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Multi-Regime Shallow Free-Surface Flow Models for Quasi-Newtonian Fluids

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Abstract

The mathematical modeling of thin free-surface flows for quasi-Newtonian fluids (power-law rheology: shear-thickening, shear-thinning and newtonian) is addressed. Asymptotic shallow flow models, one-equation and two-equations, consistent with any laminar viscous regimes, are derived formally. To do so, first reference flows, corresponding to different regimes, are considered. Second, the exact expressions of all fields (σ , \mathbf{u} , \mathbf{p}) are written as perturbations of the reference flows; all expressions are unified in the sense they are available for any viscous regimes. The asymptotic derivations are presented either in a mean slope coordinate system with local variations of the topography or in the Prandtl's coordinate system since it is shown to be equivalent. Then, all expressions and equations remain valid in presence of large topography variations. Formal error estimates proving the consistency of the derivations are stated. Next, one equation model (lubrication-type) on the flow thickness is derived at order 0 and at order 1. Then, two-equations models (shallow water type) are stated; also a gradually varied flow version is presented. All these models are unified in the sense that they remain valid for any viscous regimes, with any boundary conditions at bottom (from adherence to pure slip). The classical models from the literature are retrieved if considering the corresponding assumptions (generally a particular regime or a specific boundary condition at bottom). Preliminary 1D numerical examples illustrate the robustness of the unified one-equation model, in presence of two-regime flows (either due to a sharp change of the mean-slope topography or due to a sharp change of boundary condition at bottom).

Keywords: Thin-film flows, shallow-water, lubrication, viscous regimes, power-law, geophysical flows

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1. Introduction

This paper aims to derive mathematical models for free-surface thin layer flows, power-law rheology, and with general boundary conditions at bottom (from adherence to pure slip), hence multi-regimes. It is assumed that the applications concerned are geophysical ones, hence the effect of surface tension is not examined. A large number of geophysical flows involve viscoplastic rheology i.e. power-law fluids, from lava flows (see e.g. [1] and references therein) to mountains glaciers (inertia negligible), through ice-sheet flows (Antarctica, Greenland) (see e.g. [2] and references therein). (These two types of geophysical flows are strongly thermal dependent; nevertheless, given a temperature distribution, both present viscoplastic rheology which can be modeled by a power-law strain-stress relationship). Another large field of applications within the range of geophysical and industrial flows is paste and granular flows (from paints to debris avalanches e.g. [3]).

For Newtonian rheology and with flat bottom, gravity driven thin flows have been widely explored. The simplest model for Newtonian thin flows resulting from lubrication and shallow theory is the one equation model derived in [4], a first order viscous model with surface tension and adherence at bottom. It provides consistent models in the limit of long wave asymptotic (thin film approximation); but for large value of the Reynolds number, the Benney equation solution tends to blow up in amplitude in finite time. An evolution equation on the flow rate to get two equations models (shallow water type) can be introduced, then overpassing this limitation. For a detailed derivation of one and two equations models and related models, the reader can refer to [5] and references therein. More recently, [6, 7] have extended the two equations models to second order accuracy models.

All models cited previously are derived considering adherence at bottom, but it is possible to consider friction laws using the same approach. Considering a Navier type law at bottom with asymptotically small friction value, [8] obtained a Shallow-water type model with an extra source term; see e.g. [9] for an extension to 2D and varying topography. Using the methodology of [6], [10] presents models up to order two with Navier type friction law at bottom too.

The influence of the bottom topography has been studied using a curvilinear coordinate system by [11?] with adherence and by [12] with a Mohr-Coulomb yield. However this approach requires to have a full parametrization of the bottom surface, which is not available in a geophysical context.

For non-Newtonian rheologies, models are often elaborated empirically as follows. Starting from the standard shallow-water equations, the authors develop extra empirical terms modeling the non-linear viscous term in the averaged momentum equation, or the friction term, see e.g. [13].

In the particular case of power-law fluids, for flat bottom and no-slip condition, and starting from the Navier-Stokes equations, [14] obtained a one equation model that is of Benney's type. The corresponding two equations model has been written in [15], and even extended to yield stress fluids with a Bingham law. Another approach for deriving a two equations model was developed in [16, 17] in a weak formulation as a variational inequality. This method has been used to derive shallow models for both Bingham and Herschel-Bulkley law and a small friction at bottom.

Note that all the models found in the literature are mono regime and with either no-slip at bottom or with a Navier type law.

In the field of geophysical flows, generalized newtonian fluids, multi-regime flows are very frequent, and a unique asymptotic model for describing such flows remains a challenge. In the particular case of glaciers flows (power-law rheology, inertia negligible), the most famous models are the Shallow Ice Approximation (SIA) mode [18–20], the Shallow-Shelf Approximation (SSA) depth-integrated model, see [21]. For more details, see e.g. [22] and references therein. In [23, 24] it is shown that the so-called higher-order model established in [25] (model in primitive variables, not a asymptotic depth-integrated model) is consistent with various viscous regimes of Stokes ice flows (vanishing inertial term). In [26] (see also [27]), an extended SIA equation taking into account non vanishing velocity at bottom is presented. It is valid for a unique viscous regime (in the sense presented later) and for a vanishing mean slope. In [27], it is shown that this

last equation is valid for all friction - slip values at bottom.

In the present study, more general one-equation models and two-equations models (depth-integrated) are derived. The present derivations are new since they remain valid for any viscous regimes, for all the range of friction at bottom and for any topography variations. It leads to so-called *unified models*. The equations are unified in the sense they include all terms required for the few viscous regimes considered. To do so, a classical perturbation method is used but presented in a fixed-point like formalism (approach presented in [15, 28, 29]). The derivation of one-equation models (lubrication) consistent with the Navier-Stokes free-surface flow equation, power-law rheology, for multiple viscous regimes and multiple boundary conditions at bottom are obtained at order 0 and order 1 (the order definition is precised later). Furthermore, non-flat bottoms are taken into account. Curvilinear coordinates are not chosen in the present formalism, like it has been done in [11, 30]. In the present study, all derivations of the solutions and models are presented either in a mean slope coordinate system with local variations (mean slope may be large) or in the cartesian coordinate system without any restriction on the topography variations using the Prandtl's shift, [31], see also [32, 33]. In fact, all the equations and explicit fields derived are shown to be the same in both coordinate systems. The mean coordinate system is more restrictive (since a mean-slope with local small variations of topography is supposed to exist) but it is easier to read; while all the equations obtained are proved to remain valid without this restriction if using the Prandtl's shift. In the latter coordinate system, the free surface is flatten and the terms related to the topography in the equations can be developed.

The modeling strategy is as follows. The geometrical ratio $\varepsilon = \frac{H}{L}$ where H is a characteristic film thickness and L a characteristic streamwise length is classically defined. All asymptotic expansions are derived with respect to ε . The dimensionless Navier-Stokes equations are written in a fixed-point like form, following [15, 28, 29]. Next, the following dimensionless parameters appear naturally: $\beta = \varepsilon Re$ (balancing the inertial term), $\alpha = \varepsilon \frac{Fr^2}{Re}$ (balancing the viscous term) and $\delta = \varepsilon \frac{Re}{Fr^2}$ (balancing the pressure term). Then, the three regimes considered are as follows:

- Regime A: β , α small and $\delta = O(1)$;
- Regime B: β , δ small and $\alpha = O(1)$;
- Regime C: the three parameters small. This last regime is a direct consequence of the two previous ones since all three parameters are small. Furthermore, it includes the case β larger ($\beta = \sqrt{\varepsilon}$). This last case remains a viscous regime (it could be denoted as a Regime D). Regime C is the regime implicitly considered in the literature.

All regimes include potentially (non-linear) friction boundary conditions at bottom, from no-slip to pure slip. The present regime classification can be related to the review classification presented in [34] where the regimes considered are: inertial (which is not possible in the present approach), viscous (Regime B with small slope), visco-inertial (limit case of Regime A), nearly steady-uniform (Regime C with large λ).

A preliminary two-equations model (shallow-water type model) is derived, next the corresponding gradually varied flow equation is presented. Next, using the order 1 expression, we can derive an order 0 (but consistent) two-equations model.

If considering the same regime assumption and/or the same boundary condition at bottom (either adherence or asymptotically vanishing friction), if considering the same order in the geometrical ratio ε , then the classical shallow models from the literature are retrieved, both for power-law and newtonian fluids. Roughly, compare to the classical shallow models from literature, the present unified models contain extra terms in δ , $\beta\delta$ and in C (the slip coefficient).

Finally, let us remark that the well-known non-physical singularity at free surface for shear-thinning fluids (see e.g. [35–37]) is not explicit in the present shallow models since all asymptotic expansions are made at

order 1 both in regime and in the geometrical ratio ε . This question is discussed in Remark 16 and Section 11.

This paper is organized as follows. In Section 2, the dimensionless Navier-Stokes power-law free surface equations, with friction condition at bottom and varying topography, presented like in [15, 28]; it makes appear the dimensionless numbers β , α , δ introduced previously. The velocity scaling done leads naturally to a stress tensor scaling. In Section 3, the reference flow, corresponding to different boundary condition at bottom, are presented. The reference flows are important since all the asymptotic fields are calculated as perturbations of the reference flows. The perturbation algorithm used to calculate explicit expressions of the asymptotic solution $(\sigma, \mathbf{u}, \mathbf{p})$ up to order 1 or 2 with respect to ε is presented in sections 7 and 8. Nevertheless, if using the Prandtl's shift, see e.g. [32], then the asymptotic solutions up to the current orders are the same (excepted the vertical velocity w which is not used in the depth-integrated models); also equations are the shown to be the same, see Section 5. A by-product is all expressions and models remain valid for arbitrary large variations of the local topography at bottom.

Error estimates of the formal calculations done are presented in Section 6, validating the asymptotic derivations stated. Also, the error estimates show that the model error can be dominated by the friction model error stated a-priori. Next, the exact solution of the discharge (expression available for all regimes) is combined with the averaged mass equation in order to get the unified one-equation models (lubrification type model), at order 0 and at order 1, Section 9. Let us point out that already at order 0, the one-equation model includes distinct terms for Regime A and Regime B. The order 1 model contains extra terms in $\beta, \beta\delta$. Next, a preliminary two-equations model (shallow-water type model) is stated; and the corresponding gradually varied flow equation is deduced, Section 10. Then, in order to be able to state a consistent two-equations model, the fields expressions are required at order 1. Then, the unified two-equations models (shallow-water type) are derived in Section 11. Comparisons with existing models from the literature are done in Section 12. Finally, the present unified one-equation model order 0 is solved numerically in the cases of two regime thin flows with sharp regime transitions. The numerical results obtained demonstrate the ability of the unified model to represent accurately multi-regime flows with sharp but continuous transition; the change of regime being due to either a change of bottom boundary condition at bottom, or a change of topography (the mean slope in this case).

2. Primitive equations

In this section the dimensionless Navier-Stokes equations for power-law fluids with free surface and with a general power law friction law at bottom, are written. The scaling of the velocity naturally defines the scaling of the stress tensor. In other respect, the present scalings are compared to other ones from the literature.

2.1. Flow equations and dimensionless parameters

The fluid is assumed to be isotropic, non-Newtonian with a power-law type rheology. The power law is employed for a large range of fluid flows. The topography can be non constant. The coordinate system (x, z) is the “mean-slope” coordinate system, see figure 1. Later, it is shown that defining a mean slope is not a necessary condition; using the Prandtl's shift, the results obtained remain valid for arbitrary large topography variations.

Here, H is the fluid elevation, b the topography elevation, $h = (H - b)$ the fluid depth, $\mathbf{u} = (u, w)$ the velocity, p the pressure and θ the bottom mean slope, see figure 1. Remark that if $\theta = 0$, then (u, w) denotes the horizontal-vertical velocities components. \vec{g} is the gravity source term and a is a mass balance source term at the free surface (it may depends on space and time $a = a(x, t)$).

2.1.1. Flow equations

A two dimensional flow described by the Navier-Stokes equations is considered:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \varrho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= \nabla \cdot \sigma - \nabla p + \varrho \mathbf{g} \quad \text{in } \Omega_t \times]0, T[\end{aligned} \quad (1)$$

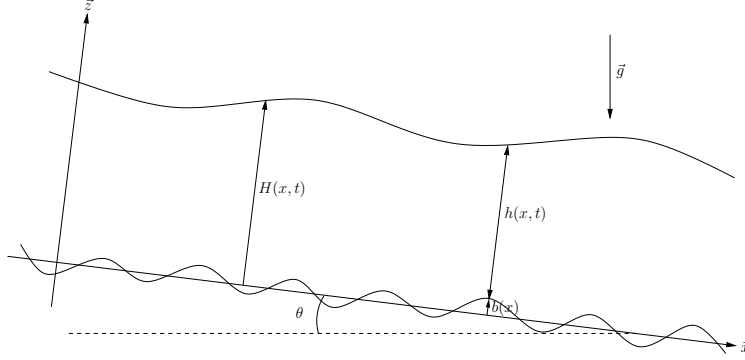


Figure 1: Geometry and notations. Mean-slope coordinate system.

with $(x, z) \in \Omega_t = \{(x, z) \in \mathbb{R}^2 | b(x) \leq z \leq H(x, t)\}$.

Here, $\mathbf{u} = (u, w)^T$ is the fluid velocity (u the streamwise and w the cross-stream component), p the pressure, $\Sigma = \sigma - p\text{Id}$ the full stress tensor with $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{pmatrix}$ the deviatoric stress tensor, ρ the fluid density and $\mathbf{g} = (g \sin \theta, -g \cos \theta)^T$ the gravity.

2.1.2. Rheology: power-law fluid

The generalized Newtonian fluid for which the shear stress τ is related to the deviatoric stress tensor σ by the Norton-Hoff Law (a viscoplastic law) is considered. This law is also known as the Ostwald-de Waele law. The Norton-Hoff law is considered in other fields such as glaciology, frozen soil mechanics or lithospheric materials.

The stress tensor Σ , is defined by:

$$\Sigma(\mathbf{u}, p) = \sigma(\mathbf{u}) - p\text{Id} = 2\mu D(\mathbf{u}) - p\text{Id} \quad (2)$$

where $\sigma(\mathbf{u}) = (\sigma_{ij})_{1 \leq i, j \leq 2}$, is the deviatoric stress tensor, μ is the apparent viscosity and $D(u)$ is the symmetric rate of strain tensor:

$$D(\mathbf{u}) = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) = \begin{pmatrix} \partial_x u & \frac{\partial_z u + \partial_x w}{2} \\ \frac{\partial_z u + \partial_x w}{2} & \partial_z w \end{pmatrix} \quad (3)$$

For a non-Newtonian fluid, the viscosity μ is a scalar function of the rate of strain tensor through the second invariant of $D(\mathbf{u})$:

$$I_2 = \|D(\mathbf{u})\|_F^2 = \sum_{i,j} D_{ij} D_{ji} = \partial_x u^2 + \frac{1}{2} (\partial_z u + \partial_x w)^2 + \partial_z w^2$$

Since the flow is incompressible, the following relation holds:

$$\partial_x u = -\partial_z w \Leftrightarrow \sigma_{xx} = -\sigma_{zz}$$

and then the expression of the second invariant can be simplified. The strain rate $\dot{\gamma}$ is defined as:

$$\dot{\gamma} = \sqrt{2I_2} = \sqrt{2} \|D(\mathbf{u})\|_F$$

where $\|D(\mathbf{u})\|_F = \sqrt{2\partial_x u^2 + \frac{1}{2} (\partial_z u + \partial_x w)^2}$. The power-law expresses the apparent viscosity using the strain rate:

$$\mu = \mu_0 \dot{\gamma}^{n-1} \quad (4)$$

Remark 1. In multi-physics flows, μ_0 may depend on an extra field. Typically, for ice or lava flows, it depends on the temperature.

Finally the deviatoric stress tensor can be written as:

$$\sigma = 2\mu_0\dot{\gamma}^{n-1}D(\mathbf{u}) \quad (5)$$

The relation (5) can be inverted. The norm of σ writes:

$$\|\sigma\|_F = 2\mu_0\dot{\gamma}^{n-1} \|D(\mathbf{u})\|_F = \sqrt{2}\mu_0\dot{\gamma}^n$$

Introduce the scalar σ_e for the effective stress defined as follows:

$$\sigma_e = \frac{1}{\sqrt{2}} \|\sigma\|_F = \mu_0\dot{\gamma}^n = \sqrt{\sigma_{xx}^2 + \sigma_{xz}^2}$$

The shear rate using the effective stress can be written:

$$\dot{\gamma} = \left(\frac{\sigma_e}{\mu_0} \right)^{\frac{1}{n}}$$

Then, the so-called Nye-Glen's flow law (the dual law of Ostwald-de Waele's law) is obtained. The latter writes:

$$D(\mathbf{u}) = A\sigma_e^{p-1}\sigma \quad (6)$$

with:

$$A = \frac{1}{2\mu_0^p} \quad \mu = \mu_0^p \sigma_e^{1-p} \quad p = \frac{1}{n} \quad \sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{xz}^2}$$

where A is the rate factor and p the stress exponent.

