*

$$f \in C_b^\infty((\varepsilon, T) \times \mathbb{R}^2), \quad \forall \varepsilon, T, \quad 0 < \varepsilon < T < T^*,$$

so that in particular it is a “classical solution” for positive time.

Proof of Lemma 2.8. For any time $t_0 \in (0, T)$ and any exponent $p \in (1, \infty)$, there exists $t'_0 \in (0, t_0)$ such that $f(t'_0) \in L^p(\mathbb{R}^2)$ thanks to (2.8), from what we deduce using (2.20) on the time interval (t'_0, T) that

$$(2.21) \quad f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0, T) \times \mathbb{R}^2).$$

Next, by writing $\mathcal{K} = \mathcal{K} \mathbf{1}_{|z| \leq 1} + \mathcal{K} \mathbf{1}_{|z| \geq 1} \in L^{3/2} + L^\infty$, it is easily checked $\|\mathcal{K} * f\|_{L^\infty} \leq C(\|f\|_{L^3} + \|f\|_{L^1})$, and then $\bar{\mathcal{K}} \in L^\infty(t_0, T; L^\infty(\mathbb{R}^2))$ because of (2.21) and (1.13). We thus have

$$(2.22) \quad \partial_t f + \Delta_x f = f^2 + \bar{\mathcal{K}} \cdot \nabla_x f \in L^2((t_0, T) \times \mathbb{R}^2), \quad \forall t_0 > 0,$$

so that the maximal regularity of the heat equation in L^2 -spaces (see Theorem X.11 stated in [9] and the quoted reference) provides the bound

$$(2.23) \quad f \in L^2(t_0, T; H^2(\mathbb{R}^2)) \cap L^\infty(t_0, T; H^1(\mathbb{R}^2)), \quad \forall t_0 > 0.$$

Thanks to (2.23), an interpolation inequality and the Sobolev inequality, we deduce that $\nabla_x f \in L^p((t_0, T) \times \mathbb{R}^2)$ for any $1 < p < \infty$, whence $\bar{\mathcal{K}} \cdot \nabla_x f \in L^p((t_0, T) \times \mathbb{R}^2)$, for all $t_0 > 0$. Then the maximal regularity of the heat equation in L^p -spaces (see Theorem X.12 stated in [9] and the quoted references) provides the bound

$$(2.24) \quad \partial_t f, \nabla_x f \in L^p((t_0, T) \times \mathbb{R}^2), \quad \forall t_0 > 0,$$

and then the Morrey inequality implies the Hölderian regularity $f \in C^{0,\alpha}((t_0, T) \times \mathbb{R}^2)$ for any $0 < \alpha < 1$, and any $t_0 > 0$. Observing that the RHS term in (2.22) has then also an Hölderian regularity, we deduce that

$$\partial_t f, \partial_x f, \partial_{x_i x_j}^2 f \in C_b^{0,\alpha}((t_0, T) \times \mathbb{R}^2), \quad \forall T, t_0; \quad 0 < t_0 < T < T^*,$$

thanks to the classical Hölderian regularity result for the heat equation (see Theorem X.13 stated in [9] and the quoted references). We conclude by (weakly) differentiating in time and space the equation (2.22), observing that the resulting RHS term is still a function with Hölderian regularity, applying again [9, Theorem X.13] and iterating the argument. \square

Proof of Theorem 1.4. We split the proof into seven steps, many of them are independent from one another.

Step 1. The regularity of f has been yet established in Lemma 2.8.

Step 2. First, we claim that the free energy functional \mathcal{F} is lsc in the sense that for any bounded sequence (f_n) of nonnegative functions of $L^1_2(\mathbb{R}^2)$ with same mass $M < 8\pi$ and such that $\mathcal{F}(f_n) \leq A$ and $f_n \rightharpoonup f$ in $\mathcal{D}'(\mathbb{R}^2)$, there holds

$$(2.25) \quad 0 \leq f \in L^1_2(\mathbb{R}^2) \quad \text{and} \quad \mathcal{F}(f) \leq \liminf \mathcal{F}(f_n).$$

The proof of (2.25) is classical (see [13, 14, 8]) and we just sketch it for the sake of completeness. Because of (1.8) and (1.9), we have $\mathcal{H}^+(f_n) + M_2(f_n) \leq A'$ for any $n \geq 1$, and we may apply the Dunford-Pettis lemma which implies that $f_n \rightharpoonup f$ in $L^1(\mathbb{R}^2)$ weak. Now, introducing the splitting $\mathcal{F} = \mathcal{F}_\varepsilon + \mathcal{R}_\varepsilon$, $\mathcal{F}_\varepsilon = \mathcal{H} + \mathcal{V}_\varepsilon$, with

$$\begin{aligned} \mathcal{V}_\varepsilon(g) &:= \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \kappa(x-y) \mathbf{1}_{|x-y| \geq \varepsilon}, \\ \mathcal{R}_\varepsilon(g) &:= \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \kappa(x-y) \mathbf{1}_{|x-y| \leq \varepsilon}, \end{aligned}$$

we clearly have that $\mathcal{F}_\varepsilon(f) \leq \liminf \mathcal{F}_\varepsilon(f_n)$ because \mathcal{H} is lsc and \mathcal{V}_ε is continuous for the L^1 weak convergence. On the other hand, using the convexity inequality $uv \leq u \log u + e^v \quad \forall u > 0, v \in \mathbb{R}$ and the elementary inequality $(\log u)_- \leq u^{-1/2} \quad \forall u \in (0, 1)$, we have for $\varepsilon \in (0, 1)$ and $\lambda > 1$

$$\begin{aligned} |\mathcal{R}_\varepsilon(g)| &= \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) \mathbf{1}_{g(x) \leq \lambda} g(y) (\log |x-y|)_- \mathbf{1}_{|x-y| \leq \varepsilon} \\ &\quad + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) \mathbf{1}_{g(x) \geq \lambda} g(y) \log(|x-y|^{-1}) \mathbf{1}_{|x-y| \leq \varepsilon} \\ &\leq \frac{\lambda}{4\pi} \int_{\mathbb{R}^2} g(y) dy \int_{|z| \leq \varepsilon} (\log |z|)_- dz \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^2} g(x) \mathbf{1}_{g(x) \geq \lambda} \int \{g(y) \log g(y) + |x-y|^{-1}\} dy \\ &\leq \frac{\lambda}{3} M \varepsilon^{3/2} + \frac{1}{4\pi} \frac{\mathcal{H}^+(g)}{\log \lambda} \{\mathcal{H}^+(g) + 2\pi\varepsilon\}, \end{aligned}$$

and we get that $\sup_n |\mathcal{R}_\varepsilon(f_n)| \rightarrow 0$ as $\varepsilon \rightarrow 0$ from which we conclude that \mathcal{F} is lsc. Now, we easily deduce that the free energy identity (1.6) holds. Indeed, since f is smooth for positive time, for any fixed $t \in (0, T^*)$ and any given sequence (t_n) of positive real numbers which decreases to 0, we clearly have

$$\mathcal{F}(f(t_n)) = \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_{\mathcal{F}}(f(s)) ds.$$

Then, thanks to the Lebesgue convergence theorem, the lsc property of \mathcal{F} and the fact that $f(t_n) \rightharpoonup f_0$ weakly in $\mathcal{D}'(\mathbb{R}^2)$, we deduce from the above free energy identity for positive time that

$$\mathcal{F}(f_0) \leq \liminf_{n \rightarrow \infty} \mathcal{F}(f(t_n)) \leq \lim_{n \rightarrow \infty} \left\{ \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_{\mathcal{F}}(f(s)) ds \right\} = \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(f(s)) ds.$$

Together with the reverse inequality (1.14) we conclude to (1.6).

