HAL open science

Cauchy-Gelfand problem for quasilinear conservation law

Guennadi Khenkine, Alexandre Shananin

To cite this version:

Guennadi Khenkine, Alexandre Shananin. Cauchy-Gelfand problem for quasilinear conservation law. 2013. hal-00877591

HAL Id: hal-00877591

https://hal.science/hal-00877591

Submitted on 28 Oct 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Cauchy-Gelfand problem for quasilinear conservation law.

G.M. Henkin[a,1,*] **and A.A.Shananin**[b,1]

Abstract. We obtain the precise asymptotic ($t \to \infty$) for solution $f(x,t)$ of Cauchy-Gelfand problem for quasilinear conservation law $\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = 0$ with initial data of bounded variation $f(x,0) = f^0(x)$. The main theorem develops results of T.-P.Liu (1981), Kruzhkov, Petrosjan (1987), Henkin, Shananin (2004), Henkin (2012). Proofs are based on vanishing viscosity method and localized Maxwell type conservation laws. The main application consists in the reconstruction of parameters of initial data responsible for location of inviscid shock waves in the solution $f(x,t)$.

AMS subject classification: 35K55; 35L65; 35Q20; 35R10; 39A; 76D

Key words: Riemann-Burgers type equations, fluid mechanics, quasilinear conservation law, Cauchy-Gelfand problem, difference- differential equations, vanishing viscocity method, shock waves

[a] Université Pierre et Marie Curie, case 247, 75252, Paris, France and CEMI, Academy of Science, 117418, Moscow, Russia

$*$ corresponding author: e-mail address: henkin@math.jussieu.fr

[b] Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Russia, e-mail address: alexshan@yandex.ru

[1] This work was partially supported by TFP No, 14.A18.21.0866 of the Ministry of Education and Science of the Russian Federation

Introduction.

We study Cauchy (and inverse Cauchy) problem for equation

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \tag{$*$}$$

with initial data $f(x,0) = f^0(x)$. The most natural (not equivalent) definitions for solutions of problem (*) consists in the existence of solutions $f \stackrel{\text{def}}{=} f_\varepsilon(x,t)$ for equations

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \varepsilon\frac{\partial^2 f}{\partial x^2}, \quad \varepsilon > 0, \quad x \in \mathbb{R}, \tag{1a}$$

or

$$\frac{df}{dt} + \varphi(f)\frac{f(x,t) - f(x-\varepsilon,t)}{\varepsilon} = 0, \quad x \in \mathbb{R}, \tag{1b}$$

with property

$$f_\varepsilon(x,0) = f^0(x), \quad \varepsilon \geq 0, \tag{2}$$

such that $f_\varepsilon(x,t) \to f_0(x,t)$, when $\varepsilon \to +0$.

Equation (1a) with linear $f \mapsto \varphi(f)$ was introduced at first by Riemann (1860) (for $\varepsilon = +0$) and later by Bateman (1915), Burgers (1939) and Hopf (1950) (for $\varepsilon > 0$) as the simplest approximation to the equations of fluid dynamics.

Equation (1a) for general $\varphi(f)$ was introduced later in a very different models: displacements of oil by water (Buckley, Leverett, 1942), consolidation of wet soil (Florin, 1948), the road trafic (Lighthill, Whitham, 1955) etc. Equation (1b) was introduced by Polterovich, Henkin, 1988, for description of a Schumpeterian evolution of industry. In physical applications of (1a) the main interest has the inviscid case, when $\varepsilon = +0$, but the application of (1a) in the transport flow theory and of (1b) in Schumpeterian dynamics the main interest presents the viscid case, when $\varepsilon > 0$.

It is important to remark that behavior of solutions of (1b) with $\varepsilon = +0$ is not the same as the behavior of solutions of (1a) with $\varepsilon = +0$, in spite that for $\varepsilon = 0$ the both equations (1a), (1b) look identical.

In fact, equation (1b) is a semi-discrete approximation of the non conservative equation

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \frac{\varepsilon}{2}\varphi(f)\frac{\partial^2 f}{\partial x^2}.$$

Assumption 1.

Let $\alpha^- < \alpha^+$, $f^0(x)$ be real-valued function of bounded variation on $\mathbb{R}$ such that $f^0(x) = \alpha^\pm$, if $\pm x \geq \pm x^\pm$, $x^- < x^+$. Let $\varphi(f)$ be a positive, continuous differentiable function of real variable f such that $\varphi'(f)$ has only isolated zeros.

Theorem ([5], [19], [10], [11]).

Under assumption 1 and $\forall \varepsilon > 0$ the following general properties of Cauchy problems (1a,b) are valid.

a) Cauchy problem (1a), (2) has a unique (weak) solution $f(x,t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$. This solution satisfies Rankine-Hugoniot conservation laws for $t \geq 0$:

$$f(x,t) \to \alpha^{\pm}, \quad \text{if} \quad x \to \pm\infty, \quad \text{and}$$

$$\frac{d}{dt}\Big[\int_{-\infty}^{0} (\alpha^- - f(x,t))dx + \int_{0}^{\infty} (\alpha^+ - f(x,t))dx = \int_{\alpha^-}^{\alpha^+} \varphi(y)dy.$$

Moreover, if the initial data $f^0(x)$ is nondecreasing in x then $f(x,t)$ is nondecreasing in $x \forall\ t \geq 0$.

b) Cauchy problem (1b), (2) has a unique (weak) solution $f(x,t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$. This solution satisfies the following conservation laws for $t \geq 0$ and $\theta \in [0,1)$:

$$f(k\varepsilon + \theta\varepsilon, t) \to \alpha^{\pm}, \quad \text{if} \quad k \to \pm\infty, \quad k \in \mathbb{Z}, \quad \text{and}$$

$$\frac{d}{dt}\Big[\sum_{-\infty}^{0} \int_{f(k\varepsilon+\theta\varepsilon,t)}^{\alpha^-} \frac{dy}{\varphi(y)} + \sum_{1}^{\infty} \int_{f(k\varepsilon+\theta\varepsilon,t)}^{\alpha^+} \frac{dy}{\varphi(y)}\Big] = \frac{1}{\varepsilon}(\alpha^+ - \alpha^-).$$

Moreover, if for some $\theta \in [0,1)$ the initial data $f^0(k\varepsilon+\theta\varepsilon,t)$ is nondecreasing in $k \in \mathbb{Z}$, then $f(k\varepsilon + \theta\varepsilon, t)$ is nondecreasing in $k \in \mathbb{Z}$ with the same θ.

Put

$$\psi(u) = -\int_{\alpha^-}^{u} \varphi(y)dy, \quad u \in [\alpha^-, \alpha^+], \quad \text{for} \quad (1.a), \tag{3a}$$

$$\psi(u) = \int_{\alpha^-}^{u} \frac{dy}{\varphi(y)}, \quad u \in [\alpha^-, \alpha^+], \quad \text{for} \quad (1.b), \tag{3b}$$

Let us introduce respectively for (3a) and for (3b) the concave function $\hat{\psi}(u)$ as the upper bound of the convex hull of the set

$$\{(u,v) : \ v \leq \psi(u), \quad u \in [\alpha^-, \alpha^+]\}.$$

Assumption 2. Suppose that for (3a) and respectively for (3b) the set $S = \{u \in [\alpha^-, \alpha^+] : \ \psi(u) < \hat{\psi}(u)\}$ has the following form

$$\begin{aligned} &S = (\alpha_0^-, \alpha_0^+) \cup (\alpha_1^-, \alpha_1^+) \cup \ldots (\alpha_L^-, \alpha_L^+), \quad \text{where} \\ &\alpha^- = \alpha_0^- < \alpha_0^+ < \alpha_1^- < \alpha_1^+ < \ldots < \alpha_{L-1}^- < \alpha_{L-1}^+ < \alpha_L^- < \alpha_L^+ = \alpha^+. \end{aligned} \tag{4a, b}$$

Let

$$c_l = \frac{1}{\alpha_l^+ - \alpha_l^-} \int_{\alpha_l^-}^{\alpha_l^+} \varphi(y)dy, \quad \text{for} \quad (1a), \quad l = 0, \ldots, L, \tag{5a}$$

$$c_l = (\alpha_l^+ - \alpha_l^-)\Big(\int_{\alpha_l^-}^{\alpha_l^+} \frac{dy}{\varphi(y)}\Big)^{-1}, \quad \text{for} \quad (1b), \quad l = 0, \ldots, L. \tag{5b}$$

Assumptions 1, 2 and notation (5a,b) imply the following important inequalities (for (1a)) and respectively for (1b):

$$\begin{aligned} &\varphi(\alpha_l^+) \le c_l \le \varphi(\alpha_l^-), \quad l = 0, \dots, L, \\ &c_l = \varphi(\alpha_l^-), \quad l = 1, \dots, L, \\ &c_l = \varphi(\alpha_l^+), \quad l = 0, \dots, L-1. \end{aligned} \tag{6}$$

Let us remark that the inequalities above are, in fact, equalities except for the cases $l = 0$ and $l = L$.