Three types of fluids, depending on the power law index value, can be defined:

- pseudoplastic fluids (shear thinning fluids) for $0 < n < 1$, apparent viscosity decreases when shear stress increases (e.g. ice, lava, mud and many industrial fluids);
- dilatant fluids (shear thickening fluids) for $n > 1$, apparent viscosity increases with shear rate (less common fluids);
- newtonian fluids for $n = 1$, the shear stress is proportional to the shear rate (we get $\mu = \mu_0$).

The system of Navier-Stokes equations (1) expanded reads:

$$\begin{aligned} \partial_x u + \partial_z w &= 0 \\ \varrho(\partial_t u + u\partial_x u + w\partial_z u) &= \partial_x(\sigma_{xx}) + \partial_z(\sigma_{xz}) - \partial_x p + \varrho g \sin \theta \quad \text{in } \Omega_t \times]0, T[\\ \varrho(\partial_t w + u\partial_x w + w\partial_z w) &= \partial_x(\sigma_{xz}) + \partial_z(\sigma_{zz}) - \partial_z p - \varrho g \cos \theta \end{aligned} \quad (7)$$

2.1.3. Boundary conditions

- At bottom, where $(x, z) \in \Gamma^b = \{(x, z) \in \mathbb{R}^2 | z = b(x)\}$, one of both cases is considered:

– no-slip condition (adherence):

$$\mathbf{u}|_{z=b} = 0$$

– friction condition. A power-law friction condition (also known as Weertman-type friction law) is considered:

$$\begin{cases} \mathbf{u} \cdot \mathbf{t}_b = -C_d |\sigma \cdot \mathbf{n}_b \cdot \mathbf{t}_b|^{p-1} \sigma \cdot \mathbf{n}_b \cdot \mathbf{t}_b \\ \mathbf{u} \cdot \mathbf{n}_b = 0 \end{cases} \quad \text{in } \Gamma^b \times]0, T[$$

with C_d the basal slipperiness (the minus sign causes C_d to be positive as the normal to the bottom is pointing outward). Vectors \mathbf{n}_b and \mathbf{t}_b are respectively the outward normal and the tangent to the bottom. Note that this general expression contains the no-slip condition when the slip coefficient C_d vanishes.

Remark 2. If there exists a heterogeneous layer between the solid at bottom (eg bedrock) and the fluid homogeneous fluid, then the friction law exponent may differ from the bulk rheology exponent.

- At the top (liquid-gas interface), where $(x, z) \in \Gamma_t^s = \{(x, z) \in \mathbb{R}^2 | z = H(x, t)\}$, the following equations hold:
 - continuity of the velocity:

$$\partial_t H + u(t, H) \partial_x H = w(t, H) + a \text{ in } I \times]0, T[$$

where I is a x -interval and a is a mass balance term given.

- continuity of the fluid stress (without taking into account capillary effects):

$$(\sigma - p\text{Id}) \mathbf{n} = 0 \text{ in } \Gamma_t^s \times]0, T[\quad (8)$$

Vectors \mathbf{n} and \mathbf{t} are respectively the normal and the tangent to the free surface. By projecting (8) on \mathbf{n} and \mathbf{t} and with some computations, the condition at free surface writes:

$$\begin{cases} \sigma_{xz} = 2\sigma_{xx} \frac{\partial_x H}{1 - \partial_x H^2} \\ p = -\sigma_{xx} \frac{1 + \partial_x H^2}{1 - \partial_x H^2} \end{cases} \text{ in } \Gamma_t^s \times]0, T[$$

Remark 3. Surface tension effects is not considered here for a sake of simplicity, also it is not necessary for geophysical flows. Furthermore, it is easy to take it into account in all the following calculations. In other respect, recall it has already been done for Regime C in [15].

2.1.4. Scaling

Let (H^*, U^*, P^*) be the characteristic values of the flow.

Remark 4. The characteristic values of the flow will be set as the mean values of the reference flows presented in the next subsection.

The dimensionless variables can be defined:

$$\begin{aligned} x &= L^* \tilde{x} & z &= H^* \tilde{z} & h &= H^* \tilde{h} & b &= H^* \tilde{b} \\ t &= \frac{L^*}{U^*} \tilde{t} & u &= U^* \tilde{u} & w &= W^* \tilde{w} & p &= P^* \tilde{p} \end{aligned} \quad (9)$$

The mass equation (incompressibility constraint) suggests to define: $W^* = \frac{H^*}{L^*} U^*$.

The pressure can be scaled by the hydrostatic pressure: $P^* = \rho g H^* \cos \theta$.

The geometric scaling parameter is defined by:

$$\varepsilon = \frac{H^*}{L^*} \quad (10)$$

Remark 5. The present mean-slope coordinate system prevents to consider large variations of the local topography $\partial_x b$. Indeed, large variations would induced large variations of the normal component of the velocity w , which is $O(\varepsilon)$ due to the mass equation. Nevertheless, in Section 5, it is shown that using the Prandtl's shift all the results remain valid for potentially large arbitrary topography variations.

Define the so-called equivalent viscosity:

$$\mu_e = \mu_0 \left(\frac{U^*}{H^*} \right)^{n-1} \quad (11)$$

It defines the natural scaling of the viscosity norm.

Remark 6. The scaling of the deviatoric stress tensor σ is a direct consequence of the scaling of the velocity (9). It gives $\sigma = \mu_e \frac{U^*}{H^*} \tilde{\sigma}$ with:

$$\tilde{\sigma} = \begin{pmatrix} \varepsilon \tilde{\sigma}_{xx} & \tilde{\sigma}_{xz} \\ \tilde{\sigma}_{xz} & -\varepsilon \tilde{\sigma}_{xx} \end{pmatrix}$$

Similarly, the following dimensionless strain rate are naturally obtained:

$$\tilde{\gamma} = \sqrt{4\varepsilon^2 \partial_{\tilde{x}} \tilde{u}^2 + (\partial_{\tilde{z}} \tilde{u} + \varepsilon^2 \partial_{\tilde{x}} \tilde{w})^2} \quad (12)$$

Note that: $\tilde{\gamma} = O(1)$. The constitutive law writes:

$$\tilde{D} = \frac{1}{2} \tilde{\sigma}_e^{p-1} \tilde{\sigma}$$

In its dimensionless form the apparent viscosity reads:

$$\mu = \mu_e \tilde{\sigma}_e^{1-p}$$

with $\tilde{\sigma}_e$ defined by $\tilde{\sigma}_e = \sqrt{\tilde{\sigma}_{xz}^2 + \varepsilon^2 \tilde{\sigma}_{xx}^2}$. The apparent viscosity writes equivalently:

$$\mu = \mu_e \tilde{\gamma}^{n-1}$$

with $\tilde{\tau}$ defined by (12).

Similarly, the basal slipperiness scaling and the accumulation scaling are given by:

$$C = \mu_0^p \frac{C_d}{H^*}, \quad a = \varepsilon U^* \tilde{a} \quad (13)$$

where $\mu_0^p C_d$ has the dimension of a length. Thus it is called the slip length and it characterizes the fluid/solid interaction.

The equivalent viscosity defined by (11) leads to the standard Reynolds number definition (also called the Ostwald number) as follows:

$$Re = \frac{\rho U^* H^*}{\mu_e} \quad (14)$$

The Froude number is defined as follows:

$$Fr = \frac{U^*}{\sqrt{g H^* \cos \theta}} \quad (15)$$

2.1.5. Small numbers and resulting regimes

Since the Navier-Stokes equations will be written as a fixed-point like form, following the method exposed in [15, 28], see Eqn(18), the dimensionless parameters which appear naturally are:

$$\beta = \varepsilon Re, \quad \alpha = \varepsilon \gamma, \quad \delta = \frac{\varepsilon}{\gamma} \quad \text{with } \gamma = \frac{Fr^2}{Re} \quad (16)$$

where β is the weight coefficient corresponding to the inertial term, δ is the weight coefficient corresponding to the viscous term and α the weight coefficient corresponding to the pressure.

Note that the following relationship holds:

$$\varepsilon^2 = \alpha \delta$$

The dimensionless gravity source term due to the bedrock mean-slope is defined by:

$$\lambda = \sqrt{\frac{\delta}{\alpha}} \tan(\theta) = \frac{1}{\gamma} \tan(\theta) \quad (17)$$

In all the sequel, it is assumed

	ε	Fr	Re	β	α	δ	γ	λ	C
“Ice-sheet”	10^{-3}	$3 \cdot 10^{-8}$	$3 \cdot 10^{-15}$	10^{-18}	10^{-4}	10^{-2}	10^{-1}	from 0 to 10^{+1}	from 0 to $O(1)$
“Ice-streams”	10^{-2}	10^{-6}	10^{-13}	10^{-14}	10^{-1}	10^{-3}	10^{+1}	from 0 to 10^{-1}	from $O(1)$ to large

Table 1: Example in glaciology. Orders of magnitude of dimensionless parameters upon the flow regime considered.

Assumption 1.

- the geometrical parameter ε is small,
- the dimensionless parameters (α, β, δ) are either small or at most $O(1)$.

The asymptotic expansions derived below will depend on the dimensionless parameters above.

2.1.6. Definition of the three flow regimes considered

In the present study, asymptotic models are derived for the following three laminar viscous flow regimes:

- Regime A: β small, α small while $\delta = O(1)$.
- Regime B: β small, δ small while $\alpha = O(1)$.
- Regime C: β small, α small and δ small.

This regime is a direct consequence of Regime A and B since all three parameters are small.

Note that a moderate inertial regime is included in the present viscous regimes since β can set to $\sqrt{\varepsilon}$ (i.e. $Re = \frac{1}{\sqrt{\varepsilon}}$). Such a regime is called the visco-inertial regime in the classification done in [34].

Example: In ice flows, different regimes appears naturally. Let us give orders of magnitude of the corresponding dimensionless parameters. The most common value for the rheological exponent is $p = 3$. Let us consider for the viscosity μ_e a “medium value” ($\mu_e \approx 10^{14} - 10^{15}$) (recall its value depends strongly on the temperature, see e.g. Greve and Blater [2]). We present below two typical regimes observed in Antarctica. First, the “ice-sheet” regime which represents typical large scale orders of magnitude of the whole ice-sheet. Second the “ice-stream” regime which represents typical orders of magnitude small scale of streams flows either (in Antarctica but also in mountains such as the Himalaya or the Alps). Since ice is extremely viscous, β is negligible for both regimes.

For the so-called “ice-sheet” regime, α is negligible compared to δ . At contrary for the so-called “ice-streams” regime, δ is negligible compared to α . In the glaciology context, the “ice-sheet” regime would correspond to Regime A with no-slip at bottom (the “slip” coefficient C defined by (13) vanish). While the “ice-stream” regime would correspond to the Regime B with variable friction boundary condition at bottom (C would vary from medium to large values).

Remark 7. A transition between Regime A and Regime B is possible only if all parameters are small, i.e. in the particular case of Regime C. It is important to remark that it is the only possible asymptotic transition between the two regimes. In fact others transitions involve non-small values of α and β , which means the geometrical parameter $\varepsilon = \sqrt{\alpha\delta}$ cannot be small, and thus there is no asymptotic expansion available. Nevertheless, in such a case, it is still possible to use the full Navier-Stokes equations to connect the two regimes, since we fully know the velocity and the pressure fields from the asymptotic solution (see their expressions in next sections).

In the sequel, the way to obtain the asymptotic models will be different depending on the regime considered, and the boundary conditions considered at bottom (either adherence or friction, from large to vanishing).

2.2. Dimensionless primitive equations

The dimensionless Navier-Stokes free-surface equations are written such that the quantities $(\partial_z \sigma_{xz}, \partial_z u, \partial_z p, \partial_z w, \sigma_{xx})$ are explicitly expressed. In all the sequel, the symbol \sim on all variables is skipped since all variables are dimensionless.

2.2.1. Equations in the bulk

From Equation (7) the dimensionless form of the mass conservation equation and the momentum equations are written. The deviatoric stress tensor components are kept as unknowns. These unknowns are related to the velocity unknowns (u, w) . By reorganizing the terms, the equations in the bulk Ω_t and for any time, are written as follows:

$$\left\{ \begin{array}{l} \partial_z \sigma_{xz} = -\lambda + \delta \partial_x p + \beta (\partial_t u + u \partial_x u + w \partial_z u) \\ \quad - \alpha \delta \partial_x \sigma_{xx} \\ \partial_z u = \sigma_e^{p-1} \sigma_{xz} - \alpha \delta \partial_x w \\ \partial_z p = -1 - \alpha \partial_z \sigma_{xx} + \alpha \partial_x \sigma_{xz} \\ \quad - \alpha \beta (\partial_t w + u \partial_x w + w \partial_z w) \\ \partial_z w = -\partial_x u \\ \sigma_{xx} = 2\sigma_e^{1-p} \partial_x u \end{array} \right. \quad \text{in } \Omega_t \times]0, T[\quad (18)$$

with the effective stress given by: $\sigma_e = \sqrt{\sigma_{xz}^2 + \alpha \delta \sigma_{xx}^2}$.

Remark 8. In the system of equations (18), the case $\beta = 0$ corresponds to the *a-priori* approximation of a quasi-static Stokes flow. This assumption can be made a-priori if modeling very viscous fluid flow, typically if modeling ice flows. Furthermore, if considering model order 1 in α or δ , and order 1 in the aspect ratio ε (recall $\varepsilon^2 = \alpha \delta$), the dimensionless primitives equations write:

$$\left\{ \begin{array}{l} \partial_z \sigma_{xz} = -\lambda + \delta \partial_x p \\ \partial_z u = |\sigma_{xz}|^{p-1} \sigma_{xz} \\ \partial_z p = -1 - \alpha \partial_z \sigma_{xx} + \alpha \partial_x \sigma_{xz} \\ \partial_z w = -\partial_x u \\ \sigma_{xx} = 2|\sigma_{xz}|^{1-p} \partial_x u \end{array} \right. \quad \text{in } \Omega_t \times]0, T[\quad (19)$$

2.2.2. Boundary conditions

In its dimensionless form, the free surface dynamics equation writes:

$$\partial_t H + u(t, H) \partial_x H = w(t, H) + a \quad \text{in } I \times]0, T[\quad (20)$$

and the continuity of the stress tensor writes:

$$\left\{ \begin{array}{l} \sigma_{xz} = 2\alpha \delta \sigma_{xx} \frac{\partial_x H}{1 - \alpha \delta \partial_x H^2} \\ p = -\alpha \sigma_{xx} \frac{1 + \alpha \delta \partial_x H^2}{1 - \alpha \delta \partial_x H^2} \end{array} \right. \quad \text{in } \Gamma_t^s \times]0, T[\quad (21)$$

At bottom the friction condition writes:

$$\left\{ \begin{array}{l} u = C \frac{|\sigma_{xz} (1 - \alpha \delta \partial_x b^2) - 2\alpha \delta \sigma_{xx} \partial_x b|^{p-1}}{(1 + \alpha \delta \partial_x b^2)^{p+\frac{1}{2}}} (\sigma_{xz} (1 - \alpha \delta \partial_x b^2) - 2\alpha \delta \sigma_{xx} \partial_x b) \\ w = u \partial_x b \\ \text{in } \Gamma^b \times]0, T[\end{array} \right. \quad (22)$$

In next section, the continuous transition from $C = 0$ (adherence) to $C \rightarrow \infty$ (pure slip) is considered.

Remark 9. At order 1 in α or δ , and order 1 in the aspect ratio ε (recall $\varepsilon^2 = \alpha\delta$), the boundary conditions write:

$$\sigma_{xz} = 0 ; \quad p = -\alpha\sigma_{xx} \quad \text{in } \Gamma_t^s \times]0, T[\quad (23)$$

$$u = C |\sigma_{xz}|^{p-1} \sigma_{xz} ; \quad w = u\partial_x b \quad \text{in } \Gamma^b \times]0, T[\quad (24)$$

Remark 10. In case of a flat-bottom ($\partial_x b = 0$), and if choosing the corresponding mean-slope coordinate system, the friction boundary condition writes:

$$\begin{cases} u = C |\sigma_{xz}|^{p-1} \sigma_{xz} \\ w = 0 \end{cases} \quad \text{in } \Gamma^b \times]0, T[\quad (25)$$

The limit case pure slip is $C \rightarrow \infty$.

3. Uniform flow as the reference flow

We define the reference flow as the stationary flow, uniform in x , solution of the zero-th order equations. It is equivalent to set: $b = 0$, $h = H = 1$ and $\alpha = \beta = \delta = 0$.

The system of equations (18) with the boundary conditions (21) and (25) give the uniform flow solution:

$$\begin{cases} \sigma_{xz} = \lambda(1-z) \\ p = (1-z) \\ u = C\lambda^p + \frac{\lambda^p}{p+1} \left(1 - (1-z)^{p+1}\right) \\ w = \sigma_{xx} = 0 \end{cases}$$

The stress tensor is a linear function of z . The pressure is hydrostatic and the vertical profile of u is a Poiseuille-like flow for $p = 1$ (Newtonian case).

Remark 11. Let us recall that close to the free surface (i.e. $z = 1$) the stress tensor vanish: $\sigma_{xz} \xrightarrow{z \rightarrow 1} 0$, therefore the apparent viscosity of the power-law fluid tends towards infinity: $\mu = \mu_e \sigma_e^{1-p} \xrightarrow{z \rightarrow 1} \infty$. In other words there is a plug like flow boundary layer close to the free surface.