Step 3. From now on, we assume that $M < 8\pi$ is subcritical and we prove the uniform in time estimates (1.19) and (1.20). We start with the a priori additional moment estimate (1.19). Because we will show the uniqueness of solution without using that additional moment estimates, these ones are rigorously justified thanks to a standard approximation argument, see [8] for details. Denoting g the rescaled solution (1.17) and

$$M_k := \int_{\mathbb{R}^2} g(x) |x|^k dx$$

we compute with $\Phi(x) = |x|^k$, $k \geq 2$, thanks to the antisymmetry of the kernel and the Hölder inequality

$$\begin{aligned} \frac{d}{dt} M_k &= k^2 M_{k-2} - k M_k - \frac{1}{2\pi} \int_{\mathbb{R}^2} \Phi'(x) g(t, x) \int_{\mathbb{R}^2} g(t, y) \frac{x-y}{|x-y|^2} dy dx \\ &= k^2 M_{k-2} - k M_k \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(t, y) g(t, x) (\Phi'(x) - \Phi'(y)) \frac{x-y}{|x-y|^2} dy dx \\ &\leq k^2 M^{2/k} M_k^{1-2/k} - k M_k, \end{aligned}$$

from which we easily conclude that (1.19) holds.

Step 4. Defining the rescaled free energy $\mathcal{E}(g)$ and the associated dissipativity of rescaled free energy $\mathcal{D}_{\mathcal{E}}(g)$ by

$$(2.26) \quad \mathcal{E}(g) := \int g(1 + \log g) + \frac{1}{2} \int g|x|^2 + \frac{1}{4\pi} \iint g(x)g(y) \log|x-y| dx dy$$

$$(2.27) \quad \mathcal{D}_{\mathcal{E}}(g) := \int g \left| \nabla \left(\log g + \frac{|x|^2}{2} + \kappa * g \right) \right|^2,$$

we have that any solution g to the rescaled equation (1.18) satisfies

$$(2.28) \quad \frac{d}{dt} \mathcal{E}(g) + \mathcal{D}_{\mathcal{E}}(g) = 0 \quad \text{on } [0, \infty).$$

On the one hand, as for (1.8), the following functional inequality

$$(2.29) \quad \int g \log g + \frac{1}{2} \int g|x|^2 \leq C_3(M) \mathcal{E}(g) + C_4(M) \quad \forall g \in L_+^1(\mathbb{R}^2)$$

holds, and together with (1.9), we find

$$(2.30) \quad \int g(\log g)_+ + \frac{1}{4} \int g|x|^2 \leq C_3(M) \mathcal{E}(g) + C_7(M) \quad \forall g \in L_+^1(\mathbb{R}^2),$$

where $C_7 := C_4 + C_5$. As a consequence of (2.28) and (2.30), we get the uniform in time upper bound on the rescaled free energy for the solution g of (1.18)

$$(2.31) \quad \sup_{t \geq 0} \int g_t(\log g_t)_+ + \frac{1}{4} \int g_t|x|^2 \leq C_3(M) \mathcal{E}(f_0) + C_7(M).$$

Step 5. As in the proof of Lemma 2.7, we easily get that the rescaled solution g of the rescaled equation (1.18) satisfies for any $p \in [2, \infty)$

$$\begin{aligned} \frac{d}{dt} \|g\|_{L^p}^p + \frac{4}{p'} \|\nabla(g^{p/2})\|_{L^2}^2 &= 2(p-1) \|g\|_{L^p}^p + (p-1) \|g\|_{L^{p+1}}^{p+1} \\ &\leq 2(p-1) M + 3(p-1) \|g\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Writing $s = s \wedge A + (s - A)_+$, so that $s^{p+1} \leq 2^{p+1}(s \wedge A)^{p+1} + 2^{p+1}(s - A)_+^{p+1}$, and using the Gagliardo-Nirenberg-Sobolev type inequality (2.5) in order to get

$$\begin{aligned} \int (g - A)_+^{p+1} &\leq C_p \int |\nabla (g - A)_+^{p/2}|^2 \int (g - A)_+ \\ &\leq C_p \int |\nabla (g^{p/2})|^2 \frac{\mathcal{H}^+(g)}{\log A} \end{aligned}$$

for any $A > 1$, we deduce

$$\begin{aligned} \frac{d}{dt} \|g\|_{L^p}^p + \|\nabla (g^{p/2})\|_{L^2}^2 &\leq 2pM + 3p2^{p+1}A^p M + 3p2^{p+1} \int (g - A)_+^{p+1} \\ &\leq C_8(M, p, A) + C_p \frac{\mathcal{H}^+(g)}{\log A} \|\nabla (g^{p/2})\|_{L^2}^2. \end{aligned}$$

Taking A large enough, we obtain

$$(2.32) \quad \frac{d}{dt} \|g\|_{L^p}^p + \frac{1}{2} \|\nabla (g^{p/2})\|_{L^2}^2 \leq C_9(M, p, \mathcal{E}_0).$$

Using the Nash inequality

$$\|w\|_{L^2(\mathbb{R}^2)}^2 \leq C_N \|w\|_{L^1(\mathbb{R}^2)} \|\nabla w\|_{L^2(\mathbb{R}^2)}$$

with $w := g^{p/2}$, we conclude with

$$\frac{d}{dt} \|g\|_{L^p}^p + \frac{1}{C_N^2} \|g\|_{L^{p/2}}^{-p} \|g\|_{L^p}^{2p} \leq C_9(M, p, \mathcal{E}_0).$$

Defining $u(t) := \|g(t)\|_{L^p}^p$ first with $p = 2$, so that $\|g(t)\|_{L^{p/2}}^{p/2} = M$, we recognize the classical nonlinear ordinary differential inequality

$$u' + c u^2 \leq C \quad \text{on } (0, \infty),$$

for some constants c and C (which only depend on M and \mathcal{E}_0) from which we deduce the bound

$$(2.33) \quad \forall \varepsilon > 0 \exists C = C(\varepsilon, c, C) \quad \sup_{t \geq \varepsilon} \|g(t)\|_{L^p}^p \leq C,$$

with $p = 2$. In order to get the same uniform estimate (2.33) in all the Lebesgue spaces L^p , $p \in (2, \infty)$, we may proceed by iterating the same argument as above with the choice $p = 2^k$, $k \in \mathbb{N}^*$. Coming back to (2.32) with $p = 2$, we also deduce that for any $\varepsilon, T > 0$ there exists $C = C(\varepsilon, T, \mathcal{E}_0)$ so that

$$\sup_{t_0 \geq \varepsilon} \int_{t_0}^{t_0+T} \|\nabla g(s)\|_{L^2(\mathbb{R}^2)}^2 ds \leq C.$$

Step 6. The function $g_i := \partial_{x_i} g$ satisfies

$$\partial_t g_i - \Delta g_i - \nabla(x g_i) = g_i + 2g g_i - \partial_{x_i}(\nabla u \cdot \nabla g),$$

from which we deduce that

$$(2.34) \quad \begin{aligned} \frac{d}{dt} \int |g_i|^p + p(p-1) \int |\nabla g_i|^2 |g_i|^{p-2} &\leq \\ &\leq (3p-2) \int |g_i|^p + 2p \int g |g_i|^p + p \int \partial_{x_i}(\nabla u \cdot \nabla g) |g_i|^{p-2}. \end{aligned}$$