Motivated by models of fluid mechanics Gelfand, [5], had formulated the following problem:

$\forall \varepsilon \ge 0$ to find asymptotic ($t \to \infty$) of solution $f(x,t)$ of equation (1a) with initial condition (2).

Gelfand had found solution of this problem for the case $\varepsilon = +0$ with special (Riemann type) initial conditions

$$f(x,0) = \begin{cases} \alpha^-, & \text{if } \ x < x_0 \\ \alpha^+, & \text{if } \ x > x_0 \end{cases}$$

and had noted that it would be interesting to prove that the main term of the asymptotic ($t \to \infty$) of $f(x,t)$, satisfying (1a),(2), coincides with the solution of (1a), (2) with $\varepsilon = +0$.

Motivated by models of economical development similar problems were considered later [11], [12] for equation (1b).

Theorem. (Gelfand, 1959).

Under assumptions 1, 2, solution of (1a) with $\varepsilon = +0$ with initial condition (2): $f(x,0) = \alpha^{\pm}$, if $\pm(x - x_0) > 0$, has the following form:

$$f(x,t) = \begin{cases} \alpha^-, & \text{if } \ x - x_0 < c_0 t \\ \alpha^+, & \text{if } \ x - x_0 \ge c_L t \\ \varphi^{(-1)}\left(\frac{x-x_0}{t}\right), & \text{if } c_l t \le x - x_0 < c_{l+1} t, \quad 0 \le l < L. \end{cases}$$

The Gelfand problem for (1a), (2) with $\varepsilon \ge 0$ and with monotonic $\varphi(f)$ was solved by Iljin and Oleinik [16].

Theorem (Iljin, Oleinik, 1960).

Let under assumptions 1, 2 f be solution of (1a), (2), $\varepsilon = +0$ and $\varphi'(f) < 0$. Then $\exists\ t_0 > 0$ such that

$$f(x,t) = \begin{cases} \alpha^-, & \text{if } x < ct + x_0 \\ \alpha^+, & \text{if } x > ct + x_0, \ t \ge t_0 \end{cases}$$

where shift parameter x_0 is determined by Maxwell formula:

$$\int_{-\infty}^{x_0} (f^0(x) - \alpha^-)dx + \int_{x_0}^{\infty} (f^0(x) - \alpha^+)dx = 0,$$

and c is determined by Rankine-Hugoniot formula

$$c = \frac{1}{\alpha^+ - \alpha^-} \int\limits_{\alpha^-}^{\alpha^+} \varphi(y)dy.$$

For semi-discrete initial problem (1b), (2) with $\varepsilon \geq 0$ the analogues of the Iljin-Oleinik results had been obtained in [11].

The following result of Kruzhkov and Petrosjan [17] gives solution of Gelfand problem for equation (1a) with $\varepsilon = +0$ and with nondecreasing initial data (2).

Theorem (Kruzhkov, Petrosjan, 1987).

Let under assumptions 1, 2, $f(x,t)$ be solution of the Cauchy problem (1a), (2) with $\varepsilon = +0$ and with nondecreasing initial data function $f_0(x)$. Let $\tilde{f}(x,t)$ be solution of the Cauchy problem (1a), (2) with $\varepsilon = +0$, where the function $\varphi \stackrel{\text{def}}{=} -\psi'$ is replaced by the function $\tilde{\varphi} = -\hat{\psi}'$ and the initial function $f_0(x)$ is replaced by the function

$$\tilde{f}_0(x) = u_1\chi_{(-\infty,x_1)}(x) + u_2\chi_{(x_1,x_2)}(x) + \ldots + u_m\chi_{(x_{m-1},+\infty)}(x),$$

where

$$x_i = \frac{F_0^*(u_{i+1}) - F_0^*(u_i)}{u_{i+1} - u_i}, \quad i = 1, \ldots, m-1, m = 2L+2,$$

$$u_1 = \alpha_0^-, u_2 = \alpha_0^+, \ldots, u_{m-1} = \alpha_L^-, u_m = \alpha_L^+,$$

$$F_0(y) = \int\limits_0^y f_0(x)dx, \quad F_0^*(p) = \sup_{p\in\mathbb{R}}\{py - F_0(y)\},$$

$\chi_{(a,b)}$ is the characteristic function of $(a,b) \subset \mathbb{R}$.

Then $\|f(\cdot,t) - \tilde{f}(\cdot,t)\|_{L^1(\mathbb{R})} \to 0$, $t \to \infty$ and the asymptotic locations $\{c_l t + d_l,\ l = 0, 1, \ldots, L\}$ of shock waves for $f(x,t)$ coincide with the asymptotic locations of shock waves for $\tilde{f}(x,t)$, and so the shifts d_l, $l = 0, \ldots, L$, can be found explicitly.

Remark 1.

The proof of Theorem in [17] is based on the explicit formula of E.Hopf [15] and M.Bardi, L.C.Evans [2] for the solutions of (1a), (2) with $\varepsilon = +0$ and nondecreasing initial data $f_0(x)$

$$f(x,t) = \frac{\partial}{\partial x} \sup_{p\in\mathbb{R}} I(t,x,p),$$

where

$$I(t,x,p) = (px + \psi(p)t) - \sup_{y\in\mathbb{R}}(py - F_0(y)), \quad F_0(y) = \int\limits_0^y f_0(x)dx.$$

Remark 2.

N.S.Petrosjan [20] had announced that under assumption 1 the result of [17] is still valid for piecewise smooth solutions of the problem (1a), (2) with not necessary monotonic initial data $f_0(x)$ with the property $f_0(x) \in (\alpha^-, \alpha^+)\ \forall\ x \in (x^-, x^+)$.

Assumption 3. Let for (1a) and respectively for (1b) the following inequalities be valid

$$\begin{aligned}
&\varphi'(\alpha_l^-) \neq 0, \quad l = 1, \dots, L,\\
&\varphi'(\alpha_l^+) \neq 0, \quad l = 0, 1, \dots, L-1,\\
&\varphi(\alpha_0^-) \neq c_0, \quad \text{if} \quad \alpha_0^- < \alpha_0^+,\\
&\varphi(\alpha_L^+) \neq c_L, \quad \text{if} \quad \alpha_L^- < \alpha_L^+.
\end{aligned}$$

By developing of [17] and of [7], we obtain here the following

Main theorem.

i) Under the assumptions 1, 2, 3, the solutions $f(x,t)$ of the Cauchy-Gelfand problem (1a,b), (2) with $\varepsilon = +0$ have the following asymptotic structure

$$\|f(\cdot,t) - \tilde{f}(\cdot,t)\|_{L^1(\mathbb{R})} \to 0, \quad t \to \infty,$$

$$\tilde{f}(x,t) = \begin{cases} \alpha^-, & \text{if } x < c_0 t + d_0 \\ \varphi^{(-1)}(x/t), & \text{if } c_l t + d_l \le x < c_{l+1} t + d_{l+1},\ l = 0, 1, \dots, L-1 \\ \alpha^+, & \text{if } x \ge c_L t + d_L, \end{cases}$$

where parameters $\{c_l\}$ determined by (5a) (respectively by (5b)), parameters $\{d_l\}$ are determined by the respective equations (1a,b) and initial data (2a, 2b).

ii) Moreover, $\exists\ t^* \ge 0$ such that parameters $\{d_l\}$ for problem (1a), (2a) are determined for $t \ge t^*$ by Maxwell type formulas

$$\int\limits_{x=y_l^-(t)}^{c_l t + d_l(t)} (f(x,t) - \alpha_l^-)dx + \int\limits_{x=c_l t + d_l(t)}^{y_l^+(t)} (f(x,t) - \alpha_l^+)dx = 0,$$

$$f(y_l^-(t), t) = \alpha_l^-, \quad f(y_l^+(t), t) = \alpha_l^+, \quad \text{where} \quad d_l(t) = d_l(t^*) \quad \text{if} \quad t \ge t^*,$$

and parameters $\{d_l\}$ for problem (1b), (2b) are determined for $t \ge t^*$ by formulas

$$\int\limits_{x=y_l^-(t)}^{c_l t + d_l(t)} [\Psi(f(x,t)) - \Psi(\alpha_l^-)]dx + \int\limits_{x=c_l t + d_l(t)}^{y_l^+(t)} [\Psi(f(x,t)) - \Psi(\alpha_l^+)]dx = 0,$$

$$\text{where} \quad \Psi(f) = \int\limits_{\alpha^-}^{f} \frac{dy}{\varphi(y)}, \quad d_l(t) = d_l(t^*) \quad \text{if} \quad t \ge t^*.$$

The crucial statement of main theorem consists in the equalities $d_l(t) \overset{!}{=} d_l(t^*)$, if $t \ge t^*$.

