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A Continuation Problem for Computing Solutions of Discretised Evolution Problems. Part II – Application to Quasi-Static Contact Problems with Friction.

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Abstract

A continuation problem for finding successive solutions of discretised plane quasi-static contact problems with friction is proposed and studied. Various formulations of the first-order system are derived and results of existence and uniqueness of its solutions are obtained. Conditions guaranteeing continuation of a solution curve along a direction from the first-order system are given. Numerical continuation of solution curves is performed on examples with non-smooth folds.

Keywords: contact, Coulomb friction, incremental problem, continuation method, first-order system.

1 Introduction

In the companion paper [16], we have proposed an abstract continuation problem and we have laid foundations for continuation of its solution curves provided that the problem is piecewise C^1 (PC^1). The present paper concerns a special case of this problem, namely, a continuation problem for discretised quasi-static plane contact problems with Coulomb friction in large deformations. As far as we know, the only papers dealing with non-smooth continuation of fully discretised contact problems with friction are [11, 8, 7], but those are all restricted to elastostatics and the small-deformation framework.

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After formulating a continuous quasi-static contact problem in large deformations in Section 2, we discretise it in Section 3. The spatial discretisation is done with a finite-element method applied to a mixed variational formulation of the problem while the time derivative is approximated by the backward difference. Following [2], we write the corresponding continuation problem so that it fits exactly the general framework.

Section 4 deals with the first aim of this paper, which is to employ a specific structure of the present problem to do more precise analysis of its first-order system. Let us recall that the first-order system consists in computing directional derivatives of a curve solving the continuation problem. In particular, we formulate the first-order system in such a way that it is close to a rate problem of a quasi-static contact problem in a sense. In the rate problem, one computes first-order right rates of change, that is, right-hand side time derivatives, of a solution of a quasi-static problem for a given rate of input data when time derivatives are not approximated. This problem can serve for direct study of tangent behaviour of solutions of the quasi-static problem. Making use of the similarity between these two problems, we adapt the analysis of the rate problem from [17] to our first-order system. Moreover, we investigate the result of continuation of solution curves in directions solving the first-order system from the abstract frame [16].

The second objective of this paper is to test the general method of numerical continuation from [16]. A similar algorithm has already been tested on static contact problems in [11, 8], but only on finite-element models with very small number of degrees of freedom. Section 5 here presents results for more realistic models.

1.1 Notation and Preliminaries

The following notation is used throughout the paper: For a vector $\boldsymbol{x} \in \mathbb{R}^N$, a matrix $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ and index sets $I \subset \{1,\dots,M\}$ and $J \subset \{1,\dots,N\}$, x_i stands for the ith component of $\boldsymbol{x}$, $\boldsymbol{x}_J$ is the sub-vector of $\boldsymbol{x}$ composed from the components x_i, $i \in J$, and $\boldsymbol{A}_{I,J}$ is the sub-matrix of $\boldsymbol{A}$ with rows and columns specified by I and J, respectively. Furthermore, $\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}^\top\boldsymbol{y}$ is the scalar product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. The gradients of a real-valued function f and a vector-valued function $\boldsymbol{f}$ at a point $\boldsymbol{x}$ are denoted by $\nabla f(\boldsymbol{x})$ and $\boldsymbol{\nabla f}(\boldsymbol{x})$, respectively, the partial gradients of f and $\boldsymbol{f}$ with respect to $\boldsymbol{y}$ at $(\boldsymbol{x},\boldsymbol{y})$ are denoted by $\nabla_{\boldsymbol{y}} f(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$, respectively, and $\nabla^2 f(\boldsymbol{x})$ is the Hessian of f at $\boldsymbol{x}$.

For reader's convenience, we recall the essentials from theory of PC^1-functions [20]:

Definition 1. (i) A function $\boldsymbol{H}\colon \mathbb{R}^M \to \mathbb{R}^N$ is PC^1 if it is continuous and for every $\bar{\boldsymbol{y}} \in \mathbb{R}^M$, there exist an open neighbourhood O of $\bar{\boldsymbol{y}}$ and a finite family of C^1-functions $\boldsymbol{H}^{(i)}\colon O \to \mathbb{R}^N$, $i \in I$, such that

$$\forall \boldsymbol{y} \in O\colon\ \boldsymbol{H}(\boldsymbol{y}) \in \{\boldsymbol{H}^{(i)}(\boldsymbol{y});\, i \in I\}.$$

The functions $\boldsymbol{H}^{(i)}$ are termed *selection functions* of $\boldsymbol{H}$ at $\bar{\boldsymbol{y}}$.

(ii) A selection function $\boldsymbol{H}^{(i)}$ of a PC^1-function $\boldsymbol{H}$ is called *active* at $\bar{\boldsymbol{y}}$ if

$$\boldsymbol{H}^{(i)}(\bar{\boldsymbol{y}}) = \boldsymbol{H}(\bar{\boldsymbol{y}}).$$

The set of indices of active selection functions of $\boldsymbol{H}$ at $\bar{\boldsymbol{y}}$ is denoted by $I_{\boldsymbol{H}}(\bar{\boldsymbol{y}})$.

Obviously, a composition of two PC^1-functions is again a PC^1-function and a vector-valued function is a PC^1-function if and only if all its component functions are PC^1. The following chain rule holds:

Proposition 1. *Let $\boldsymbol{G}\colon \mathbb{R}^M \to \mathbb{R}^N$ and $\boldsymbol{H}\colon \mathbb{R}^N \to \mathbb{R}^P$ be PC^1-functions. Then the composite function $\boldsymbol{H} \circ \boldsymbol{G}$ is directionally differentiable at any point and*

$$(\boldsymbol{H} \circ \boldsymbol{G})'(\boldsymbol{y}; \boldsymbol{z}) = \boldsymbol{H}'(\boldsymbol{G}(\boldsymbol{y}); \boldsymbol{G}'(\boldsymbol{y}; \boldsymbol{z})).$$

If the inner function is a function of a real variable, one can also prove:

Proposition 2. *If $\boldsymbol{G}\colon [s_0, s_0+\delta) \to \mathbb{R}^N$ is differentiable at s_0 from the right and $\boldsymbol{H}\colon \mathbb{R}^N \to \mathbb{R}^P$ is a PC^1-function, then $\boldsymbol{H} \circ \boldsymbol{G}$ has a right-hand side derivative at s_0 and*

$$(\boldsymbol{H} \circ \boldsymbol{G})'(s_0) = \boldsymbol{H}'(\boldsymbol{G}(s_0); \boldsymbol{G}'(s_0))$$

(for any function $\boldsymbol{f}$ of a real variable, $\boldsymbol{f}'$ denotes its right-hand side derivative for brevity of notation here and in what follows).

2 Plane Contact Problems with Coulomb Friction

We start our exposition with a brief introduction into quasi-static plane contact problems with Coulomb friction in large deformations. The reader is referred to [4] for a more detailed introduction to large deformations of elastic materials.

Let us consider deformation of an elastic body in a time interval $[0, T]$. The reference configuration of the body is represented by the closure of a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\Omega$. The boundary $\partial\Omega$ is split into three disjoint relatively open subsets Γ_D, Γ_N and Γ_c such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_c$. By $\boldsymbol{u}$, we denote the displacement of the body and the deformation $\boldsymbol{\varphi}$ of the body is defined by $\boldsymbol{\varphi} = \boldsymbol{id} + \boldsymbol{u}$, where $\boldsymbol{id}$ is the identity map. At each time $t \in [0, T]$, the deformed body $\boldsymbol{\varphi}(t, \Omega)$ is subject to body forces with the density $\boldsymbol{f}^{\boldsymbol{\varphi}}$ while surface forces with the density $\boldsymbol{h}^{\boldsymbol{\varphi}}$ act on $\boldsymbol{\varphi}(t, \Gamma_N)$. The displacement $\boldsymbol{U}_D$ is imposed on Γ_D and points from Γ_c may come into contact with a fixed curved rigid obstacle represented by a closed set $\mathscr{O} \subset \mathbb{R}^2$. It is supposed that there exist $g\colon \mathbb{R}^2 \to \mathbb{R}$ of the class C^1 and a neighbourhood O of the boundary $\partial\mathscr{O}$ such that

$$\begin{aligned} &g(\boldsymbol{x}) > 0,\ \forall \boldsymbol{x} \in \operatorname{Int} \mathscr{O} \cap O, \\ &g(\boldsymbol{x}) = 0,\ \forall \boldsymbol{x} \in \partial\mathscr{O}, \\ &g(\boldsymbol{x}) < 0,\ \forall \boldsymbol{x} \in \operatorname{Ext} \mathscr{O} \cap O, \end{aligned}$$

$\nabla g \neq \boldsymbol{0}$ in O and $\boldsymbol{\varphi}(t, \Gamma_c) \subset O$ for any $t \in [0, T]$. This enables us to extend the unit inward normal and tangent vectors to the obstacle to the vicinity O of the obstacle as follows (see Figure 1):

$$\boldsymbol{\nu}(\boldsymbol{x}) = \frac{\nabla g(\boldsymbol{x})}{\|\nabla g(\boldsymbol{x})\|}, \quad \boldsymbol{\tau}(\boldsymbol{x}) = (-\nu_2(\boldsymbol{x}), \nu_1(\boldsymbol{x})).$$

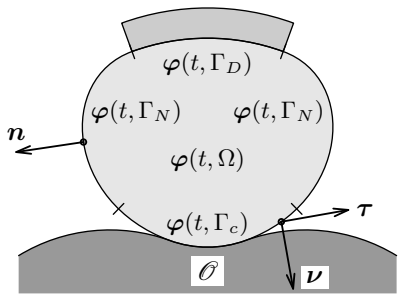


Figure 1: Geometry of the problem.

Finally, an initial displacement $\boldsymbol{u}^0$ is given at the time $t = 0$.

The classical formulation of this problem is the following:

$$\left.\begin{aligned}
&\textit{Find } \boldsymbol{u}\colon [0,T]\times\overline{\Omega}\to\mathbb{R}^2 \textit{ such that } \boldsymbol{u}(t,.)\in\mathscr{U}_{\mathrm{ad}} \textit{ for any } t\in[0,T] \textit{ and}\\
&\qquad\qquad \mathbf{div}\,\boldsymbol{\sigma}(t,\boldsymbol{x})+\boldsymbol{f}(t,\boldsymbol{x})=\boldsymbol{0}, \quad (t,\boldsymbol{x})\in(0,T)\times\Omega,\\
&\qquad\qquad \boldsymbol{\sigma}(t,\boldsymbol{x})=\hat{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x})), \quad (t,\boldsymbol{x})\in(0,T)\times\Omega,\\
&\qquad\qquad \boldsymbol{u}(t,\boldsymbol{x})=\boldsymbol{U}_D(t,\boldsymbol{x}), \quad (t,\boldsymbol{x})\in(0,T)\times\Gamma_D,\\
&\qquad\qquad \boldsymbol{\sigma}(t,\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})=\boldsymbol{h}(t,\boldsymbol{x}), \quad (t,\boldsymbol{x})\in(0,T)\times\Gamma_N,\\
&g(\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x}))\le 0,\ T_\nu(t,\boldsymbol{x})\le 0,\ g(\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x}))T_\nu(t,\boldsymbol{x})=0, \quad (t,\boldsymbol{x})\in(0,T)\times\Gamma_c,\\
&\qquad\qquad |T_\tau(t,\boldsymbol{x})|\le -\mathscr{F}(\boldsymbol{x})T_\nu(t,\boldsymbol{x}), \quad (t,\boldsymbol{x})\in(0,T)\times\Gamma_c,\\
&T_\tau(t,\boldsymbol{x})=\mathscr{F}(\boldsymbol{x})T_\nu(t,\boldsymbol{x})\frac{\dot{u}_\tau(t,\boldsymbol{x})}{|\dot{u}_\tau(t,\boldsymbol{x})|} \quad \textit{if } \dot{u}_\tau(t,\boldsymbol{x})\neq 0, \quad (t,\boldsymbol{x})\in(0,T)\times\Gamma_c,\\
&\qquad\qquad \boldsymbol{u}(0,\boldsymbol{x})=\boldsymbol{U}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega.
\end{aligned}\right\} \tag{1}$$

Here, $\boldsymbol{I}$ denotes the identity matrix, $\boldsymbol{\nabla}\boldsymbol{u}$ is the partial gradient of $\boldsymbol{u}$ with respect to $\boldsymbol{x}$ and $\mathscr{U}_{\mathrm{ad}}$ is the set of kinematically admissible displacements:

$$\mathscr{U}_{\mathrm{ad}}=\{\boldsymbol{v}\colon\overline{\Omega}\to\mathbb{R}^2 \text{ “smooth enough”; } \boldsymbol{id}+\boldsymbol{v} \text{ is injective on } \Omega,\ \det(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{v})>0 \text{ in } \overline{\Omega}\}.$$

The first Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$ is defined from the Cauchy stress tensor $\boldsymbol{\sigma}^{\boldsymbol{\varphi}}$ by

$$\boldsymbol{\sigma}(t,\boldsymbol{x})=(\det(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x})))\boldsymbol{\sigma}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x}))(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))^{-\top} \tag{2}$$

and $\hat{\boldsymbol{\sigma}}$ is a given response function for the first Piola-Kirchhoff stress of the elastic material of the body. By $\boldsymbol{n}$, we denote the unit outward normal vector along $\partial\Omega$ and

$$\begin{aligned}
\boldsymbol{f}(t,\boldsymbol{x})&=\det(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))\boldsymbol{f}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x})),\\
\boldsymbol{h}(t,\boldsymbol{x})&=\det(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))\|(\boldsymbol{I}+\boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))^{-\top}\boldsymbol{n}(\boldsymbol{x})\|\boldsymbol{h}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x}))
\end{aligned}$$

are the densities of volume and surface forces related to the reference configuration. Further,

$$T_\nu(t,\boldsymbol{x})=\boldsymbol{T}(t,\boldsymbol{x})\cdot\boldsymbol{\nu}(\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x})), \quad T_\tau(t,\boldsymbol{x})=\boldsymbol{T}(t,\boldsymbol{x})\cdot\boldsymbol{\tau}(\boldsymbol{x}+\boldsymbol{u}(t,\boldsymbol{x}))$$

are components of the first Piola-Kirchhoff stress vector $\boldsymbol{T}$ at $(t,\boldsymbol{x})$ in the directions $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, $\boldsymbol{T}(t,\boldsymbol{x}) = \boldsymbol{\sigma}(t,\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$. The friction coefficient $\mathscr{F}$ is a non-negative function of $\boldsymbol{x}$ and

$$\dot{u}_\tau(t,\boldsymbol{x}) = \frac{\partial \boldsymbol{u}}{\partial t}(t,\boldsymbol{x}) \cdot \boldsymbol{\tau}(\boldsymbol{x} + \boldsymbol{u}(t,\boldsymbol{x}))$$

is the tangent velocity of a point $\boldsymbol{x}$ at a time t.

To interpret the boundary conditions on Γ_c, let us consider $(t,\boldsymbol{x}) \in (0,T) \times \Gamma_c$ fixed and assume that $\boldsymbol{\varphi}$ is sufficiently smooth so that $\boldsymbol{\varphi}(t,\Omega)$ is a bounded domain with a Lipschitz-continuous boundary $\partial\boldsymbol{\varphi}(t,\Omega)$ and $\partial\boldsymbol{\varphi}(t,\Omega) = \boldsymbol{\varphi}(t,\partial\Omega)$. If $\boldsymbol{x}^{\boldsymbol{\varphi}} := \boldsymbol{x} + \boldsymbol{u}(t,\boldsymbol{x})$ denotes the current position of the point $\boldsymbol{x}$ at the time t, then the unit outward normal vector $\boldsymbol{n}^{\boldsymbol{\varphi}}$ along $\partial\boldsymbol{\varphi}(t,\Omega)$ can be expressed according to [4] as

$$\boldsymbol{n}^{\boldsymbol{\varphi}}(\boldsymbol{x}^{\boldsymbol{\varphi}}) = \frac{(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))^{-\top}\boldsymbol{n}(\boldsymbol{x})}{\|(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))^{-\top}\boldsymbol{n}(\boldsymbol{x})\|}.$$

From here and (2), the Cauchy stress vector $\boldsymbol{T}^{\boldsymbol{\varphi}}$, $\boldsymbol{T}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \boldsymbol{\sigma}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}})\boldsymbol{n}^{\boldsymbol{\varphi}}(\boldsymbol{x}^{\boldsymbol{\varphi}})$, can be written as

$$\boldsymbol{T}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \frac{1}{\det(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))\|(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}(t,\boldsymbol{x}))^{-\top}\boldsymbol{n}(\boldsymbol{x})\|}\boldsymbol{T}(t,\boldsymbol{x})$$

and it is readily seen that the contact conditions $(1)_{6-8}$ at $(t,\boldsymbol{x})$ are equivalent to

$$\begin{aligned} g(\boldsymbol{x}^{\boldsymbol{\varphi}}) \le 0, \quad T_\nu^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) \le 0, \quad g(\boldsymbol{x}^{\boldsymbol{\varphi}})T_\nu^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = 0, \\ |T_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}})| \le -\mathscr{F}^{\boldsymbol{\varphi}}(\boldsymbol{x}^{\boldsymbol{\varphi}})T_\nu^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}), \\ T_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \mathscr{F}^{\boldsymbol{\varphi}}(\boldsymbol{x}^{\boldsymbol{\varphi}})T_\nu^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}})\frac{v_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}})}{|v_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}})|} \quad \text{if } v_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) \neq 0. \end{aligned}$$

Here, $v_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \dot{u}_\tau(t,\boldsymbol{x})$ is the tangent velocity of the point with the current position $\boldsymbol{x}^{\boldsymbol{\varphi}}$, $\mathscr{F}^{\boldsymbol{\varphi}}(\boldsymbol{x}^{\boldsymbol{\varphi}}) = \mathscr{F}(\boldsymbol{x})$ and

$$T_\nu^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \boldsymbol{T}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) \cdot \boldsymbol{\nu}(\boldsymbol{x}^{\boldsymbol{\varphi}}), \quad T_\tau^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) = \boldsymbol{T}^{\boldsymbol{\varphi}}(t,\boldsymbol{x}^{\boldsymbol{\varphi}}) \cdot \boldsymbol{\tau}(\boldsymbol{x}^{\boldsymbol{\varphi}}).$$

In this way, we obtain the unilateral condition and the Coulomb friction law in the part $\boldsymbol{\varphi}(t,\Gamma_c)$ of the boundary $\partial\boldsymbol{\varphi}(t,\Omega)$ of the deformed configuration.

3 Discretisation and the Continuation Problem

Spatial discretisation of our problem is done by applying a Lagrange finite-element method to a mixed variational formulation of (1) with Lagrange multipliers enforcing the Dirichlet and contact boundary conditions. In particular, the contact conditions are discretised similarly as in the example of the "almost conformal" discretisation of the displacement in [12, Section 4].

Let $\{\boldsymbol{\phi}_i\}_{1\le i\le 2n_\Omega}$ be a basis of shape functions for displacements and $\{\boldsymbol{x}_i\}_{1\le i\le n_\Omega}$ be the set of the corresponding finite-element nodes such that

$$\boldsymbol{\phi}_i(\boldsymbol{x}_j) = (\delta_{i,2j-1}, \delta_{i,2j}),$$

where δ denotes the Kronecker delta. Let $\{\boldsymbol{\xi}_i\}_{1\le i\le 2n_D}$ be a basis of shape functions on $\overline{\Gamma}_D$. By $\{\boldsymbol{y}_i\}_{1\le i\le n_c}$, we denote the nodes from $\{\boldsymbol{x}_i\}_{1\le i\le n_\Omega}$ lying on $\overline{\Gamma}_c \setminus \overline{\Gamma}_D$ and by $\{\eta_i\}_{1\le i\le n_c}$, the basis of shape functions on $\overline{\Gamma}_c$ formed by the restrictions of appropriate component functions of $\boldsymbol{\phi}_i$ onto $\overline{\Gamma}_c$ such that

$$\eta_i(\boldsymbol{y}_j) = \delta_{ij}.$$

The local and global numeration of the nodes on $\overline{\Gamma}_c \setminus \overline{\Gamma}_D$ are linked by the mapping $\Theta\colon \{1,\dots,n_c\} \to \{1,\dots,n_\Omega\}$ such that $\boldsymbol{y}_i = \boldsymbol{x}_{\Theta(i)}$. In what follows, we shall assume that the finite-element partitions corresponding to the bases $\{\boldsymbol{\phi}_i\}_{1\le i\le 2n_\Omega}$ and $\{\boldsymbol{\xi}_i\}_{1\le i\le 2n_D}$ are compatible with the decomposition of $\partial\Omega$ into $\overline{\Gamma}_D$, $\overline{\Gamma}_N$ and $\overline{\Gamma}_c$.