For a geometrical parameter θ given, once settled either the Reynolds number Re or the Froude number Fr , we obtain the second one by setting λ since : $\lambda = \frac{Re}{Fr^2} \tan \theta$.

Then, the setting of the reference velocity U^* (equal to the surface velocity $u|_{z=H}$ or the mean velocity $\bar{u} = \int_b^H u(z) dz$ for example) implies a relation between λ and C :

$$\lambda (U^* = u|_{z=H}) = \left(\frac{p+1}{C(p+1)+1} \right)^{\frac{1}{p}} ; \quad \lambda (U^* = \bar{u}) = \left(\frac{p+2}{C(p+2)+1} \right)^{\frac{1}{p}} \quad (26)$$

As a first consequence, the pure slip case $C \rightarrow \infty$ corresponds necessarily to a vanishing mean slope in the sense :

$$\lambda \underset{C \rightarrow \infty}{\sim} \frac{1}{C^{1/p}}. \quad (27)$$

In the Newtonian case ($p = 1$), we recover the classical result that for a Nusselt film, the interface velocity and the mean velocity are given by $Re \tan \theta = 2Fr^2$ using the surface velocity as the reference velocity, or $Re \tan \theta = 3Fr^2$ using the mean velocity.

We plot the vertical velocity profiles corresponding to the three reference flows in figure 2 and figure 3. The reference velocity is the surface velocity. These reference flows correspond to the three cases we are interested in, which corresponds to different condition at bottom. They present a transition between the no-slip condition (a) ($C = 0$) to a pure slip condition ($C \rightarrow \infty$) through the intermediate case (b) with C at most $O(1)$.

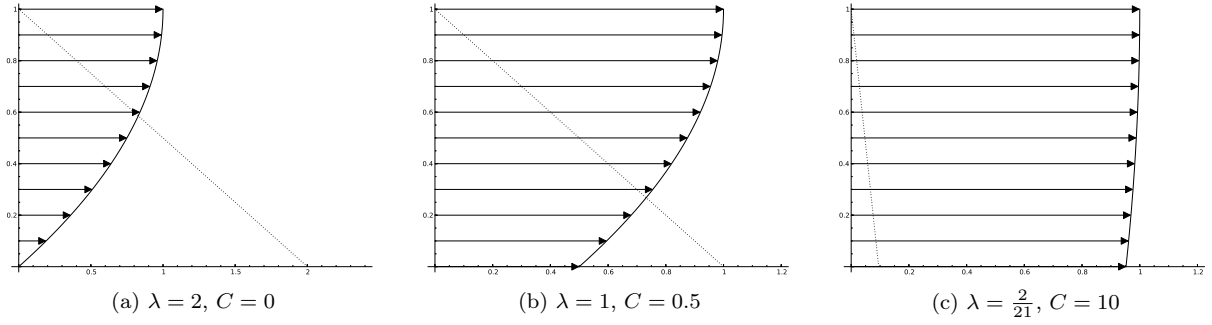


Figure 2: Reference flows vertical velocity profile in the newtonian case ($p = 1$):
 $— u$; $\cdots \partial_z u$

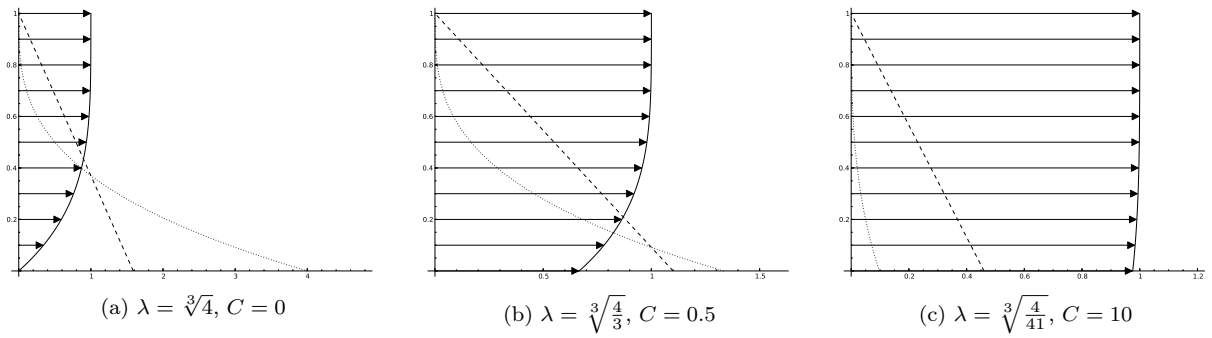


Figure 3: Reference flows vertical velocity profile for $p = 3$
 $— u$; $\cdots \partial_z u$; $-- \sigma_{xz}$

Remark 12. We can replace λ by its value (26), and passing to the limit $C \rightarrow \infty$, we obtain: $\sigma_{xz}(z) \xrightarrow{C \rightarrow \infty} 0$ and $u(z) \xrightarrow{C \rightarrow \infty} 1$. For shear thinning fluids, as the stress tensor vanishes, the apparent viscosity goes to infinity, meaning the flow moves as a solid plug:

$$\mu = \mu_e \|\sigma\|^{1-p} \xrightarrow{C \rightarrow \infty} \infty \quad (p \geq 1)$$

Therefore we will call the pure slip case $C \rightarrow \infty$ a “plug flow”. In that case, we necessarily have $\lambda = 0$. As a matter of fact, in the present model a gravitational viscous flow can not be the so-called plug flow if the mean slope θ is not vanishing.

Remark 13. If considering a steady-state uniform flow with $h = q = 1$ (hence $\partial_x H = 0$), with flat bottom, equation (26) can be seen as a relation between the regime flow parameter γ and the geometrical parameter $\tan \theta$ (the mean slope), and writes as follows:

$$\gamma = \frac{\tan \theta}{\left(\frac{p+2}{C(p+2)+1}\right)^{\frac{1}{p}}} \quad (28)$$

Since the (dimensionless) basal slipperiness C must be positive, for shear thinning fluids ($p \geq 1$) we get:

$$\tan \theta \leq 3\gamma \quad (29)$$

Let us recall that typical value of γ in Regime A is $\gamma = \varepsilon$ and in Regime B is $\gamma = \varepsilon^{-1}$. As a consequence, Regime A with sliding is possible only in case of very small mean-slope.

For shear-thickening fluids, only equation (28) holds; and in Regime A one can have any slip coefficient for any mean-slope value.

4. Asymptotic iterates

The method to calculate the exact solutions and the asymptotic models is as follows. First, we write an “explicit expression” of the unknowns $(\sigma_{xx}, \sigma_{xz}, u|_{z=b}, u, w, p)$. Second, we write the Navier-Stokes system as a fixed-point like system, where the unknown is $(\sigma_{xx}, \sigma_{xz}, u|_{z=b}, u, w, p)$. Third, by iterating (once or twice upon the asymptotic order required) formal calculations, we obtain formal higher-order expansions of the solution. This way to iterate the asymptotic expansions is presented in [15, 28].

Let us point out that a rigorous mathematical justification of such a asymptotic process is presented in [29] in the newtonian case, flat bottom, and with adherence at bottom. It is proved in [29] that the present iterative method gives rigorously the asymptotic solutions of the Navier-Stokes equations, thus obtaining (new) consistent Saint-Venant’s type equations.

Below, we develop the calculations for the three regimes considered (regimes A, B and C). Each solution and model are obtained as a perturbation of the corresponding zero-order quantity.

4.1. "Explicit expressions" of the unknowns $(\sigma_{xx}, \sigma_{xz}, u|_{z=b}, u, w, p)$

The stress tensor, the pressure and the velocity are obtained by solving the system of equations (18) with the boundary conditions (21) and (25). We use the Leibnitz rule to invert summation and derivative. The solution is given by the following relations:

$$\begin{cases} \sigma_{xz} = \psi_{xz}(\sigma_{xx}, p, u, w) \\ \sigma_{xx} = \psi_{xx}(\sigma_{xz}, \sigma_{xx}, u) \\ p = \psi_p(\sigma_{xz}, \sigma_{xx}, u, w) \\ u|_{z=b} = \psi_{bu}(\sigma_{xz}, \sigma_{xx}) \\ u = \psi_u(\sigma_{xz}, \sigma_{xx}, u|_{z=b}, w) \\ w = \psi_w(u|_{z=b}, u) \end{cases} \quad (30)$$

where the functions ψ_{\square} are defined as follows:

$$\left\{ \begin{array}{l} \psi_{xz}(\sigma_{xx}, p, u, w) = \lambda(H-z) - \delta \partial_x \left(\int_z^H p dz \right) \\ \quad - \beta \left(\partial_t \left(\int_z^H u dz \right) + \partial_x \left(\int_z^H u^2 dz \right) - uw - au|_{z=H} \right) \\ \quad + \alpha \delta \partial_x \left(\int_z^H \sigma_{xx} dz \right) \\ \\ \psi_{xx}(\sigma_{xz}, \sigma_{xx}, u) = 2 \|\sigma_e\|^{1-p} \partial_x u \\ \\ \psi_p(\sigma_{xz}, \sigma_{xx}, u, w) = (H-z) - \alpha \left(\sigma_{xx} + \partial_x \left(\int_z^H \sigma_{xz} dz \right) \right) \\ \quad + \alpha \beta \left(\partial_t \left(\int_z^H w dz \right) + \partial_x \left(\int_z^H uw dz \right) - w^2 - aw|_{z=H} \right) \\ \\ \psi_{bu}(\sigma_{xz}, \sigma_{xx}) = C \left[\frac{|\sigma_{xz}(1-\alpha\delta\partial_x b^2) - 2\alpha\delta\sigma_{xx}\partial_x b|^{p-1}}{(1+\alpha\delta\partial_x b^2)^{p+\frac{1}{2}}} (\sigma_{xz}(1-\alpha\delta\partial_x b^2) - 2\alpha\delta\sigma_{xx}\partial_x b) \right]_{|z=b} \\ \\ \psi_u(\sigma_{xz}, \sigma_{xx}, u|_{z=b}, w) = u|_{z=b} + \int_b^z \left(\|\sigma_e\|^{p-1} \sigma_{xz} \right) dz \\ \quad - \alpha \delta \left(\partial_x \left(\int_b^z w dz \right) + w|_{z=b} \partial_x b \right) \\ \\ \psi_w(u|_{z=b}, u) = u|_{z=b} \partial_x b - \int_b^z \partial_x u dz \end{array} \right. \quad (31)$$

with $\|\sigma_e\| = \sqrt{\sigma_{xz}^2 + \alpha\delta\sigma_{xx}^2}$.

It is useful to develop functions ψ_{xx} , ψ_{bu} and ψ_u in the form of Taylor series. Assuming that the quantities σ_{xz} , σ_{xx} , $\partial_x u$ and w may be written as follows:

$$\varphi = \varphi^{(1)} + O\left((\beta + \alpha\delta)^2\right)$$

with

$$\varphi^{(1)} = \varphi^{(0)} + \beta\varphi^{(1,\beta)} + \alpha\delta\varphi^{(1,\alpha\delta)}$$

Then one can expand ψ_{xx} , ψ_{bu} and ψ_u at order 1 for Regimes A and B, at order 2 for Regime C:

$$\left\{ \begin{array}{l} \psi_{xx}(\sigma_{xz}, \sigma_{xx}, u) = 2 \left| \sigma_{xz}^{(0)} \right|^{1-p} \left((\partial_x u^{(0)} + \beta \partial_x u^{(1,\beta)} + \alpha \delta \partial_x u^{(1,\alpha\delta)}) \right. \\ \quad \left. + (1-p) \frac{\partial_x u^{(0)}}{\sigma_{xz}^{(0)}} \left(\beta \sigma_{xz}^{(1,\beta)} + \alpha \delta \sigma_{xz}^{(1,\alpha\delta)} + \frac{1}{2} \alpha \delta \frac{(\sigma_{xx}^{(0)})^2}{\sigma_{xz}^{(0)}} \right) \right) + O(f(\alpha, \beta, \delta)) \\ \\ \psi_{bu}(\sigma_{xz}, \sigma_{xx}) = C \left(\left| \sigma_{xz|z=b}^{(0)} \right| \right)^{p-1} \left(\sigma_{xz|z=b}^{(0)} + p \left(\beta \sigma_{xz|z=b}^{(1,\beta)} + \alpha \delta \sigma_{xz|z=b}^{(1,\alpha\delta)} \right) \right. \\ \quad \left. - \alpha \delta \left(2p \partial_x b \sigma_{xx|z=b}^{(0)} + \sigma_{xz|z=b}^{(0)} (\partial_x b)^2 (2p + \frac{1}{2}) \right) \right) + O(f(\alpha, \beta, \delta)) \\ \\ \psi_u(\sigma_{xz}, \sigma_{xx}, u|_{z=b}, w) = u|_{z=b} + \int_b^z \left| \sigma_{xz}^{(0)} \right|^{p-1} \left(\sigma_{xz}^{(0)} + p \left(\beta \sigma_{xz}^{(1,\beta)} + \alpha \delta \sigma_{xz}^{(1,\alpha\delta)} \right) + \alpha \delta \frac{(p-1)}{2} \frac{(\sigma_{xx}^{(0)})^2}{\sigma_{xz}^{(0)}} \right) dz \\ \quad - \alpha \delta \left(\partial_x \left(\int_b^z w^{(0)} dz \right) + w|_{z=b}^{(0)} \partial_x b \right) + O(f(\alpha, \beta, \delta)) \end{array} \right.$$

where the residual function f depends on α and δ as follows:

- if α is small and δ is $O(1)$ (Regime A), $f(\alpha, \beta, \delta) = (\beta + \alpha)^2$
- if δ is small and α is $O(1)$ (Regime B), $f(\alpha, \beta, \delta) = (\beta + \delta)^2$
- if α and δ are small (Regime C), $f(\alpha, \beta, \delta) = (\beta + \alpha\delta)^2$

We define the operator:

$$\Psi_h : X = (\sigma_{xz}, \sigma_{xx}, p, u|_{z=b}, u, w) \mapsto \Psi_h(X) = (\psi_{xz}, \psi_{xx}, \psi_p, \psi_{bu}, \psi_u, \psi_w) \quad (32)$$

where $X = (\sigma_{xz}, \sigma_{xx}, p, u|_{z=0}, u, w)$ and ψ_{\square} are defined by equations (31). Then, X is solution of (18) (21) and (22) if $X = \Psi_h(X)$ (X is a fixed point of Ψ_h).

4.2. The averaged mass equation

We define the flow rate q and the mean velocity as follows:

$$q = \int_b^H u dz \text{ and } \bar{u} = \frac{1}{h} \int_b^H u dz \quad (33)$$

hence: $q = h\bar{u}$.

We assume that b is independent of time. We integrate the mass equation over the height, we use the Leibniz integration rule and the free surface equation (20), then we obtain the averaged mass equation:

$$\partial_t h + \partial_x q = a \text{ for } (x, t) \text{ in } I \times]0, T[\quad (34)$$

If the averaged mass equation is satisfied, then the free surface equation (20) is satisfied too.

4.3. Iterates and asymptotic expansions

The asymptotic expansions are derived using the following iterative algorithm (A).

- (A):
- Let $(X^{(0)}, h^{(0)})$ be the reference flow solution.
Recall that the latter depends on the flow regime considered, see the previous section.
 - $k \rightarrow (k+1)$, calculate: $X^{(k+1)} = \Psi_h(X^{(k)})$, with h given and Ψ_h defined by (32).

One need to iterate once (respectively twice i.e. $k=1$) in order to obtain 1st order (respectively 2nd order) asymptotic solutions.

Recall that in [29], it is proved (in the newtonian case) that the present algorithm leads to the Saint-Venant's equations; in other words at $k=1$ the first order asymptotical equations with respect to ε are mathematically proved.

The definition of $\Psi_h(X^{(k)})$ must depend on the regime considered.

Let us consider first **Regime A**: α and β small, $\delta = O(1)$.

The iterative scheme to compute $(\sigma_{xx}^{(k+1)}, \sigma_{xz}^{(k+1)}, u|_{z=b}^{(k+1)}, u^{(k+1)}, w^{(k+1)}, p^{(k+1)})$ from $(\sigma_{xx}^{(k)}, \sigma_{xz}^{(k)}, u|_{z=b}^{(k)}, u^{(k)}, w^{(k)}, p^{(k)})$ writes:

$$\left\{ \begin{array}{l} p^{(k+1)} = \psi_p(\sigma_{xz}^{(k)}, \sigma_{xx}^{(k)}, u^{(k-1)}, w^{(k-1)}) \\ \sigma_{xz}^{(k+1)} = \psi_{xz}(\sigma_{xx}^{(k)}, p^{(k+1)}, u^{(k)}, w^{(k)}) \\ u|_{z=b}^{(k+1)} = \psi_{bu}(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}) \\ u^{(k+1)} = \psi_u(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u|_{z=b}^{(k+1)}, w^{(k)}) \\ \sigma_{xx}^{(k+1)} = \psi_{xx}(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u^{(k+1)}) \\ w^{(k+1)} = \psi_w(u|_{z=b}^{(k+1)}, u^{(k+1)}) \end{array} \right. \quad (35)$$

Now, we consider **Regime B**: β, δ small and $\alpha = O(1)$.
As $\varepsilon^2 = \alpha\delta$, we have $\varepsilon^2 = O(\delta)$. The iterative scheme writes:

$$\begin{cases} \sigma_{xz}^{(k+1)} &= \psi_{xz} \left(\sigma_{xx}^{(k)}, p^{(k)}, u^{(k)}, w^{(k)} \right) \\ u|_{z=b}^{(k+1)} &= \psi_{bu} \left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)} \right) \\ u^{(k+1)} &= \psi_u \left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u|_{z=b}^{(k+1)}, w^{(k)} \right) \\ \sigma_{xx}^{(k+1)} &= \psi_{xx} \left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k)}, u^{(k+1)} \right) \\ p^{(k+1)} &= \psi_p \left(\sigma_{xz}^{(k+1)}, \sigma_{xx}^{(k+1)}, u^{(k)}, w^{(k)} \right) \\ w^{(k+1)} &= \psi_w \left(u|_{z=b}^{(k+1)}, u^{(k+1)} \right) \end{cases} \quad (36)$$

Recall that in both cases (Regime A and B), the algorithm is iterated once ($k = 0$) or twice ($k = 1$) depending on the asymptotic order seek.