Remark 3.

Theorem of Kruzhkov, Petrosjan [17] is the corollary of main theorem, because for nondecreasing initial data parameter t^* in the part ii) of main theorem can be taken by zero.

Remark 4.

Early T.-P.Liu [18] and A.V.Gasnikov [4] had obtained (only under assumption 1) a rough versions of part i) of main theorem with shift functions $d_l(t) = o(t)$ instead of constant shifts d_l.

1. Comparison result.

For the proof of the main theorem we need the following comparison result developing Proposition 1 from [6].

Theorem 1.

Under the assumptions 1-3 and definitions (3a,b)-(5a,b) $\forall$ solution $f = f_\varepsilon(x,t)$ of (1b), (2) (respectively (1a), (2)) $\exists\ t_0 > 0$ such that
$\forall t \geq t_0$, $\forall \varepsilon > 0$ and for $\gamma > 0$, $b_l > O(1/\gamma)$, $l = 0, \dots, L$, the following estimate is valid:

$$\varphi^{(-1)}\big(\frac{x - \gamma\sqrt{\varepsilon t}}{t}\big) \leq f_\varepsilon(x,t) \leq \varphi^{(-1)}\big(\frac{x + \gamma\sqrt{\varepsilon t}}{t}\big), \tag{1.1}$$

for $x \in [c_l t + b_l\sqrt{\varepsilon t}, c_{l+1}t - b_{l+1}\sqrt{\varepsilon t}]$.

For the proof of Theorem 1 we can not just apply rescaling of corresponding Proposition 1 from [6], because now we must take into account that under conditions of Theorem 1 initial function $f^0(x) = f(x,0)$ is independent of $\varepsilon > 0$. So, we will follow the scheme of the proof of Proposition 1 from [6], precising the dependence of all parameters on $\varepsilon > 0$. We will give detailed proof only for the case of equation (1b), (2) with $\varepsilon > 0$, $L = 1$,

$$\varphi(\alpha_0^-) > c_0 = \varphi(\alpha_0^+), \quad \varphi(\alpha_1^-) = c_1 > \varphi(\alpha_1^+).$$

The following statement generalizes essentially Proposition 1 of [8].

Lemma 1. Under assumptions of Theorem 1, let $L = 1$; $\alpha_0^- < \alpha_0^+ < \alpha_1^- < \alpha_1^+$; and let c_0, c_1 be the parameters defined by (3b), (4b), (5b). Put

$$\Delta_x f(x,t) = \frac{f_\varepsilon(x,t) - f_\varepsilon(x-\varepsilon,t)}{\varepsilon}.$$

Let $\tilde{f}_l(x - c_l t)$ be travelling wave solutions of (1b) such that $\tilde{f}_l(x) \to \alpha_l^\pm$, $l = 0,1$, $x \to \pm\infty$ and $\tilde{f}_l(0) = \frac{\alpha_l^- + \alpha_l^+}{2}$ (see Prop. 0 in [6]). Consider the following functions $f^\pm(x,t)$, depending also on parameters $\{\alpha_l^\pm\}$, $\{c_l\}$, $l = 0,1$, positive small parameters γ and δ and positive bounded functions $b_0^\pm(t)$, $b_1^\pm(t)$:

$$f^-(x,t) = \begin{cases} f_0^-(x,t) = \tilde{f}_0\big(\frac{x - c_0 t}{\varepsilon}\big), -\infty < x < c_0 t + b_0^-\sqrt{\varepsilon t}, \\ f_{01}^-(x,t) = \varphi^{(-1)}\big(\frac{x - \gamma\sqrt{\varepsilon t}}{t}\big) - \frac{c_0\varepsilon}{\varphi'(\alpha_0^+)(x - c_0 t)}, \\ \text{where } \ c_0 t + b_0^-\sqrt{\varepsilon t} \leq x \leq c_1 t + b_1^-\sqrt{\varepsilon t}, \\ f_1^-(x,t) = \tilde{f}_1\big(\frac{x - c_1 t - (2\sqrt{c_1} + \gamma + 2\delta)\sqrt{\varepsilon t}}{\varepsilon}\big), \quad c_1 t + b_1^-\sqrt{\varepsilon t} < x < +\infty. \end{cases} \tag{1.2}$$

$$f^+(x,t)=\begin{cases} f_0^+(x,t)=\tilde f_0\big(\frac{x-c_0t+(2\sqrt{c_0}+\gamma+2\delta)\sqrt{\varepsilon t}}{\varepsilon}\big),\\ \text{where}\quad -\infty<x<c_0t-b_0^+\sqrt{\varepsilon t},\\ f_{0,1}^+(x,t)=\varphi^{(-1)}\big(\frac{x+\gamma\sqrt{\varepsilon t}}{t}\big)+\frac{c_1\varepsilon}{\varphi'(\alpha_1^-)(c_1t-x)},\\ \text{where}\quad c_0t-b_0^+\sqrt{\varepsilon t}\le x\le c_1t-b_1^+\sqrt{\varepsilon t},\\ f_1^+(x,t)=\tilde f_1\big(\frac{x-c_1t}{\varepsilon}\big),\quad c_1t-b_1^+\sqrt{\varepsilon t}<x<+\infty. \end{cases} \tag{1.3}$$

Then the following statements are valid:
i) $\forall\gamma,\delta>0\ \ \exists$ functions

$$\begin{aligned} b_0^-(t)&=\gamma+o(1),\\ b_1^-(t)&=\gamma+\sqrt{c_1}+\delta+\sqrt{\delta^2+2\delta\sqrt{c_1}}+o(1),\\ b_0^+(t)&=\gamma+\sqrt{c_0}+\delta+\sqrt{\delta^2+2\delta\sqrt{c_0}}+o(1),\\ b_1^+(t)&=\gamma+o(1), \end{aligned}$$

satisfying for $t\ge\tilde t_0\varepsilon=t_0$ and $\theta\in[0,1]$ relations:

$$\begin{aligned} &f_0^-(c_0t+b_0^-\sqrt{\varepsilon t},t)=f_{01}^-(c_0t+b_0^-\sqrt{\varepsilon t},t)\\ &\Delta_x f_0^-(c_0t+b_0^-\sqrt{\varepsilon t}+\varepsilon\theta,t)<\Delta_x f_{01}^-(c_0t+b_0^-\sqrt{\varepsilon t}+\varepsilon\theta,t)\\ &f_{01}^-(c_1t+b_1^-\sqrt{\varepsilon t},t)=f_1^-(c_1t+b_1^-\sqrt{\varepsilon t},t)\\ &\Delta_x f_{01}^-(c_1t+b_1^-\sqrt{\varepsilon t}+\varepsilon\theta,t)<\Delta_x f_1^-(c_1t+b_1^-\sqrt{\varepsilon t}+\varepsilon\theta,t); \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f_0^+(c_0t-b_0^+\sqrt{\varepsilon t},t)=f_{01}^+(c_0t-b_0^+\sqrt{\varepsilon t},t)\\ &\Delta_x f_0^+(c_0t-b_0^+\sqrt{\varepsilon t}+\varepsilon\theta,t)>\Delta_x f_{01}^+(c_0t-b_0^+\sqrt{\varepsilon t}+\varepsilon\theta,t)\\ &f_{01}^+(c_1t-b_1^+\sqrt{\varepsilon t},t)=f_1^+(c_1t-b_1^+\sqrt{\varepsilon t},t)\\ &\Delta_x f_{01}^+(c_1t-b_1^+\sqrt{\varepsilon t}+\varepsilon\theta,t)>\Delta_x f_1^+(c_1t-b_1^+\sqrt{\varepsilon t}+\varepsilon\theta,t). \end{aligned} \tag{1.5}$$

ii) $\forall\gamma,\delta>0$ and with $b_0^\pm$, $b_1^\pm$ from i) $\ \exists\ \tilde t_0>0$ such that the functions $f^\mp(x,t)$, $x\in\mathbb{R}$, $t\ge\tilde t_0\varepsilon$, are sub(super)solutions for (1b), i.e.

