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THE ALGEBRA OF ESSENTIAL RELATIONS ON A FINITE SET

SERGE BOUC AND JACQUES THÉVENAZ

ABSTRACT. Let X be a finite set and let k be a commutative ring. We consider the k -algebra of the monoid of all relations on X , modulo the ideal generated by the relations factorizing through a set of cardinality strictly smaller than $\text{Card}(X)$, called inessential relations. This quotient is called the essential algebra associated to X . We then define a suitable nilpotent ideal of the essential algebra and describe completely the structure of the corresponding quotient, a product of matrix algebras over suitable group algebras. In particular, we obtain a description of the Jacobson radical and of all the simple modules for the essential algebra.

1. Introduction

Let X and Y be finite sets. A *correspondence* between X and Y is a subset R of $X \times Y$. In case $X = Y$, we say that R is a *relation* on X . Correspondences can be composed as follows. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then RS is the correspondence between X and Z defined by

$$RS = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

In particular the set of all relations on X is a monoid. Given a commutative ring k and a finite set X , let \mathcal{R} be the k -algebra of the monoid of all relations on X (having this monoid as a k -basis).

Throughout this paper, X will denote a finite set of cardinality n . We say that a relation R on X is *inessential* if there exists a set Y with $\text{Card}(Y) < \text{Card}(X)$ and two relations $S \subseteq X \times Y$ and $T \subseteq Y \times X$ such that $R = ST$. Otherwise, R is called *essential*. The set of all inessential relations on X span a two-sided ideal I of \mathcal{R} . We define $\mathcal{E} = \mathcal{R}/I$. It is clear that \mathcal{E} is a k -algebra having as a k -basis the set of all essential relations on X . The purpose of this paper is to explore the concept of essential relation and to study the structure of \mathcal{E} .

We shall define a nilpotent ideal N of \mathcal{E} and describe completely the quotient $\mathcal{P} = \mathcal{E}/N$. More precisely, \mathcal{P} is isomorphic to a product of matrix algebras over suitable group algebras, the product being indexed by the set of all (partial) order relations on X , up to permutation. Consequently, we know the Jacobson radical $J(\mathcal{E})$ and we find all the simple \mathcal{E} -modules.

The idea of passing to the quotient by all elements obtained from something smaller is widely used in the representation theory of finite groups. In the theory of G -algebras, the notion of Brauer quotient is of this kind (see [The]). In the more recent development of the theory of biset functors for finite groups (see [Bo]), the same idea plays a key role in [BST]. The analogous idea for sets instead of groups

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yields the notion of essential relation, which does not seem to have been studied. It is the purpose of this paper to fill this gap.

2. Essential relations

Given a correspondence $R \subseteq X \times Y$ between a set X and a set Y , then for every $a \in X$ and $b \in Y$ we write

$${}_aR = \{y \in Y \mid (a, y) \in R\} \quad \text{and} \quad R_b = \{x \in X \mid (x, b) \in R\}.$$

We call ${}_aR$ a *column* of R and R_b a *row* of R .

We first characterize inessential relations. Any subset of $X \times Y$ of the form $U \times V$ will be called a *block* (where $U \subseteq X$ and $V \subseteq Y$).

2.1. Lemma. *Let X, Y, Z be finite sets.*

- (a) *Let $R \subseteq X \times Z$ be a correspondence between X and Z . Then R factorizes through Y if and only if R can be decomposed as a union of blocks indexed by the set Y .*
- (b) *Let R be a relation on X , where X has cardinality n . Then R is inessential if and only if R can be decomposed as a union of at most $n - 1$ blocks.*

Proof : (a) If R factorizes through Y , then $R = ST$, where $S \subseteq X \times Y$ and $T \subseteq Y \times Z$. Then we can write

$$R = \bigcup_{y \in Y} S_y \times {}_yT,$$

as required.

Suppose conversely that $R = \bigcup_{y \in Y} U_y \times V_y$, where each U_y is a subset of X and each V_y is a subset of Z . Then we define

$$S = \bigcup_{y \in Y} U_y \times \{y\} \subseteq X \times Y \quad \text{and} \quad T = \bigcup_{y \in Y} \{y\} \times V_y \subseteq Y \times Z.$$

Then S is a correspondence between X and Y , and $U_y = S_y$, the y -th row of S . Similarly, T is a correspondence between Y and Z , and $V_y = {}_yT$, the y -th column of T .

We now claim that $R = ST$. If $(x, z) \in R$, then there exists $y \in Y$ such that $(x, z) \in U_y \times V_y$. It follows that $(x, y) \in S$ and $(y, z) \in T$, hence $(x, z) \in ST$, proving that $R \subseteq ST$. If now $(x, z) \in ST$, then there exists $y \in Y$ such that $(x, y) \in S$ and $(y, z) \in T$. It follows that $x \in S_y = U_y$ and $z \in {}_yT = V_y$, hence $(x, z) \in U_y \times V_y \subseteq R$, proving that $ST \subseteq R$. We have shown that $R = ST$, proving the claim.

(b) This follows immediately from (a). \square

2.2. Corollary. *Let R be a relation on X . If two rows of R are equal, then R is inessential. If two columns of R are equal, then R is inessential.*

Proof : Suppose that ${}_aR = {}_bR = V$. Then

$$R = (\{a, b\} \times V) \cup \left(\bigcup_{\substack{c \in X \\ c \neq a, c \neq b}} \{c\} \times {}_cR \right),$$

a union of $n - 1$ blocks, where $n = \text{Card}(X)$. The proof for rows is similar. \square

2.3. Corollary. *Let R be a relation on X . If a row of R is empty, then R is inessential. If a column of R is empty, then R is inessential.*

Proof : Assume that ${}_aR = \emptyset$. Then

$$R = \bigcup_{\substack{c \in X \\ c \neq a}} \{c\} \times {}_cR,$$

a union of $n - 1$ blocks, where $n = \text{Card}(X)$. The proof for rows is similar. \square

2.4. Corollary. *Let R be a relation on X . If R is an equivalence relation different from the equality relation (i.e. $R \neq \Delta$ where Δ is the diagonal of $X \times X$), then R is inessential.*

Proof : Suppose that a and b are equivalent and $a \neq b$. Then the rows R_a and R_b are equal and Corollary 2.2 applies. \square

We need a few basic facts about reflexive relations. Recall that a relation S on X is reflexive if S contains $\Delta = \{(x, x) \mid x \in X\}$. Moreover, a *preorder* is a relation which is reflexive and transitive, while an *order* is a preorder which is moreover antisymmetric. (Note that, throughout this paper, the word “order” stands for “partial order”.) Associated to a preorder R , there is an equivalence relation \sim_R defined by $x \sim_R y$ if and only if $(x, y) \in R$ and $(y, x) \in R$. Then \sim_R is the equality relation if and only if R is an order.

We will often use the containment of relations on X (as subsets of $X \times X$). Note that if $R \subseteq S$, then $RT \subseteq ST$ and $TR \subseteq TS$ for any relation T on X . If S is a reflexive relation, then $S \subseteq S^2 \subseteq S^3 \dots$ and there exists $m \in \mathbb{N}$ such that $S^m = S^{m+1}$, hence $S^m = S^N$ for all $N \geq m$. The relation $\overline{S} := S^m$ is called the *transitive closure* of S . It is reflexive and transitive, that is, a preorder. Note that any preorder is an idempotent relation.

2.5. Proposition. *Let R be a preorder on a finite set X of cardinality n .*

- (a) *If R is not an order, then R is inessential.*
- (b) *If R is an order and if Q is a reflexive relation contained in R , then Q is essential. In particular, if R is an order, then R is essential.*
- (c) *If R is a total order, then R is maximal among essential relations.*

Proof : (a) If R is not an order, then the associated equivalence relation \sim_R is not the equality relation. Let a and b be equivalent under \sim_R with $a \neq b$. Then, by transitivity of R , the rows R_a and R_b are equal. By Corollary 2.2, R is inessential.

(b) Suppose now that R is an order and that Q is reflexive with $Q \subseteq R$. We claim that, if $a \neq b$, then (a, a) and (b, b) cannot belong to a block contained in Q . This is because, if $(a, a), (b, b) \in U \times V \subseteq Q$, then $(a, b) \in U \times V$ (because $a \in U$ and $b \in V$) and $(b, a) \in U \times V$ (because $b \in U$ and $a \in V$), and therefore $(a, b) \in Q$ and $(b, a) \in Q$, hence $(a, b) \in R$ and $(b, a) \in R$, contrary to antisymmetry. It follows that, in any expression of Q as a union of blocks, the diagonal elements (a, a) all lie in different blocks, so that the number of blocks is at least n . This shows that Q is essential.

(c) Without loss of generality, we can assume that the total order R is the usual total order on the set $X = \{1, 2, \dots, n\}$, i.e. $(x, y) \in R \Leftrightarrow x \leq y$. Let S be a relation strictly containing R . Then $S - R \neq \emptyset$ and we choose $(j, i) \in S - R$ with i maximal, and then j maximal among all x with $(x, i) \in S - R$. In other words, $(j, i) \in S$, but $j > i$ because $(j, i) \notin R$, and moreover

$$(x, y) \in S - R \Rightarrow y \leq i \quad \text{and} \quad (x, i) \in S - R \Rightarrow x \leq j.$$

If $i = j - 1$, then the rows S_i and S_j are equal, so S is inessential by Corollary 2.2.

Assume now that $j - 1 > i$. Then we claim that

$$S = (S_i \times \{i, j\}) \cup (S_{j-1} \times \{j-1, j\}) \cup \left(\bigcup_{k \neq i, j-1, j} S_k \times \{k\} \right).$$

To show that the first block is contained in S , let $x \in S_i$. Then $x \leq j$ if $(x, i) \in S - R$, and also $x \leq j$ if $(x, i) \in R$, i.e. $x \leq i$. Hence $x \leq j$ in both cases, and therefore $(x, j) \in R \subseteq S$. This shows that $S_i \times \{j\} \subseteq S$.

To show that the second block is contained in S , let $x \in S_{j-1}$. Then $(x, j-1)$ cannot belong to $S - R$, by maximality of i (because $j-1 > i$). Thus $(x, j-1) \in R$, that is, $x \leq j-1$. Then $x < j$, hence $(x, j) \in S$. This shows that $S_{j-1} \times \{j\} \subseteq S$.

Next we show that S is contained in the union of the blocks above. This is clear for any $(x, y) \in S$ such that $y \neq j$. Now take $(x, j) \in S$. By maximality of i and since $j > i$, we have $(x, j) \in R$, that is, $x \leq j$. If $x = j$, then $(j, j) \in S_i \times \{i, j\}$ because $(j, i) \in S$, that is, $j \in S_i$. If $x < j$, then $x \leq j-1$, hence $(x, j-1) \in S$, that is $x \in S_{j-1}$, and therefore $(x, j) \in S_{j-1} \times \{j-1, j\}$.

