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How do birth and death processes come down from infinity?

Vincent Bansaye*, Sylvie Méléard†, Mathieu Richard‡

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Abstract

We finely describe the "coming down from infinity" for birth and death processes which eventually become extinct. Our biological motivation is to study the decrease of regulated populations which are initially large. Under general assumptions on the birth and death rates, we describe the behavior of the hitting time of large integers. We let two regimes appear and derive an expression of the speed of coming down from infinity. In the case of death rates with regular variations, we also get a central limit theorem and the asymptotic probability of extinction in small times. Finally, we apply our results to birth and death processes in varying environment in whose the environment influences the competition.

Key words: Birth and death processes, Coming down from infinity, Law of large numbers, Central limit theorem, Extinction probability.

MSC 2010: 60J27, 60J75, 60F15, 60F05, 60F10, 92D25.

1 Introduction

Our goal in this paper is to finely describe the "coming down from infinity" for a birth and death process which eventually becomes extinct. Our motivations come from the study of population dynamics with initially large populations. In particular we wish to describe the effect of the competition in large populations and specify persistence criteria in a possibly varying environment. For this purpose, we decompose the trajectory of the process with respect to the hitting times of large integers and we use some asymptotic results about sums of independent random variables and Tauberian theorems.

The population size is modeled by a birth and death process $(X(t), t \geq 0)$ whose birth rate (resp. death rate) at state $n \in \mathbb{N}$ is λ_n (resp. μ_n). In the whole paper, we assume that λ_n are nonnegative and μ_n are positive for $n \geq 1$ and that $\mu_0 = \lambda_0 = 0$. The latter implies that 0 is an absorbing state. These processes have been extensively studied from the pioneering works on birth and death processes [11] and on the quasi-stationary distribution [20].

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It is well known [11, 12] that

$$\sum_{i \geq 1} \frac{1}{\lambda_i \pi_i} = \infty \quad (1)$$

is a necessary and sufficient condition for almost sure absorption of the process at 0, where for $n \geq 1$,

$$\pi_n := \frac{\lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}.$$

Under (1), we let the initial population size go to infinity and focus on the case where the limiting process comes back to finite values in finite time. This behavior is captured by the notion of "coming down from infinity". Characterizations of the coming down from infinity have been given in [2, 7]. They rely on the convergence of the series

$$S := \sum_{n=1}^{+\infty} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i < +\infty \quad (2)$$

or on the finiteness of the first moment of time of absorption, uniformly in the initial condition. As proved in Van Doorn [20], this is also equivalent to the existence and uniqueness of the quasi-stationary distribution at 0. In particular, the uniqueness of the quasi-stationary distribution is deeply related to the way the process comes back into compact sets, see [15] or [7].

In Section 2, we improve this result by an additional exponential moment condition linked to the Lyapounov function $J(n) = \sum_{k=1}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{i \geq k+1} \pi_i$.

We go further in the description of the coming down from infinity, under a slightly more restrictive assumption than (1) :

$$\frac{\lambda_n}{\mu_n} \xrightarrow{n \rightarrow +\infty} l < 1. \quad (3)$$

That allows us to rigorously define the law \mathbb{P}_∞ of the process starting from infinity by a tightness argument. Assumption (3) is satisfied by the parameters of the classical models motivated by ecology, including competition between individuals or Allee effect. We also need the following technical assumption on the death rate to obtain the asymptotic behavior of the integers hitting times :

$$\sup_{k, n \geq 1} \frac{\mu_n}{\mu_{n+k}} < \infty. \quad (4)$$

Two interesting classes of sequences fulfill condition (4): the death rates which are non-decreasing for large enough integers and the regularly varying death rates (see Section A in Appendix for definitions).

When conditions (3) and (4) are both satisfied, it is easy to check (see Lemma B.1 in the Appendix) that the series S is finite if and only if

$$\sum_{i \geq 1} \frac{1}{\mu_i} < +\infty. \quad (5)$$

Under the assumptions (3), (4) and (5), we study in Section 3 the asymptotic behavior of the decreasing sequence $(T_n)_n$ of hitting times, defined by

$$T_n = \inf\{t \geq 0, X(t) = n\}.$$

Then, (1) and (2) are satisfied, X comes down from infinity a.s. at time 0. We put in light two different regimes which depend on whether the mean time to go from $n+1$ to n is negligible or not compared to the mean time to reach n . We are then able to give the asymptotic behavior of X for small times. We show that the speed of coming down from infinity is obtained from a deterministic decreasing function $t \mapsto v(t)$ tending to infinity at 0 and defined as the generalized inverse of the mapping $n \mapsto \mathbb{E}_\infty(T_n) = S - J(n-1)$. More precisely, our main result (Section 4) ensures that

$$\lim_{t \rightarrow 0} \frac{X(t)}{v(t)} = 1,$$

where the convergence is either in probability or almost surely, depending on the respective asymptotic behaviors of the birth and death rates. For that, we need to control the trajectory of the process between two successive times T_n . We also require that the rates μ_n are regularly varying to get the a.s. convergence. In addition, we derive in that case a central limit theorem (Theorem 4.2) and the probability to be absorbed exceptionally fast (Theorem 5.1). The proofs rely on a central limit theorem for the sum of independent random variables, some Tauberian results and coupling arguments. Roughly speaking, we prove that

$$\mathbb{P}_\infty(T_0 \leq t) \quad \text{behaves as} \quad \exp\left(-t^{\frac{1}{1-\rho}}\right),$$

as t tends to 0, where ρ is the index of the variations of $(\mu_n)_n$.

These results apply in particular to the logistic birth and death process and to the Kingman coalescent. In both cases, we improve the known results on the coming down from infinity. Lambert [14] characterizes the distribution of the absorption time for the logistic branching process starting from infinity. Our work extends to very general death rates, as polynomial increase, which are motivated by ecological data for competition of species, see e.g. Sibily and al [17].

The proof of the speed of coming down from infinity for Kingman coalescent has already been obtained in Aldous [1]. We complete this result by estimating the probability that the most recent common ancestor is achieved very fast. Our proof also suggests the way this rare event occurs by considering the associated successive coalescent times.