$$\pm\Big[\frac{df^\pm}{dt}+\varphi(f^\pm)\big(\frac{f^\pm(x,t)-f^\pm(x-\varepsilon,t)}{\varepsilon}\big)\Big]\ge 0. \tag{1.6}$$

Complement. Lemma 1 is also valid for equation (1a) if in definitions of $f_{01}^\pm(x,t)$ numerators c_0, c_1 are replaced by 2, the differences $\Delta_x f_0^\mp$, $\Delta_x f_1^\mp$ by derivatives $\frac{\partial f_0^\mp}{\partial x}$, $\frac{\partial f_{01}^\mp}{\partial x}$, $\frac{\partial f_1^\mp}{\partial x}$ and inequality (1.6) in ii) by inequality

$$\pm\Big[\frac{\partial f^\pm}{\partial t}+\varphi(f^\pm)\frac{\partial f^\pm}{\partial x}-\varepsilon\frac{\partial^2 f^\pm}{\partial x^2}\Big]\ge 0.$$

Proof.
Lemma 1 of this paper follows from Lemma 1 of [6], by simple rescaling

$$\tilde{t} \to \frac{t}{\varepsilon}, \quad \tilde{x} \to \frac{x}{\varepsilon}, \quad \tilde{t}_0 \to \frac{t_0}{\varepsilon}.$$

Lemma 1 is proved.

Let further function $f \mapsto \varphi(f)$ be extended outside $[\alpha^-, \alpha^+]$ keeping assumption 1 and the condition $\varphi'(f) < 0$, if $f \le \alpha^-$ or $f \ge \alpha^+$.

By results of [5], [19], [11], (see Prop. 0 in [6]) $\forall$ small $\sigma > 0$ $\exists$ travelling type sub(super)solutions of (1b) (resp. of (1a)) of the form

$$\tilde{f}^{\mp}_{l,\sigma}\big(\frac{x}{\varepsilon} - c^{\mp}_{l,\sigma}\frac{t}{\varepsilon} \mp d(\frac{t}{\varepsilon})\big) \tag{1.7}$$

with overfalls $[\alpha^-_l \mp (-1)^l\sigma, \alpha^+_l \pm (-1)^l\sigma]$ and $\tilde{f}^{\mp}_{l,\sigma}(0) = \frac{1}{2}(\alpha^-_l + \alpha^+_l)$, $l = 0, 1$. For parameters $c^{\mp}_{l,\sigma}$ we have $c^{\mp}_{l,\sigma} = c_l(1 \pm O_+(\sigma))$.

Let us replace in the definitions of $f^{\mp}(x,t)$ in the statement of Lemma 1 the travelling waves $\tilde{f}_l(x - c_l t)$, $l = 0, 1$, by σ-modified travelling sub(super)solutions (1.7) and rare type functions $f^{\mp}_{01}(x,t)$ by the σ-modified rare type sub(super)solutions for (1b) (resp. (1a)) of the form

$$\begin{aligned} f^-_{01,\sigma}(x,t) &= \varphi^{(-1)}\big(\frac{x - \gamma\sqrt{\varepsilon t}}{t}\big) - \frac{c^-_{0,\sigma}\varepsilon}{\varphi'(\alpha^{+-}_0)(x - c^-_{0,\sigma}t)}, \\ f^+_{01,\sigma}(x,t) &= \varphi^{(-1)}\big(\frac{x + \gamma\sqrt{\varepsilon t}}{t}\big) + \frac{c^+_{1,\sigma}\varepsilon}{\varphi'(\alpha^{-+}_1)(c^+_{1,\sigma}t - x)}. \end{aligned} \tag{1.8}$$

Lemma 2.
Let $f^{\mp}_{\sigma}(x,t)$ be functions of the form (1.2), (1.3), where parameters $\alpha^{\pm}_l$, c_l, $b^{\pm}_l$, $l = 0, 1$, are replaced by the σ-modified parameters:

$$\begin{aligned} &\alpha^{-\mp}_{l,\sigma} = \alpha^-_l \mp (-1)^l\sigma, \quad \alpha^{+\mp}_{l,\sigma} = \alpha^+_l \pm (-1)^l\sigma, \\ &b^{\mp}_{l,\sigma} = b^{\mp}_l \pm O_+(\sigma), \\ &c^{\mp}_{l,\sigma} = c_l(1 \pm O_+(\sigma)). \end{aligned} \tag{1.9}$$

Put $\sigma(\frac{t}{\varepsilon}) = \frac{\varepsilon}{t}$ and $d(\frac{t}{\varepsilon}) = \rho(\ln\frac{t}{\varepsilon})$, $\rho \ge \rho_0$. Then functions $f^{\mp}_{\sigma}(x,t)$ satisfy σ- modified relations i), ii) from Lemma 1, if parameter $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ are big enough.

Proof. σ- modified relation i) for $f^{\mp}_{\sigma}(x,t)$ follows from non-modified relations (1.4), (1.5), taking into account that modified parameters (1.9) coinside with non-modified parameters up to $O(\frac{\varepsilon}{t})$. Taking parameter $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ big enough permits to keep sense of strict inequalities in modified relations (1.4), (1.5), i.e. in i). σ- modified relation ii) follows from σ-modified relation i), from non-modified estimates (1.6) and from estimates of derivatives

$$\frac{\partial}{\partial t}d(\frac{t}{\varepsilon}) > 0, \quad \frac{\partial}{\partial t}\sigma(\frac{t}{\varepsilon}) < 0$$

permitting to keep sence of σ-modified inequalities (1.6), if parameter $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ is big enough.

Lemma 3.

Let $f = f(x,t)$ be solution of (1b), (2) (resp. (1a),(2)) with $L = 1$. Let $\sigma(\frac{t}{\varepsilon}) = \frac{\varepsilon}{t}$, $d(\frac{t}{\varepsilon}) = \rho(\ln \frac{t}{\varepsilon})$. If parameters $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ and ρ are big enough, then function $f(x,t)$ satisfies $\forall\ c > c_1$ the following inequalities:

$$\begin{aligned} &f_\sigma^-(x,t) < f(x,t) < f_\sigma^+(x,t), \quad x \le 0, \quad t \ge t_0, \\ &f_\sigma^-(ct,t) < f(ct,t) < f_\sigma^+(ct,t), \quad x = ct, \quad t \ge t_0. \end{aligned} \tag{1.10}$$

Proof.

For proving (1.10) it is sufficient to prove inequalities

$$f_\sigma^-(x,t) < \alpha_0^-, \quad f_\sigma^+(x,t) > \alpha_0^-, \quad \text{if} \quad x \le 0, \quad t \ge \tilde{t}_0\varepsilon, \tag{1.11}$$

$$f_\sigma^-(ct,t) < \alpha_1^-, \quad f_\sigma^+(ct,t) > \alpha_1^-, \quad \text{if} \quad t \ge \tilde{t}_0\varepsilon. \tag{1.12}$$

If $x \le 0$ and parameters $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ and ρ are big enough, then definitions above imply existence of $\lambda_0 > 0$ independent of ε such that

$$\begin{aligned} &f_\sigma^-(x,t) = \tilde{f}_{0,\sigma}^-\big(\frac{x}{\varepsilon} - c_{0,\sigma}^-\frac{t}{\varepsilon} - d(\frac{t}{\varepsilon})\big) \le \tilde{f}_{0,\sigma}^-\big(-c_{0,\sigma}^-\frac{t}{\varepsilon} - \rho\ln\frac{t}{\varepsilon}\big) \le \\ &\alpha_0^- - \sigma + O\big(\exp\big[\big(-\lambda_0 c_0\frac{t}{\varepsilon} - \lambda_0\rho\ln\frac{t}{\varepsilon}\big)\big]\big) \le \\ &\alpha_0^- - \frac{\varepsilon}{t} + O\big(\exp\big[\big(-\lambda_0 c_0\frac{t}{\varepsilon}\big]\big)O\big((\frac{\varepsilon}{t_0})^{\rho\lambda_0}\big) < \alpha_0^-, \\ &\text{if} \quad \rho > \frac{1}{\lambda_0} \quad \text{and} \quad \tilde{t}_0 = \frac{t_0}{\varepsilon} - \ \text{big enough.} \end{aligned}$$