We introduce the following vectors and matrices for $\boldsymbol{w} = (w_i)_{1\le i\le 2n_\Omega}$ and $t \in [0,T]$:

$$\boldsymbol{A}(\boldsymbol{w}) = (A_i(\boldsymbol{w}))_{1\le i\le 2n_\Omega}, \quad A_i(\boldsymbol{w}) = \int_\Omega \hat{\boldsymbol{\sigma}}\Big(\boldsymbol{x}, \boldsymbol{I} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\Big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

$$\boldsymbol{B}_D = (B_{D,ij})_{\substack{1\le i\le 2n_D,\\ 1\le j\le 2n_\Omega}}, \quad B_{D,ij} = \int_{\Gamma_D} \boldsymbol{\xi}_i(\boldsymbol{x}) \cdot \boldsymbol{\phi}_j(\boldsymbol{x})\,\mathrm{d}s,$$

$$\boldsymbol{B}_\nu(\boldsymbol{w}) = (B_{\nu,ij}(\boldsymbol{w}))_{\substack{1\le i\le n_c,\\ 1\le j\le 2n_\Omega}}, \quad B_{\nu,ij}(\boldsymbol{w}) = \begin{cases} \nu_1(\boldsymbol{y}_i + (w_{2\Theta(i)-1}, w_{2\Theta(i)})) & \text{if } j = 2\Theta(i)-1,\\ \nu_2(\boldsymbol{y}_i + (w_{2\Theta(i)-1}, w_{2\Theta(i)})) & \text{if } j = 2\Theta(i),\\ 0 & \text{otherwise,}\end{cases}$$

$$\boldsymbol{B}_\tau(\boldsymbol{w}) = (B_{\tau,ij}(\boldsymbol{w}))_{\substack{1\le i\le n_c,\\ 1\le j\le 2n_\Omega}}, \quad B_{\tau,ij}(\boldsymbol{w}) = \begin{cases} \tau_1(\boldsymbol{y}_i + (w_{2\Theta(i)-1}, w_{2\Theta(i)})) & \text{if } j = 2\Theta(i)-1,\\ \tau_2(\boldsymbol{y}_i + (w_{2\Theta(i)-1}, w_{2\Theta(i)})) & \text{if } j = 2\Theta(i),\\ 0 & \text{otherwise,}\end{cases}$$

$$\begin{aligned}
\boldsymbol{l}(t,\boldsymbol{w}) &= (l_i(t,\boldsymbol{w}))_{1\le i\le 2n_\Omega},\\
l_i(t,\boldsymbol{w}) &= \int_\Omega \Big(\det\Big(\boldsymbol{I} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\Big)\Big) \boldsymbol{f}^{\boldsymbol{\varphi}}\Big(t, \boldsymbol{x} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\phi}_j(\boldsymbol{x})\Big) \cdot \boldsymbol{\phi}_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_{\Gamma_N} \Big(\det\Big(\boldsymbol{I} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\Big)\Big) \Big\| \Big(\boldsymbol{I} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\Big)^{-\top} \boldsymbol{n}(\boldsymbol{x}) \Big\|\\
&\qquad \cdot \boldsymbol{h}^{\boldsymbol{\varphi}}\Big(t, \boldsymbol{x} + \sum_{j=1}^{2n_\Omega} w_j \boldsymbol{\phi}_j(\boldsymbol{x})\Big) \cdot \boldsymbol{\phi}_i(\boldsymbol{x})\,\mathrm{d}s,
\end{aligned}$$

$$\boldsymbol{u}_D(t) = (u_{D,i}(t))_{1\le i\le 2n_D}, \quad u_{D,i}(t) = \int_{\Gamma_D} \boldsymbol{U}_D(t,\boldsymbol{x}) \cdot \boldsymbol{\xi}_i(\boldsymbol{x})\,\mathrm{d}s,$$

$$\boldsymbol{g}(\boldsymbol{w}) = (g_i(\boldsymbol{w}))_{1\le i\le n_c}, \quad g_i(\boldsymbol{w}) = g(\boldsymbol{y}_i + (w_{2\Theta(i)-1}, w_{2\Theta(i)})),$$

$$\boldsymbol{\mathscr{F}} = (\mathscr{F}_i)_{1\le i\le n_c}, \quad \mathscr{F}_i = \mathscr{F}(\boldsymbol{y}_i).$$

Furthermore, we take $\boldsymbol{u}^0 = (u_i^0)_{1\le i\le 2n_\Omega}$ such that $\sum_{i=1}^{2n_\Omega} u_i^0 \boldsymbol{\phi}_i$ approximates $\boldsymbol{U}_0$ and we define the sets:

$$\begin{aligned}\boldsymbol{\Lambda}_\nu &= \mathbb{R}^{n_c}_- \equiv \{\boldsymbol{\mu}_\nu = (\mu_{\nu,i})_{1\le i\le n_c};\ \mu_{\nu,i} \le 0\},\\ \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\mu}_\nu) &= \{\boldsymbol{\mu}_\tau = (\mu_{\tau,i})_{1\le i\le n_c};\ |\mu_{\tau,i}| \le -\mathscr{F}_i\mu_{\nu,i}\}, \quad \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu.\end{aligned}$$

Then, the spatial semi-discretisation of (1) can be formulated as follows:

$$\left.\begin{aligned}&\textit{Find } \boldsymbol{u}\colon [0,T] \to \mathbb{R}^{2n_\Omega}, \boldsymbol{\lambda}_D\colon [0,T] \to \mathbb{R}^{2n_D}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau\colon [0,T] \to \mathbb{R}^{n_c} \textit{ such that } \boldsymbol{u}(0) = \boldsymbol{u}^0\\ &\textit{and for any } t \in (0,T), \textit{ it holds that } \boldsymbol{\lambda}_\nu(t) \in \boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau(t) \in \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu(t)) \textit{ and}\\ &\boldsymbol{A}(\boldsymbol{u}(t)) - \boldsymbol{B}_D^\top\boldsymbol{\lambda}_D(t) - \boldsymbol{B}_\nu(\boldsymbol{u}(t))^\top\boldsymbol{\lambda}_\nu(t) - \boldsymbol{B}_\tau(\boldsymbol{u}(t))^\top\boldsymbol{\lambda}_\tau(t) = \boldsymbol{l}(t,\boldsymbol{u}(t)),\\ &\qquad\qquad \boldsymbol{B}_D\boldsymbol{u}(t) = \boldsymbol{u}_D(t),\\ &\qquad\qquad (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t))^\top\boldsymbol{g}(\boldsymbol{u}(t)) \ge 0, \quad \forall\boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu,\\ &\qquad\qquad (\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau(t))^\top\boldsymbol{B}_\tau(\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t) \ge 0, \quad \forall\boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu(t)),\end{aligned}\right\} \quad (3)$$

where $\boldsymbol{u}$ is the vector of nodal displacements, $\boldsymbol{\lambda}_D$ is the Lagrange multiplier corresponding to the Dirichlet condition and $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_\tau$ are the normal and tangential Lagrange multipliers on the contact zone, respectively.

Approximating $\dot{\boldsymbol{u}}(t)$ by a backward difference and employing the idea from the abstract frame [16, Section 2], one obtains the continuation problem:

$$\begin{aligned}&\textit{Find } \gamma \in \mathbb{R}, \boldsymbol{u} \in \mathbb{R}^{2n_\Omega}, \boldsymbol{\lambda}_D \in \mathbb{R}^{2n_D}, \boldsymbol{\lambda}_\nu \in \boldsymbol{\Lambda}_\nu, \boldsymbol{\lambda}_\tau \in \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu) \textit{ such that}\\ &\boldsymbol{A}(\boldsymbol{u}) - \boldsymbol{B}_D^\top\boldsymbol{\lambda}_D - \boldsymbol{B}_\nu(\boldsymbol{u})^\top\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau(\boldsymbol{u})^\top\boldsymbol{\lambda}_\tau = \boldsymbol{L}(\gamma,\boldsymbol{u}),\\ &\qquad\qquad \boldsymbol{B}_D\boldsymbol{u} = \boldsymbol{U}_D(\gamma),\\ &\qquad\qquad (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu)^\top\boldsymbol{g}(\boldsymbol{u}) \ge 0, \quad \forall\boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu,\\ &(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau)^\top\big(\boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{u} - \boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{w} + (1-\gamma)\boldsymbol{V}_\tau\big) \ge 0, \quad \forall\boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu),\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{L}(\gamma,\boldsymbol{u}) &= \gamma\boldsymbol{l}(t_{k+1},\boldsymbol{u}) + (1-\gamma)\boldsymbol{l}(t_k,\boldsymbol{u}), \quad \boldsymbol{U}_D(\gamma) = \gamma\boldsymbol{u}_D(t_{k+1}) + (1-\gamma)\boldsymbol{u}_D(t_k),\\ \boldsymbol{w} &= \boldsymbol{u}^k, \quad \boldsymbol{V}_\tau = (t_{k+1}-t_k)\frac{\boldsymbol{B}_\tau(\boldsymbol{u}^k)\boldsymbol{u}^k - \boldsymbol{B}_\tau(\boldsymbol{u}^k)\boldsymbol{u}^{k-1}}{t_k - t_{k-1}}.\end{aligned}$$

As in the abstract frame, the benefit of this continuation problem is that any solution from the kth time level of the corresponding fully discretised problem yields always a solution of the continuation problem, which can be taken as a starting point for continuation. Moreover, any solution of the continuation problem with $\gamma = 1$ yields a solution of the discretised problem on the $(k+1)$st time level.

Due to variational characterisation of a projection onto a closed convex set, this problem is equivalent to

$$\left.\begin{aligned}&\textit{Find } (\gamma,\boldsymbol{u},\boldsymbol{\lambda}_D,\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau) \in \mathbb{R}^{1+2(n_\Omega+n_D+n_c)} \textit{ such that}\\ &\boldsymbol{H}(\gamma,\boldsymbol{u},\boldsymbol{\lambda}_D,\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau) = \boldsymbol{0},\end{aligned}\right\} \quad (\mathscr{P})$$

where $\boldsymbol{H}\colon \mathbb{R}^{1+2(n_\Omega+n_D+n_c)} \to \mathbb{R}^{2(n_\Omega+n_D+n_c)}$ is defined by

$$\boldsymbol{H}(\gamma,\boldsymbol{u},\boldsymbol{\lambda}_D,\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau) = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{u}) - \boldsymbol{L}(\gamma,\boldsymbol{u}) - \boldsymbol{B}_D^\top\boldsymbol{\lambda}_D - \boldsymbol{B}_\nu(\boldsymbol{u})^\top\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau(\boldsymbol{u})^\top\boldsymbol{\lambda}_\tau \\ \boldsymbol{B}_D\boldsymbol{u} - \boldsymbol{U}_D(\gamma) \\ \lambda_{\nu,i} - (\lambda_{\nu,i} - g_i(\boldsymbol{u}))_-, \quad i=1,\dots,n_c \\ \lambda_{\tau,i} - P_{[\mathscr{F}_i(\lambda_{\nu,i}-g_i(\boldsymbol{u}))_-,-\mathscr{F}_i(\lambda_{\nu,i}-g_i(\boldsymbol{u}))_-]}\big(\lambda_{\tau,i} \\ \quad - (\boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{u} - \boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{w})_i - (1-\gamma)V_{\tau,i}\big), \quad i=1,\dots,n_c \end{pmatrix}$$

(see also [2, Section 3]). Here, $(a)_-$ is the non-positive part of a, or equivalently, the projection of a onto the interval $(-\infty,0]$, and $P_{[a,b]}$ stands for the projection onto an interval $[a,b]$.

Since the functions $(a,b)\mapsto a-(a-b)_-$ and $(a,b,c)\mapsto a - P_{[kc_-,-kc_-]}(a-b)$, $k\geq 0$, are both PC^1 (see Figure 2) and a composition of two PC^1-functions is again PC^1, $\boldsymbol{H}$ is a PC^1-function under the following assumption:

Assumption 1. *Let $\boldsymbol{A}$ and $\boldsymbol{l}(t,.), t\in[0,T]$, be C^1-functions and g demarcating the rigid obstacle be of the class C^2.*

In this case, the general theory from [16] can be applied.

Remark 1. In the present discretisation, we treat the Dirichlet condition with the Lagrange multiplier so that it can be parametrised in a simple way. However, if the Dirichlet condition does not depend on the parameter (or more precisely, on the time), it can be prescribed directly in the discrete problem without any principal changes.

4 First-Order System

Let $(\bar\gamma,\bar{\boldsymbol{u}},\bar{\boldsymbol{\lambda}}_D,\bar{\boldsymbol{\lambda}}_\nu,\bar{\boldsymbol{\lambda}}_\tau)$ be a fixed solution of $(\mathscr{P})$. We shall now give a particular expression for the first-order system

$$\boldsymbol{H}'((\bar\gamma,\bar{\boldsymbol{u}},\bar{\boldsymbol{\lambda}}_D,\bar{\boldsymbol{\lambda}}_\nu,\bar{\boldsymbol{\lambda}}_\tau);(\gamma',\boldsymbol{u}',\boldsymbol{\lambda}_D',\boldsymbol{\lambda}_\nu',\boldsymbol{\lambda}_\tau')) = \boldsymbol{0}$$

(compare [16, problem $(\mathscr{P}')$]) provided that Assumption 1 is satisfied. The dependence upon $\bar\gamma$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{\lambda}}_D$, $\bar{\boldsymbol{\lambda}}_\nu$ and $\bar{\boldsymbol{\lambda}}_\tau$ will not be explicitly indicated in what follows usually.

After tedious applications of the chain rule, the first-order equilibrium equation corresponding to the first block of $\boldsymbol{H}$ reads

$$\boldsymbol{K}\boldsymbol{u}' - \gamma'\boldsymbol{P} - \boldsymbol{B}_D^\top\boldsymbol{\lambda}_D' - \boldsymbol{B}_\nu^\top\boldsymbol{\lambda}_\nu' - \boldsymbol{B}_\tau^\top\boldsymbol{\lambda}_\tau' = \boldsymbol{0}, \qquad (\mathscr{P}'1)$$

where we denote $\boldsymbol{B}_\nu = \boldsymbol{B}_\nu(\bar{\boldsymbol{u}})$, $\boldsymbol{B}_\tau = \boldsymbol{B}_\tau(\bar{\boldsymbol{u}})$, $\boldsymbol{P} = \frac{\partial \boldsymbol{L}}{\partial\gamma}(\bar\gamma,\bar{\boldsymbol{u}})$, and

$$\boldsymbol{K} = \boldsymbol{\nabla}\boldsymbol{A}(\bar{\boldsymbol{u}}) - \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{L}(\bar\gamma,\bar{\boldsymbol{u}}) - \boldsymbol{K}_\nu(\bar{\boldsymbol{u}},\bar{\boldsymbol{\lambda}}_\nu) - \boldsymbol{K}_\tau(\bar{\boldsymbol{u}},\bar{\boldsymbol{\lambda}}_\tau)$$

is the $2n_\Omega \times 2n_\Omega$ tangent stiffness matrix with

$$K_{\nu,ij}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu) = \begin{cases} \frac{\partial \nu_1}{\partial x_1}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\nu,k} & \text{if } i = j = 2\Theta(k) - 1, \\ \frac{\partial \nu_1}{\partial x_2}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\nu,k} & \text{if } i = 2\Theta(k) - 1,\ j = 2\Theta(k), \\ \frac{\partial \nu_2}{\partial x_1}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\nu,k} & \text{if } i = 2\Theta(k),\ j = 2\Theta(k) - 1, \\ \frac{\partial \nu_2}{\partial x_2}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\nu,k} & \text{if } i = j = 2\Theta(k), \\ 0 & \text{otherwise}, \end{cases}$$

$$K_{\tau,ij}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\tau) = \begin{cases} \frac{\partial \tau_1}{\partial x_1}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\tau,k} & \text{if } i = j = 2\Theta(k) - 1, \\ \frac{\partial \tau_1}{\partial x_2}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\tau,k} & \text{if } i = 2\Theta(k) - 1,\ j = 2\Theta(k), \\ \frac{\partial \tau_2}{\partial x_1}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\tau,k} & \text{if } i = 2\Theta(k),\ j = 2\Theta(k) - 1, \\ \frac{\partial \tau_2}{\partial x_2}\big(\boldsymbol{y}_k + (\bar{u}_{2\Theta(k)-1}, \bar{u}_{2\Theta(k)})\big)\bar{\lambda}_{\tau,k} & \text{if } i = j = 2\Theta(k), \\ 0 & \text{otherwise}, \end{cases}$$

$$\frac{\partial \nu_k}{\partial x_m} = \frac{1}{\|\nabla g\|}\sum_{l=1}^{2}\frac{\partial^2 g}{\partial x_l \partial x_m}\tau_k\tau_l, \quad \frac{\partial \tau_k}{\partial x_m} = -\frac{1}{\|\nabla g\|}\sum_{l=1}^{2}\frac{\partial^2 g}{\partial x_l \partial x_m}\nu_k\tau_l.$$

Remark 2. The partial derivatives of the normal and tangent are connected with the obstacle curvature κ: one has

$$\frac{\partial \nu_1}{\partial x_1} + \frac{\partial \nu_2}{\partial x_2} = -\kappa$$

(curvature formulae for implicit planar curves can be found, for instance, in [6, Section 3]).

Furthermore, the first-order Dirichlet condition is

$$\boldsymbol{B}_D\boldsymbol{u}' - \gamma'\boldsymbol{U}'_D = \boldsymbol{0} \qquad (\mathscr{P}'2)$$

with $\boldsymbol{U}'_D = \boldsymbol{U}'_D(\bar{\gamma})$.

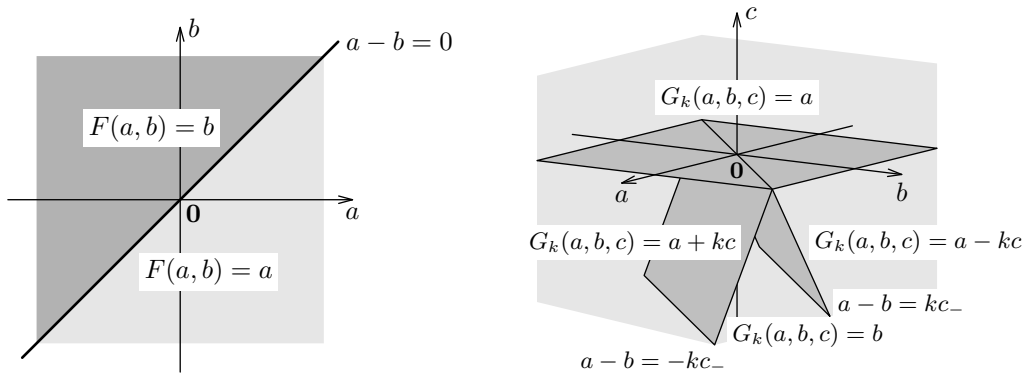
To derive the first-order contact conditions, one can show by other applications of the chain rule that

$$g'_i(\bar{\boldsymbol{u}}; \boldsymbol{u}') = (\boldsymbol{B}_\nu\boldsymbol{u}')_i, \quad \big((\boldsymbol{B}_\tau(.)\,. - \boldsymbol{B}_\tau(.)\boldsymbol{w})_i\big)'(\bar{\boldsymbol{u}}; \boldsymbol{u}') = (\boldsymbol{C}_\tau\boldsymbol{u}')_i,$$

where

$$\boldsymbol{C}_\tau = \boldsymbol{C}_\tau^{\bar{\boldsymbol{u}}}(\bar{\boldsymbol{u}}) + \boldsymbol{B}_\tau - \boldsymbol{C}_\tau^{\boldsymbol{w}}(\bar{\boldsymbol{u}}) \in \mathbb{R}^{n_c \times 2n_\Omega},$$

$$C^{\boldsymbol{w}}_{\tau,ij}(\bar{\boldsymbol{u}}) = \begin{cases} \frac{\partial \tau_1}{\partial x_1}\big(\boldsymbol{y}_i + (\bar{u}_{2\Theta(i)-1}, \bar{u}_{2\Theta(i)})\big)w_{2\Theta(i)-1} & \\ \quad + \frac{\partial \tau_2}{\partial x_1}\big(\boldsymbol{y}_i + (\bar{u}_{2\Theta(i)-1}, \bar{u}_{2\Theta(i)})\big)w_{2\Theta(i)} & \text{if } 2\Theta(i) - 1 = j, \\ \frac{\partial \tau_1}{\partial x_2}\big(\boldsymbol{y}_i + (\bar{u}_{2\Theta(i)-1}, \bar{u}_{2\Theta(i)})\big)w_{2\Theta(i)-1} & \\ \quad + \frac{\partial \tau_2}{\partial x_2}\big(\boldsymbol{y}_i + (\bar{u}_{2\Theta(i)-1}, \bar{u}_{2\Theta(i)})\big)w_{2\Theta(i)} & \text{if } 2\Theta(i) = j, \\ 0 & \text{otherwise}. \end{cases}$$



(a) $F(a,b)=a-(a-b)_-$.

(b) $G_k(a,b,c)=a-P_{[kc_-,-kc_-]}(a-b)$ for $k>0$.

Figure 2: Subdivisions for the PC^1-functions F and G_k defined by (4) and (5).