In next sections, error estimates of the present formal calculations are stated; the exact asymptotical solutions using Algorithm (\mathcal{A}) are calculated. Nevertheless, before it is shown in next section that at the present orders (1 and 2) writing the equations in the mean coordinate system (see Section 2) or writing them in the horizontal-vertical coordinate system while using the Prandtl shift [31], is equivalent.

5. Asymptotic models for arbitrary topography: the Prandtl shift

Untill now, the equations have been written in the mean slope coordinate system, hence a-priori limiting the equations validity to small local topography variation $\partial_x b$. (As a matter of fact, if not the vertical velocity w would not remain small, and the scaling $W^* = \varepsilon U^*$ would not remain true anymore).

Nevertheless, this limitation of small local topography variation is in fact void since it is shown below that if using the horizontal-vertical coordinate system and if using Prandtl's shift [31], then at the orders considered in the present study (order 1 in both regimes and order 1 in the geometrical ratio $\varepsilon = (\alpha\delta)^{1/2}$), then the terms in the equations are exactly the same. Therefore, all the order 1 equations stated below are valid even it does not exist any mean slope, and in presence of arbitrary large local topography variations.

Let us recall the Prandtl transposition theorem [31] for the boundary layer equations. It states that, if the velocity satisfies the boundary layer equations, then applying a shift on the vertical component z with an arbitrary function $f(x, t)$ will also leave the boundary layer equations unchanged. Using this shift one can flatten the interface (see e.g. [32, 33]), and develop the terms related to the topography in the equations, see figure 4.

The Prandtl shift writes:

$$\hat{z} = z - H(x, t) \quad \hat{w} = w - \partial_t H - u \partial_x H$$

The vertical shift transforms the free surface into a flat surface (see figure 4). In the present study, its main interest is that all z -independent fields are identical in both coordinate systems. In other terms, the depth-averaged models do not depend on the Prandtl shift, and there is no need to make any reverse shift.

Let us introduce the basic rules for writing derivatives and integrals using the Prandtl shift. We denote the variables in the new coordinate system with a hat symbol: $\hat{\varphi}(x, \hat{z}(x, t), t) = \varphi(x, z, t)$. The derivatives with respect to x, z and t write in the new system:

$$\begin{aligned} \partial_x \varphi(x, z, t) &= \partial_x \hat{\varphi}(x, \hat{z}, t) - \partial_z \hat{\varphi}(x, \hat{z}, t) \partial_x H(x, t) \\ \partial_z \varphi(x, z, t) &= \partial_z \hat{\varphi}(x, \hat{z}, t) \\ \partial_t \varphi(x, z, t) &= \partial_t \hat{\varphi}(x, \hat{z}, t) - \partial_z \hat{\varphi}(x, \hat{z}, t) \partial_t H(x, t) \end{aligned}$$

Since the z -derivative remains unchanged in the new coordinate system, only the interval of the integrals changes:

$$\int_z^H \varphi dz = \int_{\hat{z}}^0 \hat{\varphi} d\hat{z} \quad \int_b^z \varphi dz = \int_{-\hat{h}}^{\hat{z}} \hat{\varphi} d\hat{z} \quad \int_b^H \varphi dz = \int_{-\hat{h}}^0 \hat{\varphi} d\hat{z}$$

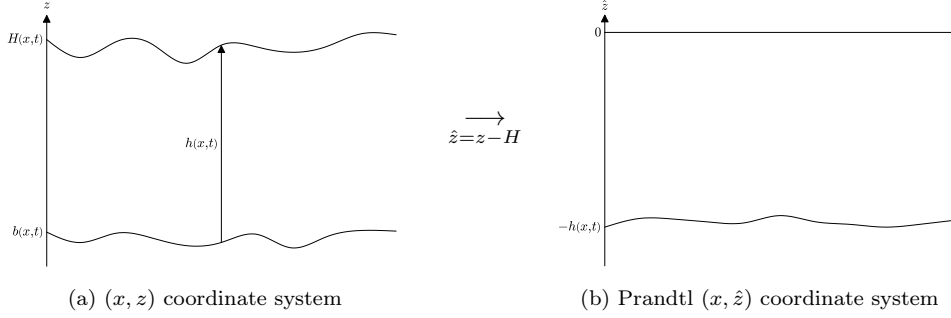


Figure 4: The Prandtl shift flattens the free surface.

But extra terms appear when deriving an integral:

$$\begin{aligned}\partial_x \left(\int_z^H \varphi dz \right) (x, z, t) &= \partial_x \left(\int_{\hat{z}}^0 \hat{\varphi} d\hat{z} \right) (x, \hat{z}, t) + \hat{\varphi} (x, \hat{z}, t) \partial_x H (x, t) \\ \partial_t \left(\int_z^H \varphi dz \right) (x, z, t) &= \partial_t \left(\int_{\hat{z}}^0 \hat{\varphi} d\hat{z} \right) (x, \hat{z}, t) + \hat{\varphi} (x, \hat{z}, t) \partial_t H (x, t)\end{aligned}$$

These relations are applied to the friction condition (22) $w|_{z=b} = u|_{z=b} \partial_x b$. The condition turns to be equivalent with the mass conservation equation (34) (which is invariant under the Prandtl shift):

$$\partial_t H + \partial_x \hat{q} = a \text{ with } \hat{q} = \int_{-h}^0 \hat{u} d\hat{z} \quad (37)$$

Likewise the functions ψ , see Eqn (31), become:

$$\begin{aligned}\hat{\psi}_{xz} (\hat{\sigma}_{xx}, \hat{p}, \hat{u}, \hat{w}) &= -\lambda \hat{z} - \delta \left(\hat{p} \partial_x H + \partial_x \left(\int_{\hat{z}}^0 \hat{p} d\hat{z} \right) \right) \\ &\quad - \beta \left(\partial_t \left(\int_{\hat{z}}^0 \hat{u} d\hat{z} \right) + \partial_x \left(\int_{\hat{z}}^0 \hat{u}^2 d\hat{z} \right) - \hat{u} \hat{w} - a \hat{u}|_{\hat{z}=0} \right) \\ &\quad + \alpha \delta \left(\partial_x \left(\int_{\hat{z}}^0 \hat{\sigma}_{xx} d\hat{z} \right) + \hat{\sigma}_{xx} \partial_x H \right)\end{aligned} \quad (38)$$

$$\begin{aligned}\psi_{xx} (\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}) &= 2 \|\hat{\sigma}\|^{1-p} (\partial_x \hat{u} - \partial_{\hat{z}} \hat{u} \partial_x H) \\ \hat{\psi}_p (\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}) &= -\hat{z} - \alpha \left(\hat{\sigma}_{xx} + \partial_x \left(\int_{\hat{z}}^0 \hat{\sigma}_{xz} d\hat{z} \right) + \hat{\sigma}_{xz} \partial_x H \right) \\ &\quad + \alpha \beta \left(\partial_t \left(\int_{\hat{z}}^0 \hat{w} d\hat{z} \right) + \partial_x \left(\int_{\hat{z}}^0 \hat{u} \hat{w} d\hat{z} \right) - \hat{w}^2 - a \hat{w}|_{\hat{z}=0} \right) \\ &\quad + \alpha \beta \partial_x H \left(\partial_t \left(\int_{\hat{z}}^0 \hat{u} d\hat{z} \right) + \partial_x \left(\int_{\hat{z}}^0 \hat{u}^2 d\hat{z} \right) - \hat{u} \hat{w} - a \hat{u}|_{\hat{z}=0} \right) \\ &\quad + \alpha \beta \left(\partial_x^2 H \int_{\hat{z}}^0 \hat{u}^2 d\hat{z} + 2 \partial_{tx} H \int_{\hat{z}}^0 \hat{u} d\hat{z} - \hat{z} \partial_t^2 H \right) \\ &\quad + \alpha \beta (\hat{u} - \hat{u}|_{\hat{z}=0}) \partial_x H \partial_t H\end{aligned} \quad (39)$$

$$\begin{aligned}
\hat{\psi}_{bu}(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}) &= C \frac{(|\hat{\sigma}_{xz}(1-\alpha\delta\partial_x b^2) - 2\alpha\delta\hat{\sigma}_{xx}\partial_x b|^{p-1})|_{\hat{z}=-h}}{(1+\alpha\delta\partial_x b^2)^{p+\frac{1}{2}} - 2\alpha\delta\hat{\sigma}_{xx}|_{\hat{z}=-h}\partial_x b} (\hat{\sigma}_{xz}|_{\hat{z}=-h} (1 - \alpha\delta\partial_x b^2)) \\
\hat{\psi}_u(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}) &= \hat{u}|_{\hat{z}=-h} + \int_{-h}^{\hat{z}} (\|\hat{\sigma}\|^{p-1} \hat{\sigma}_{xz}) d\hat{z} \\
&\quad - \alpha\delta \left(\partial_x \left(\int_{-b}^{\hat{z}} \hat{w} d\hat{z} \right) + (\hat{z} + b) \partial_{tx} H + \partial_x b \partial_t H + \partial_x^2 H \int_{-b}^{\hat{z}} \hat{u} d\hat{z} \right) \\
&\quad - \alpha\delta \partial_x H \left(\partial_x \left(\int_{-b}^{\hat{z}} \hat{u} d\hat{z} \right) - \hat{w} - 2\partial_t H - \hat{u} \partial_x H + \hat{u}|_{\hat{z}=-h} \partial_x b \right) \\
&\quad - \alpha\delta (\partial_t H + \hat{w}|_{\hat{z}=-h} \partial_x b) \\
\hat{\psi}_w(\hat{\sigma}_{xz}, \hat{\sigma}_{xx}, \hat{u}, \hat{w}) &= \int_{\hat{z}}^0 \partial_x \hat{u} d\hat{z} - a
\end{aligned} \tag{40}$$

It is important to note that all the functions $\hat{\psi}_{\square}$ are unchanged under the Prandtl shift at 0th order and at order 1 if considering 1st order terms in geometrical ratio ε (i.e. skipping terms in $\alpha\delta$, $\alpha\delta = \varepsilon^2$). Moreover the fixed point like formulation (32) and the iterative schemes (35)-(36) still apply.

Remark 14. Extra terms appear in the pressure expression at order 1 in the case $\alpha = 0(1)$ (Regime B) but this expression is useless to derive the present asymptotical models.

6. Error estimates

Let $(X^{(0)}, h^{(0)})$ be the uniform stationary solution at order 0 i.e. constant functions in (x, t) . It is the reference flow defined in Section 3.

We set:

$$\begin{aligned}
X^{(0)} &= (\sigma_{xz}, \sigma_{xx}, p, u|_{z=b}, u, w)^{(0)} \\
&= \left(\lambda(H - z), 0, H - z, C\lambda^p H^p, \left[C\lambda^p H^p + \frac{\lambda^p}{p+1} \left(H^{p+1} - (H - z)^{p+1} \right) \right], 0 \right)
\end{aligned}$$

We set the error function:

$$\varepsilon^{(k)}(\varphi) = \left[\varphi^{ex} - \varphi^{(k)} \right]$$

where φ^{ex} denotes the exact solution. We consider below the case $k = 1$, next the case $k = 2$.

6.1. Formal error estimates

We assume that if $\varepsilon^{(k)}(\varphi) = O(s)$, s being a small parameter, then $\varepsilon^{(k)}(\partial_p \varphi) = O(s)$, $\forall p \geq 0$.

First we consider **Regime A** for which α and β are small, δ is $O(1)$. At first iterate of algorithm (A) we obtain error estimates for the pressure, the velocity field and stress tensor:

$$\begin{aligned}
\varepsilon^{(1)}(p) &= O(\alpha) \\
\varepsilon^{(1)}(\sigma_{xz}) &= O(\alpha\delta + \beta) = O(\alpha + \beta) \text{ since } \delta = O(1). \\
\varepsilon^{(1)}(u|_{z=b}) &= O(C(\alpha\delta + \beta)) = O(C(\alpha + \beta)) \text{ since } \delta = O(1). \\
\varepsilon^{(1)}(u) &= O(\alpha + \varepsilon^{(1)}(\sigma_{xz}) + \varepsilon^{(1)}(u|_{z=b})) = O((1 + C)(\alpha + \beta)) \\
\varepsilon^{(1)}(\sigma_{xx}) &= O(\alpha + \varepsilon^{(1)}(u) + \varepsilon^{(1)}(\sigma_{xz})) = O((1 + C)(\alpha + \beta)) \\
\varepsilon^{(1)}(w) &= \varepsilon^{(1)}(u) = O((1 + C)(\alpha + \beta))
\end{aligned} \tag{41}$$

where we expanded nonlinearities of the norm of the stress tensor in the error function with respect to $\varepsilon^{(k)}(\varphi)$ using the fact that $\varphi^{ex} = \varphi^{(k)} + \varepsilon^{(k)}(\varphi)$. Thus, after first iterate ($k = 1$), we obtain the zero-th order solution with respect to (α, β) .

After iterate (2) of Algorithm (\mathcal{A}) we get the following error functions:

$$\begin{aligned}\varepsilon^{(2)}(p) &= O(\alpha(\varepsilon^{(1)}(\sigma_{xz}) + \varepsilon^{(1)}(\sigma_{xx}) + \beta)) \\ \varepsilon^{(2)}(\sigma_{xz}) &= O(\alpha(\varepsilon^{(1)}(\sigma_{xz}) + \varepsilon^{(1)}(\sigma_{xx}) + \beta) + \beta(\varepsilon^{(1)}(u) + \varepsilon^{(1)}(w))) \\ \varepsilon^{(2)}(u|_{z=b}) &= O(C(\alpha(\varepsilon^{(1)}(\sigma_{xz}) + \varepsilon^{(1)}(\sigma_{xx}) + \alpha + \beta) + \beta(\varepsilon^{(1)}(u) + \varepsilon^{(1)}(w))))\end{aligned}\quad (42)$$

$$\begin{aligned}\varepsilon^{(2)}(u) &= \varepsilon^{(2)}(\sigma_{xx}) = \varepsilon^{(2)}(w) \\ &= O((1+C)(\alpha(\varepsilon^{(1)}(\sigma_{xz}) + \varepsilon^{(1)}(\sigma_{xx}) + \alpha + \beta) + \beta(\varepsilon^{(1)}(u) + \varepsilon^{(1)}(w))) + \alpha\varepsilon^{(1)}(w))\end{aligned}\quad (43)$$

Again we expanded nonlinearities of the square of the velocity, and the stress tensor norm in the error function with respect to $\varepsilon^{(k)}(\varphi)$. After second iterate ($k = 2$), we obtain the first order solution with respect to (α, β) .

Using the same approach we get similar results for **Regime B**, where δ and β are small, α is $O(1)$.

Regime C (α, δ and β small) is contained in both Regime A and Regime B.

Finally, after iterate (2) of Algorithm (\mathcal{A}), the solution obtained $(X^{(2)}, h^{(2)})$ is an asymptotic solution of the free surface Navier-Stokes equations at order 1:

- in case of Regime A, it is first order with respect to (α, β) ;
- in case of Regime B it is first order with respect to (β, δ) ;
- in case of Regime C it is first order with respect to (α, β, δ) .

6.2. Relationship between the slip coefficient C , the global error and the regime

Let us consider Regime A (α, β small). If the slip coefficient C is assumed to be $O(1)$ at most, then the coefficient $C(\alpha + \beta)$ is a small parameter and Algorithm(\mathcal{A}) gives order 0 asymptotic solutions for $k = 1$ and order 1 solutions for $k = 2$.

In the case of Regime B (δ, β small), let us set: $S = \max(C\delta, C\beta)$. If S is small (i.e. $C \ll \max(1/\delta, 1/\beta)$) then, as mentioned above one can iterate Algorithm(\mathcal{A}) to obtain order 1 or order 2 asymptotic solutions. If S is $O(1)$ or larger then the a-priori friction model at bottom is the a-priori dominating error term (see Eqn (42)-(43) by replacing α by δ). Hence, following the error estimates in the case Regime B, the global error on the solution is dominated by the error on the velocity at bottom.