Note, that by estimate (6.3) from [13], parameter λ_0 can be chosen up to $O(\frac{\varepsilon}{t})$ equal to solution λ_0 of the equation

$$\lambda_0 = \varphi(\alpha_0^-)\psi(\alpha_0^-,\alpha_0^+)(1 - e^{-\lambda_0}),$$

where

$$\psi(\alpha_0^-,\alpha_0^+) = \frac{1}{\alpha_0^+ - \alpha_0^-}\int\limits_{\alpha_0^-}^{\alpha_0^+}\frac{dy}{\varphi(y)}.$$

For $f_\sigma^+(x,t)$, $x \le 0$, estimate follows more easily

$$f_\sigma^+(x,t) = \tilde{f}_{0,\sigma}^+\big(\frac{x}{\varepsilon} - c_{0,\sigma}^+\frac{t}{\varepsilon} + (2\sqrt{c_{0,\sigma}^+} + \gamma + 2\delta)(\frac{t}{\varepsilon})^{1/2} + d(\frac{t}{\varepsilon})\big) \ge \alpha_0^- + \sigma > \alpha_0^-,$$

if $\tilde{t}_0$ big enough.

Inequalities (1.11) are proved.

Let us prove (1.12) by the similar way.

For $c > c_1$, $x = ct$ and $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ we have

$$f_\sigma^-(ct,t) = \tilde{f}_{1,\sigma}^-\big(\frac{x}{\varepsilon} - c_{1,\sigma}^-\frac{t}{\varepsilon} - d(\frac{t}{\varepsilon})\big) < \alpha_1^+ - \sigma < \alpha_1^+,$$
$$f_\sigma^+(ct,t) = \tilde{f}_{1,\sigma}^+\big(c\frac{t}{\varepsilon} - c_{1,\sigma}^+\frac{t}{\varepsilon} + d(\frac{t}{\varepsilon})\big) = \tilde{f}_{1,\sigma}^+\big(c\frac{t}{\varepsilon} - c_1\big(1 - O(\frac{\varepsilon}{t})\big)\frac{t}{\varepsilon} + \rho\ln\frac{t}{\varepsilon}\big) \geq$$
$$\alpha_1^+ + \sigma - O\big(\exp\big[(-\lambda_1(c-c_1)\frac{t}{\varepsilon} + \lambda_1\rho\ln\frac{\varepsilon}{t}\big]\big) \geq$$
$$\alpha_1^+ + \frac{\varepsilon}{t} - O\big(\exp\big[(-\lambda_1(c-c_1)\frac{t}{\varepsilon}\big]\big)O\big((\frac{\varepsilon}{t})^{\rho\lambda_1}\big) > \alpha_1^+,$$

if $\rho > \frac{1}{\lambda_1}$ and $\tilde{t}_0 = \frac{t_0}{\varepsilon}$ big enough.

Inequalities (1.12) are proved.

Lemma 3 is proved.

Lemma 4.

Under conditions of Lemmas 1,2 $\exists\, T > 0$ (independent of $\varepsilon > 0$) such that for $t \geq T$ function $f = f_\varepsilon(x,t)$ satisfies inequalities

$$f_\sigma^-(x,t+T) < f_\varepsilon(x,t) < f_\sigma^+(x,t-T), \quad x \in \mathbb{R}. \tag{1.13}$$

Proof.

From Lemma 3 and from results of [22] (section 2) it follows the existence of $T > 0$ such that initial values $f(x,t_0) = f^0(x)$ satisfy (1.13) with $t_0 = T$.

From this and comparison principle for solutions of (1b) (see Lemma 7.3 in [13]) we deduce inequality (1.13) for $t \geq t_0 = T$ with T and $\tilde{t}_0$ big enough. Lemma 4 is proved.

Proof of Theorem 1.

From σ- modified versions (1.8) of (1.2), (1.3) for $f^\mp(x,t)$ we have

$$f_{01,\sigma}^-(x,t) = \varphi^{(-1)}\big(\frac{x - \gamma\sqrt{\varepsilon t}}{t}\big) - \frac{c_{0,\sigma}^-\varepsilon}{\varphi'(\alpha_0^{+-})(x - c_{0,\sigma}^- t)},$$
$$\text{if} \quad c_{0,\sigma}^- t + b_{0,\sigma}^-\sqrt{\varepsilon t} \leq x \leq c_{1,\sigma}^- t + b_{1,\sigma}^-\sqrt{\varepsilon t},$$

where

$$c_{0,\sigma}^- = \varphi(\alpha_0^{+-}) = c_0(1 + O_+(\frac{\varepsilon}{t})), \quad c_0 = \varphi(\alpha_0^+),$$
$$c_{1,\sigma}^- = \varphi(\alpha_1^{-+}) = c_1(1 + O_+(\frac{\varepsilon}{t})), \quad c_1 = \varphi(\alpha_1^-),$$
$$b_1^- = \gamma + \sqrt{c_1} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_1}} + o(1),$$
$$b_{0,\sigma}^- = b_0^- + O_+(\frac{\varepsilon}{t}), \quad b_0^- = \gamma + o(1), \quad b_{1,\sigma}^- = b_1^- + O_+(\frac{\varepsilon}{t}).$$

$$f_{01,\sigma}^+(x,t) = \varphi^{(-1)}\big(\frac{x + \gamma\sqrt{\varepsilon t}}{t}\big) + \frac{c_{1,\sigma}^+\varepsilon}{\varphi'(\alpha_1^{-+})(c_{1,\sigma}^+ t - x)},$$
$$\text{if} \quad c_{0,\sigma}^+ t - b_{0,\sigma}^+\sqrt{\varepsilon t} \leq x \leq c_{1,\sigma}^+ t - b_{1,\sigma}^+\sqrt{\varepsilon t},$$

where

$$c^+_{0,\sigma} = c_0(1 - O_+(\frac{\varepsilon}{t})), \quad c^+_{1,\sigma} = c_1(1 - O_+(\frac{\varepsilon}{t})),$$
$$b^+_{0,\sigma} = b^+_0 - O_+(\frac{\varepsilon}{t}); b^+_0 = \gamma + \sqrt{c_0} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_0}};$$
$$b^+_{1,\sigma} = b^+_1 - O_+(\frac{\varepsilon}{t}); b^+_1 = \gamma + o(1).$$

Let $\tilde{\gamma} = \gamma + \Gamma$ be such that

$$\begin{aligned} &\varphi^{(-1)}\big(\frac{x - \tilde{\gamma}\sqrt{\varepsilon t}}{t}\big) \le \varphi^{(-1)}\big(\frac{x - \gamma\sqrt{\varepsilon t}}{t}\big) - \frac{c^-_{0,\sigma}\varepsilon}{\varphi'(\alpha^{+-}_0)(x - c^-_{0,\sigma}t)} \quad \text{and} \\ &\varphi^{(-1)}\big(\frac{x + \gamma\sqrt{\varepsilon t}}{t}\big) + \frac{c^+_{0,\sigma}\varepsilon}{\varphi'(\alpha^{-+}_1)(c^+_{1,\sigma}t - x)} \le \varphi^{(-1)}\big(\frac{x + \tilde{\gamma}\sqrt{\varepsilon t}}{t}\big), \end{aligned} \tag{1.14}$$

where

$$c^-_{0,\sigma}t + b^-_{0,\sigma}\sqrt{\varepsilon t} \le x \le c^+_{1,\sigma}t - b^+_{1,\sigma}\sqrt{\varepsilon t} \tag{1.15}$$