This proves the claim about the block decomposition. It follows that S is a union of $n - 1$ blocks, so S is inessential. \square

2.6. Example. Let $n = \text{Card}(X)$. Let Δ be the diagonal of $X \times X$ and let $R = (X \times X) - \Delta$. It is not difficult to see that R is essential if $2 \leq n \leq 4$. However, for $n \geq 5$, we prove that R is inessential. Without loss of generality, we can choose $X = \{1, \dots, n\}$. If $U \subseteq X$, we write U^c for the complement of U in X . Then it is easy to prove that R is equal to

$$\left(\bigcup_{i=1}^{n-3} \{i, i+3\}^c \times \{i, i+3\} \right) \cup (\{n-2, n-1, n\}^c \times \{n-2, n-1, n\}) \cup (\{1, 2, 3\}^c \times \{1, 2, 3\}).$$

This is a union of $n - 1$ blocks, so R is inessential.

3. Permutations

As before, X denotes a finite set. We let Σ be the symmetric group on X , that is, the group of all permutations of X . For any $\sigma \in \Sigma$, we define

$$\Delta_\sigma = \{ (\sigma(x), x) \in X \times X \mid x \in X \}.$$

This is actually the graph of the map σ^{-1} , but the choice is made so that $\Delta_\sigma \Delta_\tau = \Delta_{\sigma\tau}$ for all $\sigma, \tau \in \Sigma$. With a slight abuse, we shall often call Δ_σ a permutation. We also write $\Delta = \Delta_{\text{id}}$.

The group Σ has a left action on the set of all relations, σ acting via left multiplication by Δ_σ . Similarly, Σ also acts on the right on the set of relations. It is useful to note how multiplication by Δ_σ behaves. Given any relation R on X ,

$$(x, y) \in R \iff (\sigma(x), y) \in \Delta_\sigma R \iff (x, \sigma^{-1}(y)) \in R\Delta_\sigma.$$

3.1. Lemma. *Let R be a relation on X and let Δ_σ be a permutation.*

- (a) *R is essential if and only if $\Delta_\sigma R$ is essential.*
- (b) *Δ_σ is essential.*
- (c) *The left action of Σ on the set of all essential relations is free.*

Proof : (a) If R factorizes through a set of cardinality smaller than $\text{Card}(X)$, then so does $\Delta_\sigma R$. The converse follows similarly using multiplication by $\Delta_{\sigma^{-1}}$.

(b) This follows from (a) by taking $R = \Delta$ (which is essential by Proposition 2.5 because it is an order).

(c) Suppose that $\Delta_\sigma R = R$ for some $\sigma \neq \text{id}$. Then

$$(x, y) \in R \iff (\sigma(x), y) \in R,$$

hence ${}_x R = {}_{\sigma(x)} R$. Since $\sigma \neq \text{id}$, two columns of R are equal and so R is inessential by Corollary 2.2. Thus if R is essential, $\Delta_\sigma R \neq R$ for all $\sigma \neq \text{id}$. \square

Our next result will be essential in our analysis of essential relations.

3.2. Theorem. *Any essential relation contains a permutation.*

We shall provide two different proofs. The first is direct, while the second uses a theorem of Philip Hall. In fact, a relation containing a permutation is called a Hall relation in a paper of Schwarz [Sch], because of Hall's theorem, so Theorem 3.2 asserts that any essential relation is a Hall relation.

First proof : Let R be a relation on X and let $n = \text{Card}(X)$. We have to prove that, if R is essential, then there exists $\sigma \in \Sigma$ such that R contains $\Delta_{\sigma^{-1}}$, that is, $R\Delta_\sigma$ contains Δ (or in other words $R\Delta_\sigma$ is reflexive). Let $D_\sigma = R\Delta_\sigma \cap \Delta$ and suppose that $\text{Card}(D_\sigma) < n$, for all $\sigma \in \Sigma$. Then we have to prove that R is inessential.

We choose $\tau \in \Sigma$ such that $\text{Card}(D_\sigma) \leq \text{Card}(D_\tau)$, for all $\sigma \in \Sigma$. We let $S = R\Delta_\tau$ and we aim to prove that S is inessential (hence R too by Lemma 3.1). Note that $D_\tau \subseteq S$ by construction. Define

$$A = \{a \in X \mid (a, a) \in D_\tau\}, \quad \text{in other words} \quad D_\tau = \{(a, a) \mid a \in A\}.$$

In particular $\text{Card}(A) = \text{Card}(D_\tau) < n$. By maximality of D_τ , we have the following property :

$$\text{Card}(S\Delta_\sigma \cap \Delta) \leq \text{Card}(A), \quad \forall \sigma \in \Sigma. \quad (*)$$

Given $x, y \in X$, define a *path* from x to y to be a sequence x_0, x_1, \dots, x_r of elements of X such that $x_0 = x$, $x_r = y$, and $(x_i, x_{i+1}) \in S$ for all $i = 0, \dots, r-1$. We write $x \rightsquigarrow y$ to indicate that there is a path from x to y , and also $x \rightarrow y$ whenever $(x, y) \in S$ (path of length 1). Define A^c to be the complement of A in X (so A^c is nonempty by assumption). Define also

$$\begin{aligned} A_1 &= \{a \in A \mid \text{there exists } z \in A^c \text{ and a path } z \rightsquigarrow a\}, \\ A_2 &= \{a \in A \mid \text{there exists } z \in A^c \text{ and a path } a \rightsquigarrow z\}. \end{aligned}$$

We claim that there is no path from an element of A_1 to an element of A_2 . Suppose by contradiction that there is a path $a_1 \rightsquigarrow a_2$ with $a_i \in A_i$. Then there exists $z_i \in A^c$ and paths $z_1 \rightsquigarrow a_1 \rightsquigarrow a_2 \rightsquigarrow z_2$, in particular $z_1 \rightsquigarrow a \rightsquigarrow z_2$ with $a \in A$. In the path $z_1 \rightsquigarrow a$, let w_1 be the element of A^c closest to a , so that the path $w_1 \rightsquigarrow a$ does not contain any element of A^c except w_1 . Similarly, let w_2 be the element of A^c closest to a in the path $a \rightsquigarrow z_2$, so that the path $a \rightsquigarrow w_2$ does not contain any element of A^c except w_2 . We obtain a path $w_1 \rightsquigarrow a \rightsquigarrow w_2$ having all its elements in A except the two extremities w_1 and w_2 . By suppressing cycles within A , we can assume that all elements of A in this path are distinct. We end up with a path

$$w_1 \rightarrow x_1 \rightarrow \dots \rightarrow x_r \rightarrow w_2$$

where $x_1, \dots, x_r \in A$ are all distinct.

Let $\sigma \in \Sigma$ be the cycle defined by $\sigma(w_1) = x_1$, $\sigma(x_i) = x_{i+1}$ for $1 \leq i \leq r-1$, $\sigma(x_r) = w_2$, and finally $\sigma(w_2) = w_1$ in case $w_2 \neq w_1$. In case $w_2 = w_1$, then $\sigma(w_2)$ is already defined to be $\sigma(w_2) = \sigma(w_1) = x_1$. We emphasize that $\sigma(y) = y$ for all the other elements $y \in X$. Then we obtain :

$$\begin{array}{llll} (w_1, x_1) \in S & \text{hence} & (w_1, w_1) \in S\Delta_\sigma, \\ (x_i, x_{i+1}) \in S & \text{hence} & (x_i, x_i) \in S\Delta_\sigma, \\ (x_r, w_2) \in S & \text{hence} & (x_r, x_r) \in S\Delta_\sigma, \\ (y, y) \in S & \text{hence} & (y, y) \in S\Delta_\sigma, \quad \forall y \in A - \{x_1, \dots, x_r\}. \end{array}$$

Thus we obtain $(a, a) \in S\Delta_\sigma$, $\forall a \in A$, but also $(w_1, w_1) \in S\Delta_\sigma$. Therefore $\text{Card}(S\Delta_\sigma \cap \Delta) > \text{Card}(A)$, contrary to Property (*). This proves the claim that there is no path from A_1 to A_2 .

In particular, $A_1 \cap A_2 = \emptyset$ because if $a \in A_1 \cap A_2$ we would have a path of length zero from A_1 to A_2 (since $(a, a) \in S$). Let A_3 be the complement of $A_1 \cup A_2$ in A . Thus X is the disjoint union of the 4 subsets A^c , A_1 , A_2 , and A_3 .

We now claim the following :

- (a) There is no relation between A^c and A^c , that is, $S \cap (A^c \times A^c) = \emptyset$.
- (b) There is no relation between A^c and $A_2 \cup A_3$, that is, $S \cap (A^c \times A_2) = \emptyset$ and $S \cap (A^c \times A_3) = \emptyset$.
- (c) There is no relation between A_1 and A^c , that is, $S \cap (A_1 \times A^c) = \emptyset$.
- (d) There is no relation between A_1 and A_2 , that is, $S \cap (A_1 \times A_2) = \emptyset$.
- (e) There is no relation between A_1 and A_3 , that is, $S \cap (A_1 \times A_3) = \emptyset$.

To prove (a), suppose that $(w, z) \in S$ where $w, z \in A^c$. Choose $\rho \in \Sigma$ such that $\rho(a) = a$ for all $a \in A$ and $\rho(w) = z$. Then $(a, a) \in S\Delta_\rho$ for all $a \in A$ but also $(w, w) \in S\Delta_\rho$, contrary to Property (*).

To prove (b), we note that the definition of A_1 implies that, if $(z, a) \in S$ with $z \in A^c$ and $a \in A$, then $a \in A_1$. Thus $a \notin A_2 \cup A_3$.

To prove (c), suppose that $(a, z) \in S$ with $a \in A$ and $z \in A^c$. Then $a \in A_2$ by the definition of A_2 and in particular $a \notin A_1$.

Property (d) follows immediately from the previous claim that there is no path from A_1 to A_2 .

To prove (e), suppose that $(a_1, a_3) \in S$ with $a_1 \in A_1$ and $a_3 \in A_3$. Then by the definition of A_1 , there is a path $z \rightsquigarrow a_1 \rightarrow a_3$ where $z \in A^c$, but this means that $a_3 \in A_1$, a contradiction.