Moreover, we define PC^1-functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ and $G_k\colon \mathbb{R}^3 \to \mathbb{R}$ for any $k \geq 0$ by

$$F(a,b) := a-(a-b)_- = \begin{cases} a & \text{if } a-b \geq 0, \\ b & \text{if } a-b \leq 0, \end{cases} \tag{4}$$

$$G_k(a,b,c) := a - P_{[kc_-,-kc_-]}(a-b) = \begin{cases} a & \text{if } c \geq 0, \\ b & \text{if } c \leq 0,\ |a-b| \leq -kc, \\ a-kc & \text{if } c \leq 0,\ a-b \leq kc, \\ a+kc & \text{if } c \leq 0,\ a-b \geq -kc \end{cases} \tag{5}$$

(see Figure 2). It is readily seen that for any $a,b,c,a',b',c' \in \mathbb{R}$,

$$F'((a,b);(a',b')) = \begin{cases} a' & \text{if } a-b>0, \\ b' & \text{if } a-b<0, \\ a'-(a'-b')_- & \text{if } a-b=0, \end{cases}$$

$$G_0'((a,b,c);(a',b',c')) = a'$$

and if $k > 0$,

$$G'_k((a,b,c);(a',b',c')) = \begin{cases} a' & \text{if } c > 0, \\ b' & \text{if } c < 0,\ |a-b| < -kc, \\ a' - kc' & \text{if } c < 0,\ a-b < kc, \\ a' + kc' & \text{if } c < 0,\ a-b > -kc, \\ a' - k(c')_- & \text{if } c = 0,\ a-b < 0, \\ a' + k(c')_- & \text{if } c = 0,\ a-b > 0, \\ a' - P_{[kc',+\infty)}(a'-b') & \text{if } c < 0,\ a-b = kc, \\ a' - P_{(-\infty,-kc']}(a'-b') & \text{if } c < 0,\ a-b = -kc, \\ a' - P_{[k(c')_-,-k(c')_-]}(a'-b') & \text{if } c = a-b = 0. \end{cases}$$

Hence, Proposition 1 gives the first-order contact conditions in the form:

$$\left.\begin{aligned} \lambda'_{\nu,i} &= 0 \quad \text{if } \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) > 0, \\ (\boldsymbol{B}_\nu \boldsymbol{u}')_i &= 0 \quad \text{if } \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0, \\ \lambda'_{\nu,i} - (\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)_- &= 0 \quad \text{if } \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) = 0, \end{aligned}\right\} \qquad (\mathscr{P}'3)$$

$$\left.\begin{aligned} \lambda'_{\tau,i} &= 0 \quad \text{if } \mathscr{F}_i = 0 \text{ or } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) > 0, \\ (\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i} &= 0 \quad \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0, \\ & \qquad |\bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i}| < -\mathscr{F}_i(\bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}})), \\ \lambda'_{\tau,i} - \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i) &= 0 \quad \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0, \\ & \qquad \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} < \mathscr{F}_i(\bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}})), \\ \lambda'_{\tau,i} + \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i) &= 0 \quad \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0, \\ & \qquad \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} > -\mathscr{F}_i(\bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}})), \\ \lambda'_{\tau,i} - \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)_- &= 0 \quad \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) = 0, \\ & \qquad \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} < 0, \\ \lambda'_{\tau,i} + \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)_- &= 0 \quad \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) = 0, \\ & \qquad \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} > 0, \\ \lambda'_{\tau,i} - P_{[\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i),+\infty)}&(\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}) = 0 \\ & \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0,\ \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} = \mathscr{F}_i(\bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}})), \\ \lambda'_{\tau,i} - P_{(-\infty,-\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)]}&(\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}) = 0 \\ & \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0,\ \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} = -\mathscr{F}_i(\bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}})), \\ \lambda'_{\tau,i} - P_{[\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)_-,-\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i)_-]}&(\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}) = 0 \\ & \text{if } \mathscr{F}_i > 0,\ \bar\lambda_{\nu,i} - g_i(\bar{\boldsymbol{u}}) = \bar\lambda_{\tau,i} - \bar v_{\tau,i} - (1-\bar\gamma)V_{\tau,i} = 0, \end{aligned}\right\} \qquad (\mathscr{P}'4)$$

where $\bar v_{\tau,i} = (\boldsymbol{B}_\tau(\bar{\boldsymbol{u}})\bar{\boldsymbol{u}} - \boldsymbol{B}_\tau(\bar{\boldsymbol{u}})\boldsymbol{w})_i$ and i passes $\{1,\dots,n_c\}$.

To sum up, the complete first-order system for the problem $(\mathscr{P})$ consists in finding $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau) \in \mathbb{R}^{1+2(n_\Omega+n_D+n_c)}$ satisfying $(\mathscr{P}'1)$, $(\mathscr{P}'2)$, $(\mathscr{P}'3)$ and $(\mathscr{P}'4)$.

Unfortunately, Proposition 2 from [16] guaranteeing existence and uniqueness of solutions of a general PC^1 first-order system does not seem to be very useful for analysing this particular system. We can refer the reader to [1] and [15] for attempts to employ some results from theory of PC^1-functions to discrete contact problems. For this reason, we shall present some other forms of the first-order system.

To start with, we shall reformulate the contact conditions with the aid of the following lemma, which can be proved by a brief discussion. By Sgn, we denote the multifunction defined by

$$\operatorname{Sgn}(b) = \begin{cases} \frac{b}{|b|} & \text{if } b \neq 0, \\ [-1,1] & \text{if } b = 0. \end{cases}$$

Lemma 1. *For any $a, b, c \in \mathbb{R}$ and any $k \geq 0$, it holds that*

$$\begin{aligned}
&\text{(i)} && a - (a-b)_- = 0 \iff a \leq 0,\ b \leq 0,\ ab = 0; \\
&\text{(ii)} && a - P_{[kc,+\infty)}(a-b) = 0 \iff a - kc \geq 0,\ b \geq 0,\ (a-kc)b = 0; \\
&\text{(iii)} && a - P_{(-\infty,-kc]}(a-b) = 0 \iff a + kc \leq 0,\ b \leq 0,\ (a+kc)b = 0; \\
&\text{(iv)} && a - P_{[kc_-,-kc_-]}(a-b) = 0 \iff a \in kc_- \operatorname{Sgn}(b).
\end{aligned}$$

Next, we introduce the index sets (compare ($\mathscr{P}'3$) and ($\mathscr{P}'4$)):

$$\left.\begin{aligned}
&I = \{1, \dots, n_c\}, && I_{\mathrm{f}} = \{i \in I;\ \bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}) > 0\}, \\
&I_{\mathrm{c}} = \{i \in I;\ \bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}) < 0\}, && I_{\mathrm{z}} = \{i \in I;\ \bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}) = 0\}, \\
&I_0 = \{i \in I;\ \mathscr{F}_i = 0\}, \\
&I_{\mathrm{s}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i} - \bar{v}_{\tau,i} - (1-\bar{\gamma})V_{\tau,i}| < -\mathscr{F}_i(\bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}))_-\}, \\
&I_{\mathrm{l}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i} - \bar{v}_{\tau,i} - (1-\bar{\gamma})V_{\tau,i}| > -\mathscr{F}_i(\bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}))_-\}, \\
&I_{\mathrm{i}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i} - \bar{v}_{\tau,i} - (1-\bar{\gamma})V_{\tau,i}| = -\mathscr{F}_i(\bar{\lambda}_{\nu,i} - g_i(\bar{\boldsymbol{u}}))_-\}.
\end{aligned}\right\} \tag{6}$$

Observe that the contact conditions corresponding to the last two blocks of components of $\boldsymbol{H}$ can be rewritten for $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ according to Lemma 1 as

$$\begin{aligned}
\bar{\lambda}_{\nu,i} \leq 0,\ g_i(\bar{\boldsymbol{u}}) \leq 0,\ \bar{\lambda}_{\nu,i} g_i(\bar{\boldsymbol{u}}) = 0, \quad i \in I, \\
\bar{\lambda}_{\tau,i} \in \mathscr{F}_i \bar{\lambda}_{\nu,i} \operatorname{Sgn}(\bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i}), \quad i \in I,
\end{aligned}$$

which can be viewed as contact conditions at each contact node. Consequently, one obtains

$$\begin{aligned}
&I_{\mathrm{f}} = \{i \in I;\ \bar{\lambda}_{\nu,i} = 0,\ g_i(\bar{\boldsymbol{u}}) < 0\} && \text{(nodes not in contact – \underline{f}ree)}, \\
&I_{\mathrm{c}} = \{i \in I;\ \bar{\lambda}_{\nu,i} < 0,\ g_i(\bar{\boldsymbol{u}}) = 0\} && \text{(nodes in strong \underline{c}ontact)}, \\
&I_{\mathrm{z}} = \{i \in I;\ \bar{\lambda}_{\nu,i} = g_i(\bar{\boldsymbol{u}}) = 0\} && \text{(nodes in grazing contact – with \underline{z}ero contact forces)}, \\
&I_{\mathrm{s}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i}| < -\mathscr{F}_i \bar{\lambda}_{\nu,i},\ \bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i} = 0\} && \text{(nodes in strong \underline{s}tick)}, \\
&I_{\mathrm{l}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i}| = -\mathscr{F}_i \bar{\lambda}_{\nu,i},\ \bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i} \neq 0\} && \text{(nodes in non-zero s\underline{l}ip)}, \\
&I_{\mathrm{i}} = \{i \in I;\ \mathscr{F}_i > 0,\ |\bar{\lambda}_{\tau,i}| = -\mathscr{F}_i \bar{\lambda}_{\nu,i},\ \bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i} = 0\} && \text{(nodes in \underline{i}mpending slip)}
\end{aligned}$$

and $I_{\mathrm{s}} \subset I_{\mathrm{c}}$.

To conclude, ($\mathscr{P}'3$) and ($\mathscr{P}'4$) are equivalent to

$$\left.\begin{aligned} \lambda'_{\nu,i} &= 0, \quad i \in I_{\mathrm{f}}, \\ (\boldsymbol{B}_\nu \boldsymbol{u}')_i &= 0, \quad i \in I_{\mathrm{c}}, \\ \lambda'_{\nu,i} \le 0,\ (\boldsymbol{B}_\nu \boldsymbol{u}')_i \le 0,\ \lambda'_{\nu,i}(\boldsymbol{B}_\nu \boldsymbol{u}')_i &= 0, \quad i \in I_{\mathrm{z}}, \end{aligned}\right\} \qquad (\mathscr{P}'3\text{'})$$

$$\left.\begin{aligned} \lambda'_{\tau,i} &= 0, \quad i \in I_{\mathrm{f}} \cup I_0, \\ (\boldsymbol{C}_\tau \boldsymbol{u}')_i &= \gamma' V_{\tau,i}, \quad i \in I_{\mathrm{s}}, \\ \lambda'_{\tau,i} &= \operatorname{sgn}(\bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i})\mathscr{F}_i \lambda'_{\nu,i}, \quad i \in (I_{\mathrm{c}} \cup I_{\mathrm{z}}) \cap I_{\mathrm{l}}, \\ \operatorname{sgn}(\bar{\lambda}_{\tau,i})((\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i}) \le 0,\ \operatorname{sgn}(\bar{\lambda}_{\tau,i})\lambda'_{\tau,i} &+ \mathscr{F}_i \lambda'_{\nu,i} \le 0, \\ \operatorname{sgn}(\bar{\lambda}_{\tau,i})((\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i})(\operatorname{sgn}(\bar{\lambda}_{\tau,i})\lambda'_{\tau,i} &+ \mathscr{F}_i \lambda'_{\nu,i}) = 0, \quad i \in I_{\mathrm{c}} \cap I_{\mathrm{i}}, \\ \lambda'_{\tau,i} &\in \mathscr{F}_i \lambda'_{\nu,i} \operatorname{Sgn}((\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i}), \quad i \in I_{\mathrm{z}} \cap I_{\mathrm{i}}. \end{aligned}\right\} \qquad (\mathscr{P}'4\text{'})$$

The system ($\mathscr{P}'1$), ($\mathscr{P}'2$), ($\mathscr{P}'3$') and ($\mathscr{P}'4$') is close to a rate problem of a quasi-static contact problem discretised in space only (this problem can be found in [17], for example). The main difference between the two problems is that in our continuation problem, we suppress solutions in which slipping contact nodes become stuck when loading of the body changes to unloading suddenly. Indeed, there is no equivalent of the index set I_{l} in the rate problem and all nodes in contact are treated either as in stick or as in possible slip in the near future. Of course, this causes the rate problem to be more complicated (compare ($\mathscr{P}'4$') for $i \in I_{\mathrm{l}}$ and $i \in I_{\mathrm{i}}$).

Let us also mention that due to nodal discretisation of the contact conditions considered in this paper, the continuation problem could be formulated without the term $(1-\gamma)\boldsymbol{V}_\tau$ approximating a multiple of an initial velocity in the frictional condition and $(0, \boldsymbol{u}^k, \boldsymbol{\lambda}_D^k, \boldsymbol{\lambda}_\nu^k, \boldsymbol{\lambda}_\tau^k)$ would remain its solution (which is not always the case for other types of discretisation). However, when starting continuation from this point, one would run into an analogous situation as in the rate problem. More precisely, one would have $\bar{\boldsymbol{v}}_\tau = \boldsymbol{0}$ for $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = (0, \boldsymbol{u}^k, \boldsymbol{\lambda}_D^k, \boldsymbol{\lambda}_\nu^k, \boldsymbol{\lambda}_\tau^k)$, the indices i with $(\boldsymbol{B}_\tau(\boldsymbol{u}^k)\boldsymbol{u}^k - \boldsymbol{B}_\tau(\boldsymbol{u}^k)\boldsymbol{u}^{k-1})_i \neq 0$, which correspond to slipping nodes at the time t_k, would fall into I_{i} instead of I_{l} and I_{l} would be empty.

By the way, the situation for continuation from $(0, \boldsymbol{u}^k, \boldsymbol{\lambda}_D^k, \boldsymbol{\lambda}_\nu^k, \boldsymbol{\lambda}_\tau^k)$ in the problem without $(1-\gamma)\boldsymbol{V}_\tau$ is very similar to what happens in the semi-smooth (or piecewise-smooth) Newton method applied directly to a projection formulation of the fully discretised problem for the time t_{k+1} with the initial approximation $(\boldsymbol{u}^k, \boldsymbol{\lambda}_D^k, \boldsymbol{\lambda}_\nu^k, \boldsymbol{\lambda}_\tau^k)$. Regarding the Newton method as an active-set strategy, one can employ adaptations of the index sets defined above to determine an element from the generalised gradient used in each Newton's iteration and one can see that any slipping node at the time t_k makes the initial approximation $(\boldsymbol{u}^k, \boldsymbol{\lambda}_D^k, \boldsymbol{\lambda}_\nu^k, \boldsymbol{\lambda}_\tau^k)$ a non-smooth point of the equation being solved!

Example 1. Let us consider the simple example from [9, Section 4] modified to the quasi-static case (which is nothing else than a special case of the example with Coulomb's law

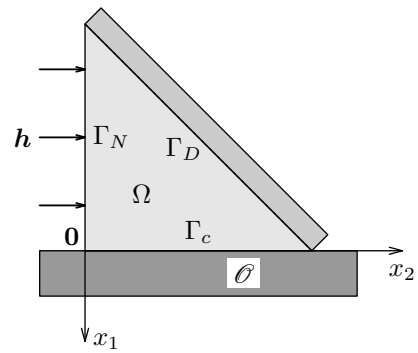


Figure 3: Geometry from Example 1.

of friction in [13]): contact of a single P_1 triangle with a flat foundation in the framework of linearised elasticity (Figure 3). The volume forces and the initial state are supposed to be null, the triangle is fixed along Γ_D and the Dirichlet condition is prescribed directly in this example.

The problem can be formulated as follows (compare (3) and ($\mathscr{P}$)):

Find $u_1, u_2, \lambda_\nu, \lambda_\tau \colon [0,T] \to \mathbb{R}$ *such that* $(u_1(0), u_2(0), \lambda_\nu(0), \lambda_\tau(0)) = (0,0,0,0)$ *and for any* $t \in (0,T)$*, it holds that*

$$\begin{aligned}
\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - \begin{pmatrix} l_1(t) \\ l_2(t) \end{pmatrix} - \begin{pmatrix} \lambda_\nu(t) \\ \lambda_\tau(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\lambda_\nu(t) - (\lambda_\nu(t) - u_1(t))_- &= 0, \\
\lambda_\tau(t) - P_{[\mathscr{F}(\lambda_\nu(t)-u_1(t))_-, -\mathscr{F}(\lambda_\nu(t)-u_1(t))_-]}(\lambda_\tau(t) - \dot{u}_2(t)) &= 0,
\end{aligned}$$

where $a := (\lambda + 3\mu)/2$ and $b := (\lambda + \mu)/2$ are given by Lamé constants λ and μ. The loading is considered to be $(l_1(t), l_2(t)) = (0, t)$.

According to the analysis in [13], the unique solution of this problem is

$$\begin{aligned}
u_1(t) = 0, \quad u_2(t) &= \frac{l_2(t) - \mathscr{F} l_1(t)}{a + \mathscr{F} b} = \frac{t}{a + \mathscr{F} b}, \\
\lambda_\nu(t) = -\frac{a l_1(t) + b l_2(t)}{a + \mathscr{F} b} &= -\frac{bt}{a + \mathscr{F} b}, \quad \lambda_\tau(t) = \mathscr{F} \lambda_\nu(t) = -\frac{\mathscr{F} bt}{a + \mathscr{F} b}
\end{aligned}$$

and it corresponds to slip of the only contact node to the right. The rate problem at each solution point for $t > 0$ is:

$$\left.\begin{aligned}
&\textit{Find } (\dot{u}_1, \dot{u}_2, \dot{\lambda}_\nu, \dot{\lambda}_\tau) \in \mathbb{R}^4 \textit{ such that} \\
&\qquad \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \dot{\lambda}_\nu \\ \dot{\lambda}_\tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&\qquad \dot{u}_1 = 0, \\
&\dot{u}_2 \geq 0, \quad -\dot{\lambda}_\tau + \mathscr{F} \dot{\lambda}_\nu \leq 0, \quad \dot{u}_2(-\dot{\lambda}_\tau + \mathscr{F} \dot{\lambda}_\nu) = 0,
\end{aligned}\right\} \tag{7}$$

that is, one has to deal with a complementarity between slip and stick at each time (in fact, the contact node would get stuck if the loading changed to unloading in the opposite direction).

On the other hand, solutions of the fully discretised (also called incremental) problems, which can be derived by analysing all possible cases similarly as in [9] for the static case, are:

$$u_1^k = 0, \quad u_2^k = \frac{l_2(t_k) - \mathscr{F} l_1(t_k)}{a + \mathscr{F} b} = \frac{t_k}{a + \mathscr{F} b},$$
$$\lambda_\nu^k = -\frac{a l_1(t_k) + b l_2(t_k)}{a + \mathscr{F} b} = -\frac{b t_k}{a + \mathscr{F} b}, \quad \lambda_\tau^k = \mathscr{F} \lambda_\nu^k = -\frac{\mathscr{F} b t_k}{a + \mathscr{F} b}$$

and the continuation problem proposed in this paper reads as follows:

$$\left.\begin{aligned} &\textit{Find } (\gamma, u_1, u_2, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^5 \textit{ such that} \\ &\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} L_1(\gamma) \\ L_2(\gamma) \end{pmatrix} - \begin{pmatrix} \lambda_\nu \\ \lambda_\tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &\lambda_\nu - (\lambda_\nu - u_1)_- = 0, \\ &\lambda_\tau - P_{[\mathscr{F}(\lambda_\nu - u_1)_-, -\mathscr{F}(\lambda_\nu - u_1)_-]}(\lambda_\tau - (u_2 - u_2^k) - (1-\gamma) V_\tau) = 0. \end{aligned}\right\} \tag{8}$$

Here, $(L_1(\gamma), L_2(\gamma)) = (0, t_k + \gamma(t_{k+1} - t_k))$ and

$$V_\tau = \frac{t_{k+1} - t_k}{t_k - t_{k-1}} (u_2^k - u_2^{k-1})$$

is positive for any $k \geq 1$. Solutions of this problem can be computed analogously as the solutions of the incremental problems and they form the set:

$$\Big\{ (\gamma, u_1, u_2, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^5;\ \gamma \in [0,1],\ u_1 = 0,\ u_2 = \frac{L_2(\gamma) - \mathscr{F} L_1(\gamma)}{a + \mathscr{F} b} = \frac{t_k + \gamma(t_{k+1} - t_k)}{a + \mathscr{F} b},$$
$$\lambda_\nu = -\frac{a L_1(\gamma) + b L_2(\gamma)}{a + \mathscr{F} b} = -\frac{b(t_k + \gamma(t_{k+1} - t_k))}{a + \mathscr{F} b},$$
$$\lambda_\tau = \mathscr{F} \lambda_\nu = -\frac{\mathscr{F} b (t_k + \gamma(t_{k+1} - t_k))}{a + \mathscr{F} b} \Big\}.$$

Then, the corresponding first-order system for any $\bar{\gamma} \in [0,1]$ becomes:

$$\begin{aligned} &\textit{Find } (\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5 \textit{ such that} \\ &\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} - \begin{pmatrix} 0 \\ \gamma'(t_{k+1} - t_k) \end{pmatrix} - \begin{pmatrix} \lambda_\nu' \\ \lambda_\tau' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &u_1' = 0, \\ &\lambda_\tau' = \mathscr{F} \lambda_\nu'. \end{aligned}$$

In comparison with the rate problem (7), the complementarity conditions are replaced by an equality here, which is apparently easier to handle.

Finally, solutions of the continuation problem without $(1-\gamma)V_\tau$ would remain the same as with it, however, the first-order system would contain the following frictional condition at $\bar{\gamma} = 0$:

$$u_2' \geq 0, \quad -\lambda_\tau' + \mathscr{F}\lambda_\nu' \leq 0, \quad u_2'(-\lambda_\tau' + \mathscr{F}\lambda_\nu') = 0.$$

Therefore, it would be more complicated to find a suitable initial tangent vector for numerical continuation.

Due to the similarity of the first-order system to the rate problem mentioned above, we shall follow steps in [17] and we shall reformulate it as a mixed complementarity inclusion and an explicit complementarity problem. To this end, we shall need the following assumption.

Assumption 2. *Let the matrix*

$$\boldsymbol{B} := \begin{pmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau \end{pmatrix}$$

have full row rank.