To obtain (1.14) we must have under condition (1.15) the following inequalities for $\Gamma > 0$:

$$\begin{aligned} &-\frac{1}{(\sup\varphi')}\frac{\Gamma\sqrt{\varepsilon t}}{t} \le -\frac{c^-_{0,\sigma}\varepsilon}{\varphi'(\alpha^{+-}_0)(x - c^-_{0,\sigma}t)} \quad \text{and} \\ &\frac{c^+_{0,\sigma}\varepsilon}{\varphi'(\alpha^{-+}_1)(c^+_{1,\sigma}t - x)} \le \frac{1}{(\sup\varphi')}\frac{\Gamma\sqrt{\varepsilon t}}{t}, \end{aligned} \tag{1.16}$$

where

$$\varphi' = \varphi'\big(\frac{x}{t} \pm (\gamma + \theta\Gamma)\sqrt{\frac{\varepsilon}{t}}\big), \quad \theta \in [0,1], \quad t \ge t_0.$$

From (1.15) and (1.16) we obtain the following condition for parameter Γ:

$$\begin{aligned} &\frac{\Gamma}{(\sup\varphi')} > \frac{c^-_{0,\sigma}}{\varphi'(\alpha^{+-}_0)b^-_{0,\sigma}} \quad \text{and} \\ &\frac{\Gamma}{(\sup\varphi')} \ge \frac{c^+_{0,\sigma}}{\varphi'(\alpha^{-+}_0)b^+_{1,\sigma}}. \end{aligned} \tag{1.17}$$

To satisfy (1.17) it is sufficient to take Γ such that

$$\begin{aligned} &\Gamma > (\sup\varphi')\frac{c^-_{0,\sigma}}{\varphi'(\alpha^{+-}_0)b^-_{0,\sigma}} \quad \text{and} \\ &\Gamma > (\sup\varphi')\frac{c^+_{0,\sigma}}{\varphi'(\alpha^{-+}_0)b^+_{1,\sigma}}. \end{aligned}$$

From Lemma 4 and inequalities (1.14), (1.15) for $t \ge T$ and

$x \in [c_0 t + b_0\sqrt{\varepsilon t}, c_1 t - b_1\sqrt{\varepsilon t}]$ we obtain

$$\varphi^{(-1)}\Big(\frac{x-\tilde{\gamma}\sqrt{\varepsilon(t+T)}}{t+T}\Big) \le f_\varepsilon(x,t) \le \varphi^{(-1)}\Big(\frac{x+\tilde{\gamma}\sqrt{\varepsilon(t-T)}}{t-T}\Big), \tag{1.18}$$

where $b_0 > \gamma + o(1)$, $b_1 > \gamma + O(1)$, $\tilde{\gamma} = \gamma + O\big(\frac{1}{b_0}\big)$.

Theorem 1 is proved.

2. Vanishing viscosity method for Cauchy-Gelfand problem.

Theorem 1, incorporated in the proof of Proposition 2 from [7], implies the following improved version of this proposition as well as of Theorem 2 of [9].

Theorem 2.

Let under assumptions and notations of Theorem 1, $\tilde{b}_l > b_l > O(1/\gamma)$, $l = 0, \dots, L$, $\gamma > 0$. Then $\exists\ t_0 > 0$ such that $\forall\ t \ge t_0$ and $\varepsilon > 0$ the difference $\Delta f = \frac{f(x,t)-f(x-\varepsilon,t)}{\varepsilon}$ for a solution $f = f_\varepsilon(x,t)$ of (1b), (2) and the derivative $\frac{\partial f}{\partial x}(x,t)$ for a solution $f = f_\varepsilon(x,t)$ of (1a), (2) satisfy the following estimates

$$\begin{pmatrix} \Delta f \\ \frac{\partial f}{\partial x} \end{pmatrix} = \frac{1}{\varphi'(\alpha_l^+)\cdot t} + O\Big(\frac{\gamma}{\varphi'(\alpha_l^+)\cdot t}\Big) \tag{2.1}$$

for $x \in [c_l t + b_l\sqrt{\varepsilon t}, c_l t + \tilde{b}_l\sqrt{\varepsilon t}]$, $l = 0, \dots, L-1$, $t \ge t_0$, and

$$\begin{pmatrix} \Delta f \\ \frac{\partial f}{\partial x} \end{pmatrix} = \frac{1}{\varphi'(\alpha_l^-)\cdot t} + O\Big(\frac{\gamma}{\varphi'(\alpha_l^-)\cdot t}\Big) \tag{2.2}$$

for $x \in [c_l t - \tilde{b}_l\sqrt{\varepsilon t}, c_l t - b_l\sqrt{\varepsilon t}]$, $l = 1, \dots, L$, $t \ge t_0$.

Corollary.

Under conditions of Theorem 2 $\exists \gamma_0 > 0$ small enough and $\exists\ t_0 > 0$ big enough such that for $t \ge t_0$, $\gamma \le \gamma_0$ and $\varepsilon > 0$ functions $x \to f_\varepsilon(x,t)$ from (1a,b), (2) are increasing functions on the intervals $x \in [c_l t + b_l\sqrt{\varepsilon t}, c_l t + \tilde{b}_l\sqrt{\varepsilon t}]$, $l = 0, \dots, L-1$, and $x \in [c_l t - \tilde{b}_l\sqrt{\varepsilon t}, c_l t - b_l\sqrt{\varepsilon t}]$, $l = 1, \dots, L$.

From theorem 1 and from corollary of theorem 2 $\forall \varepsilon > 0$ and for big enough $t_0 > 0$ and $\Gamma > 0$ we deduce existence of functions $t \mapsto y_l^\pm(t,\varepsilon)$, $l = 0, \dots, L$, $t \ge t_0$, with properties

$$f_\varepsilon(y_0^+, t) = \alpha_0^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}, \quad f_\varepsilon(y_L^-, t) = \alpha_L^- - \sqrt{\frac{\varepsilon\Gamma}{t}},$$

$$f_\varepsilon(y_l^\pm, t) = \alpha_l^\pm \pm \sqrt{\frac{\varepsilon\Gamma}{t}}, \ l = 1, \dots, L-1,$$

$$\alpha_0^- < f_\varepsilon(x,t) < \alpha_0^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}, \quad \text{if} \quad x < y_0^+(t,\varepsilon),$$

$$\alpha_L^- - \sqrt{\frac{\varepsilon\Gamma}{t}} < f_\varepsilon(x,t) < \alpha_L^+, \quad \text{if} \quad x > y_L^-(t,\varepsilon),$$

$$\alpha_l^- - \sqrt{\frac{\varepsilon\Gamma}{t}} < f_\varepsilon(x,t) < \alpha_l^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}, \quad l = 1, \dots, L-1, \quad \text{if} \quad y_l^-(t,\varepsilon) < x < y_l^+(t,\varepsilon).$$

This implies correctness of the following definition of Maxwell type shift-functions $d_l(t,\varepsilon)$ for solutions of (1a,b), (2) with $\varepsilon > 0$.

Definition 1.

Under assumptions and notations of Theorem 1 for Cauchy problem (1a), (2) $\forall\ t_0 > 0$ and $\Gamma > 0$ big enough $\exists!$ well defined functions $d_l(t,\varepsilon)$ and $y_l^{\pm}(t,\varepsilon)$, $l = 0,\dots,L$, $t \geq t_0$, $\varepsilon > 0$ such that

$$\int\limits_{-\infty}^{c_0 t + d_0(t,\varepsilon)} (f_\varepsilon(x,t) - \alpha_0^-)dx + \int\limits_{c_0 t + d_0(t,\varepsilon)}^{y_0^+(t,\varepsilon)} (f_\varepsilon(x,t) - \alpha_0^+ - \sqrt{\frac{\varepsilon\Gamma}{t}})dx = 0,$$
$$f_\varepsilon(y_0^+,t) = \alpha_0^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}, \tag{2.3}$$

$$\int\limits_{y_l^-(t,\varepsilon)}^{c_l t + d_l(t,\varepsilon)} (f_\varepsilon(x,t) - \alpha_l^- + \sqrt{\frac{\varepsilon\Gamma}{t}})dx + \int\limits_{c_l t + d_l(t,\varepsilon)}^{y_l^+(t,\varepsilon)} (f_\varepsilon(x,t) - \alpha_l^+ - \sqrt{\frac{\varepsilon\Gamma}{t}})dx = 0,$$
$$f_\varepsilon(y_l^{\pm},t) = \alpha_l^{\pm} \pm \sqrt{\frac{\varepsilon\Gamma}{t}},\ l = 1,\dots,L-1, \tag{2.4}$$

$$\int\limits_{y_L^-(t,\varepsilon)}^{c_L t + d_L(t,\varepsilon)} (f_\varepsilon(x,t) - \alpha_L^- + \sqrt{\frac{\varepsilon\Gamma}{t}})dx + \int\limits_{c_L t + d_L(t,\varepsilon)}^{\infty} (f_\varepsilon(x,t) - \alpha_L^+)dx = 0,$$
$$f_\varepsilon(y_L^-,t) = \alpha_L^- - \sqrt{\frac{\varepsilon\Gamma}{t}}. \tag{2.5}$$

To define Maxwell type shift-functions for Cauchy problem (1b), (2) it is sufficient to replace in definition 1 function $f_\varepsilon(x,t)$ and parameters $\alpha_l^{\pm}$ by functions $\Psi(f_\varepsilon(x,t))$ and parameters $\Psi(\alpha_l^{\pm})$, where $\Psi(f) = \int\limits_{\alpha^-}^{f} \frac{y}{\varphi(y)}$.