For $p = 2$, we have for any $t \geq \varepsilon$

$$\begin{aligned} \mathcal{T}(t) &:= 4 \int g |g_i|^2 + 2 \int \partial_{x_i}(\nabla u \cdot \nabla g) g_i \\ &\leq 4 \|g\|_{L^3} \|g_i\|_{L^3}^2 + 2 \|\nabla u \cdot \nabla g\|_{L^2}^2 + \frac{1}{2} \|\partial_i g_i\|_{L^2}^2 \end{aligned}$$

thanks to the Hölder inequality, an integration by part and the Young inequality. Next, we have for any $t \geq \varepsilon$

$$\mathcal{T}(t) \leq C_1 \|g_i\|_{L^2}^{4/3} \|\nabla g_i\|_{L^2}^{2/3} + C_2 \|\nabla g\|_{L^2}^2 + \frac{1}{2} \|\nabla g_i\|_{L^2}^2$$

where we have used the classical Gagliardo-Nirenberg inequality (see (85) in [9, Chapter IX] and the quoted references)

$$(2.35) \quad \|w\|_{L^r(\mathbb{R}^2)} \leq C_{GN} \|w\|_{L^q(\mathbb{R}^2)}^{1-a} \|\nabla w\|_{L^2(\mathbb{R}^2)}^a, \quad a = 1 - \frac{q}{r}, \quad 1 \leq q \leq r < \infty,$$

with $w := g_i$, $r = 3$, $q = 2$, the uniform bound established in step 5 and the fact that $\nabla u = -\mathcal{K} * g \in L^\infty((\varepsilon, \infty) \times \mathbb{R}^2)$ thanks to the same argument as in the proof of Lemma 2.8. Last, by the Young inequality we get for any $t \geq \varepsilon$

$$\mathcal{T}(t) \leq \frac{2}{3} C_1^{3/2} \|g_i\|_{L^2}^2 + \frac{1}{3} \|\nabla g_i\|_{L^2}^2 + C_2 \|\nabla g\|_{L^2}^2 + \frac{1}{2} \|\nabla g_i\|_{L^2}^2,$$

from which we deduce from (2.34)

$$\frac{d}{dt} \int |g_i|^2 + \int |\nabla g_i|^2 \leq C_3 \|\nabla g\|_{L^2}^2 \quad \text{on } (\varepsilon, \infty),$$

with $C_3 := 4 + \frac{2}{3} C_1^{3/2} + C_2$. Remarking that for any fixed $\varepsilon \in (0, 1)$ and any $t_1 \geq 2\varepsilon$, we may define $t_0 \in (t_1 - \varepsilon, t_1)$ so that

$$\|\nabla g(t_0)\|_{L^2}^2 = \inf_{(t_1 - \varepsilon, t_1)} \|\nabla g\|_{L^2}^2 \leq \frac{2}{\varepsilon} \int_{t_1 - \varepsilon}^{t_1} \|\nabla g(s)\|_{L^2}^2 ds \leq C_4$$

thanks to the bound established at the end of step 5, we deduce from the above differential inequality that

$$\|g_i(t_1)\|_{L^2}^2 \leq \|g_i(t_0)\|_{L^2}^2 + C_3 \int_{t_0}^{t_1} \|\nabla g(s)\|_{L^2}^2 ds \leq C_5,$$

where again $C_5 := C_4 + C_3 C_4 \varepsilon / 2$ only depends on ε , M and \mathcal{E}_0 . Coming back to the above differential inequality again, we easily conclude that for any $\varepsilon > 0$, there exists a constant $\mathcal{C}_\varepsilon = \mathcal{C}(\varepsilon, M, \mathcal{E}_0)$ so that

$$(2.36) \quad \sup_{t \geq \varepsilon} \left\{ \|\nabla g(t)\|_{L^2}^2 + \int_t^{t+1} \|D^2 g(s)\|_{L^2}^2 ds \right\} \leq \mathcal{C}_\varepsilon.$$

Step 7. Starting from the differential inequality (2.34) for $p \in (2, \infty)$ and using the Morrey-Sobolev inequalities

$$\|g\|_{L^\infty} \leq C \|g\|_{H^2} \quad \text{and} \quad \|D^2 u\|_{L^\infty} \leq C \|D^2 u\|_{H^2} \leq C \|g\|_{H^2},$$

we easily get

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int |\nabla g|^p &\leq C (1 + \|g\|_{L^\infty} + \|D^2 u\|_{L^\infty}) \int |\nabla g|^p \\ &\leq C (1 + \|g\|_{H^2}) \int |\nabla g|^p \quad \text{on } (\varepsilon, \infty), \end{aligned}$$

from which we deduce for any $t_1 \geq t_0 \geq \varepsilon$

$$\|\nabla g(t_1)\|_{L^p} \leq \|\nabla g(t_0)\|_{L^p} \exp\left(\int_{t_0}^{t_1} C (1 + \|g(s)\|_{H^2}) ds\right).$$

Now, arguing similarly as in step 6, we deduce from the above time integral inequality, the Sobolev inequality $\|\nabla g\|_{L^p} \leq C_p \|g\|_{H^2}$ for $p \in [2, \infty)$ and the already established bound (2.36), that for any $\varepsilon > 0$, there exists a constant $\mathcal{C}_\varepsilon = \mathcal{C}(\varepsilon, M, \mathcal{E}_0, p)$ so that

$$(2.37) \quad \sup_{t \geq \varepsilon} \|\nabla g(t)\|_{L^p} \leq \mathcal{C}_\varepsilon.$$

Step 8. Iterating twice the arguments we have presented in steps 6 and 7, it is not difficult to prove

$$\sup_{t \geq \varepsilon} \|g(t, \cdot)\|_{W^{3,p}} \leq C \quad \forall \varepsilon > 0, p \in [2, \infty),$$

for some constant $C = C(\varepsilon, p, M, \mathcal{F}_0, M_{2,0})$ from which (1.20) immediately follows. \square

3. UNIQUENESS - PROOF OF THEOREM 1.3

We split the proof into two steps. We recall that from Theorem 1.4 we already know that $\|f\|_{L^2} \in C^1(0, T)$ and $\|f\|_{L^p} \in L^\infty(t_0, T)$ for any $0 < t_0 < T < T^*$ and any $p \in [1, \infty]$.

Step 1. We establish our new main estimate, namely that any weak solution satisfies

$$(3.1) \quad t^{1/4} \|f(t, \cdot)\|_{L^{4/3}} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

First, from (1.1) and the regularity of the solution, we have

$$\frac{d}{dt} \|f\|_{L^2}^2 + 2 \|\nabla_x f\|_{L^2}^2 = \|f\|_{L^3}^3 \quad \text{on } (0, T).$$

As in the proof of Lemma 2.7, we deduce that

$$\frac{d}{dt} \|f\|_{L^2}^2 + \frac{1}{2} \|\nabla_x f\|_{L^2}^2 \leq A^2 M \quad \text{on } (0, T)$$

for A large enough. Thanks to the Nash inequality

$$\|f\|_{L^2}^2 \leq C M \|\nabla f\|_{L^2},$$

we thus obtain

$$\frac{d}{dt} \|f\|_{L^2}^2 + c_M \|f\|_{L^2}^4 \leq A^2 M \quad \text{on } (0, T).$$