One can easily verify that this assumption is equivalent to the following inf-sup condition:

$$\exists \beta > 0\colon \quad \inf_{\boldsymbol{0} \neq (\boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{2(n_D+n_c)}} \sup_{\boldsymbol{0} \neq \boldsymbol{u} \in \mathbb{R}^{2n_\Omega}} \frac{\boldsymbol{u}^\top(\boldsymbol{B}_D^\top \boldsymbol{\lambda}_D + \boldsymbol{B}_\nu^\top \boldsymbol{\lambda}_\nu + \boldsymbol{B}_\tau^\top \boldsymbol{\lambda}_\tau)}{\|\boldsymbol{u}\| \|(\boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)\|} \geq \beta.$$

This is a reasonable assumption on the used spatial discretisation because it prevents the discretised problem from being over-constrained. If it is satisfied,

$$\dim \operatorname{Im}(\boldsymbol{B}^\top) = 2(n_D + n_c), \quad \dim \ker(\boldsymbol{B}) = 2n_\Omega - 2(n_D + n_c)$$

and one can define a non-singular matrix

$$\boldsymbol{T}_{\boldsymbol{B}} := \begin{pmatrix} \boldsymbol{N} & \boldsymbol{B}^+ \end{pmatrix},$$

where the columns of the matrix $\boldsymbol{N} \in \mathbb{R}^{2n_\Omega \times (2n_\Omega - 2(n_D+n_c))}$ form a basis of $\ker(\boldsymbol{B})$ and $\boldsymbol{B}^+ = \boldsymbol{B}^\top(\boldsymbol{B}\boldsymbol{B}^\top)^{-1}$ is the right inverse of $\boldsymbol{B}$. Then

$$\boldsymbol{T}_{\boldsymbol{B}}^\top \begin{pmatrix} \boldsymbol{B}_D^\top & \boldsymbol{B}_\nu^\top & \boldsymbol{B}_\tau^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_D' \\ \boldsymbol{\lambda}_\nu' \\ \boldsymbol{\lambda}_\tau' \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}\boldsymbol{N} & \boldsymbol{B}\boldsymbol{B}^+ \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\lambda}_D' \\ \boldsymbol{\lambda}_\nu' \\ \boldsymbol{\lambda}_\tau' \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{2n_\Omega - (2n_D + 2n_c)} \\ \boldsymbol{\lambda}_D' \\ \boldsymbol{\lambda}_\nu' \\ \boldsymbol{\lambda}_\tau' \end{pmatrix}. \tag{9}$$

Remark 3. For the nodal discretisation of contact conditions considered in this paper, the rows of $\left(\begin{smallmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau \end{smallmatrix}\right)$ are always linearly independent and they are orthogonal to the rows of

$\boldsymbol{B}_D$. Therefore, Assumption 2 is satisfied provided that $\boldsymbol{B}_D$ has full row rank, which is equivalent to the inf-sup condition on the Dirichlet boundary solely:

$$\exists \beta > 0\colon \quad \inf_{\mathbf{0} \neq \boldsymbol{\lambda}_D \in \mathbb{R}^{2n_D}} \sup_{\mathbf{0} \neq \boldsymbol{u} \in \mathbb{R}^{2n_\Omega}} \frac{\boldsymbol{u}^\top \boldsymbol{B}_D^\top \boldsymbol{\lambda}_D}{\|\boldsymbol{u}\| \|\boldsymbol{\lambda}_D\|} \geq \beta.$$

If it is the case,

$$\boldsymbol{T}_{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{N} & \boldsymbol{B}_D^+ & \left(\begin{smallmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau \end{smallmatrix}\right)^+ \end{pmatrix}$$

and the columns of $\boldsymbol{N}$ and $\left(\begin{smallmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau \end{smallmatrix}\right)^+$ form a basis of $\ker(\boldsymbol{B}_D)$.

Similarly, we shall need the following additional assumption.

Assumption 3. *Let the matrix*

$$\boldsymbol{C} := \begin{pmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_\nu \\ \boldsymbol{C}_\tau \end{pmatrix}$$

have full row rank.

Let us note that if the rigid obstacle is plane, that is, $\nabla^2 g = \mathbf{0}$, then $\boldsymbol{C}_\tau = \boldsymbol{B}_\tau$ and this assumption is always satisfied provided that Assumption 2 is fulfilled. Under Assumption 3, one can introduce a non-singular transformation matrix

$$\boldsymbol{T}_{\boldsymbol{C}} := \begin{pmatrix} \boldsymbol{N} & \boldsymbol{C}^+ \end{pmatrix}$$

with the columns of $\boldsymbol{N}$ constituting a basis of $\ker(\boldsymbol{C})$ and $\boldsymbol{C}^+ = \boldsymbol{C}^\top (\boldsymbol{C}\boldsymbol{C}^\top)^{-1}$ (invoking the introduction of $\boldsymbol{C}$, one can easily verify that $\boldsymbol{N}$ may be taken the same as above). Consequently, any displacement direction $\boldsymbol{u}' \in \mathbb{R}^{2n_\Omega}$ in the first-order system can be represented by a vector $\tilde{\boldsymbol{u}}'$ of the same length in the following way:

$$\boldsymbol{u}' = \boldsymbol{T}_{\boldsymbol{C}} \tilde{\boldsymbol{u}}', \quad \tilde{\boldsymbol{u}}' = \begin{pmatrix} \boldsymbol{u}'_{\mathrm{u}} \\ \boldsymbol{u}'_D \\ \boldsymbol{u}'_\nu \\ \boldsymbol{u}'_\tau \end{pmatrix}, \tag{10}$$

where $\boldsymbol{u}'_D \in \mathbb{R}^{2n_D}$ is a part imposed to the first-order Dirichlet condition, $\boldsymbol{u}'_\nu, \boldsymbol{u}'_\tau \in \mathbb{R}^{n_c}$ correspond to the normal and tangential components on the contact zone and $\boldsymbol{u}'_{\mathrm{u}} \in \mathbb{R}^{2n_\Omega - 2(n_D + n_c)}$ is free of the first-order Dirichlet and contact conditions (unconstrained). In fact,

$$\begin{pmatrix} \boldsymbol{B}_D \boldsymbol{u}' \\ \boldsymbol{B}_\nu \boldsymbol{u}' \\ \boldsymbol{C}_\tau \boldsymbol{u}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_\nu \\ \boldsymbol{C}_\tau \end{pmatrix} \begin{pmatrix} \boldsymbol{N} & \boldsymbol{C}^+ \end{pmatrix} \tilde{\boldsymbol{u}}' = \begin{pmatrix} \boldsymbol{u}'_D \\ \boldsymbol{u}'_\nu \\ \boldsymbol{u}'_\tau \end{pmatrix}. \tag{11}$$

Making use of (9) and (10) and introducing further partitioning of $\tilde{\boldsymbol{u}}'$, $\boldsymbol{\lambda}'_\nu$ and $\boldsymbol{\lambda}'_\tau$, we can write ($\mathscr{P}'1$) as

$$\begin{pmatrix} \tilde{\boldsymbol{K}}_{\mathrm{u,u}} & \tilde{\boldsymbol{K}}_{\mathrm{u,G}} & \tilde{\boldsymbol{K}}_{\mathrm{u,N}} & \tilde{\boldsymbol{K}}_{\mathrm{u,Z}} & \tilde{\boldsymbol{K}}_{\mathrm{u},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{u},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\mathrm{u},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{u,ci}} \\ \tilde{\boldsymbol{K}}_{\mathrm{G,u}} & \tilde{\boldsymbol{K}}_{\mathrm{G,G}} & \tilde{\boldsymbol{K}}_{\mathrm{G,N}} & \tilde{\boldsymbol{K}}_{\mathrm{G,Z}} & \tilde{\boldsymbol{K}}_{\mathrm{G},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{G},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\mathrm{G},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{G,ci}} \\ \tilde{\boldsymbol{K}}_{\mathrm{N,u}} & \tilde{\boldsymbol{K}}_{\mathrm{N,G}} & \tilde{\boldsymbol{K}}_{\mathrm{N,N}} & \tilde{\boldsymbol{K}}_{\mathrm{N,Z}} & \tilde{\boldsymbol{K}}_{\mathrm{N},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{N},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\mathrm{N},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{N,ci}} \\ \tilde{\boldsymbol{K}}_{\mathrm{Z,u}} & \tilde{\boldsymbol{K}}_{\mathrm{Z,G}} & \tilde{\boldsymbol{K}}_{\mathrm{Z,N}} & \tilde{\boldsymbol{K}}_{\mathrm{Z,Z}} & \tilde{\boldsymbol{K}}_{\mathrm{Z},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{Z},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\mathrm{Z},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{Z,ci}} \\ \tilde{\boldsymbol{K}}_{\nu\mathrm{zl,u}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl,G}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl,N}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl,Z}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\nu\mathrm{zl,ci}} \\ \tilde{\boldsymbol{K}}_{\tau\mathrm{cl,u}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl,G}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl,N}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl,Z}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{cl,ci}} \\ \tilde{\boldsymbol{K}}_{\tau\mathrm{zl,u}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl,G}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl,N}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl,Z}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\tau\mathrm{zl,ci}} \\ \tilde{\boldsymbol{K}}_{\mathrm{ci,u}} & \tilde{\boldsymbol{K}}_{\mathrm{ci,G}} & \tilde{\boldsymbol{K}}_{\mathrm{ci,N}} & \tilde{\boldsymbol{K}}_{\mathrm{ci,Z}} & \tilde{\boldsymbol{K}}_{\mathrm{ci},\nu\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{ci},\tau\mathrm{cl}} & \tilde{\boldsymbol{K}}_{\mathrm{ci},\tau\mathrm{zl}} & \tilde{\boldsymbol{K}}_{\mathrm{ci,ci}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}'_{\mathrm{u}} \\ \boldsymbol{u}'_{\mathrm{G}} \\ \boldsymbol{u}'_{\mathrm{N}} \\ \boldsymbol{u}'_{\mathrm{Z}} \\ \boldsymbol{u}'_{\nu\mathrm{zl}} \\ \boldsymbol{u}'_{\tau\mathrm{cl}} \\ \boldsymbol{u}'_{\tau\mathrm{zl}} \\ \boldsymbol{u}'_{\mathrm{ci}} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\lambda}'_{\mathrm{G}} \\ \boldsymbol{\lambda}'_{\mathrm{N}} \\ \boldsymbol{\lambda}'_{\mathrm{Z}} \\ \boldsymbol{\lambda}'_{\nu\mathrm{zl}} \\ \boldsymbol{\lambda}'_{\tau\mathrm{cl}} \\ \boldsymbol{\lambda}'_{\tau\mathrm{zl}} \\ \boldsymbol{\lambda}'_{\mathrm{ci}} \end{pmatrix} + \gamma' \begin{pmatrix} \tilde{\boldsymbol{P}}_{\mathrm{u}} \\ \tilde{\boldsymbol{P}}_{\mathrm{G}} \\ \tilde{\boldsymbol{P}}_{\mathrm{N}} \\ \tilde{\boldsymbol{P}}_{\mathrm{Z}} \\ \tilde{\boldsymbol{P}}_{\nu\mathrm{zl}} \\ \tilde{\boldsymbol{P}}_{\tau\mathrm{cl}} \\ \tilde{\boldsymbol{P}}_{\tau\mathrm{zl}} \\ \tilde{\boldsymbol{P}}_{\mathrm{ci}} \end{pmatrix}, \quad (12)$$

where $\tilde{\boldsymbol{K}} = \boldsymbol{T}_{\boldsymbol{B}}^\top \boldsymbol{K} \boldsymbol{T}_{\boldsymbol{C}}$, $\tilde{\boldsymbol{P}} = \boldsymbol{T}_{\boldsymbol{B}}^\top \boldsymbol{P}$, G denotes jointly the indices D, νs, νc0, νcl and τs, N the indices νf, τf, τc0 and τz0, ci the indices νci and τci, Z the indices νz0, νzi and τzi, and s stands for the indices from I_{s}, c0 for the indices from $I_{\mathrm{c}} \cap I_0$, cl for the indices from $I_{\mathrm{c}} \cap I_{\mathrm{l}}$ and so forth.

The values of the components of $\boldsymbol{u}'_{\mathrm{G}}$ and $\boldsymbol{u}'_{\nu\mathrm{ci}}$ are given by ($\mathscr{P}'2$), ($\mathscr{P}'3$') and ($\mathscr{P}'4$') combined with (11):

$$\boldsymbol{u}'_D = \gamma' \boldsymbol{U}'_D, \quad \boldsymbol{u}'_{\nu s} = \boldsymbol{0}, \quad \boldsymbol{u}'_{\nu\mathrm{c0}} = \boldsymbol{0}, \quad \boldsymbol{u}'_{\nu\mathrm{cl}} = \boldsymbol{0}, \quad \boldsymbol{u}'_{\tau\mathrm{s}} = \gamma' \boldsymbol{V}_{\tau\mathrm{s}}, \quad \boldsymbol{u}'_{\nu\mathrm{ci}} = \boldsymbol{0}$$

and $\boldsymbol{\lambda}'_{\mathrm{G}}$ can be expressed directly from (12). In particular,

$$\begin{aligned} \boldsymbol{\lambda}'_{\nu\mathrm{cl}} =& \tilde{\boldsymbol{K}}_{\nu\mathrm{cl,u}} \boldsymbol{u}'_{\mathrm{u}} + \tilde{\boldsymbol{K}}_{\nu\mathrm{cl,N}} \boldsymbol{u}'_{\mathrm{N}} + \tilde{\boldsymbol{K}}_{\nu\mathrm{cl,Z}} \boldsymbol{u}'_{\mathrm{Z}} + \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\nu\mathrm{zl}} \boldsymbol{u}'_{\nu\mathrm{zl}} + \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\tau\mathrm{cl}} \boldsymbol{u}'_{\tau\mathrm{cl}} + \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\tau\mathrm{zl}} \boldsymbol{u}'_{\tau\mathrm{zl}} \\ &+ \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\tau\mathrm{ci}} \boldsymbol{u}'_{\tau\mathrm{ci}} - \gamma' (\tilde{\boldsymbol{P}}_{\nu\mathrm{cl}} - \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},D} \boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\tau\mathrm{s}} \boldsymbol{V}_{\tau\mathrm{s}}). \end{aligned}$$

Further, one has from ($\mathscr{P}'3$') and ($\mathscr{P}'4$') that

$$\boldsymbol{\lambda}'_{\mathrm{N}} = \boldsymbol{0}, \quad \boldsymbol{\lambda}'_{\tau\mathrm{cl}} = \boldsymbol{S}_{\mathrm{cl}} \boldsymbol{F}_{\mathrm{cl}} \boldsymbol{\lambda}'_{\nu\mathrm{cl}}, \quad \boldsymbol{\lambda}'_{\tau\mathrm{zl}} = \boldsymbol{S}_{\mathrm{zl}} \boldsymbol{F}_{\mathrm{zl}} \boldsymbol{\lambda}'_{\nu\mathrm{zl}},$$

where

$$\begin{aligned} \boldsymbol{S}_{\mathrm{cl}} &= \mathbf{Diag}(\mathrm{sgn}(\bar{v}_{\tau,i} + (1-\bar{\gamma}) V_{\tau,i})), \ i \in I_{\mathrm{c}} \cap I_{\mathrm{l}}, & \boldsymbol{F}_{\mathrm{cl}} &= \mathbf{Diag}(\mathscr{F}_i), \ i \in I_{\mathrm{c}} \cap I_{\mathrm{l}}, \\ \boldsymbol{S}_{\mathrm{zl}} &= \mathbf{Diag}(\mathrm{sgn}(\bar{v}_{\tau,i} + (1-\bar{\gamma}) V_{\tau,i})), \ i \in I_{\mathrm{z}} \cap I_{\mathrm{l}}, & \boldsymbol{F}_{\mathrm{zl}} &= \mathbf{Diag}(\mathscr{F}_i), \ i \in I_{\mathrm{z}} \cap I_{\mathrm{l}}. \end{aligned}$$

Hence, (12) can be transformed into

$$\begin{pmatrix} \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},\tau\mathrm{ci}} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,u}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,N}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,Z}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\nu\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\tau\mathrm{cl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\tau\mathrm{zl}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\nu\mathrm{ci}} & \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\tau\mathrm{ci}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}'_{\mathrm{u}} \\ \boldsymbol{u}'_{\mathrm{N}} \\ \boldsymbol{u}'_{\mathrm{Z}} \\ \boldsymbol{u}'_{\nu\mathrm{zl}} \\ \boldsymbol{u}'_{\tau\mathrm{cl}} \\ \boldsymbol{u}'_{\tau\mathrm{zl}} \\ \boldsymbol{0} \\ \boldsymbol{u}'_{\tau\mathrm{ci}} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{\lambda}'_{\mathrm{Z}} \\ \boldsymbol{\lambda}'_{\nu\mathrm{zl}} \\ \boldsymbol{0} \\ \boldsymbol{S}_{\mathrm{zl}}\boldsymbol{F}_{\mathrm{zl}}\boldsymbol{\lambda}'_{\nu\mathrm{zl}} \\ \boldsymbol{\lambda}'_{\mathrm{ci}} \end{pmatrix} + \gamma' \begin{pmatrix} \tilde{\tilde{\boldsymbol{P}}}_{\mathrm{u}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\mathrm{N}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\mathrm{Z}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\nu\mathrm{zl}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\tau\mathrm{cl}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\tau\mathrm{zl}} \\ \tilde{\tilde{\boldsymbol{P}}}_{\mathrm{ci}} \end{pmatrix} \tag{13}$$

with

$$\begin{pmatrix} \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{u},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{N},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{Z},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{zl},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{cl},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{zl},.} \\ \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},.} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{K}}_{\mathrm{u},.} \\ \tilde{\boldsymbol{K}}_{\mathrm{N},.} \\ \tilde{\boldsymbol{K}}_{\mathrm{Z},.} \\ \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},.} \\ \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},.} - \boldsymbol{S}_{\mathrm{cl}}\boldsymbol{F}_{\mathrm{cl}}\tilde{\boldsymbol{K}}_{\nu\mathrm{cl},.} \\ \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},.} \\ \tilde{\boldsymbol{K}}_{\mathrm{ci},.} \end{pmatrix},$$

$$\tilde{\tilde{\boldsymbol{P}}} = \begin{pmatrix} \tilde{\boldsymbol{P}}_{\mathrm{u}} - \tilde{\boldsymbol{K}}_{\mathrm{u},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\mathrm{u},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \\ \tilde{\boldsymbol{P}}_{\mathrm{N}} - \tilde{\boldsymbol{K}}_{\mathrm{N},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\mathrm{N},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \\ \tilde{\boldsymbol{P}}_{\mathrm{Z}} - \tilde{\boldsymbol{K}}_{\mathrm{Z},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\mathrm{Z},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \\ \tilde{\boldsymbol{P}}_{\nu\mathrm{zl}} - \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\nu\mathrm{zl},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \\ \tilde{\boldsymbol{P}}_{\tau\mathrm{cl}} - \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\tau\mathrm{cl},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} - \boldsymbol{S}_{\mathrm{cl}}\boldsymbol{F}_{\mathrm{cl}}(\tilde{\boldsymbol{P}}_{\nu\mathrm{cl}} - \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\nu\mathrm{cl},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}}) \\ \tilde{\boldsymbol{P}}_{\tau\mathrm{zl}} - \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\tau\mathrm{zl},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \\ \tilde{\boldsymbol{P}}_{\mathrm{ci}} - \tilde{\boldsymbol{K}}_{\mathrm{ci},D}\boldsymbol{U}'_D - \tilde{\boldsymbol{K}}_{\mathrm{ci},\tau\mathrm{s}}\boldsymbol{V}_{\tau\mathrm{s}} \end{pmatrix}.$$

Denoting the indices u, N, Z, νzl, τcl and τzl jointly by E, we want to eliminate $\boldsymbol{u}'_{\mathrm{E}}$ now. This leads us to the following assumption.