Section 6 is devoted to the main application of our results, which is an extinction criterion for time inhomogeneous birth and death processes. These processes have been studied in the framework of randomly varying environment, as described in Cogburn and Torrez [9], [18]. Our results allow to obtain extinction results in cases where the environment can be both unfavorable during some periods and favorable during the rest of the time. We quantify the minimal duration of the unfavorable environmental periods leading to eventual extinction. For example, this problem is relevant in epidemiology, when the environment influences the parameters of the infection (see Bacaer-Dads [3] and van den Broek-Heesterbeek [19]). The proof relies on the evaluation of the probability of extinction for time homogeneous birth and death processes starting from ∞ given above.

The paper is organized as follows. In the next section, we work under the extinction assumption (1) and gather general characterizations of the coming down from infinity, such as (2). Focusing on the subclass of birth and death processes satisfying (3), (4) and (5) in Section 3, we describe the hitting time of large integers when the process starts from infinity. Under some additional assumption, we can then derive in Section 4 the asymptotic behavior of $X(t)$ when t is small. Moreover, when the death rate has regular variations, we can also

quantify the probability of extinction before a small time (Section 5) and the minimal time of competition leading to extinction in varying environment (Section 6). Finally in Appendix, we illustrate our results by several examples and give some useful details on regularly varying functions.

2 Coming down from infinity

We first focus on the time spent by the process $(X(t), t \geq 0)$ to go from level $n+1$ to level n . We introduce the notation

$$\tau_n := \inf\{t > T_{n+1}; X(t) = n\} - T_{n+1}, \quad G_n(a) := \mathbb{E}(\exp(-a\tau_n)), \quad (a > 0).$$

By the strong Markov property, τ_n has the law of T_n under \mathbb{P}_{n+1} and the random variables $(\tau_n)_{n \geq 0}$ are independent.

Proposition 2.1. *For each $n \in \mathbb{N}$, we have*

$$\mathbb{E}(\tau_n) = \mathbb{E}_{n+1}(T_n) = \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i \quad (6)$$

and for every $a > 0$,

$$G_n(a) = \mathbb{E}_{n+1}(\exp(-aT_n)) = 1 + \frac{\mu_n + a}{\lambda_n} - \frac{\mu_n}{\lambda_n} \frac{1}{G_{n-1}(a)}. \quad (7)$$

Proof. The proof of the first part uses the Lyapounov function defined for all $m \in \mathbb{N}^*$ by

$$J(m) = \begin{cases} \sum_{n=1}^{m-1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i & \text{if } m \geq 2, \\ 0 & \text{if } m < 2. \end{cases}$$

We denote by L the infinitesimal generator of X : for any bounded function f on \mathbb{N} and any $n \in \mathbb{N}^*$,

$$L(f)(n) = (f(n+1) - f(n)) \lambda_n + (f(n-1) - f(n)) \mu_n. \quad (8)$$

One easily checks that for any $m \in \mathbb{N}^*$,

$$LJ(m) = -1.$$

Thus, the process

$$J(X(t)) - \int_0^t LJ(X(u)) du = J(X(t)) + t$$

is a martingale with respect to the natural filtration of X . Therefore, we have for all $k \geq 0$ and $t \geq 0$,

$$\mathbb{E}_{n+1}(J(X(t \wedge T_n))) + \mathbb{E}_{n+1}(t \wedge T_n) = J(n+1). \quad (9)$$

Adding that J is bounded by $S < +\infty$, we can use the bounded and monotone convergence theorems to let $t \rightarrow \infty$ in (9) and get

$$\mathbb{E}_{n+1}(J(X(T_n))) + \mathbb{E}_{n+1}(T_n) = J(n) + \mathbb{E}_{n+1}(T_n) = J(n+1).$$

Thus $\mathbb{E}_{n+1}(T_n) = J(n+1) - J(n)$ which concludes the proof of the first part of the proposition.

We consider now the Laplace transform of τ_n and follow [2, p. 264]. By the Markov property, we have

$$\tau_{n-1} \stackrel{(d)}{=} \mathbf{1}_{\{Y_n=-1\}} E_n + \mathbf{1}_{\{Y_n=1\}} (E_n + \tau_n + \tau'_{n-1})$$

where Y_n , E_n , τ'_{n-1} and τ_n are independent random variables, E_n is exponentially distributed with parameter $\lambda_n + \mu_n$ and τ'_{n-1} is distributed as τ_{n-1} and $\mathbb{P}(Y_n = 1) = 1 - \mathbb{P}(Y_n = -1) = \lambda_n / (\lambda_n + \mu_n)$. Hence, we get

$$G_{n-1}(a) = \frac{\lambda_n + \mu_n}{a + \lambda_n + \mu_n} \left(G_n(a) G_{n-1}(a) \frac{\lambda_n}{\lambda_n + \mu_n} + \frac{\mu_n}{\lambda_n + \mu_n} \right)$$

and (7) follows. \square

We now give the usual definition of coming down from infinity, which means that the state ∞ is an entrance boundary for the process X [16, p. 305].

Definition 2.2. *We say that the process $(X(t), t \geq 0)$ comes down from infinity if there exist a positive number t and a non-negative integer m such that*

$$\lim_{k \rightarrow +\infty} \mathbb{P}_k(T_m < t) > 0.$$

We give now several necessary and sufficient conditions for $(X(t), t \geq 0)$ to come down from infinity. The two first ones are directly taken from [7]. We add here an exponential moment criterion which is useful for the forthcoming proofs. Let us also mention that it is equivalent to the existence and uniqueness of a quasistationary distribution (cf. Van Doorn [20]).

Proposition 2.3. *Under condition (1), the following assertions are equivalent:*

- (i) *The process $(X(t), t \geq 0)$ comes down from infinity.*
- (ii) $S < +\infty$.
- (iii) $\sup_{k \geq 0} \mathbb{E}_k[T_0] < +\infty$.
- (iv) *For all $a > 0$, there exists $k_a \in \mathbb{N}$ such that $\sup_{k \geq k_a} \mathbb{E}_k(\exp(aT_{k_a})) < +\infty$.*

This result is the discrete counterpart of Lemma 7.4 in [7] for Feller diffusion processes $dZ_t = \sqrt{\gamma Z_t} dB_t + Z_t(r - f(Z_t))dt$ and suitable function f and $r > 0$. Recall that if (3) and (4) are satisfied, Assertion (ii), and then (i), (iii), (iv), are equivalent to Condition (5), which can be seen as the discrete counterpart of the criteria in [7, p.1953] stating that the process Z comes down from infinity if and only if $\int_1^\infty \frac{dx}{xf(x)} < +\infty$.