It is a classical trick of ordinary differential inequality to deduce that there exists a constant K (which only depends on $c_M, A^2 M$ and T) so that

$$(3.2) \quad t \|f(t, \cdot)\|_{L^2}^2 \leq K \quad \forall t \in (0, T).$$

We now prove (3.1) from (3.2) and an interpolation argument. On the one hand, introducing the notation $\widetilde{\log}_+ f := 2 + (\log f)_+$, we use the Hölder inequality in order to get

$$\begin{aligned} \int f^{4/3} &= \int f^{2/3} (\widetilde{\log}_+ f)^{2/3} f^{2/3} (\widetilde{\log}_+ f)^{-2/3} \\ &\leq \left(\int f \widetilde{\log}_+ f \right)^{2/3} \left(\int f^2 (\widetilde{\log}_+ f)^{-2} \right)^{1/3}, \end{aligned}$$

or in other words and using a similar estimate as (1.9)

$$(3.3) \quad \|f\|_{L^{4/3}} \leq C(\mathcal{H}(f), M_2(f)) \left(\int f^2 (\widetilde{\log}_+ f)^{-2} \right)^{1/4}.$$

On the other hand, we observe that for any $R \in (0, \infty)$

$$\begin{aligned} t \int f^2 (\widetilde{\log}_+ f)^{-2} &\leq t \int_{f \leq R} f^2 (\widetilde{\log}_+ f)^{-2} + t \int_{f \geq R} f^2 (\widetilde{\log}_+ f)^{-2} \\ &\leq t \frac{R}{(\widetilde{\log}_+ R)^2} \int_{f \leq R} f + \frac{t}{(\widetilde{\log}_+ R)^2} \int_{f \geq R} f^2 \\ (3.4) \quad &\leq t \frac{MR}{(\widetilde{\log}_+ R)^2} + \frac{K}{(\widetilde{\log}_+ R)^2} \leq \frac{M+K}{(\widetilde{\log}_+ 1/t)^2} \rightarrow 0, \end{aligned}$$

where we have used that $s \mapsto s/(\widetilde{\log}_+ s)^2$ is an increasing function in the second line, then the mass conservation and estimate (3.2) in the third line, and we have chosen $R := t^{-1}$ in order to get the last inequality. We conclude to (3.1) by gathering (3.3) and (3.4).

Step 3. Conclusion. We consider two weak solutions f_1 and f_2 to the Keller-Segel equation (1.1) that we write in the mild form

$$f_i(t) = e^{t\Delta} f_i(0) + \int_0^t e^{(t-s)\Delta} \nabla(V_i(s) f_i(s)) ds, \quad V_i = \mathcal{K} * f_i,$$

where $e^{t\Delta}$ stands for the heat semigroup defined in \mathbb{R}^2 by $e^{t\Delta} f := \gamma_t * f$, $\gamma_t(x) := (2\pi t)^{-1} \exp(-|x|^2/(2t))$. When we assume $f_1(0) = f_2(0)$, the difference $F := f_2 - f_1$ satisfies

$$F(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (V_2(s) F(s)) ds + \int_0^t \nabla \cdot e^{(t-s)\Delta} (W(s) f_1(s)) ds = I_1 + I_2,$$

with $W := V_2 - V_1$. For any $t > 0$, we define

$$Z_i(t) := \sup_{0 < s \leq t} s^{1/4} \|f_i(s)\|_{L^{4/3}}, \quad \Delta(t) := \sup_{0 < s \leq t} s^{1/4} \|F(s)\|_{L^{4/3}}.$$

We then compute

$$\begin{aligned} J_1 &:= t^{1/4} \|I_1(t)\|_{L^{4/3}} \\ &\leq t^{1/4} \int_0^t \|\nabla \cdot e^{(t-s)\Delta} (V_2(s) F(s))\|_{L^{4/3}} ds \\ &\leq t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \|V_2(s) F(s)\|_{L^1} ds \\ &\leq t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \|V_2(s)\|_{L^4} \|F(s)\|_{L^{4/3}} ds \\ &\leq t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \|f_2(s)\|_{L^{4/3}} \|F(s)\|_{L^{4/3}} ds \\ &\leq \int_0^t \frac{C}{(t-s)^{3/4}} \frac{t^{1/4}}{s^{1/2}} ds Z_2(t) \Delta(t) \\ &= \int_0^1 \frac{C}{(1-u)^{3/4}} \frac{du}{u^{1/2}} Z_2(t) \Delta(t), \end{aligned}$$

where we have used the regularizing effect of the heat equation

$$\|\nabla(e^{t\Delta} g)\|_{L^{4/3}} \leq \|\nabla \gamma_t\|_{L^{4/3}} \|g\|_{L^1} \leq \frac{C}{t^{3/4}} \|g\|_{L^1},$$

at the third line, the Hölder inequality at the fourth line and the critical Hardy-Littlewood-Sobolev inequality (1.16) at the fifth line.

Similarly, we have

$$\begin{aligned} J_2 &:= t^{1/4} \|I_2(t)\|_{L^{4/3}} \\ &\leq \int_0^1 \frac{C}{(1-u)^{3/4}} \frac{du}{u^{1/2}} \Delta(t) Z_1(t). \end{aligned}$$

All together, we conclude thanks to (3.1) with the inequality

$$\Delta(t) \leq \int_0^1 \frac{C}{(1-u)^{3/4}} \frac{du}{u^{1/2}} (Z_1(t) + Z_2(t)) \Delta(t) \leq \frac{1}{2} \Delta(t)$$

for $t \in (0, T)$, $T > 0$ small enough, which in turn implies $\Delta(t) \equiv 0$ on $[0, T)$. \square

4. SELF-SIMILAR BEHAVIOUR - PROOF OF THEOREM 1.5

In this section we restrict ourself to the subcritical case $M < 8\pi$ and we investigate the self-similar long time behaviour of generic solutions to the KS equation or more precisely, and equivalently, we investigate the long time convergence to the self-similar profile of the rescaled solution g defined through (1.17). We start by recalling some known results on the self-similar profile and its stability. First, we consider the stationary problem (1.22).

Theorem 4.1. *For any $M \in (0, 8\pi)$, there exists a unique nonnegative self-similar profile $G = G_M$ of mass M with finite second moment and finite entropy of the KS equation (1.1), it is the unique solution to the stationary problem (1.22) and it satisfies*

$$G \in C^\infty(\mathbb{R}^2), \quad e^{-(1+\varepsilon)|x|^2/2+C_{1,\varepsilon}} \leq G \leq e^{-(1-\varepsilon)|x|^2/2+C_{2,\varepsilon}},$$

for any $\varepsilon \in (0, 1)$ and some constants $C_{i,\varepsilon} \in (0, \infty)$. Moreover, with the definitions (2.26) of the modified free energy \mathcal{E} and (2.27) of the modified dissipation of the free energy $\mathcal{D}\mathcal{E}$, the self-similar profile G is characterized as the unique solution to the optimization problem

$$(4.1) \quad \tilde{g} \in \mathcal{Z}_M, \quad \mathcal{E}(\tilde{g}) = \min_{g \in \mathcal{Z}_M} \mathcal{E}(g),$$

where $\mathcal{Z}_M := \{g \in L^1_+ \cap L^2_2; M_0(g) = M\}$, as well as the unique function $g \in \mathcal{Z}_M$ such that $\mathcal{D}\mathcal{E}(g) = 0$.