Nevertheless, the present iterative process gives the right asymptotic solution since the mean slope must vanish for large slip coefficient following the relationship (27). And, in case of a vanishing mean slope, one has the reference solution:

$$(\sigma_{xz}, \sigma_{xx}, p, u|_{z=b}, u, w)^{(0)} = (0, 0, (H - z), C\lambda^p H^p, C\lambda^p H^p, 0)$$

Next, we obtain:

$$\psi_{bu}(\sigma_{xz}, \sigma_{xx}) = \left(1 - \frac{\lambda^p}{p+1}\right) H^p + O\left(\delta\left(1 + \frac{\lambda^p}{p+1}\right) + \beta\right)$$

Next, the following error estimate can be stated similarly as before:

$$\begin{aligned}\varepsilon^{(1)}(u|_{z=b}) &= \psi_{bu}(\sigma_{xz}, \sigma_{xx}) - \psi_{bu}(\sigma_{xz}^{(1)}, 0) \\ &= O(\delta + \beta)\end{aligned}$$

Hence the resulting velocity field at bottom u_b is order 1 with respect to small parameters; and the present iterative process give asymptotical solutions at the orders required, in both regimes.

7. Expressions of the 0th order exact solutions

We apply Algorithm (A) in order to derive one-equation models in h variable only. It gives the asymptotic models for regimes A, B, C, and it allows to derive an unique equation valid for the three regimes and with the different boundary conditions at bottom.

After one (respectively two) iterate(s) of Algorithm (A) we obtain the 0th (respectively 1st) order multi-regime model. As a first step, we need to derive the exact expression of the unknown at order 0.

For a sake of clarity, the solutions are written using the mean slope coordinate system. Recall the latter and the Prandtl's coordinate system lead to the same expressions for the terms we need here, see Remark 14.

7.1. Case of Regime A: α small

In the flow Regime A, we consider α, β small and $\delta = O(1)$. As $\varepsilon^2 = \alpha\delta$ we have $\varepsilon^2 = O(\alpha)$. We use the iterative scheme described in (35). At zero-th order the pressure is hydrostatic:

$$p^{(0)} = H - z \quad (44)$$

The shear stress σ_{xz} is linear in z :

$$\sigma_{xz}^{(0)} = \lambda(H - z) - \delta\partial_x \left(\int_z^H pdz \right) = (H - z)(\lambda - \delta\partial_x H) \quad (45)$$

To compute the velocity profile we need the modulus of the deviatoric tensor:

$$\left\| \sigma^{(0)} \right\| = \left| \sigma_{xz}^{(0)} \right| = (H - z) |\lambda - \delta\partial_x H|$$

Remark 15. The absolute value that appear in the modulus of the deviatoric tensor can not be removed. As a matter of fact for a zero mean slope λ vanishes and the sign of σ_{xz} is the sign of the local slope $\partial_x b$.

The streamwise component of the velocity writes:

$$u^{(0)} = (\lambda - \delta\partial_x H) |\lambda - \delta\partial_x H|^{p-1} \left(Ch^p + \frac{1}{p+1} (h^{p+1} - (H - z)^{p+1}) \right) \quad (46)$$

A direct integration over the depth of the fluid gives the discharge rate:

$$q_A^{(0)} = \int_b^H u^{(0)} dz = (\lambda - \delta\partial_x H) |\lambda - \delta\partial_x H|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) \quad (47)$$

7.2. Case of Regime B: δ small

In the flow regime B, we consider β, δ small and $\alpha = O(1)$. As $\varepsilon^2 = \alpha\delta$, we have $\varepsilon^2 = O(\delta)$. We use the iterative scheme described in (36). Note that unlike Regime A we have to compute the velocity before the pressure.

The shear stress is still linear in z , but as we don't have the corrective pressure term $\delta\partial_x H$, $\sigma_{xz}^{(0)}$ is always positive:

$$\sigma_{xz}^{(0)} = \lambda(H - z)$$

We easily calculate the velocity and the discharge rate:

$$\begin{aligned} u^{(0)} &= \lambda^p \left(Ch^p + \frac{1}{p+1} (h^{p+1} - (H - z)^{p+1}) \right) \\ q_B^{(0)} &= \lambda^p h^{p+1} \left(C + \frac{h}{p+2} \right) \end{aligned} \quad (48)$$

Once calculated the velocity, we obtain the pressure:

$$p^{(0)} = (H - z) \left(1 + \lambda\alpha \left(\partial_x H - 2 \frac{h^{p-1} (pC + h)}{(H - z)^p} \partial_x h \right) \right) \quad (49)$$

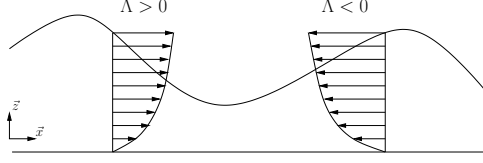


Figure 5: At order 0, Regime A, the sign of Λ (local value) indicates the local direction of the flow (equation (51)).

Remark 16. The pressure expression (49) is singular at the free surface (hence the normal stress components (σ_{xx}, σ_{zz}) too). It is due to the well-known singularity at the free-surface mentioned for example in [35–37]. This singularity, already described differently in [36], appears here at order 0 since it is the multi-regime expression. In the sequel, the singularity does not prevent to pursue the calculations (in the two-equations model in particular) since we consider order 0 in regime and order 1 in the geometrical ratio ε .

7.3. Unified expression of q^0

Let us set:

$$\Lambda = \begin{cases} \lambda - \delta \partial_x H & \text{in the case Regime A} \\ \lambda & \text{in the cases Regime B and C} \end{cases} \quad (50)$$

Then, we recover the discharge rate for Regime A and Regime B writes similarly as follows (see equations (47) and (48)):

$$q^{(0)} = \Lambda |\Lambda|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) \quad (51)$$

We recover the previous expressions of the discharge rate for Regime A and Regime B as a specific case of the equation (51):

$$\begin{aligned} q_B^{(0)} &= q^{(0)} (\Lambda = \lambda) \\ q_A^{(0)} &= q^{(0)} (\Lambda = \lambda - \delta \partial_x H) \end{aligned}$$

Remark 17. In its dimensional form (case Regime A), Λ writes:

$$\Lambda = \frac{\rho g H^* (\sin \theta - \cos \theta \partial_x H)}{\frac{\mu_e U^*}{H^*}}$$

It is the ratio between a term related to the gravity and the scaling of the deviatoric stress tensor σ . Indeed the gravity term is composed of the streamwise gravity component $\rho g H^* \sin \theta$ and the normal component of the gravity scaled by the slope of the free surface ($-\rho g H^* \cos \theta \partial_x H$). As a result, the quantity Λ represents the gravity contribution due to the local slope of the free-surface.

In Regime B, the velocity u depends on the mean-slope contribution only, while in Regime A the velocity u depends on the local slope of the free-surface too.

In Regime A, if Λ is negative, then $q^{(0)}$ is negative, see figure 5.

7.4. Case of Regime C: α and δ small

Regime C is a special case of both Regime A and Regime B. It is included in Regime A since α is small, and in Regime B since δ is small. As a consequence one recover Regime C flow values (stress, pressure, velocity and flow rate) by setting $\delta = 0$ in Regime A or $\alpha = 0$ in Regime B. Inasmuch as α does not appear in the flow rate of Regime B, the flow rate of Regime C is given by the one of Regime B, hence:

$$q_C^{(0)} = q_B^{(0)}$$

8. First order solutions

Next iteration of algorithms (35) and (36) is computed using Sage software ([38]). It gives the explicit expressions of all fields. For a sake of simplicity, we detail below the exact expression of the discharge in each regime only.

8.1. General form of solutions

Formally, first order solutions can be written in a general form as:

$$\varphi = \varphi^{(1)} + \mathcal{O}\left(\left(\sum_{\phi \in \Phi} \phi\right)^2\right) = \varphi^{(0)} + \sum_{\phi \in \Phi} \phi \varphi^{(1,\phi)} + \mathcal{O}\left(\left(\sum_{\phi \in \Phi} \phi\right)^2\right) \quad (52)$$

with

$$\Phi = \begin{cases} \{\alpha, \beta\} & \text{(regime A)} \\ \{\beta, \delta\} & \text{(regime B)} \\ \{\alpha, \beta, \delta\} & \text{(regime C)} \end{cases}$$

In all the sequel, we call first order solutions, solutions order one with respect to the regime considered and order one with respect to the geometrical parameter ε . In others words, since $\varepsilon^2 = \alpha\delta$, we omit all terms in $\alpha\delta$.

Then, any first order solution may write as follows:

$$\varphi^{(1)} = \varphi^{(0)} + \beta\varphi^{(1,\beta)} + \beta\delta\varphi^{(1,\beta\delta)} + \beta\delta^2\varphi^{(1,\beta\delta^2)} + \beta\delta^3\varphi^{(1,\beta\delta^3)} + \mathcal{O}\left(\left(\sum_{\phi \in \Phi} \phi\right)^2 + \varepsilon^2\right)$$

8.2. Discharge in Regime A (α, β small)

Following Eqn. (52), the discharge at first order writes:

$$q^{(1)} = q^{(0)} + \beta q^{(1,\beta)} + \beta\delta q^{(1,\beta\delta)} + \beta\delta^2 q^{(1,\beta\delta^2)} + \beta\delta^3 q^{(1,\beta\delta^3)} + \mathcal{O}\left((\alpha + \beta)^2 + \varepsilon^2\right) \quad (53)$$

where $q^{(0)}$ is the zero-th order term defined by Eqn. (51), and the first order coefficients are given in [Appendix A](#).

Remark 18. If one set $\delta = 1$ and $\lambda = 0$ in Equation (18), we obtain all fields calculated in [37] (which have been fixed by considering the scaling a-priori $\sigma_{xz}^* = \varepsilon \rho g h$, see [37] page 238).

8.3. Discharge in Regime B (δ, β small) and Regime C (all the three small)

Following Eqn. (52), the discharge at first order writes:

$$q^{(1)} = q_B^{(0)} + \delta q^{(1,\delta)} + \beta q^{(1,\beta)} + \mathcal{O}\left((\delta + \beta)^2 + \varepsilon^2\right) \quad (54)$$

where $q^{(0)}$ is the zero-th order term defined by Eqn. (48), $q^{(1,\delta)}$ is the only term that differs from Regime A:

$$q^{(1,\delta)} = -p\lambda^{p-1}h^{p+1} \left(C + \frac{h}{p+2}\right) \partial_x h$$

and the other terms $q^{(1,\alpha\delta)}$ and $q^{(1,\beta)}$ are given by setting $\Lambda = \lambda$ in the Regime A coefficients ([Appendix A](#)). Actually $q^{(1,\delta)}$ is not really a different term, but is the result of the linearization of $q^{(0)}$ defined by Eqn. (51):

$$q^{(0)} = q_B^{(0)} + \delta q^{(1,\delta)} + \mathcal{O}(\delta^2)$$

9. One-equation models

In this section, first, we deduce classically from the average mass equation and the exact solution expressions above, the one-equation models at order 0 and at order 1, which remain consistent for the regimes A, B, C, and for any boundary condition at bottom (from adherence to pure-slip through a non-linear friction law). In the 0th order case, we give the expression of the steady-state model both for Regime A and Regime B. Next, we derive the equation obtained if pure slip (vanishing friction) is imposed at bottom; also we precise the model obtained in the newtonian case ($p = 1$).

Let us point out that all derivations of the equations are similar for 3D flows. At contrary, the discussion presented on the steady state solution is specific to 2D flows.

9.1. The 0th order unified one-equation model

The fluid height verifies the averaged mass equation (34) which leads to the steady height equation by replacing the discharge rate by its value. Then we obtain the zero-th order unified model for both regimes A and B:

$$\partial_t h + \partial_x \left(\Lambda |\Lambda|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) \right) = a \quad (55)$$

where Λ is defined by (50).

Recall the coefficient C denotes the slip coefficient which can vary from 0 (adherence at bottom) to infinity (pure slip case, see below the resulting equations).

Once the solution of the unified model (55) is computed, the velocity profile, the pressure and the stress tensor are given by the formal expressions.

Remark 19. The 0th order unified model is the same for regimes B and C. Therefore the fluid height solution is identical. Nevertheless the pressure and the normal stress are different since they include terms in α .

In next section we develop the steady state case.

9.2. Steady-state solution

Let us calculate the steady solution order 0 in both regimes A and B. Since the flow is steady, we consider only the case of vanishing source term ($a = 0$). A direct integration gives:

$$\Lambda |\Lambda|^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) = q_0 \quad (56)$$

with q_0 the initial discharge rate. Let us develop (56) for both regimes. The steady-state equation order 0 for both regimes writes:

$$\partial_x H = \partial_x h + \partial_x b = \frac{1}{\delta} \left(\lambda - \frac{\text{sgn}(q_0)}{h} \left(\frac{|q_0|}{h \left(C + \frac{h}{p+2} \right)} \right)^{1/p} \right) \quad (57)$$

Using the expressions of δ and λ in function of γ , see (16), we have:

$$\partial_x h + \partial_x b = \frac{1}{\varepsilon} \left(\tan \theta - \gamma \frac{\text{sgn}(q_0)}{h} \left(\frac{|q_0|}{h \left(C + \frac{h}{p+2} \right)} \right)^{1/p} \right)$$

We remark that in Regime A and if $\lambda = 0$ (vanishing mean slope) then at order 0 the steady solution satisfies: $\partial_x H < 0$ for $q_0 > 0$.

In Regime B (recall δ small and $\partial_x H \leq O(\frac{1}{\delta})$), it follows from (56) that h is the solution of the following equation:

$$\lambda^p h^{p+1} \left(C + \frac{h}{p+2} \right) = q_0$$

In other words, the steady-state solution Regime B is such that the depth h is constant. That is to say the surface is a translation of the bottom shape. Indeed we recover the reference flow as defined by equation (26).

9.3. The 1st order unified one-equation model

At first order, Equation (34) writes:

$$\partial_t h + \partial_x \left(q^{(0)} + \beta \left(q^{(1,\beta)} + \delta q^{(1,\beta\delta)} + \delta^2 q^{(1,\beta\delta^2)} + \delta^3 q^{(1,\beta\delta^3)} \right) \right) = a \quad (58)$$

with $q^{(0)}$ defined by Eqn. (51) and coefficients $q^{(1,\cdot)}$ given in Appendix A.

For Regimes B and C the unified model (58) can be written more easily as follows:

$$\partial_t h + \partial_x \left(\lambda^p h^{p+1} \left(C + \frac{h}{p+2} \right) - p\delta\lambda^{p-1} h^{p+1} \left(C + \frac{h}{p+2} \right) \partial_x h + \beta q^{(1,\beta)} (\Lambda = \lambda) \right) = a$$

Remark 20. For low Reynolds flows β is negligible (eg. ice flows), then the unified model order 1 (58) equals the zero-th order model (55). In other words, in this case, Equation (55) is order 1.

9.4. Pure slip case

Let us recall that it is not worth to focus on the 1st order model with large values of the slip coefficients C since the global error on the solution is dominated by the error on the velocity at bottom (see Subsection 6.2).

Let us consider the 0th order unified model in the pure slip case i.e. $C \rightarrow \infty$. Then Equation (55) writes:

$$\frac{1}{C} \partial_t h + \partial_x \left(\Lambda |\Lambda|^{p-1} h^{p+1} + \frac{1}{C} \frac{1}{p+2} \Lambda |\Lambda|^{p-1} h^{p+2} \right) = \frac{a}{C}$$

Passing to the limit in C , one obtains the so-called plug model (see Remark 11):

$$\partial_x \left(\Lambda |\Lambda|^{p-1} h^{p+1} \right) = 0 \quad (59)$$

Note that it is a steady state solution and it depends only on the sliding law.

In Regime B, $\Lambda = \lambda$ is constant hence: $\partial_x (h^{p+1}) = 0$. Thus, in the pure slip case Regime B, the solution is a constant height. In other words, the surface shape is a simple translation of the bottom shape.