Theorem 3.

Let $\Phi(f)$ be such that $\varphi(f)\frac{\partial f}{\partial x} = \frac{\partial \Phi(f)}{\partial x}$. Then under notations of Theorem 1 and definition 1 $\forall\ t \geq t_0$ and $\varepsilon > 0$ the following formulas for Maxwell type shift-functions $d_l(t,\varepsilon)$ for solutions of Cauchy problem (1a), (2) are valid:

$$(\alpha_l^+ - \alpha_l^- + 2\sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_l t + d_l(t,\varepsilon)) = \Phi(\alpha_l^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}) - \Phi(\alpha_l^- - \sqrt{\frac{\varepsilon\Gamma}{t}}) -$$
$$\varepsilon(\frac{\partial f_\varepsilon}{\partial x}(y_l^+(t,\varepsilon),t) - \frac{\partial f_\varepsilon}{\partial x}(y_l^-(t,\varepsilon),t)) - \tag{2.6}$$
$$\frac{1}{t}\sqrt{\frac{\varepsilon\Gamma}{t}}\big(\frac{y_l^+(t,\varepsilon) + y_l^-(t,\varepsilon)}{2} - c_l t - d_l(t,\varepsilon)\big),\quad \text{if}\ \ l = 1,\dots,L-1,$$

$$
\begin{aligned}
&(\alpha_0^+ - \alpha_0^- + \sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_0 t + d_0(t,\varepsilon)) = \Phi(\alpha_0^+ + \sqrt{\frac{\varepsilon\Gamma}{t}}) - \Phi(\alpha_0^-) - \\
&\varepsilon(\frac{\partial f_\varepsilon}{\partial x}(y_0^+(t,\varepsilon),t) - \frac{1}{2t}\sqrt{\frac{\varepsilon\Gamma}{t}}(y_0^+(t,\varepsilon) - c_0 t - d_0(t,\varepsilon)), \quad \text{if} \quad l = 0,
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
&(\alpha_L^+ - \alpha_L^- + \sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_L t + d_L(t,\varepsilon)) = \Phi(\alpha_L^+) - \Phi(\alpha_L^- - \sqrt{\frac{\varepsilon\Gamma}{t}}) + \\
&\varepsilon(\frac{\partial f_\varepsilon}{\partial x}(y_L^-(t,\varepsilon) - \frac{1}{2t}\sqrt{\frac{\varepsilon}{t}}(y_L^- - c_L t - d_L(t,\varepsilon)), \quad \text{if} \quad l = L.
\end{aligned} \tag{2.8}
$$

Proof.

Let us consider firstly the case $l = 1, \ldots, L-1$. Derivation of equality (2.4) from definition 1 gives equality

$$
\begin{aligned}
&\frac{d}{dt}(c_l t + d_l(t,\varepsilon))(f_\varepsilon(c_l t + d_l(t,\varepsilon),t) - \alpha_l^- + \sqrt{\frac{\varepsilon\Gamma}{t}}) - \\
&\frac{d}{dt}(c_l t + d_l(t,\varepsilon))(f_\varepsilon(c_l t + d_l(t,\varepsilon),t) - \alpha_l^+ - \sqrt{\frac{\varepsilon\Gamma}{t}}) + \\
&\int\limits_{y_l^-(t,\varepsilon)}^{y_l^+(t,\varepsilon)} \frac{\partial f_\varepsilon(x,t)}{\partial t}dx + 2\big(c_l t + d_l - \frac{y_l^- + y_l^+}{2}\big)\frac{d}{dt}\sqrt{\frac{\varepsilon\Gamma}{t}} = 0.
\end{aligned}
$$

This equality and equation (1a) imply

$$
\begin{aligned}
&(\alpha_l^+ - \alpha_l^- + 2\sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_l t + d_l(t,\varepsilon)) + \\
&\int\limits_{y_l^-(t,\varepsilon)}^{y_l^+(t,\varepsilon)} \big(\varepsilon\frac{\partial^2 f_\varepsilon}{\partial x^2} - \varphi(f)\frac{\partial f_\varepsilon}{\partial x}\big)dx - \frac{1}{t}\sqrt{\frac{\varepsilon\Gamma}{t}}\big(c_l t + d_l - \frac{y_l^- + y_l^+}{2}\big) = 0.
\end{aligned}
$$

Using equality $\varphi(f)\frac{\partial f}{\partial x} = \frac{\partial \Phi(f)}{\partial x}$, we obtain further

$$
\begin{aligned}
&(\alpha_l^+ - \alpha_l^- + 2\sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_l t + d_l(t,\varepsilon)) = -\varepsilon\big(\frac{\partial f_\varepsilon}{\partial x}(y_l^+,t) - \frac{\partial f_\varepsilon}{\partial x}(y_l^-,t)\big) + \\
&\Phi(f_\varepsilon(y_l^+,t)) - \Phi(f_\varepsilon(y_l^-,t)) - \frac{1}{t}\sqrt{\frac{\varepsilon\Gamma}{t}}\big(\frac{y_l^+ + y_l^-}{2} - c_l t - d_l\big).
\end{aligned}
$$

This gives (2.6), taking into account equality

$$
f_\varepsilon(y_l^\pm, t) = \alpha_l^\pm \pm \sqrt{\frac{\varepsilon\Gamma}{t}}.
$$

If $l = 0$ then derivation of (2.3) implies

$$\frac{d}{dt}(c_0 t + d_0(t,\varepsilon))(f_\varepsilon(c_0 t + d_0(t,\varepsilon), t) - \alpha_0^-) -$$
$$\frac{d}{dt}(c_0 t + d_0(t,\varepsilon))(f_\varepsilon(c_0 t + d_0(t,\varepsilon), t) - \alpha_0^+ - \sqrt{\frac{\varepsilon\Gamma}{t}}) +$$
$$\int\limits_{-\infty}^{y_0^+(t,\varepsilon)} \frac{\partial f_\varepsilon(x,t)}{\partial t} dx + (c_0 t + d_0(t,\varepsilon) - y_0^+)\frac{d}{dt}\sqrt{\frac{\varepsilon\Gamma}{t}} = 0.$$

Using (1a) we have further

$$(\alpha_0^+ - \alpha_0^- + \sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_0 t + d_0(t,\varepsilon)) + \int\limits_{-\infty}^{y_0^+} \big(\varepsilon\frac{\partial^2 f_\varepsilon}{\partial x^2} - \varphi(f)\frac{\partial f_\varepsilon}{\partial x}\big) dx -$$
$$\frac{\sqrt{\varepsilon\Gamma}}{2t^{3/2}}(c_0 t + d_0(t) - y_0^+) = 0.$$

Finally,

$$(\alpha_0^+ - \alpha_0^- + \sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_0 t + d_0(t,\varepsilon)) = -\varepsilon\big(\frac{\partial f_\varepsilon}{\partial x}(y_0^+, t) - 0\big) +$$
$$\Phi(f_\varepsilon(y_0^-)) - \Phi(f_\varepsilon(-\infty)) - \frac{\sqrt{\varepsilon\Gamma}}{2t^{3/2}}(y_0^+ - c_0 t + d_0(t)).$$

This gives (2.7). Equality (2.8) can be proved by a similar way.

Theorem 3 is proved.

Remark 5.

Result similar to Theorem 3 is valid also for shift-functions for Cauchy problem (1b), (2).

From Theorems 1, 2, 3 we can deduce the following.

Theorem 4.

Under assumptions and notations of Theorem 1 and definition 1 Maxwell type shift-functions $d_l(t,\varepsilon)$ for solutions of Cauchy problem (1a,b), (2) have the following small viscosity estimate:

$$\frac{d}{dt} d_l(t,\varepsilon) = O(\sqrt{\varepsilon/t}), \quad \varepsilon > 0, \ t \geq t_0, \ l = 0, \ldots, L. \tag{2.9}$$

Proof (for problem (1a), (2)).