Assumption 4. *Let the matrix $\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E,E}}$ be invertible.*

As we shall show in the proof of Corollary 1, this holds, for example, for a model of linearised elasticity with a properly constrained body in contact with a plane rigid obstacle provided that the coefficient of friction is sufficiently small. The assumption enables us to express $\boldsymbol{u}'_{\mathrm{E}}$ from the first six rows of (13):

$$\boldsymbol{u}'_{\mathrm{E}} = \boldsymbol{E}_{\mathrm{E,Z}}\boldsymbol{\lambda}'_{\mathrm{Z}} + \bar{\boldsymbol{E}}_{\mathrm{E},\nu\mathrm{zl}}\boldsymbol{\lambda}'_{\nu\mathrm{zl}} + \gamma'\boldsymbol{E}_{\mathrm{E,E}}\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}} - \boldsymbol{E}_{\mathrm{E,E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{u}'_{\tau\mathrm{ci}}, \tag{14}$$

where

$$\boldsymbol{E}_{\mathrm{E,E}} = \tilde{\tilde{\boldsymbol{K}}}^{-1}_{\mathrm{E,E}}, \quad \bar{\boldsymbol{E}}_{\mathrm{E},\nu\mathrm{zl}} = \boldsymbol{E}_{\mathrm{E},\nu\mathrm{zl}} + \boldsymbol{E}_{\mathrm{E},\tau\mathrm{zl}}\boldsymbol{S}_{\mathrm{zl}}\boldsymbol{F}_{\mathrm{zl}}.$$

Then, substituting for it into the last row of (13), we obtain

$$\boldsymbol{\lambda}'_{\mathrm{ci}} = -\gamma'\bar{\boldsymbol{P}}_{\mathrm{ci}} + \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,E}}\boldsymbol{E}_{\mathrm{E,Z}}\boldsymbol{\lambda}'_{\mathrm{Z}} + \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,E}}\bar{\boldsymbol{E}}_{\mathrm{E},\nu\mathrm{zl}}\boldsymbol{\lambda}'_{\nu\mathrm{zl}} + \bar{\boldsymbol{K}}_{\mathrm{ci},\tau\mathrm{ci}}\boldsymbol{u}'_{\tau\mathrm{ci}}$$

with

$$\bar{\boldsymbol{P}}_{\mathrm{ci}} = \tilde{\tilde{\boldsymbol{P}}}_{\mathrm{ci}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,E}}\boldsymbol{E}_{\mathrm{E,E}}\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}}, \quad \bar{\boldsymbol{K}}_{\mathrm{ci},\tau\mathrm{ci}} = \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci},\tau\mathrm{ci}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{ci,E}}\boldsymbol{E}_{\mathrm{E,E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}.$$

From here, we have

$$\begin{aligned}&\boldsymbol{S}_{\mathrm{ci}}\boldsymbol{\lambda}'_{\tau\mathrm{ci}} - \boldsymbol{F}_{\mathrm{ci}}\boldsymbol{\lambda}'_{\nu\mathrm{ci}}\\ &\quad = -\gamma'\bar{\bar{\boldsymbol{P}}}_{\tau\mathrm{ci}} + \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\tau\mathrm{ci}}\boldsymbol{S}^2_{\mathrm{ci}}(\boldsymbol{u}'_{\tau\mathrm{ci}} - \gamma'\boldsymbol{V}_{\tau\mathrm{ci}}) + \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci,E}}\boldsymbol{E}_{\mathrm{E,Z}}\boldsymbol{\lambda}'_{\mathrm{Z}} + \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci,E}}\bar{\boldsymbol{E}}_{\mathrm{E},\nu\mathrm{zl}}\boldsymbol{\lambda}'_{\nu\mathrm{zl}},\end{aligned} \tag{15}$$

where

$$\boldsymbol{S}_{\mathrm{ci}} = -\,\mathbf{Diag}(\mathrm{sgn}(\bar{\lambda}_{\tau,i})),\ i \in I_{\mathrm{c}} \cap I_{\mathrm{i}}, \quad \boldsymbol{F}_{\mathrm{ci}} = \mathbf{Diag}(\mathscr{F}_i),\ i \in I_{\mathrm{c}} \cap I_{\mathrm{i}},$$
$$\bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\tau\mathrm{ci}} = \boldsymbol{S}_{\mathrm{ci}}\bar{\boldsymbol{K}}_{\tau\mathrm{ci},\tau\mathrm{ci}} - \boldsymbol{F}_{\mathrm{ci}}\bar{\boldsymbol{K}}_{\nu\mathrm{ci},\tau\mathrm{ci}}, \quad \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci,E}} = \boldsymbol{S}_{\mathrm{ci}}\tilde{\tilde{\boldsymbol{K}}}_{\tau\mathrm{ci,E}} - \boldsymbol{F}_{\mathrm{ci}}\tilde{\tilde{\boldsymbol{K}}}_{\nu\mathrm{ci,E}},$$
$$\bar{\bar{\boldsymbol{P}}}_{\tau\mathrm{ci}} = \boldsymbol{S}_{\mathrm{ci}}(\bar{\boldsymbol{P}}_{\tau\mathrm{ci}} - \bar{\boldsymbol{K}}_{\tau\mathrm{ci},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}) - \boldsymbol{F}_{\mathrm{ci}}(\bar{\boldsymbol{P}}_{\nu\mathrm{ci}} - \bar{\boldsymbol{K}}_{\nu\mathrm{ci},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}).$$

Taking (15), (14) with the indices from νz0, νzl, νzi and τzi, and the remaining conditions from ($\mathscr{P}'$3') and ($\mathscr{P}'$4'), we arrive at the mixed complementarity inclusion:

$$\left.\begin{aligned}&\textit{Find } (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^M \times \mathbb{R}^M \textit{ such that}\\ &\boldsymbol{D}\boldsymbol{x} = \boldsymbol{y} + \gamma'\boldsymbol{b},\\ &\boldsymbol{\zeta}_{\tau\mathrm{ci}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\tau\mathrm{ci}} \geq \mathbf{0},\ \boldsymbol{\psi}^{\top}_{\tau\mathrm{ci}}\boldsymbol{\zeta}_{\tau\mathrm{ci}} = 0,\\ &\boldsymbol{\zeta}_{\nu\mathrm{z0}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{z0}} \geq \mathbf{0},\ \boldsymbol{\psi}^{\top}_{\nu\mathrm{z0}}\boldsymbol{\zeta}_{\nu\mathrm{z0}} = 0,\\ &\boldsymbol{\zeta}_{\nu\mathrm{zl}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{zl}} \geq \mathbf{0},\ \boldsymbol{\psi}^{\top}_{\nu\mathrm{zl}}\boldsymbol{\zeta}_{\nu\mathrm{zl}} = 0,\\ &\boldsymbol{\zeta}_{\nu\mathrm{zi}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{zi}} \geq \mathbf{0},\ \boldsymbol{\psi}^{\top}_{\nu\mathrm{zi}}\boldsymbol{\zeta}_{\nu\mathrm{zi}} = 0,\\ &\boldsymbol{\psi}_{\tau\mathrm{zi}} \in -\boldsymbol{S}_{\mathrm{zi}}(\boldsymbol{\zeta}_{\tau\mathrm{zi}})\boldsymbol{F}_{\mathrm{zi}}\boldsymbol{\psi}_{\nu\mathrm{zi}},\end{aligned}\right\} \tag{16}$$

where $M = n_{\mathrm{ci}} + n_{\mathrm{z0}} + n_{\mathrm{zl}} + 2n_{\mathrm{zi}}$ with $n_{\mathrm{ci}} = \#(I_{\mathrm{c}} \cap I_{\mathrm{i}})$, $n_{\mathrm{z0}} = \#(I_{\mathrm{z}} \cap I_0)$, $n_{\mathrm{zl}} = \#(I_{\mathrm{z}} \cap I_{\mathrm{l}})$ and $n_{\mathrm{zi}} = \#(I_{\mathrm{z}} \cap I_{\mathrm{i}})$,

$$\boldsymbol{x} = \begin{pmatrix}\boldsymbol{\zeta}_{\tau\mathrm{ci}}\\ \boldsymbol{\psi}_{\nu\mathrm{z0}}\\ \boldsymbol{\psi}_{\nu\mathrm{zl}}\\ \boldsymbol{\psi}_{\nu\mathrm{zi}}\\ \boldsymbol{\psi}_{\tau\mathrm{zi}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{S}_{\mathrm{ci}}(\boldsymbol{u}'_{\tau\mathrm{ci}} - \gamma'\boldsymbol{V}_{\tau\mathrm{ci}})\\ -\boldsymbol{\lambda}'_{\nu\mathrm{z0}}\\ -\boldsymbol{\lambda}'_{\nu\mathrm{zl}}\\ -\boldsymbol{\lambda}'_{\nu\mathrm{zi}}\\ \boldsymbol{\lambda}'_{\tau\mathrm{zi}}\end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix}\boldsymbol{\psi}_{\tau\mathrm{ci}}\\ \boldsymbol{\zeta}_{\nu\mathrm{z0}}\\ \boldsymbol{\zeta}_{\nu\mathrm{zl}}\\ \boldsymbol{\zeta}_{\nu\mathrm{zi}}\\ \boldsymbol{\zeta}_{\tau\mathrm{zi}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{S}_{\mathrm{ci}}\boldsymbol{\lambda}'_{\tau\mathrm{ci}} - \boldsymbol{F}_{\mathrm{ci}}\boldsymbol{\lambda}'_{\nu\mathrm{ci}}\\ -\boldsymbol{u}'_{\nu\mathrm{z0}}\\ -\boldsymbol{u}'_{\nu\mathrm{zl}}\\ -\boldsymbol{u}'_{\nu\mathrm{zi}}\\ \boldsymbol{u}'_{\tau\mathrm{zi}} - \gamma'\boldsymbol{V}_{\tau\mathrm{zi}}\end{pmatrix},$$

$$
\boldsymbol{D} = \begin{pmatrix} \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\tau\mathrm{ci}}\boldsymbol{S}_{\mathrm{ci}} & -\bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\mathrm{E}}\boldsymbol{E}_{\mathrm{E},\nu\mathrm{z}0} & -\bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\mathrm{E}}\bar{\boldsymbol{E}}_{\mathrm{E},\nu\mathrm{zl}} & -\bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\mathrm{E}}\boldsymbol{E}_{\mathrm{E},\nu\mathrm{zi}} & \bar{\bar{\boldsymbol{K}}}_{\tau\mathrm{ci},\mathrm{E}}\boldsymbol{E}_{\mathrm{E},\tau\mathrm{zi}} \\ \boldsymbol{E}_{\nu\mathrm{z}0,\mathrm{E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{S}_{\mathrm{ci}} & \boldsymbol{E}_{\nu\mathrm{z}0,\nu\mathrm{z}0} & \bar{\boldsymbol{E}}_{\nu\mathrm{z}0,\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{z}0,\nu\mathrm{zi}} & -\boldsymbol{E}_{\nu\mathrm{z}0,\tau\mathrm{zi}} \\ \boldsymbol{E}_{\nu\mathrm{zl},\mathrm{E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{S}_{\mathrm{ci}} & \boldsymbol{E}_{\nu\mathrm{zl},\nu\mathrm{z}0} & \bar{\boldsymbol{E}}_{\nu\mathrm{zl},\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{zl},\nu\mathrm{zi}} & -\boldsymbol{E}_{\nu\mathrm{zl},\tau\mathrm{zi}} \\ \boldsymbol{E}_{\nu\mathrm{zi},\mathrm{E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{S}_{\mathrm{ci}} & \boldsymbol{E}_{\nu\mathrm{zi},\nu\mathrm{z}0} & \bar{\boldsymbol{E}}_{\nu\mathrm{zi},\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{zi},\nu\mathrm{zi}} & -\boldsymbol{E}_{\nu\mathrm{zi},\tau\mathrm{zi}} \\ -\boldsymbol{E}_{\tau\mathrm{zi},\mathrm{E}}\tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{S}_{\mathrm{ci}} & -\boldsymbol{E}_{\tau\mathrm{zi},\nu\mathrm{z}0} & -\bar{\boldsymbol{E}}_{\tau\mathrm{zi},\nu\mathrm{zl}} & -\boldsymbol{E}_{\tau\mathrm{zi},\nu\mathrm{zi}} & \boldsymbol{E}_{\tau\mathrm{zi},\tau\mathrm{zi}} \end{pmatrix},
$$

$$
\boldsymbol{b} = \begin{pmatrix} \bar{\bar{\boldsymbol{P}}}_{\tau\mathrm{ci}} \\ \boldsymbol{E}_{\nu\mathrm{z}0,\mathrm{E}}(\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}) \\ \boldsymbol{E}_{\nu\mathrm{zl},\mathrm{E}}(\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}) \\ \boldsymbol{E}_{\nu\mathrm{zi},\mathrm{E}}(\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}) \\ -\boldsymbol{E}_{\tau\mathrm{zi},\mathrm{E}}(\tilde{\tilde{\boldsymbol{P}}}_{\mathrm{E}} - \tilde{\tilde{\boldsymbol{K}}}_{\mathrm{E},\tau\mathrm{ci}}\boldsymbol{V}_{\tau\mathrm{ci}}) + \boldsymbol{V}_{\tau\mathrm{zi}} \end{pmatrix},
$$

$$
\boldsymbol{F}_{\mathrm{zi}} = \mathbf{Diag}(\mathscr{F}_i),\ i \in I_{\mathrm{z}} \cap I_{\mathrm{i}},
$$

and $\boldsymbol{S}_{\mathrm{zi}} \colon \mathbb{R}^{n_{\mathrm{zi}}} \rightrightarrows \mathbb{R}^{n_{\mathrm{zi}} \times n_{\mathrm{zi}}}$ is a matrix-set-valued map defined by

$$
\boldsymbol{S}_{\mathrm{zi}}(\boldsymbol{\zeta}_{\tau\mathrm{zi}}) = \mathbf{Diag}(\mathrm{Sgn}((\boldsymbol{\zeta}_{\tau\mathrm{zi}})_1), \ldots, \mathrm{Sgn}((\boldsymbol{\zeta}_{\tau\mathrm{zi}})_{n_{\mathrm{zi}}}).
$$

It is readily seen from the construction that there is a one-to-one correspondence between solutions of the first-order system ($\mathscr{P}'1$), ($\mathscr{P}'2$), ($\mathscr{P}'3$') and ($\mathscr{P}'4$') and of (16) for $\gamma' \in \mathbb{R}$ given.

To transform (16) into the announced explicit complementarity problem, we start with an easy observation.

Lemma 2. *For any $a, c \in \mathbb{R}$ and any $b, k \geq 0$, the following two conditions are equivalent:*

$$
\begin{aligned}
&\text{(i)}\ a \in -kb\,\mathrm{Sgn}(c);\\
&\text{(ii)}\ -|a| + kb \geq 0\ \&\ ac + kb|c| = 0.
\end{aligned}
$$

Next, we define closed convex cones of admissible values of $\boldsymbol{x}$ and $\boldsymbol{y}$ from (16) (all inequalities have to be understood component-wise):

$$
\begin{aligned}
\mathscr{C}_{\boldsymbol{x}} := \{&\boldsymbol{x} = (\boldsymbol{\zeta}_{\tau\mathrm{ci}}, \boldsymbol{\psi}_{\nu\mathrm{z}0}, \boldsymbol{\psi}_{\nu\mathrm{zl}}, \boldsymbol{\psi}_{\nu\mathrm{zi}}, \boldsymbol{\psi}_{\tau\mathrm{zi}}) \in \mathbb{R}^M;\\
&\boldsymbol{\zeta}_{\tau\mathrm{ci}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{z}0} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{zl}} \geq \mathbf{0},\ \boldsymbol{\psi}_{\nu\mathrm{zi}} \geq \mathbf{0},\ -|\boldsymbol{\psi}_{\tau\mathrm{zi}}| + \boldsymbol{F}_{\mathrm{zi}}\boldsymbol{\psi}_{\nu\mathrm{zi}} \geq \mathbf{0}\},\\
\mathscr{C}_{\boldsymbol{y}} := \{&\boldsymbol{y} = (\boldsymbol{\psi}_{\tau\mathrm{ci}}, \boldsymbol{\zeta}_{\nu\mathrm{z}0}, \boldsymbol{\zeta}_{\nu\mathrm{zl}}, \boldsymbol{\zeta}_{\nu\mathrm{zi}}, \boldsymbol{\zeta}_{\tau\mathrm{zi}}) \in \mathbb{R}^M;\\
&\boldsymbol{\psi}_{\tau\mathrm{ci}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{z}0} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{zl}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{zi}} \geq \mathbf{0}, \boldsymbol{\zeta}_{\tau\mathrm{zi}} \in \mathbb{R}^{n_{\mathrm{zi}}}\}
\end{aligned}
$$

and a mapping $\boldsymbol{h} \colon \mathbb{R}^M \to \mathbb{R}^M$ by

$$
\boldsymbol{h}(\boldsymbol{\psi}_{\tau\mathrm{ci}}, \boldsymbol{\zeta}_{\nu\mathrm{z}0}, \boldsymbol{\zeta}_{\nu\mathrm{zl}}, \boldsymbol{\zeta}_{\nu\mathrm{zi}}, \boldsymbol{\zeta}_{\tau\mathrm{zi}}) = \begin{pmatrix} \boldsymbol{\psi}_{\tau\mathrm{ci}} \\ \boldsymbol{\zeta}_{\nu\mathrm{z}0} \\ \boldsymbol{\zeta}_{\nu\mathrm{zl}} \\ \boldsymbol{\zeta}_{\nu\mathrm{zi}} + \boldsymbol{F}_{\mathrm{zi}}|\boldsymbol{\zeta}_{\tau\mathrm{zi}}| \\ \boldsymbol{\zeta}_{\tau\mathrm{zi}} \end{pmatrix}.
$$

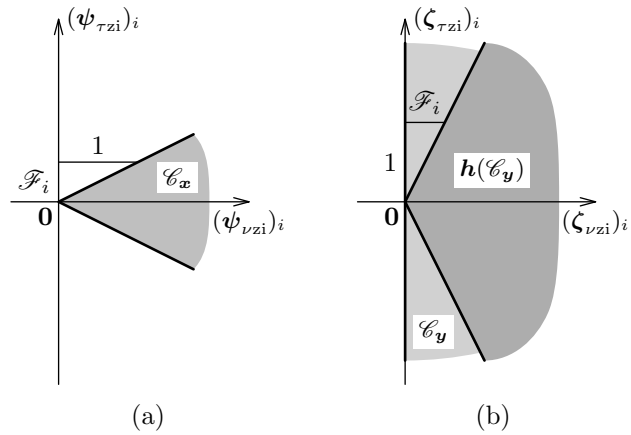


Figure 4: Components of the cones for the indices $i \in I_{\mathrm{z}} \cap I_{\mathrm{i}}$: (a) $\mathscr{C}_{\boldsymbol{x}}$; (b) $\mathscr{C}_{\boldsymbol{y}}$ and $\boldsymbol{h}(\mathscr{C}_{\boldsymbol{y}}) = (\mathscr{C}_{\boldsymbol{x}})^*$.

Clearly, these objects enjoy the properties summarised below (see also Figure 4). Recall that a dual cone $\mathscr{C}^*$ of a cone $\mathscr{C} \subset \mathbb{R}^M$ is introduced as

$$\mathscr{C}^* = \{\boldsymbol{y} \in \mathbb{R}^M;\ \boldsymbol{y}^\top \boldsymbol{x} \geq 0,\ \forall \boldsymbol{x} \in \mathscr{C}\}.$$

Lemma 3. *It holds that*
(i) $\boldsymbol{h}(\mathscr{C}_{\boldsymbol{y}}) = \{\boldsymbol{y} = (\boldsymbol{\psi}_{\tau\mathrm{ci}}, \boldsymbol{\zeta}_{\nu\mathrm{z0}}, \boldsymbol{\zeta}_{\nu\mathrm{zl}}, \boldsymbol{\zeta}_{\nu\mathrm{zi}}, \boldsymbol{\zeta}_{\tau\mathrm{zi}}) \in \mathbb{R}^M;$
$\boldsymbol{\psi}_{\tau\mathrm{ci}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{z0}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{zl}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{zi}} \geq \mathbf{0},\ \boldsymbol{\zeta}_{\nu\mathrm{zi}} - \boldsymbol{F}_{\mathrm{zi}}|\boldsymbol{\zeta}_{\tau\mathrm{zi}}| \geq \mathbf{0}\} = (\mathscr{C}_{\boldsymbol{x}})^*;$
(ii) $\boldsymbol{h}$ *is a bijection.*

Finally, we define a map $\boldsymbol{f}_{\gamma'} \colon \mathbb{R}^M \to \mathbb{R}^M$ by

$$\boldsymbol{f}_{\gamma'}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{D}\boldsymbol{x} - \gamma' \boldsymbol{b})$$

for any $\gamma' \in \mathbb{R}$ fixed.

Altogether, one can easily verify that there is a one-to-one correspondence between solutions of (16) and the following explicit complementarity problem:

$$\left.\begin{array}{l} \textit{Find } \boldsymbol{x} \in \mathscr{C}_{\boldsymbol{x}} \textit{ such that } \boldsymbol{f}_{\gamma'}(\boldsymbol{x}) \in (\mathscr{C}_{\boldsymbol{x}})^* \textit{ and} \\ \boldsymbol{x}^\top \boldsymbol{f}_{\gamma'}(\boldsymbol{x}) = 0. \end{array}\right\} \tag{17}$$

4.1 Results of Existence and Uniqueness

We are prepared to adapt the existence and uniqueness results from [17].

Proposition 3. *Let the matrix* $\boldsymbol{D}$ *be strictly co-positive in* $\mathscr{C}_{\boldsymbol{x}}$*, that is,*

$$\boldsymbol{x}^\top \boldsymbol{D}\boldsymbol{x} > 0, \quad \forall \boldsymbol{x} \in \mathscr{C}_{\boldsymbol{x}},\ \boldsymbol{x} \neq \mathbf{0}. \tag{18}$$

Then (17) *possesses a solution for any* $\gamma' \in \mathbb{R}$ *and any* $\boldsymbol{b} \in \mathbb{R}^M$.

Proof. The assertion follows directly from [10, Corollary 4.13]. Since $\mathscr{C}_{\boldsymbol{x}}$ is a closed convex cone and $\boldsymbol{f}_{\gamma'}$ is a bounded continuous finite-dimensional operator, it suffices to verify that $\boldsymbol{f}_{\gamma'}$ is coercive in the following sense:

$$\lim_{\substack{\|\boldsymbol{x}\|\to+\infty \\ \boldsymbol{x}\in\mathscr{C}_{\boldsymbol{x}}}} \frac{\boldsymbol{x}^\top \boldsymbol{f}_{\gamma'}(\boldsymbol{x})}{\|\boldsymbol{x}\|} = +\infty.$$

But this is ensured by (18) as it gives $D_0 > 0$ such that

$$\frac{\boldsymbol{x}^\top \boldsymbol{D}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} \geq D_0, \quad \forall \boldsymbol{x} \in \mathscr{C}_{\boldsymbol{x}},\ \boldsymbol{x} \neq \boldsymbol{0}.$$

Hence,

$$\forall \boldsymbol{x} \in \mathscr{C}_{\boldsymbol{x}},\ \boldsymbol{x} \neq \boldsymbol{0}\colon$$
$$\frac{\boldsymbol{x}^\top \boldsymbol{f}_{\gamma'}(\boldsymbol{x})}{\|\boldsymbol{x}\|} = \frac{1}{\|\boldsymbol{x}\|}\big(\boldsymbol{x}^\top(\boldsymbol{D}\boldsymbol{x} - \gamma'\boldsymbol{b}) + \boldsymbol{\psi}_{\nu\mathrm{zi}}^\top \boldsymbol{F}_{\mathrm{zi}}|(\boldsymbol{D}\boldsymbol{x} - \gamma'\boldsymbol{b})_{\tau\mathrm{zi}}|\big) \geq D_0\|\boldsymbol{x}\| - \|\gamma'\boldsymbol{b}\|,$$

where we have used the non-negativity of the elements of $\boldsymbol{\psi}_{\nu\mathrm{zi}}$ and $\boldsymbol{F}_{\mathrm{zi}}$, and $(\boldsymbol{D}\boldsymbol{x} - \gamma'\boldsymbol{b})_{\tau\mathrm{zi}}$ denotes the last sub-vector of $\boldsymbol{D}\boldsymbol{x} - \gamma'\boldsymbol{b}$. □

Before stating a consequence of this result, we recall the concept of positive definiteness of a matrix on a subspace: Let $\boldsymbol{M}$ be a square matrix of order n and V be an m-dimensional subspace of $\mathbb{R}^n$. We say that $\boldsymbol{M}$ is positive definite on V if $\boldsymbol{u}^\top \boldsymbol{M}\boldsymbol{u} > 0$ for any non-zero $\boldsymbol{u} \in V$, or equivalently, if the matrix $\boldsymbol{T}^\top \boldsymbol{M}\boldsymbol{T}$ is positive definite for any matrix $\boldsymbol{T} \in \mathbb{R}^{n\times m}$ whose columns form a basis of V.