In Regime A, the equation (59) writes: $\Lambda = \text{sgn}(q_0) \frac{1}{h} \left| \frac{q_0}{h} \right|^{1/p}$, with q_0 the steady uniform discharge. Since $\Lambda = (\lambda - \delta \partial_x H)$, Eqn (55) writes into an ordinary differential equation:

$$\partial_x h = \frac{1}{\delta} \left(\lambda - \text{sgn}(q_0) \frac{1}{h} \left| \frac{q_0}{h} \right|^{1/p} \right) - \partial_x b \quad (60)$$

It is the one-equation model Regime A if pure slip at bottom

9.5. Newtonian rheology case

Let us detail the newtonian case: $p = 1$. At order 0 and in the case of Regime B, with the mean velocity as referent velocity, we obtain the following non-linear advection equation:

$$\partial_t h + \lambda \partial_x \left(Ch^2 + \frac{h^3}{3} \right) = a$$

with the velocity value equal to $\lambda h (2C + h)$. At order 0 and in the case of Regime A, we obtain the advective term of Regime B plus an extra diffusive term. The model reads:

$$\partial_t h + \partial_x \left((\lambda - \delta \partial_x (h + b)) \left(Ch^2 + \frac{h^3}{3} \right) \right) = a$$

At order 1, Equation (58) with $p = 1$ writes:

$$\partial_t h + \partial_x \partial_x \left(\Lambda h^2 \left(C + \frac{h}{3} \right) + \beta \left(q^{(1,\beta)}|_{p=1} + \delta q^{(1,\beta\delta)}|_{p=1} + \delta^2 q^{(1,\beta\delta^2)}|_{p=1} \right) \right) = a$$

with:

$$q^{(1,\beta)} = \frac{1}{15} (15C^3 + 25C^2h + 12Ch^2 + 2h^3) h^3 \Lambda^2 \partial_x h - \frac{1}{24} (24C^2 + 20Ch + 5h^2) h^2 \Lambda a$$

$$q^{(1,\beta\delta)} = \left(C^2 + \frac{2}{3}Ch + \frac{2}{15}h^2 \right) h^3 \partial_x a - \frac{1}{315} \left\{ 42 (15C^3 + 25C^2h + 12Ch^2 + 2h^3) (\partial_x h)^2 + 21 (30C^3 + 35C^2h + 14Ch^2 + 2h^3) h \partial_x^2 h + (105C^2 + 56Ch + 8h^2) h^2 \partial_x^2 H \right\} h^3 \Lambda$$

$$q^{(1,\beta\delta^2)} = \frac{2}{15} (30C^3 + 35C^2h + 14Ch^2 + 2h^3) h^4 \partial_x h \partial_x^2 H + \frac{1}{45} (45C^3 + 45C^2h + 16Ch^2 + 2h^3) h^5 \partial_x^3 H$$

10. Preliminary two-equations and gradually varied flow models

In this section, we state two-equations models (Saint-Venant or shallow-water like models if using the hydraulic terminology).

First, we present how to state such models, consistent with the regime considered. Next, we detail a preliminary two-equations model (which is not consistent in some sense) but which allows to derive a gradually varied flow model, easier to use in practice.

In next section, we detail shallow water type systems compatible with the multi-regime flow description.

10.1. Fundamentals of shallow-water type system multi-regime

We consider the first momentum equation in its primal form, we integrate it over the depth, we use the boundary conditions, and together with the mass conservation equation we obtain:

$$\begin{cases} \partial_t h + \partial_x q = a \\ \partial_t q + \partial_x \left(\int_b^H u^2 dz \right) - \frac{\alpha \delta}{\beta} \partial_x \left(\int_b^H \sigma_{xx} dz \right) + \frac{\delta}{\beta} \partial_x \left(\int_b^H p dz \right) \\ + \frac{\delta}{\beta} (p|_{z=b} + \alpha \sigma_{xx}|_{z=b}) \partial_x b = \frac{1}{\beta} (\lambda h - \sigma_{xz}|_{z=b}) \end{cases} \quad (61)$$

Remark 21. In the case of steady-state uniform flow with flat bottom, Equation (61) writes: $\lambda h = \sigma_{xz}|_{z=b}$. It means that the fluid elevation is determined by the shear stress at bottom. This equation gives the law elevation - velocity (or discharge) of the steady uniform flow, like it is done by the Manning-Strickler (or Chezy) law for example.

In Equation (61), the advection term $\int_b^H u^2 dz$ can be related to the flow rate q assuming that the tangential velocity u is close enough to its average value \bar{u} , so that when integrating the advection term is “closed to” $\frac{q^2}{H}$. Thus we write :

$$\int_b^H u^2 dz = \frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)(p+2)^2}$$

The basal shear stress writes as a function of the rate of flow as follows:

$$\sigma_{xz}|_{z=b} = \text{sgn}(\Lambda) \left(\frac{|q^{(0)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \quad (62)$$

10.2. A preliminary two-equations model

Let us neglect the terms of order $\frac{\alpha\delta}{\beta}$. Also, for a sake of simplicity we set : $a = 0$ (mass balance source term). Then, we obtain the following multi-regime shallow water type model power-law rheology:

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(2p+3)(p+2)^2} \right) + \frac{\delta}{\beta} h \partial_x H \\ = \frac{1}{\beta} \left(\lambda h - \text{sgn}(\Lambda) \left(\frac{|q|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) + h.o.t. \end{cases} \quad (63)$$

where Λ is defined by (50) and *h.o.t.* denotes "higher-order terms".

Remark 22. Let us point out that the singularity, see Equation (49), does not appear in the two-equations model (63) since the terms in order $\frac{\alpha\delta}{\beta}$ are neglected (recall $\alpha\delta = \varepsilon^2$). Otherwise, this singularity due to an unphysical modeling of the rheological behavior at low shear rate, would introduce a modeling problem. Observe that few possibilities of regularization (not addressed here) would be possible, see e.g. [36, 39].

Remark 23. Let us point out that the vertical viscous term leading to the “friction term” (first term of the R.H.S.) must be developed at one extra order compared to (62), since it is multiplied by $\frac{1}{\beta}$. If not, Equation (63) is not fully consistent with the asymptotic of the corresponding Navier-Stokes equations since the basal shear stress $\sigma_{xz}|_{z=b}$ should be expanded up to the first order in β .

In the mono-regime case C (Regime C: all three parameters small), a consistent two equations model has been written by Fernandez-Nieto et al. [15]; while in the multi-regime case, consistent Shallow water equations are detailed in next Section.

10.3. Gradually varied flow equation and hydraulic like discussion

For sake of simplicity, we consider here a fluid flowing from left to right on an inclined plane, i.e. $b = 0$, $H = h$ and $\Lambda > 0$, and $a = 0$. Let us investigate the steady flow ($\partial_t h = 0$). The mass conservation implies that the flow rate is constant. The system of equations (63) reduces to the following equation:

$$\left(-\frac{q^2}{h^3} + \frac{\Lambda^{2p} h^{2p+1}}{(p+2)^2} + \frac{\delta}{\beta} \right) \partial_x h - \delta \frac{2p\Lambda^{2p-1} h^{2p+2}}{(2p+3)(p+2)^2} \partial_x^2 h = \frac{1}{\beta} \left(\lambda - \frac{1}{h} \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right)$$

with Λ defined by (50).

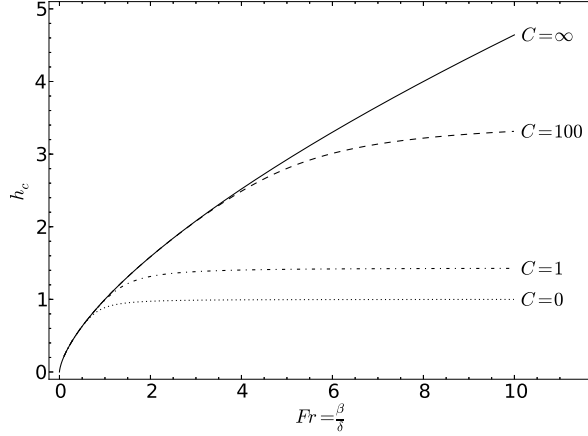


Figure 6: Critical-depth line h_c in function of the Froude number Fr for different values of the slip coefficient C defined by (66). The curves (value $h_c = 1$) separates sub-critical and supercritical flows.

We introduce the space slow variable $\xi = \eta x$, with η small (strictly positive). This classical assumption of slow space variations makes negligible the higher-order terms $\partial_{xx}^2 h$ and $(\partial_x h)^2$. Then the previous equation writes:

$$\left(1 - \frac{\beta}{\delta} \left(\frac{q^2}{h^3} - \frac{\lambda^{2p} h^{2p+1}}{(p+2)^2} \right)\right) \partial_x h = \frac{1}{\delta} \left(\lambda - \frac{1}{h} \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) + O(\eta^2) \quad (64)$$

In the hydraulic context it is so-called the gradually varied flow equation. The left-hand side and the right-hand side of equation (64) define two heights. The normal-depth line h_N is the solution of the vanishing right-hand side. It is the uniform solution of the steady-state equation (57).

The critical-depth line h_c is the solution of the vanishing left-hand side. Since $\frac{\beta}{\delta} = Fr^2$, the left-hand side can also be written in the form $(1 - F_h^2) \partial_x h$ where F_h is a dimensionless number similar to Fr but which depends on h .

Let us set the constant flow rate $q = 1$. The relation (26) allows to simplify the expression of the "hydraulic" Froude number F_h as follows:

$$F_h^2 = \frac{Fr^2}{h^3} \left(1 - \frac{h^{2(p+2)}}{(C(p+2) + 1)^2} \right) \quad (65)$$

The critical-depth line h_c defines a critical hydraulic Froude number such that:

$$F_h(h_c) = 1 \quad (66)$$

For large Froude numbers, we have $h_c^{p+2} \approx 1 + C(p+2)$, thus the critical-depth line h_c only depends on the slip coefficient. At low Froude numbers or for large slip coefficient values the critical height value $h_c \approx Fr^{\frac{2}{3}}$.

By analogy with a hydraulic description, if $F_h < 1$ (i.e. $h > h_c$) the flow is called sub-critical; if $F_h > 1$ (i.e. $h < h_c$) the flow is called supercritical.

The value of the critical height h_c is plotted on figure 6 for several values of the slip coefficient. The value $h_c = 1$ separates sub-critical and supercritical flows.

In summary, we derived a two-equation model (shallow-water type model), see (63), for multi-regime flows (but not fully consistent, see Remark 23). The latter allows to derive the corresponding gradually varied flow equation (64), hence define hydraulic-like Froude number and corresponding sub-critical, supercritical flows defined in function of (α, β, δ) and C .

These new expressions and models will be illustrated numerically in Section 13.

11. Consistent multi-regime two-equations models

In this section, we derive two-equation models consistent with all viscous regimes (i.e. we do not neglect the higher-order terms (*h.o.t.*) in Equation (61). The unified two-equations model obtained below is order 0 but contains of course more information than order 0 one-equation models. At the end of this section, in order to be able to compare with the existing models in the literature, we detail particular cases such as adherence at bottom, or the Newtonian case.

11.1. Structures of the models

In this section, we consider the Prandtl's coordinate system. The average momentum equation writes:

$$\begin{aligned} \partial_t q + \partial_x \left(\int_{-h}^0 \hat{u}^2 d\hat{z} \right) + \frac{\delta}{\beta} \int_{-h}^0 \partial_x (\hat{p} - \alpha \hat{\sigma}) d\hat{z} \\ + \frac{\delta}{\beta} (\hat{p}|_{\hat{z}=-h} - \alpha \hat{\sigma}_{xz}|_{\hat{z}=-h}) \partial_x H - a \hat{u}|_{\hat{z}=0} = \frac{1}{\beta} (\lambda h - \hat{\sigma}_{xz}|_{\hat{z}=-h}) \end{aligned} \quad (67)$$

Let us precise below the order required for each term in order to derive consistent order 0 two-equations models.

- Case of Regime C ((β, α, δ) small). Let us assume that the three parameters are of the same order of magnitude. Then, we need to derive the inertial term the viscous term and the pressure term at order 0; while the shear-stress term at bottom in the R.H.S. must be derived at order 1.

$$\begin{aligned} \partial_t q + \partial_x \left(\int_{-h}^0 (\hat{u}^{(0)})^2 d\hat{z} \right) + \frac{\delta}{\beta} \int_{-h}^0 \partial_x \hat{p}^{(0)} d\hat{z} \\ + \frac{\delta}{\beta} \hat{p}|_{\hat{z}=-h}^{(0)} \partial_x H - a \hat{u}|_{\hat{z}=0}^{(0)} = \frac{1}{\beta} (\lambda h - \hat{\sigma}_{xz}^{(1)}|_{\hat{z}=-h}) + \mathcal{O} \left(\frac{(\alpha + \beta + \delta)^2}{\beta} \right) \end{aligned} \quad (68)$$

Hence, in order to obtain the two-equations model (order 0) for Regime C, it remains to develop the calculation of the basal shear stress $\hat{\sigma}_{xz}^{(1)}|_{\hat{z}=-h}$ (order 1 term). It is done in next subsection.

- Case of Regime A ($\delta = \mathcal{O}(1)$). We need to derive the inertial and viscous terms at order 0, the shear-stress term at bottom in the R.H.S. at order 1, and the pressure term at order 1.

$$\begin{aligned} \partial_t q + \partial_x \left(\int_{-h}^0 (\hat{u}^{(0)})^2 d\hat{z} \right) + \frac{\delta}{\beta} \int_{-h}^0 \partial_x \hat{p}^{(1)} d\hat{z} \\ + \frac{\delta}{\beta} \hat{p}|_{\hat{z}=-h}^{(1)} \partial_x H - a \hat{u}|_{\hat{z}=0}^{(0)} = \frac{1}{\beta} (\lambda h - \hat{\sigma}_{xz}^{(1)}|_{\hat{z}=-h}) + \mathcal{O} \left(\frac{(\alpha + \beta)^2 + \varepsilon^2}{\beta} \right) \end{aligned} \quad (69)$$

- Case of Regime B ($\alpha = \mathcal{O}(1)$). It is the same as Regime C, we need to derive the inertial term the viscous term and the pressure term at order 0 and the shear-stress term at bottom in the R.H.S. at order 1.

$$\begin{aligned} \partial_t q + \partial_x \left(\int_{-h}^0 (\hat{u}^{(0)})^2 d\hat{z} \right) + \frac{\delta}{\beta} \int_{-h}^0 \partial_x \hat{p}^{(0)} d\hat{z} \\ + \frac{\delta}{\beta} \hat{p}|_{\hat{z}=-h}^{(0)} \partial_x H - a \hat{u}|_{\hat{z}=0}^{(0)} = \frac{1}{\beta} (\lambda h - \hat{\sigma}_{xz}^{(1)}|_{\hat{z}=-h}) + \mathcal{O} \left(\frac{(\beta + \delta)^2 + \varepsilon^2}{\beta} \right) \end{aligned} \quad (70)$$

11.2. The friction term at order 1

We derive below an unified expression for the friction term

$$\mathbf{T}_{\text{fric}} = \lambda h - \hat{\sigma}_{xz|z=-h}^{(1)} \quad (71)$$

at order 1 (for each regime). In Equation (67), the friction term \mathbf{T}_{fric} is dominating either if β is very small, or if the flow is uniform. In the latter case, we want to recover the reference flow at equilibrium $q^{(0)}$ (see Section 3). Furthermore, since we seek to derive an unified model for the three regimes, since \mathbf{T}_{fric} is the only common potentially dominating term, we seek to find a unified expression of this term.

Let us write the rate of flow as a function of the basal shear stress. We have: $q^{(0)} = \Lambda h |\Lambda h|^{p-1} h \left(C + \frac{h}{p+2} \right)$ and $\sigma_{xz|z=0}^{(0)} = \Lambda h$. Hence:

$$q^{(0)} = \sigma_{xz|z=0}^{(0)} \left| \sigma_{xz|z=0}^{(0)} \right|^{p-1} h \left(C + \frac{h}{p+2} \right) \quad (72)$$

In other respect, the sign of the shear stress is the sign of the local slope Λ :

$$\text{sgn} \left(\sigma_{xz|z=0}^{(0)} \right) = \text{sgn} \left(q^{(0)} \right) = \text{sgn} (\Lambda h) = \text{sgn} (\Lambda) = \frac{\Lambda}{|\Lambda|}$$

Therefore, the basal shear stress writes as a function of the rate of flow as follows:

$$\sigma_{xz|z=0}^{(0)} = \text{sgn} (\Lambda) \left(\frac{|q^{(0)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}}$$

Next, preserving the structure of this expression of $\sigma_{xz|z=0}$ at order 0, we seek an expression of $\sigma_{xz|z=0}$ at order 1 as follows:

$$\sigma_{xz|z=0}^{(1)} = \text{sgn} (\Lambda) \left(\frac{|q^{(1)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} + \tau$$

where τ is the corrective term to be determined. Since: $q^{(1)} = q^{(0)} + \beta q^{(1)} + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 + \varepsilon^2 \right)$, see Eqn. (52), we obtain:

$$\begin{aligned} \left(\frac{|q^{(1)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} &= \left(\frac{|q^{(0)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \left(1 + \beta \frac{1}{p} \frac{q^{(1)}}{q^{(0)}} \right) + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 + \varepsilon^2 \right) \\ &= |\Lambda| h \left(1 + \beta \frac{1}{p} \frac{q^{(1)}}{q^{(0)}} \right) + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 + \varepsilon^2 \right) \end{aligned}$$

Next, we develop the expression of the corrective term τ :

$$\begin{aligned} \tau &= \sigma_{xz|z=0}^{(1)} - \text{sgn} (\Lambda) \left(\frac{|q^{(1)}|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \\ &= \sigma_{xz|z=0}^{(0)} + \beta \sigma_{xz|z=0}^{(1)} - \text{sgn} (\Lambda) |\Lambda| h \left(1 + \beta \frac{1}{p} \frac{q^{(1)}}{q^{(0)}} \right) + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 + \varepsilon^2 \right) \\ &= \beta \left(\sigma_{xz|z=0}^{(1)} - \frac{\Lambda h}{p} \frac{q^{(1)}}{q^{(0)}} \right) + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 + \varepsilon^2 \right) \end{aligned}$$

Finally we get the full expression of the bottom stress at order 1:

$$\mathbf{T}_{\text{fric}} = \lambda h - \text{sgn} (\Lambda) \left(\frac{|q|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} - \tau + \mathcal{O} \left(\left(\sum_{\phi \in \Phi} \phi \right)^2 \right)$$

with $\tau = \beta \left(\sigma_{xz}|_{z=0} - \frac{\Lambda h}{p} \frac{q^{(1)}}{q^{(0)}} \right)$. Finally, we decompose the corrective term τ up to the each small parameter order; it gives:

$$\begin{aligned} \mathbb{T}_{\text{fric}} = & \lambda h - \text{sgn}(\Lambda) \left(\frac{|q|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \\ & - \beta \tau^{(1,\beta)} - \beta \delta \tau^{(1,\beta\delta)} - \beta \delta^2 \tau^{(1,\beta\delta^2)} - \beta \delta^3 \tau^{(1,\beta\delta^3)} \end{aligned} \quad (73)$$

where the coefficients $\tau^{(1,\phi)}$ are given in [AppendixB](#).