Proof of Proposition 2.3. Assertions (i), (ii), and (iii) are equivalent according to [7, Prop 7.10]. Let us now prove that (iv) implies that X comes down from infinity. Indeed, taking $a = 1$ in (iv), we have $M := \sup_{k \geq k_1} \mathbb{E}_k(\exp(T_{k_1})) < +\infty$. Then Markov inequality ensures that for all $k \geq k_1$ and $t \geq 0$, $\mathbb{P}_k(T_{k_1} < t) \geq 1 - \exp(-t)M$. Choosing t small enough ensures that the process comes down from infinity.

Finally, we prove that (ii) implies (iv) by adapting the proof of [7, Prop 7.6] to the discrete setting. We fix $a > 0$ and using $S < +\infty$, there exists k_a such that

$$\sum_{n \geq k_a - 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i \leq \frac{1}{a}.$$

We now define the Lyapounov function J_a as

$$J_a(m) := \begin{cases} \sum_{n=k_a-1}^{m-1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i & \text{if } m \geq k_a, \\ 0 & \text{if } m < k_a. \end{cases}$$

We note that J_a is non-decreasing, bounded and recalling the definition of the generator L from (8), $LJ_a(m) = -1$ for any $m \geq k_a$. Then,

$$M_t := e^{at} J_a(X(t)) - \int_0^t e^{au} (aJ_a(X(u)) + LJ_a(X(u))) du, \quad (t \geq 0)$$

is a martingale with respect to the natural filtration of X . Using the stopping time T_{k_a} and the fact that $J_a(X(u)) \leq J_a(\infty) \leq 1/a$, we have for all $k \geq k_a$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}_k (e^{at \wedge T_{k_a}} J_a(X(t \wedge T_{k_a}))) &= \mathbb{E}_k \left(\int_0^{t \wedge T_{k_a}} e^{au} (aJ_a(X(u)) + LJ_a(X(u))) du \right) + J_a(k) \\ &= \mathbb{E}_k \left(\int_0^{t \wedge T_{k_a}} e^{au} (aJ_a(X(u)) - 1) du \right) + J_a(k) \\ &\leq J_a(k) \end{aligned}$$

since $u \leq t \wedge T_{k_a}$ ensures that $X(u) \geq k_a$ and $LJ_a(X(u)) = -1$. Therefore, for any $k \geq k_a$, \mathbb{P}_k -a.s. $J_a(X(t \wedge T_{k_a})) \geq J_a(k_a)$ and

$$\mathbb{E}_k (e^{at \wedge T_{k_a}}) \leq \frac{J_a(k)}{J_a(k_a)}.$$

Then (iv) follows from the monotone convergence theorem and Assumption (ii). \square

Under our additional assumption (3), we can now define the process starting from infinity and check that it indeed comes down instantaneously from infinity a.s.. We set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and for any $T > 0$, we denote by $\mathbb{D}_{\overline{\mathbb{N}}}([0, T])$ the Skorohod space of càdlàg functions on $[0, T]$ with values in $\overline{\mathbb{N}}$.

Proposition 2.4. *Let $T > 0$. Under (1) and (2) and (3), the law of X under \mathbb{P}_k converges weakly as $k \rightarrow +\infty$ in $\mathbb{D}_{\overline{\mathbb{N}}}([0, T])$.*

We denote by \mathbb{P}_∞ the law of the limit and we have

$$\mathbb{P}_\infty (\inf\{t > 0 : X(t) < +\infty\} = 0) = 1.$$

Proof. First, we show that under Assumption (3) and for any $N_0 \in \mathbb{N}^*$ and $t > 0$,

$$\sup_{s \in [0, t]} \mathbb{E}_{N_0}(X(s)) < +\infty. \quad (10)$$

Indeed, there exists $N \in \mathbb{N}^*$ such that for $n \geq N$, $\lambda_n - \mu_n \leq 0$. Hence, for $s \leq t$,

$$\begin{aligned} \mathbb{E}_{N_0}(X_s) &= N_0 + \int_0^s \mathbb{E}_{N_0}(\lambda_{X(u)} - \mu_{X(u)}) \, du \\ &\leq N_0 + \int_0^s \mathbb{E}_{N_0}((\lambda_{X(u)} - \mu_{X(u)})\mathbf{1}_{X(u) \leq N}) \, du \leq N_0 + \sup_{n \leq N} |\lambda_n - \mu_n|t. \end{aligned}$$

To prove the convergence of the sequence of laws \mathbb{P}_k , we use Theorem 1 in Donnelly [10], which gives conditions under which a sequence of processes will converge (in law) to a Markov process with an entrance boundary. In our setting, the birth and death processes under \mathbb{P}_k and $\mathbb{P}_{k'}$ only differ by their initial conditions k and k' . Thus, we only need to check the equi-boundedness condition : for any $t > 0$,

$$\lim_{A \rightarrow +\infty} \liminf_{k \rightarrow +\infty} \mathbb{P}_k(X(t) \leq A) = 1. \quad (11)$$

For that purpose, we combine the first part of Proposition 2.1 and (2) to get $\mathbb{E}(\sum_{i \geq 1} \tau_i) < \infty$. Then, $\sum_{i \geq 1} \tau_i$ is a.s. finite and for any $t, \varepsilon > 0$, there exists $N_0 \geq 1$ such that

$$\mathbb{P}\left(\sum_{i=N_0}^{\infty} \tau_i \geq \frac{t}{2}\right) \leq \varepsilon. \quad (12)$$

We fix t and $\varepsilon > 0$. For $k \geq N_0$ and $A > 0$,

$$\begin{aligned} \mathbb{P}_k(X(t) \geq A) &\leq \mathbb{P}_k(T_{N_0} \geq t/2) + \mathbb{P}_k(X(t) \geq A, T_{N_0} < t/2) \\ &\leq \mathbb{P}_k\left(\sum_{i=N_0}^k \tau_i \geq t/2\right) + \sup_{s \in [0, t/2]} \mathbb{P}_{N_0}(X(t-s) \geq A) \\ &\leq \varepsilon + A^{-1} \sup_{s \in [t/2, t]} \mathbb{E}_{N_0}(X(s)), \end{aligned}$$

using (12) and the Markov inequality in the last inequality. Making A tend to infinity and recalling (10), we get (11) and the weak convergence of \mathbb{P}_k to \mathbb{P}_∞ . Using again (11) ensures that for any $t, \varepsilon > 0$, there exists A such that $\mathbb{P}_\infty(X(t) > A) \leq \varepsilon$. Therefore,

$$\mathbb{P}_\infty(\inf\{s \geq 0 : X(s) < +\infty\} > t) \leq \mathbb{P}_\infty(X(t) > A) \leq \varepsilon,$$

so that $\inf\{t \geq 0 : X(t) < +\infty\} = 0$ \mathbb{P}_∞ a.s. It ends the proof. \square