It follows that the relation S has the property that there is no relation between $A^c \cup A_1$ and $A^c \cup A_2 \cup A_3$. Therefore S is the union of the columns indexed by $(A^c \cup A_1)^c = A_2 \cup A_3$ and the lines indexed by $(A^c \cup A_2 \cup A_3)^c = A_1$, that is,

$$S = \left(\bigcup_{b \in A_2 \cup A_3} {}_b S \right) \cup \left(\bigcup_{a \in A_1} S_a \right).$$

Since $\text{Card}(A_2 \cup A_3) + \text{Card}(A_1) = \text{Card}(A)$, we obtain a union of $\text{Card}(A)$ blocks. But $\text{Card}(A) < n$ by assumption, so S is inessential, as was to be shown. \square

Second proof : Let R be an essential relation on X . For any subset A of X , define

$$R_A = \{x \in X \mid \exists a \in A \text{ such that } (x, a) \in R\} = \bigcup_{a \in A} R_a.$$

Then R decomposes as a union of blocks

$$R = \left(\bigcup_{y \notin A} (R_y \times \{y\}) \right) \cup \left(\bigcup_{x \in R_A} (\{x\} \times {}_x R) \right).$$

Since R is essential, $\text{Card}(X - A) + \text{Card}(R_A)$ cannot be strictly smaller than $\text{Card}(X)$. Therefore $\text{Card}(R_A) \geq \text{Card}(A)$, for all subsets A of X , that is

$$\text{Card} \left(\bigcup_{a \in A} R_a \right) \geq \text{Card}(A).$$

This is precisely the assumption in a theorem of Philip Hall (see Theorem 5.1.1 in [HaM], or [HaP] for the original version which is slightly different). The conclusion is that there exist elements $x_y \in R_y$, where y runs over X , which are all distinct. In other words $\sigma : y \mapsto x_y$ is a permutation and

$$(\sigma(y), y) = (x_y, y) \in R \quad \text{for all } y \in X.$$

This means that R contains Δ_σ , as required. \square

3.3. Corollary. *Let R be an essential relation on X . Then there exists $m \in \mathbb{N}$ such that R^m is a preorder.*

Proof : By Theorem 3.2, R contains Δ_σ for some $\sigma \in \Sigma$. If σ has order k in the group Σ , then R^k contains $\Delta_{\sigma^k} = \Delta$, so R^k is reflexive. Then the transitive closure of R^k is some power R^{kt} . This is reflexive and transitive, that is, a preorder. \square

We know that any order is an essential relation (Proposition 2.5), hence contains a permutation (Theorem 3.2). But in fact, we have a more precise result.

3.4. Lemma. *If R is an order on X , then R contains a unique permutation, namely Δ .*

Proof : Suppose that R is reflexive and transitive and contains a nontrivial permutation Δ_σ . Then σ contains a nontrivial k -cycle, say on x_1, \dots, x_k , for some $k \geq 2$. It follows that $(x_{i+1}, x_i) \in R$ for $1 \leq i \leq k-1$, hence $(x_k, x_1) \in R$ by transitivity of R . Now we also have $(x_1, x_k) \in R$ because $\sigma(x_k) = x_1$. Thus the relation R is not antisymmetric, hence cannot be an order. \square

In the same vein, we have the following more general result.

3.5. Lemma. *Let R be an order and let S, S' be two relations on X . The following two conditions are equivalent :*

- (a) $\Delta \subseteq S'S \subseteq R$.
- (b) *There exists a permutation Δ_σ such that :*
 $\Delta \subseteq \Delta_{\sigma^{-1}}S \subseteq R$ and $\Delta \subseteq S'\Delta_\sigma \subseteq R$.

Moreover, in condition (b), the permutation σ is unique.

Proof : If (b) holds, then

$$\Delta = \Delta^2 \subseteq (S'\Delta_\sigma)(\Delta_{\sigma^{-1}}S) = S'S \subseteq R^2 = R,$$

so (a) holds.

If (a) holds, then $S'S$ is essential, by Proposition 2.5. It follows that S is essential, and therefore S contains a permutation Δ_σ , by Theorem 3.2. Then we obtain

$$\Delta \subseteq \Delta_{\sigma^{-1}}S \quad \text{and} \quad S'\Delta_\sigma \subseteq S'S \subseteq R.$$

Similarly, S' is essential, hence contains a permutation Δ_τ , and we obtain

$$\Delta \subseteq S'\Delta_{\tau^{-1}} \quad \text{and} \quad \Delta_\tau S \subseteq S'S \subseteq R.$$

Now R contains $\Delta_\tau\Delta_\sigma = \Delta_{\tau\sigma}$ and Lemma 3.4 implies that $\tau\sigma = \text{id}$, that is, $\tau = \sigma^{-1}$. Then (b) follows.

If moreover S contains a permutation Δ_ρ (that is, $\Delta \subseteq \Delta_{\rho^{-1}}S$), then

$$\Delta_{\sigma^{-1}\rho} = \Delta_{\sigma^{-1}}\Delta_\rho \subseteq S'S \subseteq R,$$

and so $\sigma^{-1}\rho = \text{id}$ by Lemma 3.4, proving the uniqueness of σ . \square

4. The essential algebra

Let X be a finite set and let k be a commutative ring. We shall be mainly interested in the cases where k is either the ring \mathbb{Z} of integers or a field, but it is convenient to work with an arbitrary commutative ring.

Let \mathcal{R} be the k -algebra of the monoid of all relations on X . This monoid is a k -basis of \mathcal{R} and the product in the monoid defines the algebra structure. The set of all inessential relations on X spans a two-sided ideal I of \mathcal{R} . We define $\mathcal{E} = \mathcal{R}/I$ and call it the *essential algebra*. It is clear that \mathcal{E} is a k -algebra having as a k -basis

the set of all essential relations on X . Moreover, if R and S are essential relations but RS is inessential, then $RS = 0$ in \mathcal{E} .

Both \mathcal{R} and \mathcal{E} have an anti-automorphism, defined on the basis elements by $R \mapsto R^{op}$, where $(x, y) \in R^{op}$ if and only if $(y, x) \in R$. It is easy to see that $(RS)^{op} = S^{op}R^{op}$.

As before, we let Σ be the symmetric group of all permutations of X . We first describe an obvious quotient of \mathcal{E} .

4.1. Lemma. *Let H be the k -submodule of the essential algebra \mathcal{E} spanned by the set of all essential relations which strictly contain a permutation. Then H is a two-sided ideal of \mathcal{E} and $\mathcal{E}/H \cong k\Sigma$, the group algebra of the symmetric group Σ .*

Proof : Let us write \subset for the strict containment relation. Let R be an essential relation such that $\Delta_\sigma \subset R$ and let S be any essential relation. Then S contains a permutation Δ_τ , by Theorem 3.2. We obtain

$$\Delta_{\sigma\tau} = \Delta_\sigma \Delta_\tau \subset R \Delta_\tau \subseteq RS,$$

showing that $RS \in H$. Similarly $SR \in H$ and therefore H is a two-sided ideal of \mathcal{E} .

The quotient \mathcal{E}/H has a k -basis consisting of all the permutations Δ_σ , for $\sigma \in \Sigma$. Moreover, they multiply in the same way as permutations, so \mathcal{E}/H is isomorphic to the group algebra of the symmetric group Σ . \square

If k is a field, it follows, not surprisingly, that every irreducible representation of the symmetric group Σ gives rise to a simple \mathcal{E} -module. In short, the representation theory of the symmetric group Σ is part of the representation theory of \mathcal{E} .

We now want to describe another \mathcal{E} -module, which is simple when k is a field. We fix a total order T on X (e.g. the usual total order on $X = \{1, \dots, n\}$). Then any other total order on X is obtained by permuting the elements of X . Since permuting via σ corresponds to conjugation by Δ_σ , we see that $\{T_\sigma := \Delta_\sigma T \Delta_{\sigma^{-1}} \mid \sigma \in \Sigma\}$ is the set of all total orders on X . All of them are maximal essential relations on X , by Proposition 2.5.

4.2. Lemma. *Let T be a total order on X .*

- (a) *If $\rho \in \Sigma$, then $T \Delta_\rho T = 0$ in \mathcal{E} if $\rho \neq \text{id}$ and otherwise $T \Delta T = T^2 = T$.*
- (b) *The set $\{T_\sigma := \Delta_\sigma T \Delta_{\sigma^{-1}} \mid \sigma \in \Sigma\}$ is a set of pairwise orthogonal idempotents of \mathcal{E} .*

Proof : (a) TT_ρ contains both T and T_ρ (because both T and T_ρ contain Δ). Since $T \neq T_\rho$ if $\rho \neq \text{id}$, the product TT_ρ contains strictly T and is therefore inessential by Proposition 2.5. Thus $T \Delta_\rho T$ is also inessential, that is, $T \Delta_\rho T = 0$. On the other hand $T^2 = T$ because any preorder is idempotent.

(b) It follows from (a) that

$$T_\sigma T_\tau = \Delta_\sigma T \Delta_{\sigma^{-1}\tau} T \Delta_{\tau^{-1}} = \begin{cases} 0 & \text{if } \sigma \neq \tau, \\ \Delta_\sigma T \Delta_{\sigma^{-1}} = T_\sigma & \text{if } \sigma = \tau, \end{cases}$$

as was to be shown. \square

4.3. Proposition. *Fix a total order T on X . Let L be the k -submodule of the essential algebra \mathcal{E} spanned by the set $\{\Delta_\sigma T \mid \sigma \in \Sigma\}$. Then L is a left ideal of \mathcal{E} and is free of rank $n!$ as a k -module, where $n = \text{Card}(X)$. If k is a field, then L is a simple \mathcal{E} -module of dimension $n!$.*

Proof : Write $S_\sigma = \Delta_\sigma T$, for all $\sigma \in \Sigma$. Let R be an essential relation on X . Then $\Delta_\tau \subseteq R$ for some $\tau \in \Sigma$ by Theorem 3.2. Therefore $\Delta \subseteq \Delta_{\tau^{-1}}R$ and this implies that

$$S_\sigma = \Delta S_\sigma \subseteq \Delta_{\tau^{-1}}RS_\sigma \quad \text{and} \quad T = \Delta_{\sigma^{-1}}S_\sigma \subseteq \Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}RS_\sigma.$$

If this containment is strict, then $\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}RS_\sigma$ is inessential (by Proposition 2.5) and so RS_σ is inessential too (by Lemma 3.1). Otherwise $S_\sigma = \Delta_{\tau^{-1}}RS_\sigma$, hence $RS_\sigma = \Delta_\tau S_\sigma = \Delta_\tau \sigma T = S_{\tau\sigma}$. Therefore, in the algebra \mathcal{E} , either $RS_\sigma = 0$ or $RS_\sigma = S_{\tau\sigma}$. This proves that L is a left ideal of \mathcal{E} .

Clearly L has rank $n!$ with basis $\{S_\sigma \mid \sigma \in \Sigma\}$. The action of \mathcal{E} on L induces a k -algebra map

$$\phi : \mathcal{E} \longrightarrow M_{n!}(k)$$

and L can be viewed as an $M_{n!}(k)$ -module (consisting of column vectors with entries in k). By Lemma 4.2, the action of $\Delta_\tau T \Delta_{\rho^{-1}}$ on basis elements is given by

$$(\Delta_\tau T \Delta_{\rho^{-1}}) \cdot S_\sigma = (\Delta_\tau T \Delta_{\rho^{-1}}) \cdot \Delta_\sigma T = \begin{cases} 0 & \text{if } \rho \neq \sigma, \\ S_\tau & \text{if } \rho = \sigma. \end{cases}$$

This means that $\phi(\Delta_\tau T \Delta_{\rho^{-1}})$ is the elementary matrix with a single nonzero entry 1 in position (τ, ρ) . Therefore the map ϕ is surjective. This implies that, if k is a field, the module L is simple as an \mathcal{E} -module, because the space of column vectors is a simple $M_{n!}(k)$ -module. \square

5. A nilpotent ideal

The purpose of this section is to construct a suitable nilpotent ideal N of the essential algebra \mathcal{E} . We shall later pass to the quotient by N and describe the quotient \mathcal{E}/N . In order to find nilpotent ideals, the following well-known result is often useful.