That theorem follows by a combination of known results. On the one hand, as a consequence of the fact that $U := -\mathcal{K} * G$ satisfies (1.24) together with the elementary inequality

$$(4.2) \quad \forall x \in \mathbb{R}^2 \quad |U(x) + \frac{M}{2\pi} (\log |x|)_+| \leq C,$$

where C only depends on M , $M_2(G)$ and $\mathcal{H}(G)$ (see [8, Lemma 23] and the argument presented in order to bound $\mathcal{R}_\varepsilon(g)$ in step 2 of the proof of Theorem 1.4), and the Naito's variant [33] of the famous Gidas, Ni, Nirenberg radial symmetry result on solutions to Poisson type equations, it has been established in [8, Lemma 25] that U is radially symmetric. It follows that any self-similar profile G is radially symmetric. On the other hand, the uniqueness of radially symmetric self-similar profiles has been proved in [5, Theorem 3.1] (see also [15, Theorem 1.2]) and that concludes the proof of the uniqueness of the solution to the stationary problem (1.22). The smoothness property is established in [8, Lemma 25] and the behaviour for large values of $|x|$ is an immediate consequence of (4.2). It is clear from (2.28) that any solution \tilde{g} to the minimization problem (4.1) also satisfies $\mathcal{D}\mathcal{E}(\tilde{g}) = 0$ which in turns implies that $\log \tilde{g} + |x|^2/2 + \kappa * \tilde{g} = 0$ and then \tilde{g} is a solution to the stationary problem (1.22).

Second, the profile G is a stationary solution to the evolution equation (1.18) and the associated linearized equation reads

$$\partial_t h = \Lambda h := \operatorname{div}_x (\nabla h + x h + (\mathcal{K} * G) h + (\mathcal{K} * h) G).$$

We briefly explain the spectral analysis of Λ in the Hilbert space $E := L^2(G^{-1/2})$ of self-adjointness performed in [17]. Defining $h_{0,0} := \partial G_M / \partial M$, it is (formally) clear that $h_{0,0}$ is a first eigenfunction of the operator Λ associated to the first eigenvalue $\lambda = 0$, and it has been furthermore shown in [17, Lemma 8] that the null space $N(\Lambda) = \operatorname{vect}(h_{0,0})$. Moreover, defining the bilinear form

$$\langle f, g \rangle := \int_{\mathbb{R}^2} f g G^{-1} dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) g(y) \kappa(x-y) dx dy,$$

and the associated quadratic form $Q_1[f] := \langle f, f \rangle$, it has been shown in [17, Section 4.3] that Q_1 is nonnegative, that $Q_1[h_{0,0}] = 0$ and that

$$Q_1[f] = 0 \text{ and } \langle f, h_{0,0} \rangle = 0 \text{ imply } f = 0.$$

As a consequence $Q_1[\cdot]$ defines an Hilbert norm on the linear submanifold

$$E_0^\perp := \{f \in E; \langle f, h_{0,0} \rangle = 0\} = \{f \in E; M(f) = 0\}$$

which is equivalent to the initial norm $\|\cdot\|_E$. That new norm is suitable for exhibiting a spectral gap for the operator Λ and to make the stability analysis of the associated semigroup $e^{t\Lambda}$.

Theorem 4.2 ([17]). *For any $g \in E_0^\perp$ which belongs to the domain of Λ , there holds*

$$(4.3) \quad \langle \Lambda g, g \rangle \leq -Q_1[g].$$

Moreover, there exists $a^* < -1$ and $C > 0$ so that

$$(4.4) \quad \|e^{t\Lambda} h - e^{-t} \Pi_1 h - \Pi_0 h\|_E \leq C e^{a^* t} \|h - (\Pi_1 + \Pi_0)h\|_E \quad \forall t \geq 0, \forall h \in E,$$

where Π_0 is the (Q_1 -orthogonal) projection on $\text{Vect}(h_{0,0})$, also defined as $\Pi_0 h := M(h) h_{0,0}$, and Π_1 is the (Q_1 -orthogonal) projection on $\text{Vect}(h_{1,1}, h_{1,2})$ where $h_{1,i} := \partial_{x_i} G$.

Inequality (4.3) is nothing but [17, Theorem 15] and (4.4) is a consequence of the fact that the spectrum of Λ is discrete and included in the real line and that the second (larger) eigenvalue of Λ is -1 , see [17, Section 4].

Our first main result in this section is a linearized stability result in a large space \mathcal{E} , namely we consider

$$\mathcal{E} := L_k^{4/3}(\mathbb{R}^2), \quad k > 3/2.$$

We consider that space because it is the larger space in terms of moment decay in which we are able to prove a (optimal) spectral gap on the linearized semigroup. For such a general Banach space framework and the associated spectral analysis issue, we adopt the classical notations of [35, 26] used in [25], for more details we refer to [25, Section 2.1] and the references therein (in particular [26, 35, 22]).

Theorem 4.3. *For any $k > 3/2$ and any $a > \bar{a} := \max(a^*, a(k))$, $a(k) := 1/2 - k$ (so that $a(k) < -1$) there exists a constant $C_{k,a}$ so that*

$$\|e^{t\Lambda} h - e^{-t} \Pi_1 h - \Pi_0 h\|_{\mathcal{E}} \leq C e^{at} \|h - \Pi_1 h - \Pi_0 h\|_{\mathcal{E}} \quad \forall t \geq 0, \forall h \in \mathcal{E},$$

where again Π_0 stands for projection on the eigenspace $\text{Vect}(h_{0,0})$ associated to the eigenvalue 0 and Π_1 stands for projection on the eigenspace $\text{Vect}(h_{1,1}, h_{1,2})$ associated to the eigenvalue -1 . Both operators are defined through the Dunford formula (see [25, Section 2.1] or better [26, III-(6.19)])

$$\Pi_{\xi} := -\frac{1}{2i\pi} \int_{|z-\xi|=r} (\Lambda - z)^{-1} dz, \quad \xi = 0, -1, \quad r > 0 \text{ (small enough)},$$

but also in a simpler manner $\Pi_0 h = M(h) h_{0,0}$ for any $h \in \mathcal{E}$.

The proof is a straightforward adaptation of arguments of “functional extension of semigroup spectral gap estimates” developed in [25] for the Fokker-Planck equation.

Lemma 4.4. *For any $k \geq 0$ fixed, there exists a constant C_k such that for any $g \in D(\Lambda)$, there holds*

$$(4.5) \quad \langle \Lambda g, g^{\dagger} \rangle_{\mathcal{E}} \leq C_k \int |g|^{4/3} \langle x \rangle^{\frac{4}{3}k-1} + \left(\frac{1}{2} - k\right) \int |g|^{4/3} \langle x \rangle^{\frac{4}{3}k},$$

where $g^{\dagger} := \bar{g} |g|^{-2/3}$ (here \bar{g} stands for the complex conjugate of g).

Proof of Lemma 4.4. For the sake of simplicity we assume $g \geq 0$ so that $g^{\dagger} = g^{1/3}$, we set $\ell := 4k/3$, we write

$$\langle \Lambda g, g^{\dagger} \rangle_{\mathcal{E}} = \int_{\mathbb{R}^2} (\Delta g) g^{1/3} \langle x \rangle^{\ell} = T_1 + \dots + T_4,$$

and we compute each term T_i separately. First, performing two integrations by part, we have

$$\begin{aligned} T_1 &:= \int_{\mathbb{R}^2} (\Delta g) g^{1/3} \langle x \rangle^{\ell} dx \\ &= -\frac{1}{3} \int_{\mathbb{R}^2} |\nabla g|^2 g^{-2/3} \langle x \rangle^{\ell} dx + \frac{3}{4} \int_{\mathbb{R}^2} g^{4/3} \Delta \langle x \rangle^{\ell} dx. \end{aligned}$$

Second, performing one integration by part, we have

$$\begin{aligned} T_2 &:= \int_{\mathbb{R}^2} (2g + x \cdot \nabla g) g^{1/3} \langle x \rangle^{\ell} dx \\ &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \langle x \rangle^{\ell-2} + \left(\frac{1}{2} - k\right) \langle x \rangle^{\ell} \right\} g^{4/3} dx. \end{aligned}$$

Third, performing one integration by part, we have

$$\begin{aligned} T_3 &:= \int_{\mathbb{R}^2} (2Gg + (\mathcal{K} * G) \cdot \nabla g) g^{1/3} \langle x \rangle^\ell dx \\ &= \frac{5}{4} \int_{\mathbb{R}^2} G g^{4/3} \langle x \rangle^\ell dx - \frac{3}{4} \int_{\mathbb{R}^2} g^{4/3} (\mathcal{K} * G) \cdot \nabla_x \langle x \rangle^\ell dx \\ &\leq C \int_{\mathbb{R}^2} g^{4/3} \langle x \rangle^{\ell-1} dx, \end{aligned}$$

for some constant $C \in (0, \infty)$.