3 Behavior of T_n under \mathbb{P}_∞

From now on, we assume that the sequences $(\lambda_n)_n$ and $(\mu_n)_n$ satisfy the hypotheses (3), (4) and (5). Thus, according to Propositions 2.3 and 2.4, X comes down from infinity and \mathbb{P}_∞ is well-defined.

In this section, we study the asymptotic behavior of T_n as $n \rightarrow +\infty$ under \mathbb{P}_∞ by establishing a law of large numbers and a central limit theorem. Let us note that under \mathbb{P}_∞ , $T_n = \sum_{i \geq n} \tau_i$, so that (6) yields

$$\mathbb{E}_\infty(T_n) = \sum_{i \geq n} \frac{1}{\lambda_i \pi_i} \sum_{j \geq i+1} \pi_j.$$

Then $S < +\infty$ ensures that $\mathbb{E}_\infty(T_n)$ decreases to 0 as $n \rightarrow +\infty$.

In the following theorem, we prove that T_n behaves as its mean $\mathbb{E}_\infty(T_n)$ as $n \rightarrow +\infty$. Two regimes appear depending on whether the ratio of mean times $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n)$ vanishes or not. In the first case, the time T_n can be seen as the contribution of independent random variables and a law of large numbers holds. In the second case, the time T_n is essentially given by the sums of τ_i for i close to n and renormalizing T_n by its mean yields a random limit.

Theorem 3.1. *We assume that (3), (4) and (5) hold.*

(i) *If $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \xrightarrow{n \rightarrow +\infty} 0$, then*

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow{n \rightarrow +\infty} 1 \quad \text{in } \mathbb{P}_\infty\text{-probability.} \quad (13)$$

Assuming further that $\sum_{n \geq 0} (\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n))^2 < +\infty$, then (13) holds \mathbb{P}_∞ -a.s.

(ii) *If $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \xrightarrow{n \rightarrow +\infty} \alpha$ with $\alpha \in (0, 1]$, then*

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow{n \rightarrow +\infty} \stackrel{(d)}{Z} := \sum_{k \geq 0} \alpha (1 - \alpha)^k Z_k$$

where $(Z_k)_k$ is a sequence of i.i.d. random variables whose common Laplace transform $G(a) := \mathbb{E}_\infty(\exp(-aZ_0))$ is the unique function $[0, +\infty) \rightarrow [0, 1]$ that satisfies

$$\forall a > 0, \quad G(a) [l(1 - G(a(1 - \alpha))) + 1 + a(1 - l(1 - \alpha))] = 1. \quad (14)$$

We refer to Appendix C for some examples and counterexamples. For instance, if $\lambda_k = k$, then $\mu_k = k^\gamma \log(k)^\beta$ with $\gamma > 1$ obeys to the regime (i), whereas $\mu_k = \exp(\beta k)$ corresponds to the regime (ii). We also stress that $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n)$ may not converge (Example 3) and that the a.s. convergence can fail under the assumption (i) (Example 4).

Before proving Theorem 3.1, we need a lemma dealing with the asymptotic behaviors of the first moments of τ_n as $n \rightarrow +\infty$.

Lemma 3.2. *Under hypotheses (3), (4) and (5), there exist positive constants C_1, C_2, C_3 such that for $n \geq 1$*

$$\frac{i!}{\mu_{n+1}^i} \leq \mathbb{E}_\infty(\tau_n^i) \leq \frac{C_i}{\mu_{n+1}^i}, \quad i = 1, 2, 3.$$

Moreover, under the additional assumption $l = 0$, we have

$$\mathbb{E}_\infty(\tau_n^i) \underset{n \rightarrow +\infty}{\sim} \frac{i!}{\mu_{n+1}^i}, \quad i = 1, 2, 3.$$

Proof. By rewriting (6), we have $\mathbb{E}_\infty(\tau_n) = \sum_{i \geq n} \frac{1}{\mu_{i+1}} \prod_{j=n}^{i-1} \frac{\lambda_{j+1}}{\mu_{j+1}}$, with the convention $\prod_{j=n}^{n-1} \frac{\lambda_{j+1}}{\mu_{j+1}} = 1$. Thus, according to Lemma B.2 applied to $a_i = \lambda_{i+1}/\mu_{i+1}$ and $b_i = 1/\mu_{i+1}$, under (4), we obtain the expected bounds for the first moment ($i = 1$). Moreover

$$\sup_{n, k \geq 0} \frac{\mathbb{E}_\infty(\tau_{n+k})}{\mathbb{E}_\infty(\tau_n)} < \infty \quad (15)$$

and we can now deal with the second moment of τ_n . Differentiating (7) twice at $a = 0$, we get

$$\mathbb{E}_\infty(\tau_{n-1}^2) = \frac{\lambda_n}{\mu_n} \mathbb{E}(\tau_n^2) + 2 \mathbb{E}_\infty(\tau_{n-1})^2, \quad n \geq 1.$$

Adding that $(\mathbb{E}_\infty(\tau_n^2))_n$ is bounded from point (iv) of Proposition 2.3, that $\lim_{n \rightarrow +\infty} \lambda_n/\mu_n < 1$ and that $(\mathbb{E}_\infty(\tau_n))_n$ satisfies (15), another use of Lemma B.2 ensures the desired result for $i = 2$. Similarly, the case $i = 3$ is obtained by differentiating (7) three times. \square

Proof of Theorem 3.1(i). We use the notation

$$m_n = \mathbb{E}_{n+1}(T_n), \quad r_n := \frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} = \frac{m_n}{\mathbb{E}_\infty(T_n)}.$$