5.1. Lemma. *Let k be a commutative ring and let \mathcal{B} be a k -algebra which is finitely generated as a k -module. Let I be a two-sided ideal of \mathcal{B} which is k -linearly spanned by a set of nilpotent elements of \mathcal{B} .*

- (a) *If k is a field, then I is a nilpotent ideal of \mathcal{B} .*
- (b) *If $k = \mathbb{Z}$ and if \mathcal{B} is a finitely generated free \mathbb{Z} -module, then I is a nilpotent ideal of \mathcal{B} .*
- (c) *Suppose that \mathcal{B} is defined over \mathbb{Z} , that is, $\mathcal{B} \cong k \otimes_{\mathbb{Z}} \mathcal{B}_{\mathbb{Z}}$ for some \mathbb{Z} -algebra $\mathcal{B}_{\mathbb{Z}}$ which is finitely generated free as a \mathbb{Z} -module. Suppose that I is defined over \mathbb{Z} , that is, $I \cong k \otimes_{\mathbb{Z}} I_{\mathbb{Z}}$, where $I_{\mathbb{Z}}$ is a two-sided ideal of $\mathcal{B}_{\mathbb{Z}}$ which is \mathbb{Z} -linearly spanned by a set of nilpotent elements of $\mathcal{B}_{\mathbb{Z}}$. Then I is a nilpotent ideal of \mathcal{B} .*

Proof : (a) The assumption still holds after extending scalars to an algebraic closure of k . Therefore we can assume that k is algebraically closed. Let $J(\mathcal{B})$ be the Jacobson radical of \mathcal{B} . Then $J(\mathcal{B}) = \bigcap_{i=1}^r M_i$, where M_i is a maximal two-sided ideal of \mathcal{B} . Moreover, by Wedderburn's theorem, \mathcal{B}/M_i is isomorphic to a matrix algebra $M_{n_i}(k)$, because k is algebraically closed. We will show that $I \subseteq M_i$, for all $i = 1, \dots, r$. It then follows that $I \subseteq J(\mathcal{B})$, so I is nilpotent (because it is well-known that the Jacobson radical of a finite-dimensional k -algebra is nilpotent).

Let \bar{I} be the image of I in \mathcal{B}/M_i . Then \bar{I} is spanned by nilpotent elements of $M_{n_i}(k)$. But any nilpotent matrix has trace zero (because its characteristic polynomial is X^{n_i} and the coefficient of X^{n_i-1} is the trace, up to sign). It follows that \bar{I} is contained in $\text{Ker}(\text{tr})$, which is a proper subspace of $M_{n_i}(k)$. Now \bar{I} is a two-sided proper ideal of the simple algebra $M_{n_i}(k)$, hence $\bar{I} = \{0\}$, proving that $I \subseteq M_i$.

(b) Let F be a basis of \mathcal{B} as a \mathbb{Z} -module. Extending scalars to \mathbb{Q} , we see that F is a \mathbb{Q} -basis of the \mathbb{Q} -algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{B}$ and \mathcal{B} embeds in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{B}$. By part (a), the ideal $\mathbb{Q} \otimes_{\mathbb{Z}} I$ is nilpotent in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{B}$. Since I embeds in $\mathbb{Q} \otimes_{\mathbb{Z}} I$, it follows that I is nilpotent.

(c) By part (b), $I_{\mathbb{Z}}$ is a nilpotent ideal of $\mathcal{B}_{\mathbb{Z}}$. Extending scalars to k , we see that I is a nilpotent ideal of \mathcal{B} . \square

Recall that Σ denotes the symmetric group on X and that, if R is a reflexive relation, then \bar{R} denotes the transitive closure of R .

5.2. Lemma. *If $S = \Delta_{\tau^{-1}} R \Delta_{\tau}$ where $\tau \in \Sigma$ and R is a reflexive relation, then $\bar{S} = \Delta_{\tau^{-1}} \bar{R} \Delta_{\tau}$.*

Proof : We have $\bar{R} = R^m$ for some m and we obtain

$$S^m = (\Delta_{\tau^{-1}} R \Delta_{\tau})^m = \Delta_{\tau^{-1}} R^m \Delta_{\tau} = \Delta_{\tau^{-1}} \bar{R} \Delta_{\tau}.$$

Therefore S^m is a preorder, because it is conjugate to a preorder, and so $\bar{S} = S^m = \Delta_{\tau^{-1}} \bar{R} \Delta_{\tau}$. \square

5.3. Theorem. *Let \mathcal{F} be the set of all reflexive essential relations on X . Let N be the k -submodule of the essential algebra \mathcal{E} generated by all elements of the form $(S - \bar{S})\Delta_{\sigma}$ with $S \in \mathcal{F}$ and $\sigma \in \Sigma$ (where \bar{S} denotes the transitive closure of S).*

- (a) N is a nilpotent two-sided ideal of \mathcal{E} . In particular, N is contained in the Jacobson radical $J(\mathcal{E})$.
- (b) The quotient algebra $\mathcal{P} = \mathcal{E}/N$ has a k -basis consisting of all elements of the form $S\Delta_{\sigma}$, where S runs over the set of all orders on X and σ runs over the symmetric group Σ .

Proof : (a) Let \mathcal{E}_1 be the subalgebra of \mathcal{E} which is k -linearly generated by the set \mathcal{F} of all reflexive essential relations. It is clearly a subalgebra since the product of two reflexive relations is reflexive. Let N_1 be the k -submodule of \mathcal{E}_1 generated by all elements of the form $S - \bar{S}$ with $S \in \mathcal{F}$. We claim that N_1 is a two-sided ideal of \mathcal{E}_1 .

If $T \in \mathcal{F}$, then $\overline{T\overline{S}} = \overline{T\overline{S}}$ (because $\overline{T\overline{S}}$ contains both T and \overline{S} , hence $\overline{T\overline{S}}$, and $\overline{T\overline{S}}$ contains both T and S , hence $\overline{T\overline{S}}$). Therefore

$$T(S - \overline{S}) = (TS - \overline{T\overline{S}}) - (T\overline{S} - \overline{T\overline{S}}) = (TS - \overline{T\overline{S}}) - (T\overline{S} - \overline{T\overline{S}}).$$

Note that if TS is inessential (hence zero in \mathcal{E}_1), then its transitive closure \overline{TS} cannot be an order by Proposition 2.5 and is therefore also zero in \mathcal{E}_1 (again by Proposition 2.5). Thus, in the expression above, we obtain either generators of N_1 or zero. The same argument works for right multiplication by T (or use the anti-automorphism of \mathcal{E}) and this proves the claim.

The ideal N_1 is invariant under conjugation by Σ because, for every $\sigma \in \Sigma$,

$$\Delta_{\sigma^{-1}}(S - \overline{S})\Delta_\sigma = \Delta_{\sigma^{-1}}S\Delta_\sigma - \Delta_{\sigma^{-1}}\overline{S}\Delta_\sigma = \Delta_{\sigma^{-1}}S\Delta_\sigma - \overline{\Delta_{\sigma^{-1}}S\Delta_\sigma}$$

by Lemma 5.2. Therefore the generators of N can also be written $\Delta_\sigma(S' - \overline{S'})$ with $S' \in \mathcal{F}$ and $\sigma \in \Sigma$ (namely $S' = \Delta_{\sigma^{-1}}S\Delta_\sigma$). It follows that $N = N_1\Delta_\Sigma = \Delta_\Sigma N_1$, where we write for simplicity $\Delta_\Sigma = \{\Delta_\sigma \mid \sigma \in \Sigma\}$.

If R is an essential relation on X , then R contains a permutation Δ_σ (for some $\sigma \in \Sigma$) by Theorem 3.2, so $R = Q\Delta_\sigma$ with $Q \in \mathcal{F}$, and also $R = \Delta_\sigma Q'$ where $Q' = \Delta_{\sigma^{-1}}Q\Delta_\sigma$. Since N_1 is an ideal of \mathcal{E}_1 , it follows that N is invariant by right and left multiplication by R . Thus N is a two-sided ideal of \mathcal{E} .

The generators of N_1 are nilpotent, because if $\overline{S} = S^m$, then

$$\begin{aligned} (S - \overline{S})^m &= (S - S^m)^m = \sum_{j=0}^m \binom{m}{j} (-1)^j S^{m-j} S^{mj} \\ &= \left(\sum_{j=0}^m \binom{m}{j} (-1)^j \right) S^m = (1 - 1)^m S^m = 0. \end{aligned}$$

Thus N_1 is a nilpotent ideal of \mathcal{E}_1 , by Lemma 5.1 (because clearly \mathcal{E}_1 and N_1 are defined over \mathbb{Z}). Since N_1 is invariant under conjugation by Σ , we obtain $N^n = (N_1\Delta_\Sigma)^n = N_1^n\Delta_\Sigma$ for every $n \in \mathbb{N}$. Since $N_1^m = 0$ for some m , the ideal N is nilpotent.

(b) In the quotient algebra \mathcal{E}/N , any reflexive relation Q is identified with its transitive closure \overline{Q} . Moreover, by Theorem 3.2, any essential relation R on X can be written $R = Q\Delta_\sigma$, with Q reflexive, and $Q\Delta_\sigma$ is identified with $\overline{Q}\Delta_\sigma$ in the quotient algebra \mathcal{E}/N . Note that \overline{Q} is a preorder and that \overline{Q} is zero in \mathcal{E} if it is not an order, by Proposition 2.5.

On each basis element R of \mathcal{E} , the effect of passing to the quotient by N consists of just two possibilities.

- If R can be written $R = Q\Delta_\sigma$, with Q reflexive and \overline{Q} is not an order, then R is identified with $\overline{Q}\Delta_\sigma$, so R is zero in \mathcal{E}/N because \overline{Q} is zero.
- If R can be written $R = Q\Delta_\sigma$, with Q reflexive and \overline{Q} is an order, then R is identified with an element of the form $S\Delta_\sigma$ where S is an order (namely $S = \overline{Q}$).

In the second case, Δ_σ is the unique permutation contained in R (or in other words the expression $R = Q\Delta_\sigma$ is the unique decomposition of R as a product of a reflexive relation and a permutation). This is because if $\Delta_{\sigma'} \subseteq R$, we obtain

$$\Delta_{\sigma'\sigma^{-1}} = \Delta_{\sigma'}\Delta_{\sigma^{-1}} \subseteq R\Delta_{\sigma^{-1}} = Q \subseteq \overline{Q},$$

so that $\sigma'\sigma^{-1} = \text{id}$ since \overline{Q} is an order (Lemma 3.4). Thus $\sigma' = \sigma$. This uniqueness property shows, on the one hand, that both possibilities cannot occur simultaneously and, on the other hand, that in the second case the order $S = \overline{Q}$ is uniquely determined by R .