Corollary 1 (Existence)**.** *Let* ($\mathscr{P}$) *correspond to a model of linearised elasticity where the body is properly constrained by the Dirichlet condition, that is, the rows of* $\boldsymbol{B}_D$ *are linearly independent and the stiffness matrix* $\boldsymbol{A}$ *is positive definite on* $\ker(\boldsymbol{B}_D)$*. If* g *satisfies Assumption 1 and* $\|\nabla^2 g\|/\|\nabla g\|$ *and* $\mathscr{F}_i$*,* $i \in ((I_{\mathrm{c}} \cup I_{\mathrm{z}}) \cap I_{\mathrm{l}}) \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}})$*, are small, then the first-order system* ($\mathscr{P}'1$)*,* ($\mathscr{P}'2$)*,* ($\mathscr{P}'3$') *and* ($\mathscr{P}'4$') *has at least one solution for any* $\gamma' \in \mathbb{R}$*,* $\boldsymbol{P} \in \mathbb{R}^{2n_\Omega}$ *and* $\boldsymbol{U}'_D \in \mathbb{R}^{2n_D}$*.*

Proof. Suppose for a while that the assumptions are satisfied even with $\nabla^2 g = \boldsymbol{0}$ and $\mathscr{F}_i = 0$ for $i \in ((I_{\mathrm{c}} \cup I_{\mathrm{z}}) \cap I_{\mathrm{l}}) \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}})$. Then $\boldsymbol{K} = \boldsymbol{A}$, $\boldsymbol{C}_\tau = \boldsymbol{B}_\tau$ and both Assumptions 2 and 3 hold according to Remark 3. Furthermore, $\tilde{\tilde{\boldsymbol{K}}}$ is a principal sub-matrix of $\boldsymbol{N}_D^\top \boldsymbol{K}\boldsymbol{N}_D$, where the columns of $\boldsymbol{N}_D$ are formed by the columns of $\boldsymbol{N}$ and $\left(\begin{smallmatrix}\boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau\end{smallmatrix}\right)^+$ constituting a basis of $\ker(\boldsymbol{B}_D)$. Thus, $\tilde{\tilde{\boldsymbol{K}}}$ is symmetric positive definite and so are $\tilde{\tilde{\boldsymbol{K}}}_{E,E}$ and its inverse $\boldsymbol{E}_{\mathrm{E,E}}$. Consequently, Assumption 4 is fulfilled, the first-order system can be transformed into (17) by the procedure described above and

$$\begin{aligned}\boldsymbol{x}^\top \boldsymbol{D}\boldsymbol{x} &= \boldsymbol{\zeta}_{\tau\mathrm{ci}}^\top \boldsymbol{S}_{\mathrm{ci}} \bar{\boldsymbol{K}}_{\tau\mathrm{ci},\tau\mathrm{ci}} \boldsymbol{S}_{\mathrm{ci}} \boldsymbol{\zeta}_{\tau\mathrm{ci}} \\ &\quad + \begin{pmatrix}\boldsymbol{\psi}_{\nu\mathrm{z0}}^\top & \boldsymbol{\psi}_{\nu\mathrm{zl}}^\top & \boldsymbol{\psi}_{\nu\mathrm{zi}}^\top & -\boldsymbol{\psi}_{\tau\mathrm{zi}}^\top\end{pmatrix} \begin{pmatrix} \boldsymbol{E}_{\nu\mathrm{z0},\nu\mathrm{z0}} & \boldsymbol{E}_{\nu\mathrm{z0},\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{z0},\nu\mathrm{zi}} & \boldsymbol{E}_{\nu\mathrm{z0},\tau\mathrm{zi}} \\ \boldsymbol{E}_{\nu\mathrm{zl},\nu\mathrm{z0}} & \boldsymbol{E}_{\nu\mathrm{zl},\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{zl},\nu\mathrm{zi}} & \boldsymbol{E}_{\nu\mathrm{zl},\tau\mathrm{zi}} \\ \boldsymbol{E}_{\nu\mathrm{zi},\nu\mathrm{z0}} & \boldsymbol{E}_{\nu\mathrm{zi},\nu\mathrm{zl}} & \boldsymbol{E}_{\nu\mathrm{zi},\nu\mathrm{zi}} & \boldsymbol{E}_{\nu\mathrm{zi},\tau\mathrm{zi}} \\ \boldsymbol{E}_{\tau\mathrm{zi},\nu\mathrm{z0}} & \boldsymbol{E}_{\tau\mathrm{zi},\nu\mathrm{zl}} & \boldsymbol{E}_{\tau\mathrm{zi},\nu\mathrm{zi}} & \boldsymbol{E}_{\tau\mathrm{zi},\tau\mathrm{zi}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_{\nu\mathrm{z0}} \\ \boldsymbol{\psi}_{\nu\mathrm{zl}} \\ \boldsymbol{\psi}_{\nu\mathrm{zi}} \\ -\boldsymbol{\psi}_{\tau\mathrm{zi}} \end{pmatrix}.\end{aligned}$$

The matrix $\boldsymbol{S}_{\rm ci}\bar{\boldsymbol{K}}_{\tau{\rm ci},\tau{\rm ci}}\boldsymbol{S}_{\rm ci} = \boldsymbol{S}_{\rm ci}^{-1}\bar{\boldsymbol{K}}_{\tau{\rm ci},\tau{\rm ci}}\boldsymbol{S}_{\rm ci}$ in the first term above is similar to $\bar{\boldsymbol{K}}_{\tau{\rm ci},\tau{\rm ci}} = \tilde{\tilde{\boldsymbol{K}}}_{\tau{\rm ci},\tau{\rm ci}} - \tilde{\tilde{\boldsymbol{K}}}_{\tau{\rm ci},{\rm E}}\boldsymbol{E}_{\rm E,E}\tilde{\tilde{\boldsymbol{K}}}_{{\rm E},\tau{\rm ci}}$, which is a principal sub-matrix of the Schur complement of $\tilde{\tilde{\boldsymbol{K}}}_{\rm E,E}$ in $\tilde{\tilde{\boldsymbol{K}}}$. Thus, it is positive definite and so is the principal sub-matrix of $\boldsymbol{E}$ in the second term above. This implies the positive definiteness of $\boldsymbol{D}$, a fortiori fulfilment of (18) and application of the previous proposition is justified under the stronger assumptions.

Continuous dependence of the elements of $\boldsymbol{C}_\tau$, $\tilde{\tilde{\boldsymbol{K}}}$ and $\boldsymbol{D}$ on the elements of $\nabla^2 g/\|\nabla g\|$ and $\mathscr{F}_i$, $i \in ((I_{\rm c} \cup I_{\rm z}) \cap I_{\rm l}) \cup (I_{\rm c} \cap I_{\rm i})$, completes the claim. □

The uniqueness result uses the notion of a P-matrix, which is a matrix whose all principal minors are positive.

Proposition 4. *A sufficient condition for uniqueness of a solution of* (16) *for* $\gamma' \in \mathbb{R}$ *and* $\boldsymbol{b} \in \mathbb{R}^M$ *given is that the matrix*

$$\boldsymbol{D}\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\tilde{\boldsymbol{S}}_{\rm zi}\boldsymbol{F}_{\rm zi} & \boldsymbol{I} \end{pmatrix}$$

is a P-matrix for any $\tilde{\boldsymbol{S}}_{\rm zi} = \mathbf{Diag}(\tilde{s}_1, \ldots, \tilde{s}_{n_{\rm zi}})$ *with* $\tilde{s}_i \in [-1, 1]$.

Proof. Let $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ and $(\boldsymbol{x}^2, \boldsymbol{y}^2)$ be two solutions of (16) for the same $\gamma'\boldsymbol{b}$ and

$$\boldsymbol{\psi}^i_{\tau{\rm zi}} = -\tilde{\boldsymbol{S}}^i_{\rm zi}\boldsymbol{F}_{\rm zi}\boldsymbol{\psi}^i_{\nu{\rm zi}},\ \tilde{\boldsymbol{S}}^i_{\rm zi} \in \boldsymbol{S}_{\rm zi}(\boldsymbol{\zeta}^i_{\tau{\rm zi}}), \quad i = 1, 2, \tag{19}$$

(compare $(16)_7$). One can write

$$\boldsymbol{\psi}^1_{\tau{\rm zi}} - \boldsymbol{\psi}^2_{\tau{\rm zi}} = -\tilde{\boldsymbol{S}}^1_{\rm zi}\boldsymbol{F}_{\rm zi}(\boldsymbol{\psi}^1_{\nu{\rm zi}} - \boldsymbol{\psi}^2_{\nu{\rm zi}}) - (\tilde{\boldsymbol{S}}^1_{\rm zi} - \tilde{\boldsymbol{S}}^2_{\rm zi})\boldsymbol{F}_{\rm zi}\boldsymbol{\psi}^2_{\nu{\rm zi}} \tag{20}$$

and taking the difference for the two solutions in $(16)_2$, one obtains

$$\begin{pmatrix} \boldsymbol{\psi}^1_{\tau{\rm ci}} - \boldsymbol{\psi}^2_{\tau{\rm ci}} \\ \boldsymbol{\zeta}^1_{\nu{\rm z}0} - \boldsymbol{\zeta}^2_{\nu{\rm z}0} \\ \boldsymbol{\zeta}^1_{\nu{\rm zl}} - \boldsymbol{\zeta}^2_{\nu{\rm zl}} \\ \boldsymbol{\zeta}^1_{\nu{\rm zi}} - \boldsymbol{\zeta}^2_{\nu{\rm zi}} \\ \boldsymbol{\zeta}^1_{\tau{\rm zi}} - \boldsymbol{\zeta}^2_{\tau{\rm zi}} \end{pmatrix} = \boldsymbol{D}\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\tilde{\boldsymbol{S}}^1_{\rm zi}\boldsymbol{F}_{\rm zi} & \boldsymbol{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{\zeta}^1_{\tau{\rm ci}} - \boldsymbol{\zeta}^2_{\tau{\rm ci}} \\ \boldsymbol{\psi}^1_{\nu{\rm z}0} - \boldsymbol{\psi}^2_{\nu{\rm z}0} \\ \boldsymbol{\psi}^1_{\nu{\rm zl}} - \boldsymbol{\psi}^2_{\nu{\rm zl}} \\ \boldsymbol{\psi}^1_{\nu{\rm zi}} - \boldsymbol{\psi}^2_{\nu{\rm zi}} \\ -(\tilde{\boldsymbol{S}}^1_{\rm zi} - \tilde{\boldsymbol{S}}^2_{\rm zi})\boldsymbol{F}_{\rm zi}\boldsymbol{\psi}^2_{\nu{\rm zi}} \end{pmatrix}.$$

By the complementarity conditions $(16)_{3-6}$, one has

$$(\boldsymbol{\psi}^1_{\tau{\rm ci}} - \boldsymbol{\psi}^2_{\tau{\rm ci}})^\top(\boldsymbol{\zeta}^1_{\tau{\rm ci}} - \boldsymbol{\zeta}^2_{\tau{\rm ci}}) \le 0, \qquad (\boldsymbol{\zeta}^1_{\nu{\rm z}0} - \boldsymbol{\zeta}^2_{\nu{\rm z}0})^\top(\boldsymbol{\psi}^1_{\nu{\rm z}0} - \boldsymbol{\psi}^2_{\nu{\rm z}0}) \le 0,$$
$$(\boldsymbol{\zeta}^1_{\nu{\rm zl}} - \boldsymbol{\zeta}^2_{\nu{\rm zl}})^\top(\boldsymbol{\psi}^1_{\nu{\rm zl}} - \boldsymbol{\psi}^2_{\nu{\rm zl}}) \le 0, \qquad (\boldsymbol{\zeta}^1_{\nu{\rm zi}} - \boldsymbol{\zeta}^2_{\nu{\rm zi}})^\top(\boldsymbol{\psi}^1_{\nu{\rm zi}} - \boldsymbol{\psi}^2_{\nu{\rm zi}}) \le 0$$

and by (19), monotonicity of the multifunction Sgn and non-negativity of $\boldsymbol{F}_{\rm zi}$ and $\boldsymbol{\psi}^2_{\nu{\rm zi}}$,

$$-(\boldsymbol{\zeta}^1_{\tau{\rm zi}} - \boldsymbol{\zeta}^2_{\tau{\rm zi}})^\top(\tilde{\boldsymbol{S}}^1_{\rm zi} - \tilde{\boldsymbol{S}}^2_{\rm zi})\boldsymbol{F}_{\rm zi}\boldsymbol{\psi}^2_{\nu{\rm zi}} \le 0.$$

Since a P-matrix does not reverse the sign of any non-zero vector ([5, 3.3.4 Theorem]), the imposed assumption and (20) imply that $\boldsymbol{x}^1 = \boldsymbol{x}^2$ and $\boldsymbol{y}^1 = \boldsymbol{y}^2$. □

Invoking that a symmetric positive definite matrix is a P-matrix and arguing as for Corollary 1, one gets the following:

Corollary 2 (Uniqueness)**.** *Let* $(\mathscr{P})$ *correspond to a model of linearised elasticity where the body is properly constrained by the Dirichlet condition. If* g *satisfies Assumption 1 and* $\|\nabla^2 g\|/\|\nabla g\|$ *and* $\mathscr{F}_i$, $i \in (I_{\mathrm{c}} \cup I_{\mathrm{z}}) \cap (I_{\mathrm{l}} \cup I_{\mathrm{i}})$, *are small, then the first-order system* $(\mathscr{P}'1)$, $(\mathscr{P}'2)$, $(\mathscr{P}'3')$ *and* $(\mathscr{P}'4')$ *has a unique solution for any* $\gamma' \in \mathbb{R}$, $\boldsymbol{P} \in \mathbb{R}^{2n_\Omega}$ *and any* $\boldsymbol{U}'_D \in \mathbb{R}^{2n_D}$.

Example 2 (Large coefficient of friction)**.** (i) Let us consider the simple continuation problem (8) from Example 1 with $\mathscr{F} = a/b$, $(L_1(\gamma), L_2(\gamma)) = (1, -\gamma)$ and $u_2^k = V_\tau = 0$. Similarly as in [9], one can show that this problem has solutions corresponding to stick of the only contact node for $\gamma \in [0, \mathscr{F}]$, which form the set

$$\begin{aligned}\{(\gamma, u_1, u_2, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^5;\ \gamma \in [0, \mathscr{F}],\ u_1 = u_2 = 0,\\ \lambda_\nu = -L_1(\gamma) = -1,\ \lambda_\tau = -L_2(\gamma) = \gamma\},\end{aligned} \tag{21}$$

solutions corresponding to non-positive slip for $\gamma = \mathscr{F}$ forming the set

$$\begin{aligned}\Big\{(\gamma, u_1, u_2, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^5;\ \gamma = \mathscr{F},\ u_1 = 0,\ u_2 \in \Big[-\frac{L_1(\gamma)}{b}, 0\Big] = \Big[-\frac{1}{b}, 0\Big],\\ \lambda_\nu = -L_1(\gamma) - bu_2 = -1 - bu_2,\ \lambda_\tau = -\mathscr{F}\lambda_\nu = \mathscr{F}(1 + bu_2)\Big\}\end{aligned}$$

and solutions corresponding to separation for $\gamma \geq \mathscr{F}$, which form the set

$$\begin{aligned}\Big\{(\gamma, u_1, u_2, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^5;\ \gamma \geq \mathscr{F},\ u_1 = \frac{aL_1(\gamma) + bL_2(\gamma)}{a^2 - b^2} = \frac{a - b\gamma}{a^2 - b^2},\\ u_2 = \frac{bL_1(\gamma) + aL_2(\gamma)}{a^2 - b^2} = \frac{b - a\gamma}{a^2 - b^2},\ \lambda_\nu = \lambda_\tau = 0\Big\}.\end{aligned}$$

Taking $(\bar{\gamma}, \bar{u}_1, \bar{u}_2, \bar{\lambda}_\nu, \bar{\lambda}_\tau) = (\mathscr{F}, 0, 0, -1, \mathscr{F})$ (strong contact with impending slip), we get the first-order system:

$$\begin{gathered}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} - \gamma'\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} - \begin{pmatrix} \lambda_\nu' \\ \lambda_\tau' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\\ u_1' = 0,\ u_2' \leq 0,\ \lambda_\tau' + \mathscr{F}\lambda_\nu' \leq 0,\ u_2'(\lambda_\tau' + \mathscr{F}\lambda_\nu') = 0\end{gathered} \tag{22}$$

with $(P_1, P_2) = (0, -1)$. Its solutions in the two cases ($u_2' = 0$ & $\lambda_\tau' + \mathscr{F}\lambda_\nu' \leq 0$) and ($u_2' \leq 0$ & $\lambda_\tau' + \mathscr{F}\lambda_\nu' = 0$) form the sets

$$\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' \leq 0,\ u_1' = u_2' = 0,\ \lambda_\nu' = -\gamma' P_1 = 0,\ \lambda_\tau' = -\gamma' P_2 = \gamma'\},$$

and

$$\begin{aligned}\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' = 0,\ u_1' = 0,\ u_2' \leq 0,\\ \lambda_\nu' = -\gamma' P_1 - bu_2' = -bu_2',\ \lambda_\tau' = -\mathscr{F}\lambda_\nu' = \mathscr{F}bu_2'\},\end{aligned}$$

that is, there is no solution for $\gamma' > 0$ but infinity of solutions for $\gamma' = 0$.

The first-order system for $(\bar{\gamma}, \bar{u}_1, \bar{u}_2, \bar{\lambda}_\nu, \bar{\lambda}_\tau) = (\mathscr{F}, 0, \beta, -1 - b\beta, \mathscr{F}(1 + b\beta))$, $\beta \in (-1/b, 0)$ (strong contact with positive slip) consists of (22) and

$$u_1' = 0,\ \lambda_\tau' + \mathscr{F}\lambda_\nu' = 0$$

and its solution set is

$$\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' = 0,\ u_1' = 0,\ u_2' \in \mathbb{R},\\ \lambda_\nu' = -\gamma' P_1 - bu_2' = -bu_2',\ \lambda_\tau' = -\mathscr{F}\lambda_\nu' = \mathscr{F}bu_2'\}.$$

It has no solution for $\gamma' \neq 0$ but infinity of solutions for $\gamma' = 0$.

Finally, the first-order system for $(\bar{\gamma}, \bar{u}_1, \bar{u}_2, \bar{\lambda}_\nu, \bar{\lambda}_\tau) = (\mathscr{F}, 0, -1/b, 0, 0)$ (grazing contact with positive slip) is composed of (22) and

$$u_1' \le 0,\ \lambda_\nu' \le 0,\ u_1'\lambda_\nu' = 0,\ \lambda_\tau' + \mathscr{F}\lambda_\nu' = 0$$

and its solution subsets in the two cases ($u_1' = 0$ & $\lambda_\nu' \le 0$) and ($u_1' \le 0$ & $\lambda_\nu' = 0$) are

$$\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' = 0,\ u_1' = 0,\ u_2' \ge 0,\\ \lambda_\nu' = -\gamma' P_1 - bu_2' = -bu_2',\ \lambda_\tau' = -\mathscr{F}\lambda_\nu' = \mathscr{F}bu_2'\}$$

and

$$\Big\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' \ge 0,\ u_1' = \frac{a\gamma' P_1 + b\gamma' P_2}{a^2 - b^2} = \frac{-b\gamma'}{a^2 - b^2},\\ u_2' = \frac{b\gamma' P_1 + a\gamma' P_2}{a^2 - b^2} = \frac{-a\gamma'}{a^2 - b^2},\ \lambda_\nu' = \lambda_\tau' = 0\Big\}.$$

There is no solution for $\gamma' < 0$ but infinity of solutions for $\gamma' = 0$.

(ii) Now, let us consider the problem (8) with $\mathscr{F} = a/b$, $(L_1(\gamma), L_2(\gamma)) = (b\gamma, -a\gamma)$ and $u_2^k = V_\tau = 0$ and choose $(\bar{\gamma}, \bar{u}_1, \bar{u}_2, \bar{\lambda}_\nu, \bar{\lambda}_\tau) = (0, 0, 0, 0, 0)$ (grazing contact with impending slip). In this case, the first-order system comprises (22) with $(P_1, P_2) = (b, -a)$ and

$$u_1' \le 0,\ \lambda_\nu' \le 0,\ u_1'\lambda_\nu' = 0,\ \lambda_\tau' \in \mathscr{F}\lambda_\nu' \operatorname{Sgn}(u_2').$$

Trying all possible cases, one can find the following solution set for $\gamma' \ge 0$:

$$\Big\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ (\gamma' \ge 0,\ u_1' = u_2' = 0,\ \lambda_\nu' = -\gamma' P_1 = -b\gamma',\ \lambda_\tau' = -\gamma' P_2 = a\gamma')\\ \vee \Big(\gamma' \ge 0,\ u_1' = 0,\ u_2' \in \Big[-\frac{\gamma' P_1}{b}, 0\Big] = [-\gamma', 0],\\ \lambda_\nu' = -\gamma' P_1 - bu_2' = -b(\gamma' + u_2'),\ \lambda_\tau' = -\mathscr{F}\lambda_\nu' = \mathscr{F}b(\gamma' + u_2')\Big)\\ \vee \Big(\gamma' \ge 0,\ u_1' = \frac{a\gamma' P_1 + b\gamma' P_2}{a^2 - b^2} = 0,\ u_2' = \frac{b\gamma' P_1 + a\gamma' P_2}{a^2 - b^2} = -\gamma',\ \lambda_\nu' = \lambda_\tau' = 0\Big)\Big\}.$$

It consists of a continuous branch of solutions for any $\gamma' > 0$ fixed.