Let us consider the case $l = 1, \ldots, L-1$. In order to obtain estimate (2.9) for this case we must estimate all terms of (2.6). For these estimates we note that equalities $f_\varepsilon(y_l^\pm, t) = \alpha_l^\pm \pm \sqrt{\frac{\varepsilon\Gamma}{t}}$ and Theorem 1 for Γ big enough imply the following inequality

$$c_l t - \Gamma\sqrt{\varepsilon t} \leq y_l^- \leq y_l^+ \leq c_l t + \Gamma\sqrt{\varepsilon t}, \quad t \geq t_0. \tag{2.10}$$

Using (2.10) and Theorem 2 we obtain the following equality

$$\begin{aligned}&\varepsilon\big(\frac{\partial f_\varepsilon}{\partial x}(y_l^+(t,\varepsilon),t)-\frac{\partial f_\varepsilon}{\partial x}(y_l^-(t,\varepsilon),t)\big)=\\&\frac{\varepsilon}{t}\big(\frac{1}{\varphi'(\alpha_l^+)}-\frac{1}{\varphi'(\alpha_l^-)}\big)\big(1+O\big(\frac{\gamma}{\varphi'(\alpha_l^+)t}+\frac{\gamma}{\varphi'(\alpha_l^-)t}\big)\big).\end{aligned} \tag{2.11}$$

Definition 1 and (2.10) imply inequality

$$\frac{1}{t}\sqrt{\frac{\varepsilon\Gamma}{t}}\big(\frac{y_l^+(t,\varepsilon)+y_l^-(t,\varepsilon)}{2}-c_lt-d_l(t,\varepsilon)\big)=O\big(\frac{1}{t}\sqrt{\varepsilon/t}\sqrt{\varepsilon t}\big)=O(\varepsilon/t). \tag{2.12}$$

Formula (5a) gives equality

$$(\alpha_l^+-\alpha_l^-)c_l=\Phi(\alpha_l^+)-\Phi(\alpha_l^-). \tag{2.13}$$

Using (2.13), we obtain

$$\begin{aligned}&(\alpha_l^+-\alpha_l^-+2\sqrt{\frac{\varepsilon\Gamma}{t}})\frac{d}{dt}(c_lt+d_l(t,\varepsilon))-\Phi(\alpha_l^++\sqrt{\frac{\varepsilon\Gamma}{t}})+\Phi(\alpha_l^--\sqrt{\frac{\varepsilon\Gamma}{t}})=\\&\frac{d}{dt}d_l(t,\varepsilon)+O(\sqrt{\varepsilon/t}).\end{aligned} \tag{2.14}$$

Finally, (2.11), (2.12), (2.14) imply (2.9). The cases $l=0$ and $l=L$ can be obtained by a similar way.

Theorem 4 is proved.

Corollary.
Under conditions of Theorem 1 $\forall\ t\geq t_0$

$$\exists\ \lim_{\varepsilon\to 0}d_l(t,\varepsilon)=d_l(t,0)=d_l(t_0,0),\quad l=0,\dots,L.$$

Proof of the main theorem.

The first statement of the main theorem was obtained in [7], theorem 1. The second statement of the main theorem follows directly from the first statement and corollary of Theorem 4.

Remark 6.
Note that in [7] in the proof of theorem 1 there are misprints, which we correct here: "From (10) and (11) of [7], Proposition 3 with $A=(\frac{\varepsilon}{t})^{1/4}$ we obtain

$$\frac{dA}{dt}=-\frac{\varepsilon}{4}\frac{A}{t},$$

$$\frac{\partial}{\partial A}d_l(\frac{t}{\varepsilon},A)=\begin{cases}O(\frac{1}{A}), & \text{if}\ \ A\geq\sqrt{\frac{\varepsilon}{t}}\\ O(\sqrt{\frac{t}{\varepsilon}}), & \text{if}\ \ A\leq\sqrt{\frac{\varepsilon}{t}}.\end{cases}$$

$$\frac{d}{dt}d_l(\frac{t}{\varepsilon},(\frac{\varepsilon}{t})^{1/4}) = O(\frac{\varepsilon}{t}) + O(\frac{\varepsilon}{t})^{3/4}).$$

This gives property (28) in [7] and as a consequence a strong version of theorem 1 in [7]."

References.

[1] H.Bateman, Some recent researches on the motion of fluids, Monthly Weather Rev. **43** (1915), 163-170

[2] M.Bardi, L.C.Evans, On Hopf's formulas for solutions of Hamilton-Jacobi equations, Nonlinear Anal. **8** (1984), 1373-1381

[3] J.M.Burgers, Application of a model system to illustrate some points of the statistical theory of free turbulence, Proc.Acad.Sci. Amsterdam **43** (1940), 2-12

[4] A.V.Gasnikov, Time asymptotic behavior of the solution of the initial Cachy problem for a conservation law with non-linear divirgent viscosity, Izv. Ross. Akad. Nauk Ser. Mat. **73** (2009), 39-76

[5] I.M.Gelfand, Some problems in the theory of quasilinear equations, Usp.Mat.Nauk **14** (1959), 87-158 (in Russian); Amer.Math.Soc. Translations **33** (1963), 295-381

[6] G.M.Henkin, Asymptotic structure for solutions of the Cauchy problem for Burgers type equations, J. Fixed Point Theory Appl. **1** (2007), 239-291

[7] G.M.Henkin, Burgers type equations, Gelfand's problem and Schumpeterian dynamics, J. Fixed Point Theory Appl. **11** (2012), 199-223

[8] G.M.Henkin, A.A.Shananin, Asymptotic behaviour of solutions of the Cauchy problem for Burgers type equations, J.Math.Pure Appl. **83** (2004), 1457-1500

[9] G.M.Henkin, A.A.Shananin, A.E.Tumanov, Estimates for solutions of Burgers type equations and some applications, J.Math.Pures Appl. **84** (2005), 717-752

[10] G.M.Henkin, V.M.Polterovich (=V.M.Polterovich and G.M.Khenkin), An evolutionnary model with interaction between development and adoption of new technologies, Econom. Math.Methods **24** (1988), 1071-1083 (in Russian); Matecon (1988) **24**(6), p.3-19 (in English)

[11] G.M.Henkin, V.M.Polterovich, Schumpeterian dynamics as a nonlinear wave theory, J.Math.Econom. **20** (1991), 551-590

[12] G.M.Henkin, V.M.Polterovich, A difference-differential analogue of the Burgers equation: stability of the two-wave behaviour, J.Nonlinear Sci. **4** (1994), 497-517

[13] G.M.Henkin, V.M.Polterovich, A difference-differential analogue of the Burgers equation and some models of economic development, Discrete Contin.Dynam.Systems **4** (1999), 697-728

[14] E.Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm.Pure Appl.Math. **3** (1950), 201-230

[15] E.Hopf, Generalized solutions of non-linear equations of first order, J.Math.Mech. **14** (1965), 951-973

[16] A.M.Iljin, O.A.Oleinik, Asymptotic long-time behaviour of the Cauchy problem for some quasilinear equation, Mat.Sbornik **51** (1960), 191-216 (in Russian)

[17] S.N.Kruzhkov, N.S.Petrosjan, Asymptotic behavior of the solutions of the Cauchy problem for the non-linear first order equations, Russ.Math.Surveys **42**(5) (1987), 1-47

[18] T.-P.Liu, Admissible solutions of hyperbolic conservation laws, Memoirs of the Amer. Math.Soc. **30**(240) (1981), 1-78

[19] O.A.Oleinik, Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation, Usp.Mat.Nauk **14** (1959), 167-170 (in Russian); Amer. Math.Soc. Translations **33** (1963), 285-290

[20] N.S.Petrosjan, About asymptotics of solutions of the Cauchy problem for quasilinear equation of first order with nonconvex state function, Russian Math. Surveys **38** (1983), 213-214

[21] B.Riemann, Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungs weite. Göttingen Abh. Math. Cl **8** (1860), 43-65

[22] H.F.Weinberger, Long-time behaviour for a regularized scalar conservation law in the absence of genuine nonlinearity, Ann.Inst.H.Poincare Anal.Nonlinéaire **7** (1990), 407-425