It follows that the nonzero images in \mathcal{E}/N of the basis elements of \mathcal{E} form a k -basis of \mathcal{E}/N consisting of (the images of) the elements $S\Delta_\sigma$, where S is an order and $\sigma \in \Sigma$. \square

The quotient algebra $\mathcal{P} = \mathcal{E}/N$ will be called the *algebra of permuted orders* on X , because every basis element $S\Delta_\sigma$ is obtained from the order S by applying a permutation σ to the rows of S . Moreover, Δ_σ is the unique permutation contained in $S\Delta_\sigma$, because Δ is the unique permutation contained in S by Lemma 3.4. This defines a Σ -grading on \mathcal{P} :

$$\mathcal{P} = \bigoplus_{\sigma \in \Sigma} \mathcal{P}_\sigma,$$

where \mathcal{P}_σ is spanned by the set of all permuted orders containing Δ_σ . Clearly $\mathcal{P}_\sigma \cdot \mathcal{P}_\tau = \mathcal{P}_{\sigma\tau}$, so we have indeed a Σ -grading. We also write $\mathcal{P}_1 := \mathcal{P}_{\text{id}}$ and call it the *algebra of orders* on X . Moreover, $\mathcal{P}_\sigma = \Delta_\sigma \mathcal{P}_1 = \mathcal{P}_1 \Delta_\sigma$, so that the product in \mathcal{P} is completely determined by the product in the subalgebra \mathcal{P}_1 and the product in the symmetric group Σ . Hence we first need to understand the subalgebra \mathcal{P}_1 .

6. The algebra of orders

Let \mathcal{P}_1 be the algebra of orders on X defined above. It has a k -basis \mathcal{O} consisting of all orders on X . The product of basis elements $R, S \in \mathcal{O}$ will be written $R \cdot S$ and is described as follows.

6.1. Lemma. *Let \cdot be the product in the k -algebra \mathcal{P}_1 .*

- (a) *Let $R, S \in \mathcal{O}$. Then the product $R \cdot S$ is equal to the transitive closure of $R \cup S$ if this closure is an order, and zero otherwise.*
- (b) *The product \cdot is commutative.*

Proof : (a) By definition of the ideal N , the product RS in the algebra $\mathcal{P} = \mathcal{E}/N$ is identified with the transitive closure \overline{RS} , which is also the transitive closure of $R \cup S$, because the inclusions

$$R \cup S \subseteq RS \subseteq (R \cup S)^2 \subseteq \overline{R \cup S} \subseteq \overline{RS}$$

force the equality $\overline{R \cup S} = \overline{RS}$. Now \overline{RS} is a preorder. If this is an order, then $R \cdot S = \overline{RS}$. If this preorder is not an order, then it is zero in \mathcal{E} (by Proposition 2.5), hence also zero in \mathcal{P}_1 .

- (b) This follows from (a) and the fact that $R \cup S = S \cup R$. \square

6.2. Theorem.

- (a) *There exists a k -basis $\{f_R \mid R \in \mathcal{O}\}$ of \mathcal{P}_1 , consisting of mutually orthogonal idempotents whose sum is 1, and such that, for every $R \in \mathcal{O}$, the ideal generated by f_R is free of rank one as a k -module.*
- (b) *\mathcal{P}_1 is isomorphic to a product of copies of k , indexed by \mathcal{O} :*

$$\mathcal{P}_1 \cong \prod_{R \in \mathcal{O}} k \cdot f_R.$$

Proof : We know that \mathcal{P}_1 is commutative, with a basis \mathcal{O} consisting of all orders on X . Any such basis element is idempotent. Moreover \mathcal{O} is a partially ordered set with respect to the containment relation and we make it a lattice by adding an element ∞ and defining $R \vee S = \infty$ whenever the transitive closure of $R \cup S$ is not an order, while $R \vee S$ is the transitive closure of $R \cup S$ otherwise. The greatest lower bound of R and S is just the intersection $R \cap S$.

Now define $g_R = R$ if $R \in \mathcal{O}$ and $g_\infty = 0$. By Lemma 6.1, these elements satisfy the condition $g_R \cdot g_S = g_{R \vee S}$. Therefore Theorem 10.1 of the appendix applies. We let

$$f_R = \sum_{\substack{S \in \mathcal{O} \\ R \subseteq S}} \mu(R, S) S,$$

where μ denotes the Möbius function of the poset \mathcal{O} , so by Möbius inversion, we have

$$R = \sum_{\substack{S \in \mathcal{O} \\ R \subseteq S}} f_S.$$

The transition matrix from $\{R \in \mathcal{O}\}$ to $\{f_R \mid R \in \mathcal{O}\}$ is upper-triangular, with 1 along the main diagonal, hence invertible over \mathbb{Z} . It follows that $\{f_R \mid R \in \mathcal{O}\}$ is a k -basis of \mathcal{P}_1 . By Theorem 10.1 of the appendix, $\{f_R \mid R \in \mathcal{O}\}$ is a set of mutually orthogonal idempotents in \mathcal{P}_1 whose sum is 1. Moreover, by the same theorem,

$$f_R \cdot T = \begin{cases} f_R & \text{if } T \subseteq R, \\ 0 & \text{if } T \not\subseteq R. \end{cases}$$

Since T runs over a k -basis of \mathcal{P}_1 , this proves that the ideal $\mathcal{P}_1 f_R$ generated by f_R is equal to the rank one submodule $k \cdot f_R$ spanned by f_R . Thus we obtain

$$\mathcal{P}_1 \cong \prod_{R \in \mathcal{O}} \mathcal{P}_1 f_R = \prod_{R \in \mathcal{O}} k \cdot f_R,$$

as was to be shown. \square

Note that if k is a field, then each idempotent f_R is primitive.

7. The algebra of permuted orders

We know from the end Section 5 that the algebra \mathcal{P} of permuted orders is Σ -graded

$$\mathcal{P} = \bigoplus_{\sigma \in \Sigma} \mathcal{P}_\sigma.$$

If $R, S \in \mathcal{O}$ and $\sigma, \tau \in \Sigma$, then the product in \mathcal{P} satisfies

$$(R \Delta_\sigma)(S \Delta_\tau) = (R \cdot (\Delta_\sigma S \Delta_{\sigma^{-1}})) \Delta_\sigma \Delta_\tau = (R \cdot (\Delta_\sigma S \Delta_{\sigma^{-1}})) \Delta_{\sigma\tau},$$

where \cdot denotes the product in \mathcal{P}_1 described in Lemma 6.1. Note that this definition makes sense because $\Delta_\sigma S \Delta_{\sigma^{-1}}$ is an order, since S is. Note also that we can write the basis elements as $\Delta_\sigma S$ with $S \in \mathcal{O}$, because $R \Delta_\sigma = \Delta_\sigma (\Delta_{\sigma^{-1}} R \Delta_\sigma)$ and $\Delta_{\sigma^{-1}} R \Delta_\sigma \in \mathcal{O}$.

Instead of \mathcal{O} , we can use the basis $\{f_R \mid R \in \mathcal{O}\}$ of \mathcal{P}_1 , consisting of the idempotents of \mathcal{P}_1 defined in Theorem 6.2. The group Σ acts by conjugation on the set \mathcal{O} of all orders, hence also on the set $\{f_R \mid R \in \mathcal{O}\}$. We first record the following easy observation.

7.1. Lemma. *Let R be an order and let f_R be the corresponding idempotent of \mathcal{P}_1 . For every $\sigma \in \Sigma$,*

$$\Delta_\sigma f_R \Delta_{\sigma^{-1}} = f_{\sigma R},$$

where ${}^\sigma R := \Delta_\sigma R \Delta_{\sigma^{-1}}$.

Proof : This follows immediately from the definition of f_R in Section 6. \square

Since $\mathcal{P} = \bigoplus_{\sigma \in \Sigma} \mathcal{P}_\sigma$, this has a k -basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma, R \in \mathcal{O}\}$. We now describe the product in \mathcal{P} with respect to this basis.

7.2. Lemma. *The product of basis elements of \mathcal{P} is given by :*

$$(\Delta_\tau f_S)(\Delta_\sigma f_R) = \begin{cases} 0 & \text{if } S \neq {}^\sigma R, \\ \Delta_{\tau\sigma} f_R & \text{if } S = {}^\sigma R, \end{cases}$$

for all $S, R \in \mathcal{O}$ and all $\tau, \sigma \in \Sigma$.

Proof : $(\Delta_\tau f_S)(\Delta_\sigma f_R) = \Delta_\tau f_S f_{\sigma R} \Delta_\sigma$. This is zero if $S \neq {}^\sigma R$. Otherwise we obtain $\Delta_\tau f_{\sigma R} \Delta_\sigma = \Delta_\tau \Delta_\sigma f_R = \Delta_{\tau\sigma} f_R$. \square

7.3. Corollary. *Let R be an order and let f_R be the corresponding idempotent of \mathcal{P}_1 . The left ideal $\mathcal{P} f_R$ has a k -basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma\}$.*

Proof : We have $(\Delta_\tau f_S) f_R = \begin{cases} 0 & \text{if } S \neq {}^\sigma R, \\ \Delta_\tau f_R & \text{if } S = {}^\sigma R, \end{cases}$ and we know that the set $\{\Delta_\sigma f_R \mid \sigma \in \Sigma\}$ is part of the basis of \mathcal{P} . \square

Now we introduce central idempotents in \mathcal{P} . Let R be an order, let Σ_R be the stabilizer of R in Σ , and denote by $[\Sigma/\Sigma_R]$ a set of coset representatives. By Lemma 7.1, Σ_R is also the stabilizer of f_R and we define

$$e_R = \sum_{\sigma \in [\Sigma/\Sigma_R]} \Delta_\sigma f_R \Delta_{\sigma^{-1}} = \sum_{\sigma \in [\Sigma/\Sigma_R]} f_{\sigma R},$$

the sum of the Σ -orbit of f_R .

7.4. Lemma. *Let $[\Sigma \setminus \mathcal{O}]$ be a set of representatives of the Σ -orbits in \mathcal{O} . The set $\{e_R \mid R \in [\Sigma \setminus \mathcal{O}]\}$ is a set of orthogonal central idempotents of \mathcal{P} , whose sum is $1_{\mathcal{P}} = \Delta$.*

Proof : We compute

$$\Delta_{\tau} f_S e_R = \sum_{\sigma \in [\Sigma/\Sigma_R]} \Delta_{\tau} f_S f_{\sigma R} = \begin{cases} 0 & \text{if } S \text{ does not belong to the orbit of } R, \\ \Delta_{\tau} f_S & \text{if } S = \sigma R. \end{cases}$$

On the other hand

$$e_R \Delta_{\tau} f_S = \sum_{\sigma \in [\Sigma/\Sigma_R]} f_{\sigma R} \Delta_{\tau} f_S = \sum_{\sigma \in [\Sigma/\Sigma_R]} f_{\sigma R} f_{\tau S} \Delta_{\tau}.$$

This is zero if τS does not belong to the Σ -orbit of σR , that is, if S does not belong to the Σ -orbit of R , while if $\tau S = \sigma R$, then we get $f_{\tau S} \Delta_{\tau} = \Delta_{\tau} f_S$. This shows that e_R is central.