Fourth and last, thanks to the Hölder inequality and the critical Hardy-Littlewood-Sobolev inequality (1.16), we have

$$\begin{aligned} T_4 &:= \int_{\mathbb{R}^2} (\mathcal{K} * g) \cdot \nabla G g^{1/3} \langle x \rangle^\ell dx \\ &\leq \|\nabla G \langle x \rangle^\ell\|_\infty \|g\|_{L^{4/3}}^{1/3} \|\mathcal{K} * g\|_{L^4} \leq C \|g\|_{L^{4/3}}^{4/3}. \end{aligned}$$

Gathering all these estimates, we get (4.5). \square

We define

$$\mathcal{A}g := N\chi_R g \quad \text{and} \quad \mathcal{B}g = \Lambda g - \mathcal{A}g,$$

for some truncation function $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$, and some constants $N, R > 0$.

We clearly have

$$(4.6) \quad \mathcal{A} \in \mathcal{B}(L^2, E) \subset \mathcal{B}(E) \quad \text{and} \quad \mathcal{A} \in \mathcal{B}(L^{4/3}, \mathcal{E}) \subset \mathcal{B}(\mathcal{E}).$$

From lemma 4.4 we easily have that for any $a > a(k)$ there exist N and R large enough so that $\mathcal{B} - a$ is dissipative in \mathcal{E} (see [35, Chapter I, Definition 4.1]) in the sense that

$$(4.7) \quad \langle g^*, (\mathcal{B} - a)g \rangle_{\mathcal{E}', \mathcal{E}} \leq 0,$$

where $g^* := \bar{g} |g|^{-2/3} \|g\|_{\mathcal{E}}^{2/3} \in \mathcal{E}'$.

Lemma 4.5. *There exist some constants $C > 0$ and $b \in \mathbb{R}$ such that the semigroup $S_{\mathcal{B}}(t) = e^{\mathcal{B}t}$ satisfies*

$$(4.8) \quad \|S_{\mathcal{B}}(t)h\|_{L_1^2} \leq \frac{C e^{bt}}{t^{1/2}} \|h\|_{L_1^{4/3}} \quad \forall h \in L_1^{4/3}, \forall t > 0.$$

Proof of Lemma 4.5. The proof of the hypercontractivity property as stated in Lemma 4.5 is a classical consequence of the Gagliardo-Nirenberg inequality. For the sake of completeness we sketch it. Arguing similarly as in the proof of Lemma 2.7 and Lemma 4.4 and denoting $h_t := e^{t\mathcal{B}}h$, we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |h_t|^2 \langle x \rangle^2 &= - \int |\nabla h_t \langle x \rangle|^2 + \int h_t (\mathcal{K} * h_t) \cdot \nabla G \langle x \rangle^2 \\ &\quad + \int |h_t|^2 \left\{ 1 - |\nabla \langle x \rangle|^2 + \langle x \rangle^2 \left(\frac{3}{2} G - N \chi_R \right) + \langle x \rangle \mathcal{K} * G \cdot \nabla \langle x \rangle \right\}. \end{aligned}$$

On the one hand, thanks to the Gagliardo-Nirenberg inequality (2.35) with $q = 4/3$, $r = 2$ and $a = 1/3$, we know that

$$\int |\nabla(h \langle x \rangle)|^2 \geq C_{GN}^{-6} \left(\int |h \langle x \rangle|^2 \right)^3 \left(\int |h \langle x \rangle|^{4/3} \right)^{-3}.$$

On the other hand, introducing the splitting $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_\infty$ with $\mathcal{K}_0 := \mathcal{K} \mathbf{1}_{|z| \leq 1}$ and $\mathcal{K}_\infty := \mathcal{K} \mathbf{1}_{|z| \geq 1}$ and using the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \int h_t (\mathcal{K} * h_t) \cdot \nabla G \langle x \rangle^2 &\leq \|\nabla G \langle x \rangle^2\|_{L^\infty} \|h\|_2 \|\mathcal{K}_0 * h\|_{L^2} + \|\nabla G \langle x \rangle^2\|_{L^2} \|h\|_{L^2} \|\mathcal{K}_\infty * h\|_{L^\infty} \\ &\leq C (\|\mathcal{K}_0\|_{L^1} \|h\|_{L^2}^2 + \|h\|_{L^2} \|\mathcal{K}_0\|_{L^3} \|h\|_{L^{3/2}}) \\ &\leq C \|h\|_{L^2}^2. \end{aligned}$$

We also bound the last term by $C \|h_t\|_{L^2_1}^2$. All together and using the notations $X(t) := \|h_t\|_{L^2_1}^2$ and $Y(t) := \|h_t\|_{L^{4/3}_1}^{4/3}$ and the fact that $Y(t) \leq Y(0)$ thanks to Lemma 4.4, we get

$$X' \leq -\alpha (X/Y(0))^3 + \beta X$$

for some constants $\alpha, \beta > 0$. The estimate (4.8) is then a classical consequence to the above differential inequality. \square

Proof of Theorem 4.3. We immediately deduce from (4.6) and Lemma 4.5 that

$$\|\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}, E)} \leq \frac{C}{t^{1/2}} e^{bt} \quad \forall t > 0,$$

for some constants $C > 0$ and $b \in \mathbb{R}$. As a consequence, proceeding as in [25, section 3] or [29, Lemma 2.4], we deduce that the time convolution function $(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*\ell)}$ defined iteratively by $(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*1)} := (\mathcal{A}\mathcal{S}_\mathcal{B})$, $(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*\ell)} := (\mathcal{A}\mathcal{S}_\mathcal{B})^{(*(\ell-1))} * (\mathcal{A}\mathcal{S}_\mathcal{B})$, for any $\ell \geq 2$, satisfies

$$(4.9) \quad \|(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*\ell)}(t)\|_{\mathcal{B}(\mathcal{E}, E)} \leq C_\ell e^{b_\ell t} \quad \forall t > 0,$$

for some constants $C_\ell > 0$ and $b_\ell \in (a(k), -1)$ for $k > 3/2$ and $\ell \geq 2$ large enough. Putting together Theorem 4.2 and the properties (4.6), (4.7) and (4.9) we observe that $\Lambda = \mathcal{A} + \mathcal{B}$ satisfies all the assumptions of [25, Theorem 2.13]. As a consequence, the conclusions of Theorem 4.2 hold true by a straightforward application of [25, Theorem 2.13]. \square

Before going to the proof of Theorem 1.5 we present two results that will be useful during the proof of that Theorem.