Case of δ small (i.e. Regime B or C). In this case, the coefficients are given by setting $\Lambda = \lambda$ in the general coefficients formulas; and the basal shear stress writes:

$$\mathbb{T}_{\text{fric}} = \lambda h - \lambda \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} - \beta \frac{p}{(C(p+2)+h)} \left(\frac{1}{2(p+1)^2} h \lambda^p a - \frac{(Cp+h)}{(p+2)(2p+3)} \lambda^{2p} \partial_x h \right) h^{2(p+1)} \quad (74)$$

11.2.1. Advective, pressure and viscous terms

Concerning the advective, pressure and viscous terms, their expression is the same as those presented in the previous section.

Concerning the advective term, since we seek to preserve the relation:

$$\int_{-h}^0 (\hat{u})^2 d\hat{z} \approx \int_{-h}^0 (\bar{u})^2 d\hat{z} = h \bar{u}^2 = \frac{q^2}{h}$$

We write it as follows:

$$\int_{-h}^0 (\hat{u}^{(0)})^2 d\hat{z} = \frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(p+2)^2 (2p+3)}$$

In the Prandtl's coordinate system, the other terms write as follows:

$$\frac{\delta}{\beta} \left(\int_{-h}^0 \partial_x \hat{p}^{(1)} d\hat{z} + \hat{p}|_{\hat{z}=-h}^{(1)} \partial_x H \right) = h \partial_x H + \mathcal{O} \left(\frac{\varepsilon^2}{\beta} \right)$$

11.3. The unified two-equations model

We write below the resulting two-equations model, valid for any viscous regime, and consistent in an asymptotic point of view with respect to the primitive free-surface flow equations. The model, in the mean-slope coordinate system, writes:

$$\begin{aligned} & \partial_t h + \partial_x q = a \\ & \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(p+2)^2 (2p+3)} \right) + \frac{\delta}{\beta} h \partial_x H - a \Lambda |\Lambda|^{p-1} h^p \left(C + \frac{h}{p+1} \right) \\ & = \frac{1}{\beta} \left(\lambda h - \text{sgn}(\Lambda) \left(\frac{|q|}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) - \tau^{(1,\beta)} - \delta \tau^{(1,\beta\delta)} - \delta^2 \tau^{(1,\beta\delta^2)} - \delta^3 \tau^{(1,\beta\delta^3)} \end{aligned} \quad (75)$$

where Λ is defined by [\(50\)](#) and the coefficients $\tau^{(1,\phi)}$ are given in [AppendixB](#).

11.4. Particular cases

11.4.1. Case of Regime B or C

In the case δ small (regimes B and C) the unified two-equations model (75) writes:

$$\begin{aligned} \partial_t h + \partial_x q &= a \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{\lambda^{2p} h^{2p+3}}{(p+2)^2 (2p+3)} \right) + \frac{\delta}{\beta} h \partial_x H - a \lambda^p h^p \left(C + \frac{h}{p+1} \right) \\ &= \frac{1}{\beta} \left(\lambda h - \left(\frac{q}{h \left(C + \frac{h}{p+2} \right)} \right)^{\frac{1}{p}} \right) - \frac{p}{(C(p+2)+h)} \left(\frac{1}{2(p+1)^2} h \lambda^p a - \frac{(Cp+h)}{(p+2)(2p+3)} \lambda^{2p} \partial_x h \right) h^{2(p+1)} \end{aligned} \quad (76)$$

11.4.2. No-slip case

In the case no-slip (adherence) at bottom, the unified two-equations model (75) simplifies by setting $C = 0$ as follows:

$$\begin{aligned} \partial_t h + \partial_x q &= a \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{\Lambda^{2p} h^{2p+3}}{(p+2)^2 (2p+3)} \right) + \frac{\delta}{\beta} h \partial_x H - a \Lambda |\Lambda|^{p-1} \frac{h^{p+1}}{p+1} \\ &= \frac{1}{\beta} \left(\lambda h - \operatorname{sgn}(\Lambda) \left((p+2) \frac{|q|}{h^2} \right)^{\frac{1}{p}} \right) - \tau^{(1,\beta)} - \delta \tau^{(1,\beta\delta)} - \delta^2 \tau^{(1,\beta\delta^2)} - \delta^3 \tau^{(1,\beta\delta^3)} \end{aligned}$$

where the coefficients write:

$$\begin{aligned} \tau^{(1,\beta)} &= \left(\frac{p}{2(p+1)^2} |\Lambda|^{p-1} \Lambda a - \frac{p}{(p+2)(2p+3)} \Lambda^{2p} \partial_x h \right) h^{2(p+1)} \\ \tau^{(1,\beta\delta)} &= \frac{p}{(p+2)(2p+3)} \left((p+1) (\partial_x h)^2 + h \partial_x^2 h + \frac{(3p-2)}{(3p+4)} h \partial_x^2 H \right) h^{2(p+1)} \Lambda^{(2p-1)} \\ &\quad - \frac{p}{(p+2)(2p+3)} |\Lambda|^{p-1} h^{p+2} \partial_x a \\ \tau^{(1,\beta\delta^2)} &= - \frac{p^2}{(p+2)(2p+3)} \left(2 \partial_x h \partial_x^2 H + \frac{h}{(p+2)} \partial_x^3 H \right) h^{2p+3} \Lambda^{2(p-1)} \\ \tau^{(1,\beta\delta^3)} &= \frac{(p-1)p^2}{(p+2)^2 (2p+3)} h^{2(p+2)} \Lambda^{2p-3} (\partial_x^2 H)^2 \end{aligned}$$

11.4.3. Newtonian case

In the Newtonian case ($p = 1$), the unified two-equations model (75) simplifies as follows:

$$\begin{aligned} \partial_t h + \partial_x q &= a \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{\Lambda^2 h^5}{45} \right) + \frac{\delta}{\beta} h \partial_x H - a \Lambda h \left(C + \frac{h}{2} \right) \\ &= \frac{1}{\beta} \left(\lambda h - \operatorname{sgn}(\Lambda) \left(\frac{3|q|}{h(3C+h)} \right) \right) - \tau^{(1,\beta)} - \delta \tau^{(1,\beta\delta)} - \delta^2 \tau^{(1,\beta\delta^2)} \end{aligned}$$

where the coefficients write:

$$\begin{aligned}
\tau^{(1,\beta)}|_{p=1} &= \frac{h^3}{8(3C+h)}\Lambda a - \frac{(C+h)h^4}{15(3C+h)}\Lambda^2\partial_x h \\
\tau^{(1,\beta\delta)}|_{p=1} &= \frac{h^4}{15(3C+h)}\left(\left(\frac{h^2}{7}\partial_x^2 H + 2(C+h)(\partial_x h)^2 + h(2C+h)\partial_x^2 h\right)\Lambda - \partial_x a\right) \\
\tau^{(1,\beta\delta^2)}|_{p=1} &= -\frac{h^5}{15(3C+h)}\left(2(2C+h)\partial_x h\partial_x^2 H - \frac{h(3C+h)}{3}\partial_x^3 H\right)
\end{aligned}$$

12. Comparison to the literature

We show below that the one-equation models (55) and (58) and the two-equations model (75) contain all terms of the classical models from the literature if considering the same order, the same regime given (Regime C generally) and if considering the same basal condition (generally no-slip or asymptotically vanishing friction at bottom).

12.1. Thin-film flows literature - Power-law case

If considering the present unified one-equation model order 1 (58) in the particular case: Regime C, $C = 0$ at bottom (adherence) and $a = 0$ (no mass source term), then we recover the equation stated in [14] (see also [15]), but apart from the surface tension term (κ) since it is not considered here.

The one-equation model stated in [14] (see also [15]) writes:

$$\begin{aligned}
&\partial_t h + \partial_x \left(\frac{nh^2(\tilde{\lambda}h)^{1/n}}{2n+1} \right) = \\
&\partial_x \left(-\frac{\kappa\tilde{\delta}}{n\tilde{\lambda}}\partial_x^3 h + \left(\frac{\tilde{\delta}_c}{n\tilde{\lambda}}h^2(\tilde{\lambda}h)^{1/n} \frac{n}{2n+1} - \frac{2\beta}{(2n+1)(3n+2)\tilde{\lambda}}\tilde{\lambda}^{3/n}h^{3/n}h^3 \right) \partial_x h \right)
\end{aligned} \tag{77}$$

with $n = \frac{1}{p}$ and the definition of the dimensionless parameters $\tilde{\lambda}$, $\tilde{\delta}$ differ slightly from the present ones.

$$\partial_t h + \partial_x \left(\frac{\lambda^p}{p+2}h^{p+2} \left(1 - \frac{\delta p}{\lambda}\partial_x h \right) + \beta \frac{p^2}{(p+2)} \left(\frac{2}{(2p+3)}\lambda^{3p-1}h^{(3p+3)}\partial_x h - \frac{(2p+3)}{2(p+1)^2}h^{2(p+1)}\lambda^{2p-1}a \right) \right) = 0$$

Still in the same case (Regime C with $C = 0$ and $a = 0$), the second equation of the unified model (75) may writes in a conservative form as follows:

$$\partial_t q + \partial_x \left(\frac{2(p+2)q^2}{(2p+3)H} + \frac{1}{2}\frac{\delta}{\beta}H^2 - \frac{p}{(p+2)(2p+3)^2}\lambda^{2p}H^{2p+3} \right) = \frac{1}{\beta} \left(\lambda H - \left((p+2)\frac{q}{H^2} \right)^{\frac{1}{p}} \right)$$

And we retrieve the two-equations model stated in [15].

12.2. Newtonian case

The classical Benney equation [4] is a first order one-equation model, taking into account surface tension. This equation writes as follows, see e.g. [5]:

$$\partial_t h + \partial_x \left(h^3 + \left(\frac{6}{5}Reh^6 - \cot\theta h^3 \right) \partial_x h + Wh^3\partial_x^3 h \right) = 0 \tag{78}$$

with W the Weber number.

Using the average velocity as the characteristic velocity we have a relation for λ , Eqn. (26) which writes in the particular newtonian case $\lambda = 3$. Thus the flow rate writes:

$$q_{(p=1, C=0, \lambda=3, \delta=\varepsilon \lambda \cot \theta)}^{(1)} = h^3 + \varepsilon \left(\left(\frac{6}{5} R_e h^3 - \cot \theta \right) h^3 \partial_x h - \frac{5}{8} R_e h^4 a \right)$$

And we recover the Benney equation except the surface tension term which is not considered here.

Still in the same case (Regime C, $C = 0 = a$ and $p = 1$), the second equation of the unified model (75) simplifies as follows:

$$\partial_t q + \partial_x \left(\frac{6}{5} \frac{q^2}{H} + \frac{1}{2} \frac{\delta}{\beta} H^2 \right) = \frac{1}{\beta} \left(\lambda H - 3 \frac{q}{H^2} \right) + \frac{1}{15} \lambda^2 H^4 \partial_x H$$

Hence we retrieve the equation stated in [40] (except the surface tension term which is not considered here).

In other respect, the model derived in [6] may be retrieved using some algebra, and using the property that at order 0 the flow rate and the height of fluid satisfy: $q^{(0)} = \frac{1}{3} \lambda h^3$. As a matter of fact, it has been proved in [40] that the two-equations model presented in [40] is formally identical to those presented in [6].

Asymptotically vanishing friction case. The present unified two-equations model include the order one terms of the model presented in [8]. In [8], are considered: a vanishing mean slope and flat bottom ($\lambda = 0$ and $b = 0$), Regime C ($\alpha = \beta = \delta = \varepsilon$), and an asymptotically vanishing friction at bottom ($C = \frac{1}{\varepsilon \kappa}$). In this particular case, the second (momentum) equation of the unified model (75) writes:

$$\partial_t q + \partial_x \left(\frac{q^2}{H} + \frac{\delta}{\beta} \frac{1}{2} H^2 \right) = - \frac{\kappa}{(1 + \varepsilon \frac{\kappa H}{3})} \frac{q}{H}$$

Hence it includes the first order term derived in [8].

Let us note that there is an extra term in the model derived in [8], term order 1 in ε (i.e. in $\mathcal{O}(\varepsilon^2)$). We cannot obtain it straightforwardly here since we did not derive a second order model. Let us point out that for general configurations like those considered in the present article (mean slope not vanishing, and potentially non flat bottom), the second order accurate model would be too complex. Nevertheless under the assumptions made in [8], one can derive quite easily second order accurate models using the method presented in Section 4.3.

12.3. Glacier flow models

In the glaciology literature, the classical Shallow Ice Approximation (SIA) is derived with no-slip boundary condition at bottom, in the horizontal-vertical coordinate system ($\lambda = 0$), and with the rheology exponent $p = 3$ (Glen's law). As mentioned previously, if modeling ice flows the inertial term can be neglected, hence we set : $\beta = 0$. The SIA equation is known to be valid for high shear rate flow, corresponding to the "ice-sheet" case i.e. Regime A, see Table 1. Classically, it is derived by assuming *a-priori* scalings on the normal stresses ($\sigma_{xx} = \sigma_{zz} = -p$) and under the assumption of negligible topography variations, see e.g. [2]. The SIA equation writes, see e.g. [2, 18, 20] :

$$\partial_t h - \partial_x \left(\Gamma \frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} \right) = a$$

with Γ a constant depending on μ_0 and ρg . Therefore, Equation (55) includes the SIA model since for $\lambda = 0$ (vanishing mean-slope), Regime A, no-slip at bottom, it writes:

$$\partial_t h - \delta^p \partial_x \left(\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} \right) = a$$

with $\delta = \frac{\rho g H^{n+2}}{\mu_0 U^n L}$.

In the case of a non-vanishing velocity at bottom imposed (Dirichlet's condition $u = u_b$ at bottom), and under the assumption of non "abrupt" ice surface variations, the SIA model has been extended in [26] (Chapter 10 p 633) as follows:

$$\partial_t h - \partial_x \left(\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} + u_b h \right) = a \quad (79)$$

Equation (55), for $\lambda = 0$, Regime A, friction at bottom, writes:

$$\partial_t h - \delta^p \partial_x \left(\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} \right) - C \delta^p \partial_x H |\partial_x H|^{p-1} h^{p+1} = a \quad (80)$$

Using (46), the expression of u_b at order 0 writes:

$$u_b^{(0)} = -C \delta^p h^p \partial_x H |\partial_x H|^{p-1} \quad (81)$$

If replacing in Equation (80), we obtain:

$$\partial_t h - \delta^p \partial_x \left(\frac{h^{p+2}}{p+2} \partial_x H |\partial_x H|^{p-1} + h u_b^{(0)} \right) = a$$

Thus, Equation (55) includes the SIA model derived in [26], in the case velocity at bottom u_b imposed (if setting $\delta = 1$, hence a particular Regime A).

Note that a similar equation to (79) is considered in [27], and it is shown formally that it is to valid for all friction - slip regimes. To do so, the authors introduce a slip ratio R (R tends to 0 if pure slip at bottom (pseudo-plug flow) and R tends to 1 if no-slip at bottom).

Observe that in the latter references, the shallow models are derived based on a-priori scaling and with a-priori weak variations of topography and/or free-surface. Also, the asymptotic are not done fully in the non-linear terms (eg the power-law term depth-integrated).

In [37], the authors introduce classically an *a-priori* double scaling on the velocity and the stress components, leading to one particular Regime A (case $\lambda = 0$ and $\delta = 1$). Next, the authors obtain the equations (44)-(45) for this particular case, and the exact solutions order 0. The sketch of the derivation of the higher-order terms (hence higher order shallow ice models) up to order 2 in the geometrical ratio ε is presented (still in the same case). Almost all fields are calculated at order 2 in ε excepted those singular (i.e. the normal stress components and the pressure), see [37] section "second order stress formulas".

In summary, both the unified one-equation model (58) and the unified two-equations model (75) includes all the classical equations mentioned above, if considering the same order, the same regime given. Roughly, compare to the classical shallow models from literature, the present unified models contain extra terms in $\delta, \beta\delta$ and in C (the slip coefficient).

13. Numerical examples: two-regime flows with sharp transition

In this section we illustrate numerically the capabilities of the unified model (55) by performing the multi-regime numerical solution in two cases; both presenting sharp but continuous changes of regime.

In Case 1 the change of regime is due to the change of the bottom boundary condition.

In Case 2 the change of regime is due to a change of the topography (presented here in term of mean slope).