Assumption (i) means that $r_n \rightarrow 0$. Let $\varepsilon > 0$. Using Bienaymé-Tchebychev inequality and the independence of the random variables $(\tau_n)_n$, we have

$$\mathbb{P}_\infty \left(\left| \frac{T_n}{\mathbb{E}_\infty(T_n)} - 1 \right| > \varepsilon \right) \leq \frac{\text{Var}(T_n)}{\varepsilon^2 \mathbb{E}_\infty(T_n)^2} = \frac{\sum_{k \geq n} \text{Var}(\tau_k)}{\varepsilon^2 \mathbb{E}_\infty(T_n)^2}. \quad (16)$$

As $\mathbb{E}_\infty(T_{n+1})/m_n = 1/r_n - 1 \rightarrow +\infty$ as $n \rightarrow +\infty$, for all $A > 0$, there exists an integer n_0 such that, for $n \geq n_0$, $\mathbb{E}_\infty(T_{n+1}) \geq Am_n$ and

$$\mathbb{E}_\infty(T_n)^2 = \left(\sum_{k \geq n} m_k \right)^2 \geq 2 \sum_{k \geq n} m_k \sum_{l > k} m_l \geq 2A \sum_{k \geq n} m_k^2,$$

since $\sum_{l > k} m_l = \mathbb{E}_\infty(T_{k+1}) \geq Am_k$. Coming back to (16), for $n \geq n_0$, we have

$$\mathbb{P}_\infty \left(\left| \frac{T_n}{\mathbb{E}_\infty(T_n)} - 1 \right| > \varepsilon \right) \leq \frac{1}{2A\varepsilon^2} \frac{\sum_{k \geq n} \text{Var}(\tau_k)}{\sum_{k \geq n} m_k^2}. \quad (17)$$

Moreover, according to Lemma 3.2, for $n \geq 1$

$$\text{Var}_\infty(\tau_n) \leq \frac{C_2 - 1}{\mu_n^2} \leq (C_2 - 1)m_n^2.$$

Hence, the r.h.s. of (17) goes to 0 as $A \rightarrow +\infty$ and the proof of the convergence in probability is complete.

We now prove the a.s. convergence when the series $\sum_n r_n^2$ converges. According to the law of large numbers of Proposition 1 in [13], we just need to check that

$$\sum_{n \geq 0} \frac{\text{Var}(\tau_n)}{\mathbb{E}_\infty(T_n)^2} < +\infty. \quad (18)$$

From the first part of the proof, we know that $\text{Var}(\tau_n) \leq \widehat{C} \mathbb{E}(\tau_n)^2$ for some positive constant \widehat{C} . So $\sum_{n \geq 1} r_n^2 < +\infty$ ensures (18) and the proof is complete. \square

Before proving point (ii) of Theorem 3.1, we prove the following key lemma focusing on the asymptotic behavior of the time τ_n (recall that we denote its mean by m_n).

Lemma 3.3. *If $\lim_{n \rightarrow +\infty} r_n = \alpha \in (0, 1]$, we have*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)} = \lim_{n \rightarrow +\infty} \frac{m_{n+1}}{m_n} = 1 - \alpha, \quad \lim_{n \rightarrow +\infty} \mu_n m_{n-1} = \frac{1}{1 - l(1 - \alpha)}. \quad (19)$$

and

$$\frac{\tau_n}{m_n} \xrightarrow[n \rightarrow +\infty]{(d)} \zeta$$

where the Laplace transform of ζ is the unique solution of (14).

Proof. We obtain the first part of (19) by noticing that $\mathbb{E}_\infty(T_{n+1})/\mathbb{E}_\infty(T_n) = 1 - r_n$ and that

$$\frac{m_{n+1}}{m_n} = \frac{r_{n+1}}{r_n} \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)}.$$

Moreover, differentiating (7) at $a = 0$ yields

$$1 = \frac{\lambda_n}{\mu_n} \frac{m_n}{m_{n-1}} + \frac{1}{\mu_n m_{n-1}},$$

which gives the second part of (19) thanks to (3).

Let us now prove the uniqueness of the function satisfying Equation (14). For any bounded function $g : [0, +\infty) \rightarrow [0, 1]$, we define the function $H(g) : [0, +\infty) \rightarrow [0, 1]$ as

$$H(g) : a \mapsto \frac{1}{1 + l(1 - g(a(1 - \alpha))) + a(1 - l(1 - \alpha))}.$$

Let g_1 and g_2 be two solutions of (14). We then have $H(g_1) = g_1$, $H(g_2) = g_2$ and

$$\begin{aligned} |g_1(a) - g_2(a)| &= |H(g_1)(a) - H(g_2)(a)| = H(g_1)(a)H(g_2)(a)l |g_1(a(1 - \alpha)) - g_2(a(1 - \alpha))| \\ &\leq l |g_1(a(1 - \alpha)) - g_2(a(1 - \alpha))| \end{aligned}$$

where we have used that for any $a > 0$, $H(g_1)(a) \leq 1$. We then have $\|g_1 - g_2\|_\infty \leq l \|g_1 - g_2\|_\infty$ with $l < 1$, which entails that $g_1 = g_2$ and yields the expected uniqueness.

We can now prove the convergence in distribution of τ_n as $n \rightarrow +\infty$ by a tightness criterion. Indeed, for $n \geq 0$, let $H_n : [0, +\infty) \rightarrow [0, 1]$ be the function defined as

$$H_n(a) = \mathbb{E}_\infty(\exp(-a\tau_n/m_n)), \quad a > 0.$$

The sequence $(H_n)_n$ is uniformly bounded since $0 \leq H_n(a) \leq 1$ for every $n \geq 0$ and every $a > 0$. Moreover, for $n \geq 0$, H_n is differentiable and for $a > 0$,

$$|H'_n(a)| = \mathbb{E}_\infty\left(\frac{\tau_n}{m_n} \exp\left(-a\frac{\tau_n}{m_n}\right)\right) \leq \frac{\mathbb{E}_\infty(\tau_n)}{m_n} = 1.$$

Hence, the family $(H_n)_n$ is equicontinuous since all these functions are 1-Lipschitzian functions. Then, thanks to Arzelà-Ascoli theorem, $(H_n)_{n \geq 0}$ is relatively compact.