We know that $\{f_R \mid R \in \mathcal{O}\}$ is a set of orthogonal idempotents with sum 1. Since we have just grouped together the Σ -orbits, it is clear that the set $\{e_R \mid R \in [\Sigma \setminus \mathcal{O}]\}$ is also a set of orthogonal idempotents of \mathcal{P} , whose sum is $1_{\mathcal{P}} = \Delta$. \square

It follows from Lemma 7.4 that $\mathcal{P} \cong \prod_{R \in [\Sigma \setminus \mathcal{O}]} \mathcal{P} e_R$ and we have to understand the structure of each term.

7.5. Theorem. *Let R be an order on X and let Σ_R be its stabilizer in Σ . Then*

$$\mathcal{P} e_R \cong M_{|\Sigma:\Sigma_R|}(k\Sigma_R),$$

a matrix algebra of size $|\Sigma:\Sigma_R|$ on the group algebra $k\Sigma_R$. In other words

$$\mathcal{P} \cong \prod_{R \in [\Sigma \setminus \mathcal{O}]} M_{|\Sigma:\Sigma_R|}(k\Sigma_R).$$

Proof : By Corollary 7.3, the left ideal $\mathcal{P} f_R$ is a free k -submodule of \mathcal{P} spanned by the set $\{\Delta_{\sigma} f_R \mid \sigma \in \Sigma\}$. The group Σ_R acts on the right on this set, because $f_R \Delta_h = \Delta_h f_R$ for every $h \in \Sigma_R$. It follows that $\mathcal{P} f_R$ is a free right $k\Sigma_R$ -module with basis $\{\Delta_{\sigma} f_R \mid \sigma \in [\Sigma/\Sigma_R]\}$.

Clearly, the left action of \mathcal{P} commutes with the right action of $k\Sigma_R$. The left action of \mathcal{P} on this free right $k\Sigma_R$ -module induces a k -algebra map

$$\phi_R : \mathcal{P} \longrightarrow M_{|\Sigma:\Sigma_R|}(k\Sigma_R).$$

By Lemma 7.2, e_R acts as the identity on $\mathcal{P} f_R$, while e_S acts by zero if S does not belong to the Σ -orbit of R . Therefore we get a k -algebra map

$$\phi_R : \mathcal{P} e_R \longrightarrow M_{|\Sigma:\Sigma_R|}(k\Sigma_R),$$

because $\phi_R(e_S) = 0$ whenever S does not belong to the Σ -orbit of R . Putting all these maps together, we obtain a k -algebra map

$$\phi = \prod_{R \in [\Sigma \setminus \mathcal{O}]} \phi_R : \mathcal{P} \longrightarrow \prod_{R \in [\Sigma \setminus \mathcal{O}]} M_{|\Sigma:\Sigma_R|}(k\Sigma_R).$$

Lemma 7.2 shows that, if $\rho, \sigma, \tau \in [\Sigma/\Sigma_R]$ and $g \in \Sigma_R$, we have

$$(\Delta_{\tau} f_R \Delta_g \Delta_{\rho^{-1}}) \cdot \Delta_{\sigma} f_R = \begin{cases} 0 & \text{if } \rho \neq \sigma, \\ \Delta_{\tau} f_R \Delta_g & \text{if } \rho = \sigma, \end{cases}$$

using the fact that $f_R \Delta_g = \Delta_g f_R$. This means that $\phi_R(\Delta_\tau f_R \Delta_g \Delta_{\rho^{-1}})$ is the elementary matrix with a single nonzero entry equal to Δ_g in position (τ, ρ) . Moreover, we also have $\phi_S(\Delta_\tau f_R \Delta_g \Delta_{\rho^{-1}}) = 0$ whenever S does not belong to the Σ -orbit of R . Therefore the map ϕ is surjective.

Finally, we prove that ϕ is an isomorphism. It suffices to do this in the case where $k = \mathbb{Z}$, because all the algebras are defined over \mathbb{Z} (that is, they are obtained by extending scalars from \mathbb{Z} to k) and the algebra map ϕ is also defined over \mathbb{Z} . Now if $k = \mathbb{Z}$, then all algebras under consideration are finitely generated free \mathbb{Z} -modules and we know that the map ϕ is surjective. So it suffices to show that the source and the target of ϕ have the same rank as \mathbb{Z} -modules. The rank of \mathcal{P} is $|\Sigma| \text{Card}(\mathcal{O})$. On the other hand,

$$\text{rank}(M_{|\Sigma: \Sigma_R|}(k\Sigma_R)) = |\Sigma : \Sigma_R|^2 |\Sigma_R| = |\Sigma : \Sigma_R| |\Sigma|.$$

Summing over $R \in [\Sigma \setminus \mathcal{O}]$, we obtain

$$\sum_{R \in [\Sigma \setminus \mathcal{O}]} |\Sigma : \Sigma_R| |\Sigma| = |\Sigma| \sum_{R \in [\Sigma \setminus \mathcal{O}]} \text{Card}(\text{orbit of } R) = |\Sigma| \text{Card}(\mathcal{O}),$$

as was to be shown. \square

7.6. Remark. Since a matrix algebra $M_r(A)$ is Morita equivalent to A (for any k -algebra A), it follows from Theorem 7.5 that the algebra \mathcal{P} is Morita equivalent to a product of group algebras, namely $B = \prod_{R \in [\Sigma \setminus \mathcal{O}]} k\Sigma_R$. The bimodule which provides the Morita equivalence is $M = \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} \mathcal{P}f_R$, which is clearly a left \mathcal{P} -module by left multiplication, and a right module for each group algebra $k\Sigma_R$, acting by right multiplication on the summand $\mathcal{P}f_R$, and acting by zero on the other summands $\mathcal{P}f_S$, where $S \neq R$ in $[\Sigma \setminus \mathcal{O}]$. Notice that $\mathcal{P}f_R$ is the bimodule appearing in the proof of Theorem 7.5.

The bimodule inducing the inverse Morita equivalence is $M^\vee = \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} f_R \mathcal{P}$. Indeed we, obtain

$$M \otimes_B M^\vee \cong \bigoplus_{S \in [\Sigma \setminus \mathcal{O}]} \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} \mathcal{P}f_S f_R \mathcal{P} = \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} \mathcal{P}f_R \mathcal{P} = \mathcal{P}$$

and on the other hand

$$\begin{aligned} M^\vee \otimes_{\mathcal{P}} M &\cong \bigoplus_{S \in [\Sigma \setminus \mathcal{O}]} \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} f_S \mathcal{P} f_R = \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} f_R \mathcal{P} f_R \\ &= \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} k\Sigma_R \cdot f_R \cong \bigoplus_{R \in [\Sigma \setminus \mathcal{O}]} k\Sigma_R = B. \end{aligned}$$

Each module $\mathcal{P}f_R$ has rank $|\Sigma|$, because it has a k -basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma\}$, but we view it as a free right $k\Sigma_R$ -module of rank $|\Sigma : \Sigma_R|$.

8. Simple modules for the essential algebra

By standard commutative algebra, any simple \mathcal{E} -module is actually a module over $k/m \otimes_k \mathcal{E}$, where m is a maximal ideal of k . Replacing k by the quotient k/m , we assume from now on that k is a field. Let \mathcal{E} be the essential algebra of Section 5 and let $\mathcal{P} = \mathcal{E}/N$ be the algebra of permuted orders. Since N is a nilpotent ideal and since nilpotent ideals act by zero on simple modules, any simple \mathcal{E} -module can

be viewed as a simple \mathcal{P} -module. So we work with \mathcal{P} and we wish to describe all simple left \mathcal{P} -modules.

There is a general procedure for constructing all simple modules for the algebra of a semigroup S , using equivalence classes of maximal subgroups of S , see Theorem 5.33 in [CP], or Section 3 of [HK] for a short presentation. But our previous results allow for a very direct and easy approach, so we do not need to follow the method of [CP].

First notice that the simple \mathcal{P}_1 -modules are easy to describe, because \mathcal{P}_1 is a product of copies of k (by Theorem 6.2). More precisely, $\mathcal{P}_1 \cong \prod_{R \in \mathcal{O}} k \cdot f_R$ and each one-dimensional space $k \cdot f_R$ is a simple \mathcal{P}_1 -module (where R runs through the set \mathcal{O} of all orders).

8.1. Theorem. *Assume that k is a field. Let \mathcal{W} be the set of all pairs (R, V) , where R is an order on X and V is a simple $k\Sigma_R$ -module up to isomorphism. The group Σ acts on \mathcal{W} via $\sigma(R, V) := (\sigma R, \sigma V)$, where $\sigma R = \Delta_\sigma R \Delta_{\sigma^{-1}}$ and σV is the conjugate module, a module for the group algebra $k\Sigma_{\sigma R} = k[\sigma \Sigma_R \sigma^{-1}]$.*

- (a) *The set of isomorphism classes of simple \mathcal{P} -modules is parametrized by the set $\Sigma \backslash \mathcal{W}$ of Σ -conjugacy classes of pairs $(R, V) \in \mathcal{W}$.*
- (b) *The simple module corresponding to (R, V) under the parametrization of part (a) is*

$$S_{R,V} = W_R \otimes_k V,$$

where W_R is the unique (up to isomorphism) simple module for the matrix algebra $M_{|\Sigma:\Sigma_R|}(k)$ and $W_R \otimes V$ is viewed as a module for the algebra

$$M_{|\Sigma:\Sigma_R|}(k) \otimes_k k\Sigma_R \cong M_{|\Sigma:\Sigma_R|}(k\Sigma_R),$$

which is one of the factors of the decomposition of \mathcal{P} in Theorem 7.5.

- (c) *The simple \mathcal{P} -module $S_{R,V}$ is also isomorphic to $\mathcal{P}f_R \otimes_{k\Sigma_R} V$, with its natural structure of \mathcal{P} -module under left multiplication.*
- (d) *The simple \mathcal{P} -module $S_{R,V}$ has dimension $|\Sigma:\Sigma_R| \cdot \dim(V)$.*

Proof : By Theorem 7.5, any simple \mathcal{P} -module is a simple module for one of the factors $M_{|\Sigma:\Sigma_R|}(k\Sigma_R)$, where R belongs to a set $[\Sigma \backslash \mathcal{O}]$ of representatives of Σ -orbits in \mathcal{O} . In view of the isomorphism

$$M_{|\Sigma:\Sigma_R|}(k\Sigma_R) \cong M_{|\Sigma:\Sigma_R|}(k) \otimes_k k\Sigma_R,$$

any such simple module is isomorphic to a tensor product $W_R \otimes_k V$ as in the statement. This proves (a) and (b).