Lemma 4.6. *For any $M \in (0, 8\pi)$, $k' > 2 \geq k > 3/2$, $M_{k'} \geq (k' - 1)^{k'/2} M$ and $\mathcal{C} > 0$, there exists an increasing function $\eta : [0, \infty) \rightarrow [0, \infty)$, $\eta(0) = 0$, $\eta(u) > 0$ for any $u > 0$, such that*

$$(4.10) \quad \forall g \in \mathcal{Z} \quad \mathcal{D}_\mathcal{E}(g) \geq \eta(\|g - G\|_{L_k^{4/3}})$$

where

$$\mathcal{Z} := \{g \in L^1_+(\mathbb{R}^2), M(g) = M, M_{k'}(g) \leq M_{k'}, \|g\|_{W^{2,\infty}} \leq \mathcal{C}\}.$$

Proof of Lemma 4.6. We proceed by contradiction. If (4.10) does not hold, there exists a sequence (g_n) in \mathcal{Z} and a real $\delta > 0$ such that

$$\mathcal{D}_\mathcal{E}(g_n) \rightarrow 0 \text{ as } n \rightarrow \infty \quad \text{and} \quad \|g_n - G\|_{L_k^{4/3}} \geq \delta.$$

Therefore, on the one hand, there exists $\bar{g} \in \mathcal{Z}$ such that, up to the extraction of the subsequence, there holds $g_n \rightarrow \bar{g}$ strongly in $L_k^{4/3}$, so that $\|\bar{g} - G\|_{L_k^{4/3}} \geq \delta$. Using again $g_n \rightarrow \bar{g}$ and the critical Hardy-Littlewood-Sobolev inequality (1.16), we deduce that $\sqrt{g_n} \mathcal{K} * g_n \rightarrow \sqrt{\bar{g}} \mathcal{K} * \bar{g}$ strongly in $L^1_{loc}(\mathbb{R}^2)$ and then $2\nabla \sqrt{g_n} + \sqrt{g_n} \mathcal{K} * g_n \rightarrow 2\nabla \sqrt{\bar{g}} + \sqrt{\bar{g}} \mathcal{K} * \bar{g}$ in $\mathcal{D}'(\mathbb{R}^2)$. Since $(\nabla \sqrt{g_n} + \sqrt{g_n} \mathcal{K} * g_n)$ is bounded in L^2 , that implies that $2\nabla \sqrt{g_n} + \sqrt{g_n} \mathcal{K} * g_n \rightarrow 2\nabla \sqrt{\bar{g}} + \sqrt{\bar{g}} \mathcal{K} * \bar{g}$ weakly in $L^2(\mathbb{R}^2)$ and then

$$\mathcal{D}_\mathcal{E}(\bar{g}) = \|2\nabla \sqrt{\bar{g}} + \sqrt{\bar{g}} \mathcal{K} * \bar{g}\|_{L^2}^2 \leq \liminf \mathcal{D}_\mathcal{E}(g_n) = 0.$$

We easily conclude thanks to the mass condition $M_0(\bar{g}) = M$ and the uniqueness Theorem 4.1 that $\bar{g} = G$. That is our contradiction. \square

Lemma 4.7. *Define $\mathcal{E}_2 := R(I - \Pi_0 - \Pi_1)$ the supplementary linear submanifold to the eigenspaces associated to the eigenvalues 0 and -1 . There exists a norm $\|\cdot\|$ on \mathcal{E}_2 equivalent to the initial one $\|\cdot\|_{\mathcal{E}}$ so that*

$$(4.11) \quad \frac{d}{dt} \|e^{t\Lambda} f\|^2 \leq -2 \|e^{t\Lambda} f\|^2 \quad \forall t \geq 0, \forall f \in \mathcal{E}_2.$$

Proof of Lemma 4.7. This result is nothing but [25, Proposition 5.14]. For the sake of completeness and because we will need to use the same computations at the nonlinear level, we just check it below. First recall that from Theorem 4.3, we know that for any $a \in (\bar{a}, -1)$ there exists $C = C(a)$ such that

$$\|e^{\Lambda t} f\|_{\mathcal{E}} \leq C e^{at} \|f\|_{\mathcal{E}}, \quad \forall t \geq 0, \forall f \in \mathcal{E}_2,$$

and on the other hand, from Lemma 4.4 there exists some constant $b \in \mathbb{R}$ such that

$$\langle \Lambda f, f^* \rangle \leq b \|f\|_{\mathcal{E}}^2.$$

We define

$$(4.12) \quad \|f\|^2 := \eta \|f\|_{\mathcal{E}}^2 + \int_0^\infty \|e^{\tau\Lambda} e^\tau f\|_{\mathcal{E}}^2 d\tau$$

with $\eta \in (0, (b+1)^{-1})$. The norm $\|\cdot\|$ is clearly well defined and it is equivalent to $\|\cdot\|_{\mathcal{E}}$ because

$$\forall f \in \mathcal{E}_2, \quad \eta \|f\|_{\mathcal{E}}^2 \leq \|f\|^2 \leq \eta \|f\|_{\mathcal{E}}^2 + \int_0^\infty \|e^{\Lambda\tau} e^\tau f\|_{\mathcal{E}}^2 d\tau \leq \left(\eta + \int_0^\infty C^2 e^{2(a+1)\tau} d\tau \right) \|f\|_{\mathcal{E}}^2.$$

Next, for $f \in \mathcal{E}_2$ and with the notation $f_t := e^{\Lambda t} f$, we compute

$$\begin{aligned} \frac{d}{dt} \|e^{\Lambda t} f\|^2 &= \eta \frac{d}{dt} \|f_t\|^2 + \int_0^\infty \frac{d}{dt} \|e^{\Lambda(t+\tau)+\tau} f\|^2 d\tau \\ &= 2\eta \langle f_t^*, \Lambda f_t \rangle + \int_0^\infty \left\{ \frac{d}{d\tau} \|e^{\Lambda(t+\tau)+\tau} f\|^2 - 2 \|e^{\Lambda(t+\tau)+\tau} f\|^2 \right\} d\tau \\ &\leq 2\eta b \|f_t\|^2 + \left[\|e^{\Lambda(t+\tau)+\tau} f\|^2 \right]_0^\infty - 2 \int_0^\infty \|e^{\Lambda\tau} e^\tau f_t\|^2 d\tau \\ &= \left\{ 2\eta(b-a) - 1 \right\} \|f_t\|^2 - 2 \left\{ \eta \|f_t\|^2 + \int_0^\infty \|e^{\Lambda\tau} e^\tau f_t\|^2 d\tau \right\} \\ &\leq -2 \|e^{\Lambda t} f\|^2, \end{aligned}$$

so that (4.11) is proved. \square

We conclude with the proof of the long time convergence result.

Proof of Theorem 1.5. The proof follows the same strategy as in [32, 30, 25] (see also [2, 1, 36] where similar proof is carried on in the context of the Boltzmann equation). We split the proof into four steps.

Step 1. We consider a solution g to the rescaled equation (1.18) with initial datum $f_0 \neq G$. Thanks to Theorem 1.4 there holds $g(t) \in \mathcal{Z}$ for any $t \geq 1$. For any $\delta > 0$ and $T := (\mathcal{E}(f_0) - \mathcal{E}(G)) / \eta^{-1}(\delta) + 1$ there exists $t_0 \in [1, T]$ so that

$$(4.13) \quad \mathcal{D}_{\mathcal{E}}(g(t_0)) \leq \eta^{-1}(\delta)$$

because on the contrary we would have from (2.28)

$$\frac{d}{dt} (\mathcal{E}(g(t)) - \mathcal{E}(G)) \leq -\eta^{-1}(\delta) \quad \text{on } (1, T),$$

and then

$$\mathcal{E}(g(T)) - \mathcal{E}(G) \leq -(\mathcal{E}(f_0) - \mathcal{E}(G)) < 0$$

which is in contradiction with the fact that G satisfies $\mathcal{E}(G) < \mathcal{E}(f) \forall f \in \mathcal{Z} \setminus \{G\}$ from Theorem 4.1. We deduce from (4.13) and Lemma 4.6 that

$$\|g(t_0) - G\|_{L_{k'}^{4/3}} \leq \delta.$$

Step 2. The function $h := g - G$ satisfies the equation

$$\partial_t h = \Lambda h + \operatorname{div}(h \mathcal{K} * h).$$

We introduce the splitting

$$h = h_0 + h_1 + h_2, \quad h_{12} = h_1 + h_2$$

with

$$h_0 := \Pi_0 h, \quad h_1 := \Pi_1 h,$$

so that the evolution of h_1 and h_2 are given by

$$(4.14) \quad \partial_t h_1 = -h_1 + \Pi_1[\operatorname{div}(h \mathcal{K} * h)]$$

and

$$(4.15) \quad \partial_t h_2 = \Lambda h_2 + \mathcal{Q}_2, \quad \mathcal{Q}_2 := \Pi_2[\operatorname{div}(h \mathcal{K} * h)].$$

Because of the mass conservation $M(g(t)) = M(G)$, there holds $h_0(t) = \Pi_0 h(t) = h_{0,0}$ $M(h(t)) = 0$.