We now need to check that $(H_n)_n$ has a unique limit point. Let us prove that if a subsequence of (H_n) converges to H uniformly on any compact set of $[0, +\infty)$, then H satisfies (14) and is then uniquely defined. For that purpose, we use (7), so for all $a > 0$ and $n \geq 1$, we have

$$G_{n-1} \left(\frac{a}{m_{n-1}} \right) = \left[1 + \frac{a}{\mu_n m_{n-1}} + \frac{\lambda_n}{\mu_n} \left(1 - G_n \left(\frac{a}{m_{n-1}} \right) \right) \right]^{-1}$$

that is,

$$H_{n-1}(a) = \left[1 + \frac{a}{\mu_n m_{n-1}} + \frac{\lambda_n}{\mu_n} \left(1 - H_n \left(a \frac{m_n}{m_{n-1}} \right) \right) \right]^{-1}. \quad (20)$$

According to Lemma 3.3, $m_n/m_{n-1} \rightarrow 1 - \alpha$ as $n \rightarrow +\infty$. Thus, if a subsequence (also denoted by H_n for simplicity) converges to H uniformly, we have for every $a > 0$

$$\lim_{n \rightarrow +\infty} H_n \left(a \frac{m_n}{m_{n-1}} \right) = H(a(1 - \alpha)).$$

Letting $n \rightarrow +\infty$ in (20), since $\mu_n m_{n-1} \rightarrow 1/(1 - l(1 - \alpha))$ and $\lambda_n/\mu_n \rightarrow l$, H satisfies (14) and for every $a > 0$

$$\lim_{n \rightarrow +\infty} \mathbb{E}_\infty (\exp(-a\tau_n/m_n)) = H(a).$$

Finally, we check that H is the Laplace transform of some random variable by proving that $H(0^+) := \lim_{a \rightarrow 0} H(a) = 1$. From (14), $H(0^+)$ is a solution of $lH(0^+)^2 - (1+l)H(0^+) + 1 = 0$. If $l = 0$, this equation has only 1 as a solution. If $l > 0$, the two solutions are 1 and $1/l$. But $1/l > 1$ and obviously $H(0^+) \leq 1$, so that 1 is the only possible solution. Hence, in all cases, $H(0^+) = 1$ and that ends the proof. \square

We can now proceed with the proof of the second part of the theorem.

Proof of Theorem 3.1 (ii). Let $Z = \sum_{k \geq 0} \alpha(1 - \alpha)^k Z_k$ be defined as in the statement of the theorem. We use that $T_n = \sum_{k \geq n} \tau_k$ where the τ_k 's are independent and that for all $a_1, a_2, \dots, a_n, b_1, \dots, b_n \in [0, 1]$, a simple recursion ensures that

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right| \leq \sum_{i=1}^n |a_i - b_i|. \quad (21)$$

Then, for every $a > 0$

$$\begin{aligned} & \left| \mathbb{E}_\infty \left(\exp \left(-a \frac{T_n}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}_\infty (\exp(-aZ)) \right| \\ &= \left| \prod_{k \geq n} \mathbb{E}_\infty \left(\exp \left(-a \frac{\tau_k}{\mathbb{E}_\infty(T_n)} \right) \right) - \prod_{k \geq 0} \mathbb{E}_\infty \left(\exp \left(-a\alpha(1 - \alpha)^k Z_k \right) \right) \right| \\ &\leq \sum_{k \geq 0} \left| \mathbb{E}_\infty \left(\exp \left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}_\infty \left(\exp \left(-a\alpha(1 - \alpha)^k Z_k \right) \right) \right|. \end{aligned} \quad (22)$$

From Lemma 3.3, we know that in \mathbb{P}_∞ -distribution, τ_n/m_n converges to ζ . Then, thanks to (19) and the fact that $r_n \rightarrow \alpha$, we have for $k \geq 0$

$$\frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} = \frac{m_{n+k}}{\mathbb{E}_\infty(T_{n+k})} \prod_{i=1}^k \frac{\mathbb{E}_\infty[T_{n+i}]}{\mathbb{E}_\infty[T_{n+i-1}]} \cdot \frac{\tau_{k+n}}{m_{n+k}} \xrightarrow[n \rightarrow +\infty]{(d)} \alpha(1 - \alpha)^k \zeta.$$

The uniqueness in (14) ensures that the variables $(Z_k)_k$ are distributed as ζ . Then, with the last display, we get that all the terms of the sum in (22) vanish as $n \rightarrow +\infty$. We proceed by bounded convergence. Using that $1 - \exp(-x) \leq x$ for any $x \geq 0$, we get for $k, n \geq 0$

$$\begin{aligned} & \left| \mathbb{E}_\infty \left(\exp \left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}_\infty \left(\exp \left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right| \\ & \leq \left| 1 - \mathbb{E}_\infty \left(\exp \left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) \right| + \left| 1 - \mathbb{E}_\infty \left(\exp \left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right| \\ & \leq a \frac{m_{k+n}}{\mathbb{E}_\infty(T_n)} + a\alpha(1-\alpha)^k \mathbb{E}_\infty(Z_0). \end{aligned} \quad (23)$$

By differentiating (14) at 0, one finds $\mathbb{E}_\infty[Z_0] = 1$. Moreover,

$$\frac{m_{n+k}}{\mathbb{E}_\infty(T_n)} = \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)} \frac{\mathbb{E}_\infty(T_{n+2})}{\mathbb{E}_\infty(T_{n+1})} \cdots \frac{\mathbb{E}_\infty(T_{n+k})}{\mathbb{E}_\infty(T_{n+k-1})} \frac{m_{n+k}}{\mathbb{E}_\infty(T_{n+k})}.$$

Since $m_{k+n}/\mathbb{E}_\infty[T_{k+n}] \leq 1$ and $\mathbb{E}_\infty(T_{n+1})/\mathbb{E}_\infty(T_n) \rightarrow 1 - \alpha < 1$ as $n \rightarrow +\infty$, there exist $n_0 \in \mathbb{N}$, $\beta < 1$ and $C > 0$ such that $m_{k+n}/\mathbb{E}_\infty(T_n) \leq C\beta^k$ for all $k \geq 0, n \geq n_0$. Thus, coming back to (23), for $n \geq n_0$, we have

$$\left| \mathbb{E}_\infty \left(\exp \left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}_\infty \left(\exp \left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right| \leq C\beta^k + a\alpha(1-\alpha)^k.$$

Since the r.h.s. in the last display is summable, the proof is complete. \square

We end this section by giving a central limit theorem (C.L.T.) satisfied by the sequence $(T_n)_n$.