By Theorem 7.5, $\mathcal{P}e_R \cong M_{|\Sigma:\Sigma_R|}(k\Sigma_R)$ and its identity element e_R decomposes as a sum of orthogonal idempotents $e_R = \sum_{\sigma \in [\Sigma/\Sigma_R]} f_{\sigma R}$. Cutting by the idempotent f_R , we obtain the left ideal $\mathcal{P}f_R$, which is a free right $k\Sigma_R$ -module, isomorphic to the space of column vectors with coefficients in $k\Sigma_R$. Now $\mathcal{P}f_R$ is the bimodule providing the Morita equivalence between $M_{|\Sigma:\Sigma_R|}(k\Sigma_R)$ and $k\Sigma_R$ (see Remark 7.6). Therefore, for any simple left $k\Sigma_R$ -module V , the corresponding simple module for $\mathcal{P}e_R \cong M_{|\Sigma:\Sigma_R|}(k\Sigma_R)$ is the left \mathcal{P} -module $\mathcal{P}f_R \otimes_{k\Sigma_R} V$. Since $\mathcal{P}f_R$ is the space of column vectors with coefficients in $k\Sigma_R$, while W_R is the space of column vectors with coefficients in k , we get $\mathcal{P}f_R \cong W_R \otimes_k k\Sigma_R$. Therefore our simple \mathcal{P} -module is

$$\mathcal{P}f_R \otimes_{k\Sigma_R} V \cong W_R \otimes_k k\Sigma_R \otimes_{k\Sigma_R} V \cong W_R \otimes_k V \cong S_{R,V},$$

proving (c).

Finally, the dimension is

$$\dim(S_{R,V}) = \dim(W_R \otimes_k V) = \dim(W_R) \cdot \dim(V) = |\Sigma : \Sigma_R| \cdot \dim(V),$$

proving (d). \square

8.2. Example. Consider the trivial order Δ . Then $\Sigma_\Delta = \Sigma$ and the matrix algebra reduces to

$$M_{|\Sigma:\Sigma_\Delta|}(k\Sigma_\Delta) \cong M_1(k) \otimes_k k\Sigma \cong k\Sigma.$$

The simple module W_Δ for the algebra $M_1(k)$ is just k and the simple module $S_{\Delta,V} = W_\Delta \otimes_k V \cong V$ is just a simple $k\Sigma$ -module. In that case, the central idempotent e_S of \mathcal{P} acts by zero on V for any order $S \neq \Delta$, hence f_S too (because $f_S e_S = f_S$). Then $R = \sum_{R \subseteq S} f_S$ also acts by zero for any order $R \neq \Delta$. For any essential reflexive relation Q with $Q \neq \Delta$, the action of Q is equal to the action of \overline{Q} (because $Q - \overline{Q}$ belongs to the nilpotent ideal N), and therefore Q also acts by zero on V . Then so does the action of the essential relation $\Delta_\sigma Q$ containing the permutation Δ_σ . This shows that the simple modules $S_{\Delta,V} \cong V$ are just the modules for $k\Sigma$ viewed as a quotient algebra as in Lemma 4.1.

8.3. Example. Consider a total order T . Then $\Sigma_T = \{\text{id}\}$ and the matrix algebra reduces to

$$M_{|\Sigma:\Sigma_T|}(k\Sigma_T) \cong M_{n!}(k) \otimes_k k \cong M_{n!}(k).$$

In that case, there is unique simple $k\Sigma_T$ -module, namely $V = k$, the trivial module for the trivial group. We obtain the single simple module $S_{T,k} = W_T \otimes_k k \cong W_T$ for the algebra $M_{n!}(k)$. Equivalently, with the approach of part (c) of Theorem 8.1, we have $f_T = T$ (by maximality of T in \mathcal{O}) and so

$$S_{T,k} = \mathcal{P}f_T \otimes_{k\Sigma_T} V = \mathcal{P}T \otimes_k k \cong \mathcal{P}T.$$

So we obtain just the left ideal $\mathcal{P}f_T = \mathcal{P}T$, which turns out to be simple in that case. But it is also the left ideal L appearing in Proposition 4.3. So we have recovered the simple module of Proposition 4.3.

We also mention another byproduct of Theorem 7.5.

8.4. Theorem. *If the characteristic of the field k is zero or $> n$, then \mathcal{P} is a semi-simple k -algebra.*

Proof : It suffices to see that each factor in the decomposition of Theorem 7.5 is semi-simple. Now we have the isomorphism

$$M_{|\Sigma:\Sigma_R|}(k\Sigma_R) \cong M_{|\Sigma:\Sigma_R|}(k) \otimes_k k\Sigma_R,$$

and $M_{|\Sigma:\Sigma_R|}(k)$ is a simple algebra. Moreover the group algebra $k\Sigma_R$ is semi-simple by Maschke's theorem, because the characteristic of k does not divide the order of the group Σ_R , by assumption. The result follows. \square

Every simple \mathcal{P} -module $S_{R,V}$ is a simple \mathcal{R} -module, because of the successive quotients $\mathcal{R} \rightarrow \mathcal{E} \rightarrow \mathcal{P}$. We now give a direct description of the action on $S_{R,V}$ of an arbitrary relation in \mathcal{R} . Since $S_{R,V} \cong \mathcal{P}f_R \otimes_{k\Sigma_R} V$, it suffices to describe the action on $\mathcal{P}f_R$, and for this we can work again with an arbitrary commutative base ring k . Recall that $\mathcal{P}f_R$ has a basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma\}$.

8.5. Proposition. *Let k be a commutative ring. Let R be an order on X and let Q be an arbitrary relation (in the k -algebra \mathcal{R}). The action of Q on $\mathcal{P}f_R$ is described on the basis elements as follows :*

$$Q \cdot \Delta_\sigma f_R = \begin{cases} \Delta_\tau \sigma f_R & \text{if } \exists \tau \in \Sigma \text{ such that } \Delta \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

Proof : Suppose first that S is an order. By Lemma 7.2, the action of f_S is given by

$$f_S \cdot \Delta_\sigma f_R = \begin{cases} \Delta_\sigma f_R & \text{if } S = {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

Now $S = \sum_{\substack{T \in \mathcal{O} \\ S \subseteq T}} f_T$ and the action of f_T is nonzero only if $T = {}^\sigma R$. So we obtain the action of S as follows :

$$S \cdot \Delta_\sigma f_R = \begin{cases} \Delta_\sigma f_R & \text{if } S \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

Next we suppose that S is reflexive and that its transitive closure \overline{S} is an order. Since $S - \overline{S}$ belongs to the nilpotent ideal N of Section 5, which acts by zero because $\mathcal{P} = \mathcal{E}/N$, the action of S coincides with the action of \overline{S} . Moreover, the condition $\overline{S} \subseteq {}^\sigma R$ is equivalent to $S \subseteq {}^\sigma R$, because ${}^\sigma R$ is transitive. Therefore the action of S is the following :

$$S \cdot \Delta_\sigma f_R = \begin{cases} \Delta_\sigma f_R & \text{if } S \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

Now suppose that S is reflexive and that \overline{S} is not an order. Then \overline{S} is inessential by Proposition 2.5, hence zero in \mathcal{E} . So \overline{S} acts by zero, and since $S - \overline{S}$ acts by zero, the action of S is also zero. On the other hand, S cannot be contained in ${}^\sigma R$, otherwise $\overline{S} \subseteq {}^\sigma R$, which would force \overline{S} to be an order since ${}^\sigma R$ is an order. Therefore the condition $S \subseteq {}^\sigma R$ is never satisfied in that case. So the previous formula still holds, because we have zero on both sides :

$$S \cdot \Delta_\sigma f_R = \begin{cases} \Delta_\sigma f_R & \text{if } S \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

Now suppose that Q contains a permutation Δ_τ . Then $S = \Delta_{\tau^{-1}} Q$ is reflexive and $Q = \Delta_\tau S$. Thus the action of Q is :

$$\begin{aligned} Q \cdot \Delta_\sigma f_R &= \begin{cases} \Delta_\tau \Delta_\sigma f_R & \text{if } S \subseteq {}^\sigma R, \\ 0 & \text{otherwise ,} \end{cases} \\ &= \begin{cases} \Delta_\tau \sigma f_R & \text{if } \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise ,} \end{cases} \\ &= \begin{cases} \Delta_\tau \sigma f_R & \text{if } \Delta \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

The last equality holds because the condition $\Delta_{\tau^{-1}} Q \subseteq {}^\sigma R$ is equivalent to $\Delta \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R$, since $S = \Delta_{\tau^{-1}} Q$ is reflexive. This proves the result for such a relation Q .

Finally if Q does not contain a permutation, then Q is inessential by Theorem 3.2, hence acts by zero. On the other hand the condition that there exists $\tau \in \Sigma$ such

that $\Delta \subseteq \Delta_{\tau^{-1}}Q$ cannot be satisfied since Q does not contain a permutation. Therefore the previous formula still holds, because we have zero on both sides :

$$Q \cdot \Delta_{\sigma}f_R = \begin{cases} \Delta_{\tau\sigma}f_R & \text{if } \Delta \subseteq \Delta_{\tau^{-1}}Q \subseteq {}^{\sigma}R, \\ 0 & \text{otherwise .} \end{cases}$$

This proves the result in all cases. \square

8.6. Remark. In the description of the algebra \mathcal{P} (Theorem 7.5) and in the description of its simple modules (Theorem 8.1), we may wonder which groups appear as Σ_R for some order R . The answer is that the group Σ_R is arbitrary. More precisely, by a theorem of Birkhoff [Bi] (and further improvements by Thornton [Tho] and Barmak–Minian [BaM]), any finite group is isomorphic to Σ_R for some order R on a suitable finite set X . However, for a given finite set X , it is not clear which isomorphism classes of groups Σ_R appear.

Another question is to determine whether or not the simple modules of Theorem 8.1 are absolutely simple. But again this depends on the group Σ_R , because the field k may or may not be a splitting field for the group algebra $k\Sigma_R$.

9. A branching rule

In this section, we let $X = \{1, \dots, n\}$ for simplicity. In order to let n vary, we use a superscript (n) for all objects depending on n , such as $X^{(n)}$ for the set X , $\Sigma^{(n)}$ for the symmetric group on $X^{(n)}$, $\mathcal{O}^{(n)}$ for the set of all orders on $X^{(n)}$, $\mathcal{P}_1^{(n)}$ for the algebra of orders on $X^{(n)}$, $\mathcal{P}^{(n)}$ for the algebra of permuted orders on $X^{(n)}$, etc.

In the representation theory of the symmetric group $\Sigma^{(n)}$, there are well-known branching rules, describing the restriction of simple modules to the subgroup $\Sigma^{(n-1)}$ of all permutations of $X^{(n-1)}$, and the induction of simple modules from $\Sigma^{(n-1)}$ to $\Sigma^{(n)}$. In a similar fashion, working again over an arbitrary commutative base ring k , we will describe how modules for $\mathcal{P}^{(n-1)}$ behave under induction to $\mathcal{P}^{(n)}$. For this we need to view the former as a subalgebra of the latter. We first define

$$\phi : \mathcal{P}_1^{(n-1)} \longrightarrow \mathcal{P}_1^{(n)}, \quad \phi(R) = R \cup \{(n, n)\},$$

for any order R on $X^{(n-1)}$. It is clear that $\phi(R)$ is an order on $X^{(n)}$. Since $\Sigma^{(n-1)}$ is a subgroup of $\Sigma^{(n)}$ (by fixing the last letter n), the map ϕ clearly extends to an injective algebra homomorphism $\phi : \mathcal{P}^{(n-1)} \longrightarrow \mathcal{P}^{(n)}$.