To sum up, we see that the first-order system can lack solutions for some γ' if $\mathscr{F}_i$ are large for $i \in I_{\mathrm{c}} \cap I_{\mathrm{i}}$, $i \in I_{\mathrm{c}} \cap I_{\mathrm{l}}$, or $i \in I_{\mathrm{z}} \cap I_{\mathrm{l}}$. On the other hand, it can possess multiple solutions for some γ' if $\mathscr{F}_i$ are large for $i \in I_{\mathrm{c}} \cap I_{\mathrm{i}}$, $i \in I_{\mathrm{c}} \cap I_{\mathrm{l}}$, $i \in I_{\mathrm{z}} \cap I_{\mathrm{l}}$ or $i \in I_{\mathrm{z}} \cap I_{\mathrm{i}}$.

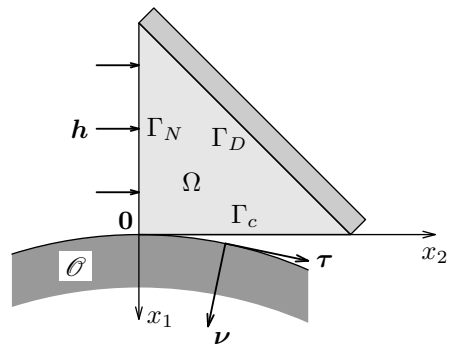


Figure 5: Geometry of Example 3.

Example 3 (Large $\|\nabla^2 g\|/\|\nabla g\|$)**.** Let us modify the simple continuation problem (8) to the case where the rigid foundation is demarcated by

$$g\colon \boldsymbol{x} = (x_1, x_2) \mapsto r^2 - (x_1 - r)^2 - x_2^2$$

for $r > 0$ fixed (Figure 5). Then

$$\nabla g(\boldsymbol{x}) = -2\begin{pmatrix} x_1 - r \\ x_2 \end{pmatrix}, \quad \|\nabla g(\boldsymbol{x})\| = 2\sqrt{(x_1 - r)^2 + x_2^2},$$

$$\boldsymbol{\nu}(\boldsymbol{x}) = \frac{-1}{\sqrt{(x_1 - r)^2 + x_2^2}}\begin{pmatrix} x_1 - r \\ x_2 \end{pmatrix}, \quad \boldsymbol{\tau}(\boldsymbol{x}) = \frac{1}{\sqrt{(x_1 - r)^2 + x_2^2}}\begin{pmatrix} x_2 \\ -(x_1 - r) \end{pmatrix}$$

and the continuation problem becomes

$$\begin{aligned} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\boldsymbol{u} - \begin{pmatrix} L_1(\gamma) \\ L_2(\gamma) \end{pmatrix} - \lambda_\nu \boldsymbol{\nu}(\boldsymbol{u}) - \lambda_\tau \boldsymbol{\tau}(\boldsymbol{u}) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \lambda_\nu - (\lambda_\nu - g(\boldsymbol{u}))_- &= 0, \\ \lambda_\tau - P_{[\mathscr{F}(\lambda_\nu - g(\boldsymbol{u}))_-, -\mathscr{F}(\lambda_\nu - g(\boldsymbol{u}))_-]}\big(\lambda_\tau - (\boldsymbol{\tau}(\boldsymbol{u})^\top \boldsymbol{u} - \boldsymbol{\tau}(\boldsymbol{u})^\top \boldsymbol{w}) - (1-\gamma)V_\tau\big) &= 0. \end{aligned}$$

Here, we consider $(L_1(\gamma), L_2(\gamma)) = (1, -\gamma)$, $\boldsymbol{w} = \mathbf{0}$, $V_\tau = 0$ as in (i) of the previous example and $\mathscr{F} > 0$ arbitrarily fixed.

It is readily seen that the set given by (21) is a part of the solution set in this case too. If one takes $(\bar\gamma, \bar u_1, \bar u_2, \bar\lambda_\nu, \bar\lambda_\tau) = (\mathscr{F}, 0, 0, -1, \mathscr{F})$ (strong contact with impending slip), one has

$$\boldsymbol{\nu}(\bar{\boldsymbol{u}}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\tau}(\bar{\boldsymbol{u}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nabla^2 g(\bar{\boldsymbol{u}}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad \frac{\nabla^2 g(\bar{\boldsymbol{u}})}{\|\nabla g(\bar{\boldsymbol{u}})\|} = -\frac{1}{r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{K}_\nu(\bar{\boldsymbol{u}}, \bar\lambda_\nu) = \frac{1}{r}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{K}_\tau(\bar{\boldsymbol{u}}, \bar\lambda_\tau) = \frac{1}{r}\begin{pmatrix} 0 & \mathscr{F} \\ 0 & 0 \end{pmatrix}$$

and the corresponding first-order system reads

$$\begin{pmatrix} a & -b - \frac{\mathscr{F}}{r} \\ -b & a - \frac{1}{r} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma' \end{pmatrix} - \begin{pmatrix} \lambda_\nu' \\ \lambda_\tau' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$u_1' = 0,\ u_2' \le 0,\ \lambda_\tau' + \mathscr{F}\lambda_\nu' \le 0,\ u_2'(\lambda_\tau' + \mathscr{F}\lambda_\nu') = 0.$$

As in the previous example, the case ($u_2' = 0$ & $\lambda_\tau' + \mathscr{F}\lambda_\nu' \le 0$) leads to the solution subset

$$\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' \le 0,\ u_1' = u_2' = 0,\ \lambda_\nu' = 0,\ \lambda_\tau' = \gamma'\},$$

whereas if one supposes that ($u_2' \le 0$ & $\lambda_\tau' + \mathscr{F}\lambda_\nu' = 0$) and r is small enough so that $a - b\mathscr{F} - (1+\mathscr{F}^2)/r < 0$, one obtains solutions forming the set

$$\Big\{(\gamma', u_1', u_2', \lambda_\nu', \lambda_\tau') \in \mathbb{R}^5;\ \gamma' \le 0,\ u_1' = 0,\ u_2' = \frac{-\gamma'}{a - b\mathscr{F} - (1+\mathscr{F}^2)/r},$$
$$\lambda_\nu' = \frac{(b + \mathscr{F}/r)\gamma'}{a - b\mathscr{F} - (1+\mathscr{F}^2)/r},\ \lambda_\tau' = \frac{-\mathscr{F}(b + \mathscr{F}/r)\gamma'}{a - b\mathscr{F} - (1+\mathscr{F}^2)/r}\Big\}.$$

Hence, this first-order system has no solution for $\gamma' > 0$ and two different solutions for $\gamma' < 0$. This shows that large $\|\nabla^2 g\|/\|\nabla g\|$ can cause both lack and multiplicity of solutions for some γ'.

Let us point out that the first-order system can lack solutions for some γ' for larger coefficients of friction even in the case of linearised elasticity with the body properly constrained by the Dirichlet condition and in contact with a flat obstacle. This is in stark contrast to the corresponding incremental problem, where existence of a solution is guaranteed for any coefficient of friction [14, Section 6]. Nevertheless, it is in accordance with the analysis of the rate and quasi-static problem. It was shown in [17, Subsection 5.2] that solvability of the rate problem requires smallness of the coefficient of friction for nodes in strong contact with possible slip in near future and an example of non-existence of solutions of a quasi-static problem for high values of the coefficient of friction can be found in [13].

To add, the uniqueness result for the rate problem presented in [17] involves an assumption on smallness of the coefficient of friction for nodes in contact with possible slip in near future and for nodes in grazing contact. In [13], an example with multiple solutions of a quasi-static problem with high values of the coefficient of friction was shown.

4.2 Continuation in a Direction Solving the First-Order System

To show the possibility of continuation of a solution curve in a direction solving the first-order system, we shall apply Theorem 2 from the companion paper [16]. First of all, we restate the theorem, taking into account Remark 2 from the same paper.

Theorem 1. *Let $\boldsymbol{H}\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a PC^1-function and $(\bar\gamma, \bar{\boldsymbol{x}})$, $(\gamma', \boldsymbol{x}') \in \mathbb{R}\times\mathbb{R}^N$ satisfy $\boldsymbol{H}(\bar\gamma, \bar{\boldsymbol{x}}) = \boldsymbol{0}$ and $\boldsymbol{H}'((\bar\gamma, \bar{\boldsymbol{x}}); (\gamma', \boldsymbol{x}')) = \boldsymbol{0}$. If $\{\boldsymbol{H}^{(i)}\}_{i\in I}$ is a family of C^1 selection functions of $\boldsymbol{H}$ at $(\bar\gamma, \bar{\boldsymbol{x}})$ and there exists $i_0 \in I$ such that*

(i) $\boldsymbol{H}^{(i_0)}(\bar\gamma, \bar{\boldsymbol{x}}) = \boldsymbol{H}(\bar\gamma, \bar{\boldsymbol{x}})$;

(ii) $\boldsymbol{\nabla H}^{(i_0)}(\bar\gamma, \bar{\boldsymbol{x}})(\gamma', \boldsymbol{x}') = \boldsymbol{H}'((\bar\gamma, \bar{\boldsymbol{x}}); (\gamma', \boldsymbol{x}')) \ne \boldsymbol{\nabla H}^{(i)}(\bar\gamma, \bar{\boldsymbol{x}})(\gamma', \boldsymbol{x}'),\ \forall i \ne i_0$;

(iii) $\boldsymbol{\nabla H}^{(i_0)}(\bar\gamma, \bar{\boldsymbol{x}})$ *has the maximal rank* N,

then there are $\delta > 0$ and a C^1-curve $\boldsymbol{c}\colon [\bar{s}, \bar{s}+\delta) \to \mathbb{R} \times \mathbb{R}^N$ such that

$$\text{(j) } \boldsymbol{c}(\bar{s}) = (\bar{\gamma}, \bar{\boldsymbol{x}}); \quad \text{(jj) } \forall s \in [\bar{s}, \bar{s}+\delta)\colon\ \boldsymbol{H}(\boldsymbol{c}(s)) = \boldsymbol{0}; \quad \text{(jjj) } \boldsymbol{c}'(\bar{s}) = (\gamma', \boldsymbol{x}').$$

Moreover, if $(\gamma', \boldsymbol{x}') \neq (0, \boldsymbol{0})$, the image of any other curve $\tilde{\boldsymbol{c}}\colon [\bar{s}, \bar{s}+\tilde{\delta}) \to \mathbb{R} \times \mathbb{R}^N$ with $\tilde{\delta} > 0$ and such that

$$\widetilde{\text{(j)}}\ \tilde{\boldsymbol{c}}(\bar{s}) = (\bar{\gamma}, \bar{\boldsymbol{x}}); \quad \widetilde{\text{(jj)}}\ \forall s \in [\bar{s}, \bar{s}+\tilde{\delta})\colon\ \boldsymbol{H}(\tilde{\boldsymbol{c}}(s)) = \boldsymbol{0}; \quad \widetilde{\text{(jjj)}}\ \tilde{\boldsymbol{c}}'(\bar{s}) \in \bigcup_{r>0} r(\gamma', \boldsymbol{x}') \tag{23}$$

coincides with the image of $\boldsymbol{c}$ in a neighbourhood of $(\bar{\gamma}, \bar{\boldsymbol{x}})$.

Now, let $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)(= (\bar{\gamma}, \bar{\boldsymbol{x}}))$ solve $(\mathscr{P})$, that is, $\boldsymbol{H}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = \boldsymbol{0}$. Taking into account the continuity of $\boldsymbol{H}$ and using the index sets defined by (6), one can construct the set $\{\boldsymbol{H}^{(j)}\}$ of selection functions of $\boldsymbol{H}$ at $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ such that, after eventual reordering of components, it consists of functions (compare the functions F and G_k introduced by (4) and (5)):

$$\boldsymbol{H}^{(j)}(\gamma, \boldsymbol{u}, \boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{u}) - \boldsymbol{L}(\gamma, \boldsymbol{u}) - \boldsymbol{B}_D^\top \boldsymbol{\lambda}_D - \boldsymbol{B}_\nu(\boldsymbol{u})^\top \boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau(\boldsymbol{u})^\top \boldsymbol{\lambda}_\tau \\ \boldsymbol{B}_D \boldsymbol{u} - \boldsymbol{U}_D(\gamma) \\ \lambda_{\nu,i}, \quad i \in I_{\mathrm{f}} \cup I_{\mathrm{z}}^{\mathrm{f}} \\ g_i(\boldsymbol{u}), \quad i \in I_{\mathrm{c}} \cup I_{\mathrm{z}}^{\mathrm{c}} \\ \lambda_{\tau,i}, \quad i \in I_{\mathrm{f}} \cup I_{\mathrm{z}}^{\mathrm{f}} \cup I_0 \\ (\boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{u} - \boldsymbol{B}_\tau(\boldsymbol{u})\boldsymbol{w})_i + (1-\gamma)V_{\tau,i}, \quad i \in I_{\mathrm{s}} \cup I_{\mathrm{ci}}^{\mathrm{s}} \cup I_{\mathrm{zi}}^{\mathrm{cs}} \\ \lambda_{\tau,i} + s_i \mathscr{F}_i(\lambda_{\nu,i} - g_i(\boldsymbol{u})), \quad i \in (I_{\mathrm{c}} \cap I_{\mathrm{l}}) \cup I_{\mathrm{zl}}^{\mathrm{c}} \cup I_{\mathrm{ci}}^{\mathrm{l}} \cup I_{\mathrm{zi}}^{\mathrm{cl}+} \cup I_{\mathrm{zi}}^{\mathrm{cl}-} \end{pmatrix}.$$

Here, in a similar spirit as in [3], the index sets $I_{\mathrm{z}}^{\mathrm{c}}$, $I_{\mathrm{ci}}^{\mathrm{l}}$, $I_{\mathrm{zi}}^{\mathrm{cl}+}$ and $I_{\mathrm{zi}}^{\mathrm{cl}-}$ pass all possible combinations such that

$$I_{\mathrm{z}}^{\mathrm{c}} \subset I_{\mathrm{z}},\ I_{\mathrm{ci}}^{\mathrm{l}} \subset (I_{\mathrm{c}} \cap I_{\mathrm{i}}),\ I_{\mathrm{zi}}^{\mathrm{cl}+}, I_{\mathrm{zi}}^{\mathrm{cl}-} \subset (I_{\mathrm{z}}^{\mathrm{c}} \cap I_{\mathrm{i}}),\ I_{\mathrm{zi}}^{\mathrm{cl}+} \cap I_{\mathrm{zi}}^{\mathrm{cl}-} = \emptyset \tag{24}$$

and

$$I_{\mathrm{z}}^{\mathrm{f}} = I_{\mathrm{z}} \setminus I_{\mathrm{z}}^{\mathrm{c}},\ I_{\mathrm{ci}}^{\mathrm{s}} = (I_{\mathrm{c}} \cap I_{\mathrm{i}}) \setminus I_{\mathrm{ci}}^{\mathrm{s}},\ I_{\mathrm{zi}}^{\mathrm{cs}} = (I_{\mathrm{z}}^{\mathrm{c}} \cap I_{\mathrm{i}}) \setminus (I_{\mathrm{zi}}^{\mathrm{cl}+} \cup I_{\mathrm{zi}}^{\mathrm{cl}-}),\ I_{\mathrm{zl}}^{\mathrm{c}} = I_{\mathrm{z}}^{\mathrm{c}} \cap I_{\mathrm{l}}, \tag{25}$$

$$s_i = \begin{cases} -\operatorname{sgn}(\bar{v}_{\tau,i} + (1-\bar{\gamma})V_{\tau,i}) & \text{if } i \in (I_{\mathrm{c}} \cap I_{\mathrm{l}}) \cup I_{\mathrm{zl}}^{\mathrm{c}}, \\ \operatorname{sgn}(\bar{\lambda}_{\tau,i}) & \text{if } i \in I_{\mathrm{ci}}^{\mathrm{l}}, \\ -1 & \text{if } i \in I_{\mathrm{zi}}^{\mathrm{cl}+}, \\ 1 & \text{if } i \in I_{\mathrm{zi}}^{\mathrm{cl}-}. \end{cases} \tag{26}$$

It is worth noticing that $I_{\mathrm{z}}^{\mathrm{f}}$ and $I_{\mathrm{z}}^{\mathrm{c}}$ constitute a decomposition of I_{z}, $I_{\mathrm{ci}}^{\mathrm{s}}$ and $I_{\mathrm{ci}}^{\mathrm{l}}$ constitute a decomposition of $I_{\mathrm{c}} \cap I_{\mathrm{i}}$ and $I_{\mathrm{zi}}^{\mathrm{cs}}$, $I_{\mathrm{zi}}^{\mathrm{cl}+}$ and $I_{\mathrm{zi}}^{\mathrm{cl}-}$ constitute a decomposition of $I_{\mathrm{z}}^{\mathrm{c}} \cap I_{\mathrm{i}}$.

Gradients of these selection functions at $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ are given by

$$\boldsymbol{\nabla} \boldsymbol{H}^{(j)}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$$
$$= \begin{pmatrix} \boldsymbol{K}\boldsymbol{u}' - \gamma' \boldsymbol{P} - \boldsymbol{B}_D^\top \boldsymbol{\lambda}'_D - \boldsymbol{B}_\nu^\top \boldsymbol{\lambda}'_\nu - \boldsymbol{B}_\tau^\top \boldsymbol{\lambda}'_\tau \\ \boldsymbol{B}_D \boldsymbol{u}' - \gamma' \boldsymbol{U}'_D \\ \lambda'_{\nu,i}, \quad i \in I_{\mathrm{f}} \cup I_{\mathrm{z}}^{\mathrm{f}} \\ (\boldsymbol{B}_\nu \boldsymbol{u}')_i, \quad i \in I_{\mathrm{c}} \cup I_{\mathrm{z}}^{\mathrm{c}} \\ \lambda'_{\tau,i}, \quad i \in I_{\mathrm{f}} \cup I_{\mathrm{z}}^{\mathrm{f}} \cup I_0 \\ (\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i}, \quad i \in I_{\mathrm{s}} \cup I_{\mathrm{ci}}^{\mathrm{s}} \cup I_{\mathrm{zi}}^{\mathrm{cs}} \\ \lambda'_{\tau,i} + s_i \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i), \quad i \in (I_{\mathrm{c}} \cap I_{\mathrm{l}}) \cup I_{\mathrm{zl}}^{\mathrm{c}} \cup I_{\mathrm{ci}}^{\mathrm{l}} \cup I_{\mathrm{zi}}^{\mathrm{cl}+} \cup I_{\mathrm{zi}}^{\mathrm{cl}-} \end{pmatrix} \tag{27}$$

(compare ($\mathscr{P}'1$), ($\mathscr{P}'2$), ($\mathscr{P}'3$') and ($\mathscr{P}'4$')).

Next, assume that $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ solves the first-order system at $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$, or equivalently,

$$\boldsymbol{H}'((\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau); (\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)) = \boldsymbol{0}. \tag{28}$$

If we want to find a selection function $\boldsymbol{H}^{(j)}$ with

$$\boldsymbol{\nabla} \boldsymbol{H}^{(j)}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$$
$$= \boldsymbol{H}'((\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau); (\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)) = \boldsymbol{0}, \tag{29}$$

one can see from ($\mathscr{P}'3$) and ($\mathscr{P}'4$) that it suffices to choose the determining index sets so that the following conditions are fulfilled:

$$\left.\begin{aligned} \lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i &\geq 0, \quad i \in I_{\mathrm{z}}^{\mathrm{f}}, \\ \lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i &\leq 0, \quad i \in I_{\mathrm{z}}^{\mathrm{c}}, \\ s_i(\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}) &\leq -\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i), \quad i \in I_{\mathrm{ci}}^{\mathrm{s}}, \\ |\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}| &\leq -\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i), \quad i \in I_{\mathrm{zi}}^{\mathrm{cs}}, \\ s_i(\lambda'_{\tau,i} - (\boldsymbol{C}_\tau \boldsymbol{u}')_i + \gamma' V_{\tau,i}) &\geq -\mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i), \quad i \in I_{\mathrm{ci}}^{\mathrm{l}} \cup I_{\mathrm{zi}}^{\mathrm{cl}+} \cup I_{\mathrm{zi}}^{\mathrm{cl}-}, \end{aligned}\right\} \tag{30}$$

with s_i introduced by (26) and $s_i = \mathrm{sgn}(\bar{\lambda}_{\tau,i})$ for $i \in I_{\mathrm{ci}}^{\mathrm{s}}$.

We are prepared to state the continuation result based on Theorem 1.