Theorem 3.4. *We suppose that assumptions (3), (4) and (5) hold. Then, if*

$$\lim_{n \rightarrow +\infty} \frac{\text{Var}_\infty(\tau_n)}{\text{Var}_\infty(T_n)} = 0 \quad (24)$$

and if

$$\lim_{n \rightarrow +\infty} \text{Var}_\infty(T_n)^{-3/2} \sum_{k \geq n} \mathbb{E}_\infty \left(|\tau_k - \mathbb{E}_\infty(\tau_k)|^3 \right) = 0, \quad (25)$$

we have

$$\frac{T_n - \mathbb{E}_\infty(T_n)}{\text{Var}_\infty(T_n)^{1/2}} \xrightarrow[n \rightarrow +\infty]{(d)} N,$$

where N follows a standard normal distribution.

Notice that by applying Lemma 3.2 and by using assumption (4), there is $C > 0$ such that $\frac{\mathbb{E}_\infty(\tau_n)}{\mathbb{E}_\infty(T_n)} \leq C \frac{\text{Var}_\infty(\tau_n)}{\text{Var}_\infty(T_n)}$. Thus, hypothesis (24) implies that we are in the regime (i) of Theorem 3.1. We refer to the first example in Appendix.

Proof of Theorem 3.4. First, Lemma 3.2 gives that for every $n \geq 0$, $\text{Var}_\infty(\tau_n) \leq \frac{C_2 - 1}{\mu_{n+1}^2}$. Recalling Assumption (4), we get for $k, n \geq 0$

$$\text{Var}_\infty(\tau_{k+n}) \leq \frac{C_2 - 1}{\mu_{n+k+1}^2} \leq \frac{(C_2 - 1) K^2}{\mu_{n+1}^2} \leq (C_2 - 1) K^2 \text{Var}_\infty(\tau_n)$$

where $K = \sup_{k,n \geq 1} \mu_n / \mu_{n+k}$. Thus assumption (24) entails the uniform convergence

$$\sup_{k \geq 0} \frac{\text{Var}_\infty(\tau_{k+n})}{\text{Var}_\infty(T_n)} \xrightarrow{n \rightarrow +\infty} 0. \quad (26)$$

Let us now prove that

$$Z_n := \frac{T_n - \mathbb{E}_\infty(T_n)}{\text{Var}_\infty(T_n)^{1/2}}$$

converges in distribution as $n \rightarrow +\infty$ toward a standard normal random variable. We follow ideas of the proof of Theorem 27.2 in [5] where Billingsley establishes a central limit theorem for partial sums of independent random variables thanks to Lévy theorem. Let t be a fixed real number. We note that by (26),

$$\sum_{k \geq 0} \log \left(1 - \frac{t^2 \text{Var}_\infty(\tau_{k+n})}{2 \text{Var}_\infty(T_n)} \right) \underset{n \rightarrow +\infty}{\sim} -\frac{t^2}{2} \sum_{k \geq 0} \frac{\text{Var}_\infty(\tau_{k+n})}{\text{Var}_\infty(T_n)} = -\frac{t^2}{2},$$

so we just need to prove that

$$U_n := \mathbb{E}_\infty(\exp(itZ_n)) - \prod_{k \geq 0} \left(1 - \frac{t^2 \text{Var}_\infty(\tau_{k+n})}{2 \text{Var}_\infty(T_n)} \right)$$

vanishes as $n \rightarrow +\infty$ to conclude. First, since the τ_n 's are independent, for all $t \in \mathbb{R}, n \geq 0$

$$|U_n| = \left| \prod_{k \geq 0} \mathbb{E}_\infty \left(\exp \left(it \frac{\tau_{k+n} - \mathbb{E}_\infty(\tau_{k+n})}{\text{Var}_\infty(T_n)^{1/2}} \right) \right) - \prod_{k \geq 0} \left(1 - \frac{t^2 \text{Var}_\infty(\tau_{k+n})}{2 \text{Var}_\infty(T_n)} \right) \right|. \quad (27)$$

According to (26), for n large enough and for any k , all the factors of the second product of (27) are less than 1. Hence, thanks to (21), we have the inequality

$$|U_n| \leq \sum_{k \geq 0} \left| \mathbb{E}_\infty \left(\exp \left(it \frac{\tau_{k+n} - \mathbb{E}_\infty(\tau_{k+n})}{\text{Var}_\infty(T_n)^{1/2}} \right) \right) - 1 + \frac{t^2 \text{Var}_\infty(\tau_{k+n})}{2 \text{Var}_\infty(T_n)} \right|. \quad (28)$$

According to equation (27.11) in [5, p.369], for any centered random variable ξ with a finite second moment, we have $|\mathbb{E}(\exp(it\xi)) - 1 + \text{Var}(\xi)t^2/2| \leq \mathbb{E}(\min(|t\xi|^2, |t\xi|^3))$, $t \geq 0$. Using this inequality with the random variables $\tau_{n+k} - \mathbb{E}_\infty(\tau_{n+k})$, we obtain from (28) that

$$|U_n| \leq |t|^3 \sum_{k \geq 0} \frac{\mathbb{E}_\infty(|\tau_{k+n} - \mathbb{E}_\infty(\tau_{k+n})|^3)}{\text{Var}_\infty(T_n)^{3/2}}.$$

and using assumption (25), U_n goes to 0 as $n \rightarrow +\infty$. Is completes the proof. \square

4 Behavior of $X(t)$ as t goes to 0

From the results of Section 3, we can describe the behavior of X for small times, when it starts at $+\infty$.

4.1 Law of large numbers

We first prove that under \mathbb{P}_∞ , $X(t)$ behaves as $v(t)$ as $t \rightarrow 0$ where

$$v(t) := \inf\{n \geq 0; \mathbb{E}_\infty(T_n) \leq t\} \quad (29)$$

is the generalized inverse function of $n \mapsto \mathbb{E}_\infty(T_n) = \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i$.

The function v is a non-increasing function which tends to infinity when t tends to 0.

Two asymptotic behaviors appear, which are inherited from Theorem 3.1. First, we assume that $\lambda_n/\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and that the death rate regularly varies in the neighborhood of $+\infty$, see Section A for details. Indeed, it ensures that the a.s. convergence of Theorem 3.1(i) holds and these assumptions are then essential to derive the behavior of X from that $(T_n : n \in \mathbb{N})$.