Moreover, from (4.14) and with the notation $h_1^* = h_1 |h_1|^{-1/3} \|h_1\|_{L_k^{4/3}}^{2/3}$, we clearly have

$$(4.16) \quad \begin{aligned} \frac{d}{dt} \|h_1\|_{L_k^{4/3}}^2 &= 2 \langle -h_1 + \Pi_1[\operatorname{div}(h \mathcal{K} * h)], h_1^* \rangle \\ &\leq -2 \|h_1\|_{L_k^{4/3}}^2 + 2 \|h_1\|_{L_k^{4/3}} \|\Pi_1[\operatorname{div}(h \mathcal{K} * h)]\|_{L_k^{4/3}} \\ &= -2 \|h_1\|_{L_k^{4/3}}^2 + C \|h_1\|_{L_k^{4/3}} \|\operatorname{div}(h \mathcal{K} * h)\|_{L_k^{4/3}}. \end{aligned}$$

Step 3. Estimate on the nonlinear term. We make the splitting

$$\|\operatorname{div}(h \mathcal{K} * h)\|_{L_k^{4/3}} \leq I_1 + I_2, \quad I_1 := \|h^2\|_{L_k^{4/3}}, \quad I_2 := \|\nabla h \cdot \mathcal{K} * h\|_{L_k^{4/3}},$$

and we compute each term separately. On the one hand, using the Hölder inequality and the Gagliardo-Nirenberg inequality (see [9, Chapter IX, inequality (86)]) in dimension 2

$$\|u\|_{L^p} \leq C \|u\|_{L^q}^{1-a} \|u\|_{W^{1,r}}^a, \quad a = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{2} - \frac{1}{r}},$$

with $r = p = \infty$, $q = 4/3$ and $a = 3/5$, we have

$$I_1 \leq \|h\|_{L^\infty} \|h\|_{L_k^{4/3}} \leq C \|h\|_{L_k^{4/3}}^{7/5} \|h\|_{W^{1,\infty}}^{3/5}.$$

On the other hand, thanks to the critical Hardy-Littlewood-Sobolev inequality (1.16), the elementary inequality

$$\|\nabla u\|_{L_k^2}^2 = - \int_{\mathbb{R}^2} u \operatorname{div}(\langle x \rangle^{2k} \nabla u) \leq C \|u\|_{W^{2,\infty}} \|u\|_{L_{2k}^1},$$

and the Hölder inequality

$$\|u\|_{L_{2k}^1} \leq \|\langle x \rangle^{-1}\|_{L^{4\gamma/\alpha}}^{1/\gamma} \|u\|_{L_k^{4/3}}^\alpha \|u\|_{L_{k'}^1}^{1-\alpha},$$

with $0 < \alpha < 1$, $2\gamma > \alpha$ and $k' = k'(\alpha, \gamma) := ((2 - \alpha)k + \gamma)/(1 - \alpha)$, we have

$$I_2 \leq \|\nabla h\|_{L_k^2} \|\mathcal{K} * h\|_{L^4} \leq C_{\alpha,\gamma} \|h\|_{L_k^{4/3}}^{1+\alpha/2} \|h\|_{W^{2,\infty}}^{1/2} \|h\|_{L_{k'}^1}^{(1-\alpha)/2}.$$

To make the computations simpler, when $k' = 4$, we can take $k = 8/5 > 3/2$, $\gamma/\alpha = 5/8 > 1/2$ and we get $\alpha = 32/121 \in (0, 1)$ and $\alpha/2 < 2/5$. All together we find

$$\forall h \in \mathcal{Z}, \quad \|\operatorname{div}(h \mathcal{K} * h)\|_{L_k^{4/3}} \leq C \|h\|_{L_k^{4/3}}^{1+\alpha/2}.$$

Thanks to Theorem 1.4 we have $h(t) \in \mathcal{Z}$ for all $t \geq 1$ (where in the definition \mathcal{Z} the constant C is given by (1.20)), and we conclude with

$$(4.17) \quad \forall t \geq 1, \quad \|\operatorname{div}(h(t) \mathcal{K} * h(t))\|_{L_k^{4/3}} \leq C \|h(t)\|_{L_k^{4/3}}^{1+\alpha/2}.$$

It is worth noticing that in the limit $k \rightarrow 3/2$, $\gamma/\alpha \rightarrow 1/2$ and $\alpha \rightarrow 0$, we find $k' = 3$. In other words, one can easily verify that (4.17) still holds for any $k' > 3$ (with another choice of $\alpha \in (0, 1)$).

Step 4. Estimate on the remaining term and conclusion. From (4.15), using the norm $\|\cdot\|$ defined in (4.12) and the notation $S_\tau := e^{\tau\Lambda} e^\tau$, we compute

$$\begin{aligned} \frac{d}{dt} \|h_2\|^2 &= \eta \langle h_2^*, \Lambda h_2 \rangle + \int_0^\infty \langle (S_\tau h_2)^*, S_\tau \Lambda h_2 \rangle d\tau \\ &\quad + \eta \langle h_2^*, \mathcal{Q}_2 \rangle + \int_0^\infty \langle (S_\tau h_2)^*, \mathcal{Q}_2 \rangle d\tau \\ (4.18) \quad &\leq -2 \|h_2\|^2 + C \|h_2\|_{L_k^{4/2}} \|\operatorname{div}(h \mathcal{K} * h)\|_{L_k^{4/2}}, \end{aligned}$$

where we have used Lemma 4.7 in order to bound the first (linear) term and the equivalence between the two norms $\|\cdot\|$ and $\|\cdot\|_{L_k^{4/2}}$ in order to estimate the second one (which involves the nonlinear quantity). Gathering (4.16), (4.18), (4.17), we clearly see that

$$u(t) := \|h_1\|_{\mathcal{E}}^2 + \|h_2\|^2$$

satisfies the differential inequality

$$u' \leq -2u + C \|h\|^{2+\alpha} \quad \text{on } (0, \infty),$$

and then thanks to the first step

$$(4.19) \quad u' \leq -2u + C u^{1+\alpha/2} \quad \text{on } (t_0, \infty), \quad u(t_0) \leq K_2 \delta.$$

Taking $\delta > 0$ small enough in the first step, we classically deduce that

$$(4.20) \quad u(t) \leq C_a e^{2a t} \quad \forall t \geq t_0$$

for any $a > -1$, so that for a close enough to -1 , we deduce from (4.19)-(4.20) that

$$u' \leq -2u + K_2 e^{-2t} \quad \text{on } (t_0, \infty),$$

from which we easily conclude $u \leq C e^{-2t}$. \square

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