Theorem 2. *Let Assumption 1 be satisfied, $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ be a solution of ($\mathscr{P}$) with non-empty $I_{\mathrm{z}} \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}})$, and $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ solve the first-order system ($\mathscr{P}'1$), ($\mathscr{P}'2$), ($\mathscr{P}'3$') and ($\mathscr{P}'4$'). If one can find index sets $I_{\mathrm{z}}^{\mathrm{c}}$, $I_{\mathrm{ci}}^{\mathrm{l}}$, $I_{\mathrm{zi}}^{\mathrm{cl}+}$, $I_{\mathrm{zi}}^{\mathrm{cl}-}$, $I_{\mathrm{z}}^{\mathrm{f}}$, $I_{\mathrm{ci}}^{\mathrm{s}}$, $I_{\mathrm{zi}}^{\mathrm{cs}}$ and $I_{\mathrm{zl}}^{\mathrm{c}}$ in accordance with (24) and (25) and such that*

(i) *all the inequalities in (30) are satisfied in strict sense;*

(ii) *the gradient of the corresponding selection function $\boldsymbol{H}^{(j)}$ at $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ given by (27) has the maximal rank,*

then there exist $\delta > 0$ *and a* C^1*-curve* $\boldsymbol{c}\colon [\bar{s}, \bar{s}+\delta) \to \mathbb{R}^{1+2(n_\Omega+n_D+n_c)}$ *such that*

$$\begin{aligned}
&\text{(j)}\ \boldsymbol{c}(\bar{s}) = (\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau);\\
&\text{(jj)}\ \boldsymbol{H}(\boldsymbol{c}(s)) = \boldsymbol{0},\ \forall s \in [\bar{s}, \bar{s}+\delta);\\
&\text{(jjj)}\ \boldsymbol{c}'(\bar{s}) = (\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau).
\end{aligned}$$

Moreover, the image of any other curve $\tilde{\boldsymbol{c}}\colon [\bar{s}, \bar{s}+\tilde{\delta}) \to \mathbb{R}^{1+2(n_\Omega+n_D+n_c)}$ *with* $\tilde{\delta} > 0$ *and such that*

$$\left.\begin{aligned}
&\widetilde{\text{(j)}}\ \tilde{\boldsymbol{c}}(\bar{s}) = (\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau);\\
&\widetilde{\text{(jj)}}\ \boldsymbol{H}(\tilde{\boldsymbol{c}}(s)) = \boldsymbol{0},\ \forall s \in [\bar{s}, \bar{s}+\tilde{\delta});\\
&\widetilde{\text{(jjj)}}\ \tilde{\boldsymbol{c}}'(\bar{s}) \neq \boldsymbol{0}\ \textit{satisfies}\ (30)
\end{aligned}\right\} \tag{31}$$

coincides with the image of $\boldsymbol{c}$ *around* $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$.

Proof. Let $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ and $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ meet the imposed assumptions and $\boldsymbol{H}^{(j)}$ be the selection function corresponding to the particular index sets from the assertion. We shall verify the assumptions of Theorem 1 for $i_0 = j$.

First, the construction of the selection functions above guarantees that all selection functions corresponding to various combinations of the index sets restricted by (24) and (25) are active at $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$. In particular,

$$\boldsymbol{H}^{(j)}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = \boldsymbol{H}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = \boldsymbol{0}.$$

Second, since (30) holds for $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$, (29) is valid for our selection function $\boldsymbol{H}^{(j)}$. Finally, if a selection function $\boldsymbol{H}^{(\tilde{j})}$ corresponds to $\tilde{I}^{\mathrm{c}}_{\mathrm{z}}$, $\tilde{I}^{\mathrm{l}}_{\mathrm{ci}}$, $\tilde{I}^{\mathrm{cl+}}_{\mathrm{zi}}$, $\tilde{I}^{\mathrm{cl-}}_{\mathrm{zi}}$, $\tilde{I}^{\mathrm{f}}_{\mathrm{z}}$, $\tilde{I}^{\mathrm{s}}_{\mathrm{ci}}$, $\tilde{I}^{\mathrm{cs}}_{\mathrm{zi}}$ and $\tilde{I}^{\mathrm{c}}_{\mathrm{zl}}$ that fulfil the same conditions as in (24) and (25) and at least one of the sets $\tilde{I}^{\mathrm{c}}_{\mathrm{z}}$, $\tilde{I}^{\mathrm{l}}_{\mathrm{ci}}$, $\tilde{I}^{\mathrm{cl+}}_{\mathrm{zi}}$, $\tilde{I}^{\mathrm{cl-}}_{\mathrm{zi}}$ differs from $I^{\mathrm{c}}_{\mathrm{z}}$, $I^{\mathrm{l}}_{\mathrm{ci}}$, $I^{\mathrm{cl+}}_{\mathrm{zi}}$ or $I^{\mathrm{cl-}}_{\mathrm{zi}}$, respectively, then we claim that

$$\boldsymbol{\nabla}\boldsymbol{H}^{(\tilde{j})}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau) \neq \boldsymbol{0}.$$

Indeed, making use of (29) and the assumption (i), one gets either

$$i \in \tilde{I}^{\mathrm{c}}_{\mathrm{z}} \cap I^{\mathrm{f}}_{\mathrm{z}} \implies (\boldsymbol{B}_\nu \boldsymbol{u}')_i = -\lambda'_{\nu,i} + (\boldsymbol{B}_\nu \boldsymbol{u}')_i < 0$$

or

$$i \in \tilde{I}^{\mathrm{f}}_{\mathrm{z}} \cap I^{\mathrm{c}}_{\mathrm{z}} \implies \lambda'_{\nu,i} = \lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i < 0$$

if $\tilde{I}^{\mathrm{c}}_{\mathrm{z}} \neq I^{\mathrm{c}}_{\mathrm{z}}$. Similarly, assuming that $\tilde{I}^{\mathrm{c}}_{\mathrm{z}} = I^{\mathrm{c}}_{\mathrm{z}}$ but another couple of the index sets does not coincide, one can show that

$$\begin{aligned}
i \in \tilde{I}^{\mathrm{l}}_{\mathrm{ci}} \cap I^{\mathrm{s}}_{\mathrm{ci}} &\implies \tilde{s}_i \lambda'_{\tau,i} + \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i) < 0,\\
(i \in \tilde{I}^{\mathrm{s}}_{\mathrm{ci}} \cap I^{\mathrm{l}}_{\mathrm{ci}}) \vee (i \in \tilde{I}^{\mathrm{cs}}_{\mathrm{zi}} \cap (I^{\mathrm{cl+}}_{\mathrm{zi}} \cup I^{\mathrm{cl-}}_{\mathrm{zi}})) &\implies s_i((\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i}) < 0,\\
(i \in \tilde{I}^{\mathrm{cl+}}_{\mathrm{zi}} \cap I^{\mathrm{cl-}}_{\mathrm{zi}}) \vee (i \in \tilde{I}^{\mathrm{cl-}}_{\mathrm{zi}} \cap I^{\mathrm{cl+}}_{\mathrm{zi}}) \vee (i \in (\tilde{I}^{\mathrm{cl+}}_{\mathrm{zi}} \cup \tilde{I}^{\mathrm{cl-}}_{\mathrm{zi}}) \cap I^{\mathrm{cs}}_{\mathrm{zi}})&\\
\implies \lambda'_{\tau,i} + \tilde{s}_i \mathscr{F}_i(\lambda'_{\nu,i} - (\boldsymbol{B}_\nu \boldsymbol{u}')_i) &\neq 0,
\end{aligned}$$

where s_i are defined by (26) and

$$\tilde{s}_i = \begin{cases} \operatorname{sgn}(\bar{\lambda}_{\tau,i}) & \text{if } i \in \tilde{I}^{\mathrm{l}}_{\mathrm{ci}}, \\ -1 & \text{if } i \in \tilde{I}^{\mathrm{cl}+}_{\mathrm{zl}}, \\ 1 & \text{if } i \in \tilde{I}^{\mathrm{cl}-}_{\mathrm{zl}}. \end{cases}$$

Therefore, the assumptions of Theorem 1 hold and existence of the announced curve $\boldsymbol{c}$ follows.

Now, let $\tilde{\boldsymbol{c}}$ be a curve from the second part of the assertion. Comparing (23) and (31), we have to show (23)($\widetilde{\mathrm{jjj}}$):

$$\tilde{\boldsymbol{c}}'(\bar{s}) \in \bigcup_{r>0} r(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau). \tag{32}$$

From (31)($\widetilde{\mathrm{jj}}$), we see that $(\boldsymbol{H} \circ \tilde{\boldsymbol{c}})'(\bar{s}) = \boldsymbol{0}$. In addition, we deduce from Proposition 2 that $(\boldsymbol{H} \circ \tilde{\boldsymbol{c}})'(\bar{s}) = \boldsymbol{H}'(\tilde{\boldsymbol{c}}(\bar{s}); \tilde{\boldsymbol{c}}'(\bar{s}))$. Taking into account (31)($\widetilde{\mathrm{j}}$) and ($\widetilde{\mathrm{jjj}}$), we thus have similarly as before that

$$\boldsymbol{\nabla} \boldsymbol{H}^{(j)}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) \tilde{\boldsymbol{c}}'(\bar{s}) = \boldsymbol{H}'(\tilde{\boldsymbol{c}}(\bar{s}); \tilde{\boldsymbol{c}}'(\bar{s})) = \boldsymbol{0}.$$

This combined with (29) and the assumption (ii) implies that

$$\tilde{\boldsymbol{c}}'(\bar{s}) \in \operatorname{span}\{(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)\}.$$

But $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ satisfies all the inequalities in (30) strictly, so it is clear that these conditions are violated for any $r(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ with $r < 0$ and any $i \in I^{\mathrm{f}}_{\mathrm{z}} \cup I^{\mathrm{c}}_{\mathrm{z}} \cup I^{\mathrm{s}}_{\mathrm{ci}} \cup I^{\mathrm{cs}}_{\mathrm{zi}} \cup I^{\mathrm{l}}_{\mathrm{ci}} \cup I^{\mathrm{cl}+}_{\mathrm{zi}} \cup I^{\mathrm{cl}-}_{\mathrm{zi}} = I_{\mathrm{z}} \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}})$, which is assumed to be non-empty. This together with (31)($\widetilde{\mathrm{jjj}}$) lead to (32) and the uniqueness part of Theorem 1 applies. □

In the trivial case when $I_{\mathrm{z}} \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}}) = \emptyset$, $\boldsymbol{H}$ is smooth in some vicinity of $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ and the classical implicit function theorem can be employed.

Theorem 3. *Let Assumption 1 hold, $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ be a solution of $(\mathscr{P})$ with $I_{\mathrm{z}} \cup (I_{\mathrm{c}} \cap I_{\mathrm{i}}) = \emptyset$ and $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ be a non-zero vector solving the first-order system $(\mathscr{P}'1)$, $(\mathscr{P}'2)$, $(\mathscr{P}'3')$ and $(\mathscr{P}'4')$. If $\boldsymbol{\nabla} \boldsymbol{H}(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ has the maximal rank, then there exist $\delta > 0$ and a C^1-curve $\boldsymbol{c}\colon (\bar{s} - \delta, \bar{s} + \delta) \to \mathbb{R}^{1+2(n_\Omega + n_D + n_c)}$ such that*

$$\boldsymbol{c}(\bar{s}) = (\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau), \quad \boldsymbol{H}(\boldsymbol{c}(s)) = \boldsymbol{0}, \ \forall s \in (\bar{s} - \delta, \bar{s} + \delta), \quad \boldsymbol{c}'(\bar{s}) = (\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau).$$

Moreover, the image of any other solution curve passing through $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ is in some neighbourhood of $(\bar{\gamma}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ contained in the image of $\boldsymbol{c}$.

Remark 4. Assume that the vector $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ satisfies (28). Then one can easily verify that (30) holds strictly if and only if

$$\left.\begin{array}{rll} \lambda'_{\nu,i} = 0, \ (\boldsymbol{B}_\nu \boldsymbol{u}')_i < 0, & i \in I^{\mathrm{f}}_{\mathrm{z}} & (\to \text{no contact}); \\ (\boldsymbol{B}_\nu \boldsymbol{u}')_i = 0, \ \lambda'_{\nu,i} < 0, & i \in I^{\mathrm{c}}_{\mathrm{z}} & (\to \text{strong contact}); \\ (\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i} = 0, \ s_i \lambda'_{\tau,i} < -\mathscr{F}_i \lambda'_{\nu,i}, & i \in I^{\mathrm{s}}_{\mathrm{ci}} & (\to \text{strong stick}); \\ (\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i} = 0, \ |\lambda'_{\tau,i}| < -\mathscr{F}_i \lambda'_{\nu,i}, & i \in I^{\mathrm{cs}}_{\mathrm{zi}} & (\to \text{strong stick}); \\ \lambda'_{\tau,i} + s_i \mathscr{F}_i \lambda'_{\nu,i} = 0, \ s_i((\boldsymbol{C}_\tau \boldsymbol{u}')_i - \gamma' V_{\tau,i}) < 0, & i \in I^{\mathrm{l}}_{\mathrm{ci}} \cup I^{\mathrm{cl}+}_{\mathrm{zi}} \cup I^{\mathrm{cl}-}_{\mathrm{zi}} & (\to \text{non-zero slip}) \end{array}\right\} \tag{33}$$

(compare ($\mathscr{P}'$3') and ($\mathscr{P}'$4')). This means that the direction $(\gamma', \boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_\nu, \boldsymbol{\lambda}'_\tau)$ satisfying the assumption (i) in Theorem 2 leads to solutions with strict contact modes of all nodes, without any grazing contact or impending slip.

Similarly, non-strict inequalities in (30) correspond to (33) with non-strict inequalities above. Thus, the uniqueness part of Theorem 2 means that there is only one branch of solutions with the nodal contact modes corresponding to (30) according to the interpretation above with non-strict modes. From this mechanical point of view, the claim of Theorem 2 resembles the one of [17, Proposition 8].

Proposition 5 (Satisfaction of the assumption (ii) in Theorem 2)**.** *Let* ($\mathscr{P}$) *correspond to a model of linearised elasticity with a body properly constrained by the Dirichlet condition. If g satisfies Assumption 1 and $\|\nabla^2 g\|/\|\nabla g\|$ and $\mathscr{F}_i$, $i \in (I_\mathrm{c} \cup I_\mathrm{z}) \cap (I_\mathrm{l} \cup I_\mathrm{i})$ are small, then the gradients of all selection functions $\boldsymbol{H}^{(j)}$ given by* (27) *have the maximal rank for any $\boldsymbol{P} \in \mathbb{R}^{2n_\Omega}$ and any $\boldsymbol{U}'_D \in \mathbb{R}^{2n_D}$.*

Proof. We shall argue similarly as in [3, proof of Proposition 3.2]. By (27),

$$\boldsymbol{\nabla} \boldsymbol{H}^{(j)}(\bar\gamma, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = \begin{pmatrix} -\boldsymbol{P} & \boldsymbol{K} & -\boldsymbol{B}_D^\top & -\boldsymbol{B}_\nu^\top & -\boldsymbol{B}_\tau^\top \\ -\boldsymbol{U}'_D & \boldsymbol{B}_D & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{\mathrm{F},I} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{B}_\nu)_{\mathrm{C},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{\mathrm{N},I} \\ -(\boldsymbol{V}_\tau)_\mathrm{S} & (\boldsymbol{C}_\tau)_{\mathrm{S},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -(\boldsymbol{S}\boldsymbol{F}\boldsymbol{B}_\nu)_{\mathrm{L},J} & \boldsymbol{0} & (\boldsymbol{S}\boldsymbol{F})_{\mathrm{L},I} & \boldsymbol{I}_{\mathrm{L},I} \end{pmatrix},$$

where $J = \{1, \dots, 2n_\Omega\}$, F denotes jointly the indices from I_f and I_z^f, C the indices from I_c and I_z^c, N the indices from I_f, I_z^f and I_0, S the indices from I_s, I_ci^s and $I_\mathrm{zi}^\mathrm{cs}$, and L the indices from $(I_\mathrm{c} \cap I_\mathrm{l})$, I_zl^c, I_ci^l, $I_\mathrm{zi}^\mathrm{cl+}$ and $I_\mathrm{zi}^\mathrm{cl-}$. Further,

$$\boldsymbol{S} = \mathbf{Diag}(s_{11}, \dots, s_{n_c n_c}), \quad s_{ii} = \begin{cases} s_i & \text{if } i \in (I_\mathrm{c} \cap I_\mathrm{l}) \cup I_\mathrm{zl}^\mathrm{c} \cup I_\mathrm{ci}^\mathrm{l} \cup I_\mathrm{zi}^\mathrm{cl+} \cup I_\mathrm{zi}^\mathrm{cl-}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\boldsymbol{F} = \mathbf{Diag}(\mathscr{F}_1, \dots, \mathscr{F}_{n_c}).$$

Suppose first that $\nabla^2 g = \boldsymbol{0}$ and $\mathscr{F}_i = 0$ for any $i \in (I_\mathrm{c} \cup I_\mathrm{z}) \cap (I_\mathrm{l} \cup I_\mathrm{i})$. Then $\boldsymbol{K}$ is positive definite on $\ker(\boldsymbol{B}_D)$, $\boldsymbol{C}_\tau = \boldsymbol{B}_\tau$, $\boldsymbol{S}\boldsymbol{F} = \boldsymbol{0}$ and we claim that the partial gradient $\boldsymbol{\nabla}_{(\boldsymbol{u}, \boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)} \boldsymbol{H}^{(j)}(\bar\gamma, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ is non-singular for any $\boldsymbol{H}^{(j)}$. Indeed, one has

$$\boldsymbol{\nabla}_{(\boldsymbol{u}, \boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)} \boldsymbol{H}^{(j)}(\bar\gamma, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_D, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) = \begin{pmatrix} \boldsymbol{K} & -\boldsymbol{B}_D^\top & -\boldsymbol{B}_\nu^\top & -\boldsymbol{B}_\tau^\top \\ \boldsymbol{B}_D & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{\mathrm{F},I} & \boldsymbol{0} \\ (\boldsymbol{B}_\nu)_{\mathrm{C},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{\mathrm{N},I} \\ (\boldsymbol{B}_\tau)_{\mathrm{S},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{\mathrm{L},I} \end{pmatrix},$$

which is obviously a non-singular matrix iff the matrix

$$\boldsymbol{J} := \begin{pmatrix} \boldsymbol{K} & -\boldsymbol{B}_D^\top & -(\boldsymbol{B}_\nu^\top)_{J,\mathrm{C}} & -(\boldsymbol{B}_\tau^\top)_{J,\mathrm{S}} \\ \boldsymbol{B}_D & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{B}_\nu)_{\mathrm{C},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{B}_\tau)_{\mathrm{S},J} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},$$

is non-singular. But taking any $(\boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_{\nu\mathrm{C}}, \boldsymbol{\lambda}'_{\tau\mathrm{S}})$ such that $\boldsymbol{J}(\boldsymbol{u}', \boldsymbol{\lambda}'_D, \boldsymbol{\lambda}'_{\nu\mathrm{C}}, \boldsymbol{\lambda}'_{\tau\mathrm{S}}) = \boldsymbol{0}$, one gets from the last three blocks of lines that

$$\boldsymbol{u}' \in (\ker(\boldsymbol{B}_D) \cap \ker((\boldsymbol{B}_\nu)_{\mathrm{C},J}) \cap \ker((\boldsymbol{B}_\tau)_{\mathrm{S},J})) \subset \ker(\boldsymbol{B}_D).$$

This together with the first block of lines of the system gives

$$0 = \boldsymbol{u}'^\top(\boldsymbol{K}\boldsymbol{u}' - \boldsymbol{B}_D^\top\boldsymbol{\lambda}'_D - (\boldsymbol{B}_\nu^\top)_{J,\mathrm{C}}\boldsymbol{\lambda}'_{\nu\mathrm{C}} - (\boldsymbol{B}_\tau^\top)_{J,\mathrm{S}}\boldsymbol{\lambda}'_{\tau\mathrm{S}}) = \boldsymbol{u}'^\top\boldsymbol{K}\boldsymbol{u}'$$

and positive definiteness of $\boldsymbol{K}$ on $\ker(\boldsymbol{B}_D)$ implies that $\boldsymbol{u}' = \boldsymbol{0}$. Consequently,

$$\boldsymbol{B}_D^\top\boldsymbol{\lambda}'_D + (\boldsymbol{B}_\nu^\top)_{J,\mathrm{C}}\boldsymbol{\lambda}'_{\nu\mathrm{C}} + (\boldsymbol{B}_\tau^\top)_{J,\mathrm{S}}\boldsymbol{\lambda}'_{\tau\mathrm{S}} = \boldsymbol{0}$$

and taking into account Remark 3, we have necessarily $\boldsymbol{\lambda}'_D = \boldsymbol{0}$, $\boldsymbol{\lambda}'_{\nu\mathrm{C}} = \boldsymbol{0}$ and $\boldsymbol{\lambda}'_{\tau\mathrm{S}} = \boldsymbol{0}$ due to full row rank of $\begin{pmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_\nu \\ \boldsymbol{B}_\tau \end{pmatrix}$. This shows that $\boldsymbol{J}$ is non-singular and the conclusion is valid in the special case.

Continuous dependence on the elements of $\nabla^2 g/\|\nabla g\|$ and $\mathscr{F}_i$ completes the assertion. □

5 Numerical Experiments

This section presents results computed with the predictor-corrector continuation method described in [16, Section 4] and performed with the finite-element library GetFEM++ [19].

5.1 V-Shaped Body

The first experiment is done for a V-shaped body from [17, Subsection 8.2], which is depicted in Figure 7(a). It is approximately 0.82 m wide, 0.8 m high and we consider a plane-strain model of linearised elasticity with the Lamé constants $\lambda = 100$ GN/m^2 and $\mu = 82$ GN/m^2. The body is subject to volume forces of the density $\boldsymbol{f}(\gamma) = (-5\,\mathrm{GN/m^3}, 1-8(1-\gamma)\,\mathrm{GN/m^3})$ and it is fixed along both parts of Γ_D and free of surface tractions on Γ_N, that is, $\boldsymbol{U}_D = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$. The continuation problem is taken with $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{V}_\tau = \boldsymbol{0}$.

Three unstructured meshes with 2220, 8880 and 35520 linear triangles (M2220, M8880 and M35520) are used for the finite-element discretisation. The numerical continuation is initialised with $\varepsilon = \varepsilon' = 5 \cdot 10^{-12}$, $c_{\min} = 0.99999$, $j_{\mathrm{thr}} = 4$, $j_{\max} = 5$, $h_{\mathrm{dec}} = 0.5$, $h_{\mathrm{inc}} = 1.3$, $h_{\min} = 5 \cdot 10^{-7}$, $h_{\max} = 0.1$ and after a change of the tangent performed by Algorithm 2 from [16], the predictor-corrector is restarted with $h = 5 \cdot 10^{-4}$.