In both cases the change of the boundary condition (resp. the mean slope) are sharp but continuous. The goal is to capture accurately this multi-regime flow (with a sharp change) using the single one-equation model (55).

We consider the steady state solution; it satisfies (57). Assuming the fluid flows from left to right (i.e. $q_0 > 0$), the steady solution writes:

$$\partial_x H = \partial_x h + \partial_x b = \frac{1}{\delta} \left(\lambda - \frac{1}{h} \left(\frac{1}{h \left(C + \frac{h}{p+2} \right)} \right)^{1/p} \right) \quad (82)$$

At the left part of the domain, let us assume the equilibrium is reached, that is to say a steady state with a constant height. We use this equilibrium to define the reference flow: $q_0 = 1$, $h_0 = 1$, $C = C_0$. λ is defined by the relation (26) that means $\lambda = \left(C_0 + \frac{1}{p+2} \right)^{-1/p}$.

We want to consider the transition between two regimes occurring when either the boundary condition changes (from adherence to slip) or the mean slope changes.

Let us point out we have fixed the reference flow. Thus we obtain a set of dimensionless parameters. If we choose another reference height, we obtain a different set of dimensionless parameters. This ‘‘change of reference basis’’ can be expressed in terms of the previous dimensionless parameters through the following relations:

$$\alpha^{new}(h) = \frac{\alpha}{h^{1+\frac{2}{p}}} \quad \delta^{new}(h) = h^{1+\frac{2}{p}} \delta \quad (83)$$

If $h = 1$ then $\delta^{new} = \delta$ and $\alpha^{new} = \alpha$. If we have $h \neq 1$, then it is a new regime.

Any equilibrium solution satisfies (82). We expect to reach an equilibrium at infinity: $h = h_\infty$, $\partial_x b = \tan \theta_\infty$, $C = C_\infty$; therefore equation (82) writes:

$$h_\infty^{p+1} \left(C_\infty + \frac{h_\infty}{p+2} \right) = \frac{1}{(\lambda - \delta \tan \theta_\infty)^p} \quad (84)$$

As mentioned before we consider two cases:

- Case 1: the mean slope is constant, hence $\tan \theta_\infty = 0$ and the change of regime is due to a change of the bottom boundary condition ($C_0 \neq C_\infty$). The equilibrium at infinity is given by:

$$h_\infty^{p+1} \left(C_\infty + \frac{h_\infty}{p+2} \right) = C_0 + \frac{1}{p+2} \quad (85)$$

Case 1 is separated in two sub-cases:

- Case 1a: constant mean slope unperturbed $b(x) = 0$.
- Case 1b: constant mean slope with arbitrary varying topography $b(x) \neq 0$.

- Case 2: the bottom boundary condition is constant ($C_0 = C_\infty$). The change of regime is due to a change of mean slope $\tan \theta_\infty \neq 0$. The equilibrium at infinity is given by:

$$h_\infty^{p+1} \left(C_0 + \frac{h_\infty}{p+2} \right) = \frac{1}{\left(\left(C_0 + \frac{1}{p+2} \right)^{-1/p} - \delta \tan \theta_\infty \right)^p} \quad (86)$$

Then we compute the corresponding height solution of the unified model (82). Also, we show that the steady-state multi-regime solution (solution of (82)) corresponds to a change of regime in the sense sub-critical to supercritical flow (see calculation presented in Section 10).

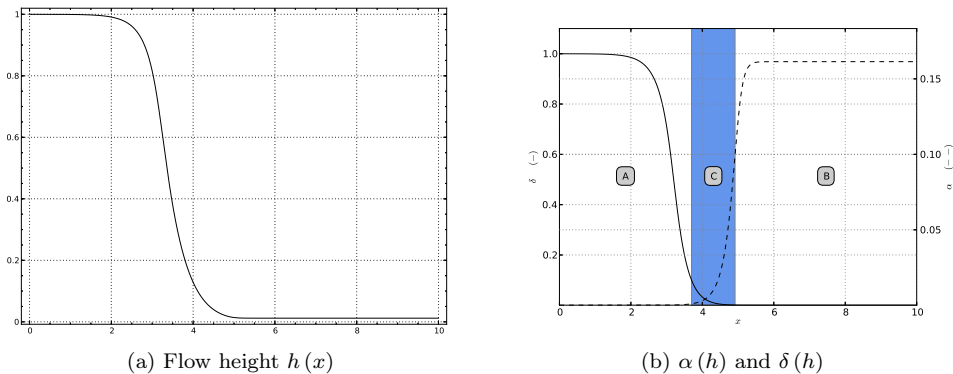


Figure 7: Case 1-a: Multi-regime transition due to a change of the bottom boundary condition

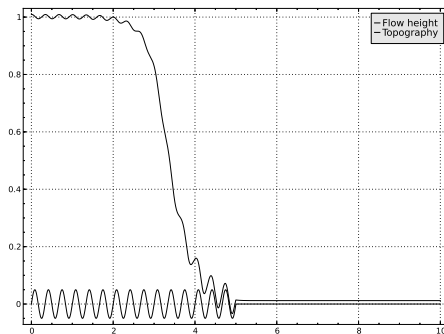


Figure 8: Case 1-b: Flow height and topography of the multi-regime transition over an arbitrary topography

13.1. Case 1: Multi-regime transition due to a change of the bottom boundary condition

Let us consider a domain in two parts with a sharp but continuous transition between them. We set the geometric scaling parameter $\varepsilon=10^{-2}$, the power-law exponent $p = 3$. At the left half of the domain ($x < 4$), we set $\gamma = \varepsilon$ and $Fr^2 = \varepsilon$ in order to obtain a Regime A ($\alpha = \varepsilon^2$ and $\delta = 1$) with adherence at bottom ($C = 0$). At the right half of the domain part ($6 < x$), we consider a slip boundary condition with $C = 10^7$ with a smooth transition ($4 \leq x \leq 6$) (see figure 7a).

Case 1a.

The local topography variations $b(x)$ are null (non-horizontal flat bottom).

We solve numerically the steady unified model (82) using a classic RK4 method. The resulting flow height is plotted on figure 7a. On figure 7b the dimensionless parameters α and δ are expressed in terms of the local height $h(x)$ using relations (83). Figure 7a clearly shows the transition between the Regime A with $\delta = 1$ to the Regime B with $\delta = O(\varepsilon^2)$ through a Regime C (transition zone).

The corresponding hydraulic Froude number (defined by equation (65)) varies from $F_h \approx 1.7 \cdot 10^{-4}$ upstream to $F_h \approx 77$ downstream. The reduction of the wall friction transforms the sub-critical upstream flow ($F_h < 1$) to a super-critical flow ($F_h > 1$).

Case 1b.

Next we disturb the previous flow by adding at the bottom a varying topography i.e. $b(x) \neq 0$. The solution (see figure 8) shows that the steady state solution of Regime B is a translation of the bottom, whereas in Regime A there is a diffusion effect induced by the gravity contribution due to the local slope of the free-surface $\delta \partial_x H$.

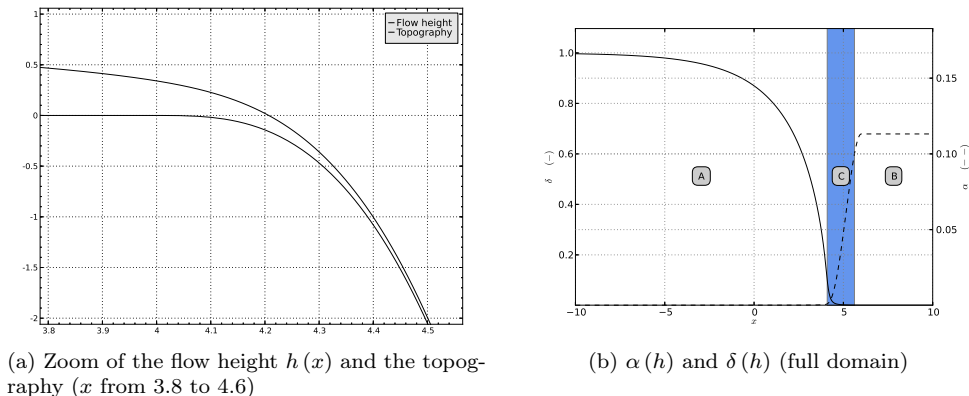


Figure 9: Case 2: Multi-regime transition due to a change of the mean slope

13.2. Case 2: Multi-regime transition due to a change of the mean slope

We set the geometric scaling parameter $\varepsilon=10^{-2}$, the power-law exponent $p = 3$ and a constant friction coefficient at bottom $C = 50$. At the left half of the domain ($x < 4$), we set $\gamma = \varepsilon$ in order to obtain a Regime A ($\alpha = \varepsilon^2$ and $\delta = 1$). At the right half of the domain part ($6 < x$), we consider a different slope $\tan \theta_\infty = 75$. The slope varies smoothly from 0 to $\tan \theta_\infty$ in $4 \leq x \leq 6$ (see figure 9a). The resulting flow height is plotted on figure 9a, and the local dimensionless parameters are plotted on figure 9b.

The unified model allows to shift from one regime to another continuously due to a change of the local slope from weak to large.

The corresponding "hydraulic" Froude number defined by equation (65), varies from $F_h \approx 0.1$ upstream to $F_h \approx 56$ downstream. The increase of the wall slope transforms the sub-critical upstream flow ($F_h < 1$) to a super-critical flow ($F_h > 1$).

These test cases present sharp variations of regime due to either a change of boundary condition at bottom, or a change of mean slope. In each case, the "unified model" (55) allows to compute the solution in all the domain, in other words the bi-regime solution. At contrary, to our knowledge no other single equation from the literature allows to handle such configurations.

14. Conclusion

In this paper we have derived one-equation models (lubrication type) and two-equations models (depth-integrated, shallow-water type) consistent with any viscous flows and multiple boundary conditions at bottom (from no-slip to pure slip through friction), multi-regime flows. The multi-regime (unified) model obtained at order 0 may be of practicable use, while its complexity at order 1 may prevent to implement it for complex multi-scale flows. Nevertheless it may be interesting for finer flows (on regular topography or not). Also, the field expressions at order 1 are compulsory to derive a consistent depth-integrated two-equations models (it cannot be achieved using 0th order expressions).

The present models and explicit expressions of all fields (velocity, pressure and stress components) are valid either in a mean slope coordinate system with local topography variations, or in the Cartesian coordinate system using the Prandtl shift, hence for arbitrary large topography variations.

We have stated formal error estimates on the solution (velocity, pressure, stress). These error estimates enhance the potentially dominating error due to the friction law at bottom; more precisely in the bulk the asymptotic expansions may be derived at order one or two while the global asymptotic error may remain at order zero with respect to the friction law at bottom.

We have detailed a two-equations model not fully consistent (as the basal shear stress should be expanded up to the first order) in order to derive a simple gradually varied flow model. The latter defines a hydraulic-like Froude number and the corresponding sub-critical and supercritical flows.

We perform numerical test cases which enhance the features of the new unified model, while no single equation from the literature would allow to handle the flow configuration considered.

The unified models derived here include all classical models of the literature if considering the same order, the particular regime considered for each of them, or the same condition at bottom (no-slip, or asymptotically vanishing friction at bottom).

The asymptotic equations have been derived using a standard perturbation method of the Navier-Stokes equations free-surface, power-law, based on the iterative algorithm formulated like in [15, 28]. The latter may allow to derive formally the asymptotic fields and equations up to order two, for all viscous regimes. Nevertheless, observe that at higher order (order 2 or more), on one hand the expressions become very heavy (excepted may be in particular cases such as flat bottom with no-slip etc), and more important the power-law rheology in the case $p \geq 2$ breaks down since the solution (pressure and normal stress components) presents an unphysical singularity.

Other rheology laws could be investigated such as yield stress fluids. In [15], the authors derived shallow equations for a Bingham fluid but at Regime C and with no-slip at bottom only. Following the present approach and the treatment of the yield stress presented in [15], one could obtain a multi-regime, multi boundary conditions at bottom model for Bingham type fluids.

The surface tension terms could be added easily to the present unified models (as it may be useful for industrial thin-film flows) using the method presented here. Note it has already been written in the mono-regime case in [15] (Regime C). Finally, we point out that all derivations of the equations would be similar (but heavier) for 3D flows.

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Appendix A. Coefficients of the first order corrections for the discharge

We present here the first order terms of the discharge function; the expressions are given in the mean slope coordinate system.

$$\begin{aligned}
q^{(1,\beta)} &= -p^2 \left(C^2 + \frac{(2p+3)C}{(p+1)(p+2)}h + \frac{(2p+3)}{2(p+1)^2(p+2)}h^2 \right) h^{2p} \Lambda^{2p-1} a \\
&\quad + p^2 \left(C^3 p + \frac{(3p+2)C^2}{p+2}h + \frac{6(p+1)C}{(p+2)(2p+3)}h^2 + \frac{2}{(p+2)(2p+3)}h^3 \right) h^{3p} |\Lambda|^{p-1} \Lambda^{2p} \partial_x h \\
q^{(1,\beta\delta)} &= p^2 \left(C^2 + \frac{2C}{p+2}h + \frac{2}{(p+2)(2p+3)}h^2 \right) h^{2p+1} \Lambda^{2(p-1)} \partial_x a \\
&\quad - p^2 \left\{ (p+1) \left(C^3 p + \frac{(3p+2)C^2}{p+2}h + \frac{6(p+1)C}{(p+2)(2p+3)}h^2 + \frac{2}{(p+2)(2p+3)}h^3 \right) (\partial_x h)^2 \right. \\
&\quad + \left((p+1) C^3 + \frac{(3p+4)C^2}{p+2}h + \frac{2(3p+4)C}{(p+2)(2p+3)}h^2 + \frac{2}{(p+2)(2p+3)}h^3 \right) h \partial_x^2 h \\
&\quad \left. + \left((p-1) C^3 + \frac{(3p-2)C^2}{p+2}h + \frac{2(3p^2+4p-3)C}{(p+2)^2(2p+3)}h^2 + \frac{2(3p^2+4p-3)}{(p+2)^2(2p+3)(3p+4)}h^3 \right) h \partial_x^2 H \right\} h^{3p} |\Lambda|^{p-1} \Lambda^{(2p-1)} \\
q^{(1,\beta\delta^2)} &= p^3 \left\{ 2 \left((p+1) C^3 + \frac{(3p+4)C^2}{p+2}h + \frac{2(3p+4)C}{(p+2)(2p+3)}h^2 + \frac{2}{(p+2)(2p+3)}h^3 \right) \partial_x h \partial_x^2 H \right. \\
&\quad \left. + \left(C^3 + \frac{3C^2}{p+2}h + \frac{2(3p+5)C}{(p+2)^2(2p+3)}h^2 + \frac{2}{(p+2)^2(2p+3)}h^3 \right) h \partial_x^3 H \right\} h^{3p+1} |\Lambda|^{p-1} \Lambda^{2(p-1)}
\end{aligned}$$

$$q^{(1,\beta\delta^3)} = -(p-1)p^3 \left(C^3 + \frac{3C^2}{p+2}h + \frac{2(3p+5)C}{(p+2)^2(2p+3)}h^2 + \frac{2}{(p+2)^2(2p+3)}h^3 \right) h^{3p+2} |\Lambda|^{p-1} \Lambda^{2p-3} (\partial_x^2 H)^2$$

Recall the newtonian case corresponds to the case $p = 1$, and no-slip at bottom corresponds to the case $C = 0$. Λ is defined by (50).

Appendix B. Coefficients of the unified two-equations model

We present here the basal shear stress first order corrective terms; the expressions are given in the mean slope coordinate system.

$$\begin{aligned} \tau^{(1,\beta)} &= \frac{p}{(C(p+2)+h)} \left(\frac{1}{2(p+1)^2} h |\Lambda|^{p-1} \Lambda a - \frac{(Cp+h)}{(p+2)(2p+3)} \Lambda^{2p} \partial_x h \right) h^{2(p+1)} \\ \tau^{(1,\beta\delta)} &= \frac{p}{(p+2)(2p+3)(C(p+2)+h)} \left\{ (p+1)(Cp+h)(\partial_x h)^2 \right. \\ &\quad \left. + ((p+1)C+h)h\partial_x^2 h + \left((p-1)C + \frac{(3p-2)}{(3p+4)}h \right) h\partial_x^2 H \right\} h^{2(p+1)} \Lambda^{2(p-1)} \\ &\quad - \frac{p}{(p+2)(2p+3)(C(p+2)+h)} |\Lambda|^{p-1} h^{p+3} \partial_x a \\ \tau^{(1,\beta\delta^2)} &= -\frac{p^2}{(p+2)(2p+3)(C(p+2)+h)} \left(2((p+1)C+h)\partial_x h \partial_x^2 H + \left(C + \frac{h}{(p+2)} \right) h \partial_x^3 H \right) h^{2p+3} \Lambda^{2(p-1)} \\ \tau^{(1,\beta\delta^3)} &= \frac{(p-1)p^2}{(p+2)^2(2p+3)} h^{2(p+2)} \Lambda^{2p-3} (\partial_x^2 H)^2 \end{aligned}$$

Recall the newtonian case corresponds to the case $p = 1$, and no-slip at bottom corresponds to the case $C = 0$. Λ is defined by (50).

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