Now we want to compute the image under ϕ of the idempotents $f_R^{(n-1)}$. For a given order R on $X^{(n-1)}$, we define

$$\mathcal{S}_R = \{S \in \mathcal{O}^{(n)} \mid S \cap (X^{(n-1)} \times X^{(n-1)}) = R\}.$$

9.1. Lemma. *Let R be an order on $X^{(n-1)}$ and let $f_R^{(n-1)}$ be the corresponding idempotent of $\mathcal{P}_1^{(n-1)}$.*

(a) *If S is an order on $X^{(n)}$ and if $f_S^{(n)}$ is the corresponding idempotent of $\mathcal{P}_1^{(n)}$, then*

$$\phi(f_R^{(n-1)}) \cdot f_S^{(n)} = \begin{cases} f_S^{(n)} & \text{if } S \in \mathcal{S}_R, \\ 0 & \text{otherwise .} \end{cases}$$

$$(b) \quad \phi(f_R^{(n-1)}) = \sum_{S \in \mathcal{S}_R} f_S^{(n)}.$$

Proof : We have $f_R^{(n-1)} = \sum_{\substack{Y \in \mathcal{O}^{(n-1)} \\ R \subseteq Y}} \mu(R, Y)Y$ and $\phi(Y) = Y \cup \{(n, n)\}$. Therefore, using part (a) of Theorem 10.1, we obtain

$$\begin{aligned} \phi(f_R^{(n-1)}) \cdot f_S^{(n)} &= \sum_{\substack{Y \in \mathcal{O}^{(n-1)} \\ R \subseteq Y}} \mu(R, Y)(Y \cup \{(n, n)\}) \cdot f_S^{(n)} \\ &= \left(\sum_{\substack{Y \in \mathcal{O}^{(n-1)} \\ R \subseteq Y, Y \cup \{(n, n)\} \subseteq S}} \mu(R, Y) \right) f_S^{(n)} \\ &= \left(\sum_{\substack{Y \in \mathcal{O}^{(n-1)} \\ R \subseteq Y \subseteq S \cap (X^{(n-1)} \times X^{(n-1)})}} \mu(R, Y) \right) f_S^{(n)}. \end{aligned}$$

We get zero if $R \not\subseteq S \cap (X^{(n-1)} \times X^{(n-1)})$ and also if $R \subset S \cap (X^{(n-1)} \times X^{(n-1)})$ (by the definition of the Möbius function). If now $R = S \cap (X^{(n-1)} \times X^{(n-1)})$, that is, if $S \in \mathcal{S}_R$, then the sum reduces to $\mu(R, R) = 1$ and we obtain $f_S^{(n)}$, proving (a).

Now we have

$$\phi(f_R^{(n-1)}) = \sum_{S \in \mathcal{O}^{(n)}} \phi(f_R^{(n-1)}) \cdot f_S^{(n)} = \sum_{S \in \mathcal{S}_R} f_S^{(n)},$$

proving (b). \square

9.2. Theorem. *Let R be an order on $X^{(n-1)}$, let $f_R^{(n-1)}$ be the corresponding idempotent of $\mathcal{P}_1^{(n-1)}$, and let V be a $k\Sigma_R^{(n-1)}$ -module. Then, inducing to $\mathcal{P}^{(n)}$ the $\mathcal{P}^{(n-1)}$ -module $S_{R,V} = \mathcal{P}^{(n-1)} f_R^{(n-1)} \otimes_{k\Sigma_R^{(n-1)}} V$, we obtain*

$$\begin{aligned} &\mathcal{P}^{(n)} \otimes_{\mathcal{P}^{(n-1)}} (\mathcal{P}^{(n-1)} f_R^{(n-1)} \otimes_{k\Sigma_R^{(n-1)}} V) \\ &\cong \bigoplus_{S \in \mathcal{S}_R} \mathcal{P}^{(n)} f_S^{(n)} \otimes_{k\Sigma_S^{(n)}} \text{Ind}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_S^{(n)}} \text{Res}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_R^{(n-1)}} V. \end{aligned}$$

Proof : Using Lemma 9.1, we obtain

$$\begin{aligned} &\mathcal{P}^{(n)} \otimes_{\mathcal{P}^{(n-1)}} (\mathcal{P}^{(n-1)} f_R^{(n-1)} \otimes_{k\Sigma_R^{(n-1)}} V) \\ &= \mathcal{P}^{(n)} \phi(f_R^{(n-1)}) \otimes_{k\Sigma_R^{(n-1)}} V \\ &\cong \bigoplus_{S \in \mathcal{S}_R} \mathcal{P}^{(n)} f_S^{(n)} \otimes_{k\Sigma_R^{(n-1)}} V \\ &\cong \bigoplus_{S \in \mathcal{S}_R} \mathcal{P}^{(n)} f_S^{(n)} \otimes_{k\Sigma_S^{(n)}} k\Sigma_S^{(n)} \otimes_{k[\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}]} k[\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}] \otimes_{k\Sigma_R^{(n-1)}} V \\ &\cong \bigoplus_{S \in \mathcal{S}_R} \mathcal{P}^{(n)} f_S^{(n)} \otimes_{k\Sigma_S^{(n)}} \text{Ind}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_S^{(n)}} \text{Res}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_R^{(n-1)}} V, \end{aligned}$$

proving the result. \square

Assume for simplicity that the base ring k is a field of characteristic zero and let V be a simple $k\Sigma_R^{(n-1)}$ -module. The $k\Sigma_S^{(n)}$ -module $\text{Ind}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_S^{(n)}} \text{Res}_{\Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}}^{\Sigma_R^{(n-1)}} V$ is a direct sum of simple modules W , and each W gives rise to a simple $\mathcal{P}^{(n)}$ -module $\mathcal{P}^{(n)} f_S^{(n)} \otimes_{k\Sigma_S^{(n)}} W$. Moreover, every such simple $\mathcal{P}^{(n)}$ -module occurs with multiplicities, appearing for instance whenever we have σ running in a set of representatives of cosets $[\Sigma_R^{(n-1)} / \Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}]$. For any such σ , we have ${}^\sigma S \in \mathcal{S}_R$ and also ${}^\sigma V \cong V$, because V is a $k\Sigma_R^{(n-1)}$ -module and $\sigma \in \Sigma_R^{(n-1)}$. Therefore the corresponding term in the direct sum is

$$\mathcal{P}^{(n)} f_{\sigma S}^{(n)} \otimes_{k\Sigma_R^{(n-1)}} V \cong \mathcal{P}^{(n)} f_{\sigma S}^{(n)} \otimes_{k\Sigma_R^{(n-1)}} {}^\sigma V,$$

but this gives rise to the same simple $\mathcal{P}^{(n)}$ -modules as the ones coming from S , by Theorem 8.1. Thus the multiplicity of these simple $\mathcal{P}^{(n)}$ -modules is at least $|\Sigma_R^{(n-1)} / \Sigma_R^{(n-1)} \cap \Sigma_S^{(n)}|$.

10. Appendix on Möbius inversion

In this appendix, we prove a general result on Möbius inversion involving idempotents in a ring. This was already used by the first author in other contexts (see Section 6.2 of [Bo]) and can be of independent interest.

Let (P, \leq) be a finite lattice. Write 0 for the minimal element of P and write $x \vee y$ for the least upper bound of x and y in P .

10.1. Theorem. *Let P be a finite lattice. Let $\{g_x \mid x \in P\}$ be a family of elements in a ring A such that $g_0 = 1$ and $g_x g_y = g_{x \vee y}$ for all $x, y \in P$. For every $x \in P$, define*

$$f_x = \sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_y,$$

where μ denotes the Möbius function of the poset P .

- (a) For all $x, y \in P$, we have $g_z f_x = f_x g_z = \begin{cases} f_x & \text{if } z \leq x, \\ 0 & \text{if } z \not\leq x. \end{cases}$
- (b) The set $\{f_x \mid x \in P\}$ is a set of mutually orthogonal idempotents in P whose sum is 1.

Note that our assumption implies that every g_x is idempotent, because $x \vee x = x$.

Proof : By Möbius inversion, we have

$$g_x = \sum_{\substack{y \in P \\ x \leq y}} f_y,$$

and in particular $1 = g_0 = \sum_{y \in P} f_y$.

Next we compute products. If $x, z \in P$, then

$$\begin{aligned} f_x g_z &= \left(\sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_y \right) g_z = \sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_y g_z = \sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_{y \vee z} \\ &= \sum_{\substack{w \in P \\ x \vee z \leq w}} \left(\sum_{\substack{y \in P \\ x \leq y, y \vee z = w}} \mu(x, y) \right) g_w. \end{aligned}$$

Note that $g_z f_x = f_x g_z$ because $g_z g_y = g_{z \vee y} = g_{y \vee z} = g_y g_z$. If x is strictly smaller than $x \vee z$, then the inner sum runs over the set of all elements y in the interval $[x, w] := \{v \in P \mid x \leq v \leq w\}$ such that $y \vee (x \vee z) = w$. But we have

$$\sum_{\substack{x \leq y \\ y \vee z = w}} \mu(x, y) = \sum_{\substack{x \leq y \\ y \vee (x \vee z) = w}} \mu(x, y) = 0,$$

by a well-known property of the Möbius function (Corollary 3.9.3 in [St]). Thus $f_x g_z = 0$ if x is strictly smaller than $x \vee z$, that is, if $z \not\leq x$.

If now $x = x \vee z$, that is, $z \leq x$, we get $y = y \vee z$ (because $z \leq x \leq y$), hence $y = w$, so that the inner sum has a single term for $y = w$. In that case, we get

$$f_x g_z = \sum_{\substack{w \in P \\ x \leq w}} \mu(x, w) g_w = f_x.$$

Therefore

$$f_x g_z = \begin{cases} f_x & \text{if } z \leq x, \\ 0 & \text{if } z \not\leq x, \end{cases}$$

proving (a).

If now $x, u \in P$, then

$$f_x f_u = \sum_{\substack{y \in P \\ u \leq y}} \mu(u, y) f_x g_y = \sum_{\substack{y \in P \\ u \leq y \leq x}} \mu(u, y) f_x.$$

If $u \not\leq x$, the sum is empty and we get zero. If $u < x$, then $\sum_{u \leq y \leq x} \mu(u, y) = 0$ by

the very definition of the Möbius function. This shows that $f_x f_u = 0$ if $u \neq x$. Finally, if $u = x$, then we get $f_x f_x = f_x$, thus f_x is idempotent, and the proof is complete. \square

10.2. Corollary. *Let P be a finite lattice. Write t for the maximal element of P and write $x \wedge y$ for the greatest lower bound of x and y in P . Let $\{g_x \mid x \in P\}$ be a family of elements in a ring A such that $g_t = 1$ and $g_x g_y = g_{x \wedge y}$ for all $x, y \in P$. For every $x \in P$, define*

$$f_x = \sum_{\substack{y \in P \\ y \leq x}} \mu(y, x) g_y,$$

where μ denotes the Möbius function of the poset P . Then the set $\{f_x \mid x \in P\}$ is a set of mutually orthogonal idempotents in P whose sum is 1.

Proof : This follows from Theorem 10.1 by using the opposite ordering on P . \square

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