Theorem 4.1. (i) *If $\lim_{n \rightarrow +\infty} \lambda_n/\mu_n = 0$ and $(\mu_n)_n$ regularly varies with index $\rho > 1$, we have*

$$\lim_{t \rightarrow 0} \frac{X(t)}{v(t)} = 1 \quad \mathbb{P}_\infty - \text{a.s.}$$

(ii) *Under assumptions (3), (4) and (5) and if $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \rightarrow \alpha > 0$ as $n \rightarrow +\infty$, then*

$$\lim_{t \rightarrow 0} \frac{X(t)}{v(t)} = 1 \quad \text{in } \mathbb{P}_\infty - \text{probability.}$$

We refer to Appendix for some examples. Further, we remark that if $\lambda_n = 0$ and $\mu_n = n(n-1)/2$, $X(t)$ is the number of blocks of the Kingman coalescent at time t . We recover here from (i) the speed of coming down from infinity obtained for these processes by Aldous in paragraph 4.2. of [1]: $tX(t) \xrightarrow[t \rightarrow 0]{} 2$ a.s..

The extension to the general case of Λ -coalescent has been solved by Berestycki, Berestycki and Limic [4], but it is not directly included in our work for simultaneous deaths.

Proof of Theorem 4.1(i). First, we notice that the hypotheses of point (i) imply that assumptions (3), (4) and (5) with $l = 0$ are all satisfied. We now prove that $\sum_n (\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n))^2 < +\infty$ to get the a.s. convergence from Theorem 3.1(i).

Since $l = 0$ and according to Lemma 3.2, $m_n \sim_{n \rightarrow +\infty} 1/\mu_{n+1}$, which implies that $(m_n)_n$ regularly varies at $+\infty$ with index $-\rho < -1$. Then, according to Lemma A.4 in Appendix applied to $(1/\mu_n)_n$, we have

$$\frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} \underset{n \rightarrow +\infty}{\sim} \left(\mu_{n+1} \sum_{k \geq n+1} \frac{1}{\mu_k} \right)^{-1} \underset{n \rightarrow +\infty}{\sim} \frac{\rho - 1}{n + 1},$$

which entails that $\sum_n \left(\frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} \right)^2 < +\infty$. So Theorem 3.1(i) yields

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \underset{n \rightarrow +\infty}{\longrightarrow} 1 \quad \mathbb{P}_\infty - \text{a.s.} \quad (30)$$

The proof is now organized as follows: firstly we consider the a.s. non-increasing process Y

defined by

$$Y(t) = n \quad \text{if } t \in [T_n, T_{n-1})$$

and prove that this (more regular) process comes down from infinity at speed $v(t)$. Secondly, we compare the process $X(t)$ to $Y(t)$ as $t \rightarrow 0$ to get the result.

Thanks to Proposition A.3, v regularly varies at 0 with index $1/(1 - \rho)$ and $v(\mathbb{E}_\infty(T_n)) \sim n$ as $n \rightarrow +\infty$. Thus, from (30) and Lemma A.5 we obtain that almost surely

$$v(T_n) \underset{n \rightarrow +\infty}{\sim} v(\mathbb{E}_\infty(T_n)) \underset{n \rightarrow +\infty}{\sim} n.$$

Adding that v is non-increasing, we get a.s. that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$1 - \varepsilon \leq \frac{n}{v(T_n)} \leq \frac{n}{v(T_{n-1})} \leq 1 + \varepsilon.$$

Let $t < T_{n_0}$ so that $Y(t) > n_0$, then if $t \in [T_n, T_{n-1})$,

$$1 - \varepsilon \leq \frac{n}{v(T_n)} \leq \frac{Y(t)}{v(t)} \leq \frac{n}{v(T_{n-1})} \leq 1 + \varepsilon.$$

That ensures

$$\lim_{t \rightarrow 0} \frac{Y(t)}{v(t)} = 1 \quad \text{a.s.} \quad (31)$$

Let us now check that $X(t) \sim Y(t)$ as $t \rightarrow 0$ by proving that the heights of the excursions of X between T_n and T_{n-1} are negligible compared to n . For that purpose, we introduce the number of birth events between the times T_n and T_{n-1} :

$$H_n := \#\{s \in [T_n, T_{n-1}) : X(s) - X(s-) > 0\}, \quad n \geq 1.$$

For any $t \in [T_n, T_{n-1})$, $Y(t) = n$ and $0 \leq X(t) - Y(t) \leq H_n$, so

$$0 \leq \frac{X(t)}{v(t)} - \frac{Y(t)}{v(t)} \leq \frac{H_n}{v(t)} = \frac{H_n}{Y(t)} \frac{Y(t)}{v(t)}. \quad (32)$$

Using (31), we just need to prove that $H_n/n \rightarrow 0$ a.s. as $n \rightarrow \infty$ to conclude that $X(t)/v(t) \rightarrow 1$ as $t \rightarrow 0$.

For that purpose, we consider $\widehat{G}_n(a) = \mathbb{E}_\infty(\exp(-aH_n))$ the Laplace transform of H_n . In the same vein as we have obtained (7), by applying the strong Markov property at the first time when X jumps after T_n , we have the recursion formula

$$\widehat{G}_n(a) = \frac{\mu_n}{\lambda_n + \mu_n} + \frac{\lambda_n}{\lambda_n + \mu_n} e^{-a} \widehat{G}_n(a) \widehat{G}_{n+1}(a), \quad a \geq 0, \quad n \geq 1. \quad (33)$$

Differentiating (33) twice at $a = 0$, the second moment of H_n satisfies the following recursion formula

$$\frac{\mu_n}{\lambda_n} \mathbb{E}_\infty(H_n^2) = \mathbb{E}_\infty(H_{n+1}^2) + 1 + 2(\mathbb{E}_\infty(H_n) + \mathbb{E}_\infty(H_{n+1}) + \mathbb{E}_\infty(H_n) \mathbb{E}_\infty(H_{n+1})). \quad (34)$$

Let us prove that the right hand side of the latter is uniformly bounded in $n \geq 0$. Notice that H_n equals the number of positive jumps between time T_n and T_{n-1} of a random walk whose

transition probabilities are given by $p_{i,i+1} = \lambda_i/(\lambda_i + \mu_i)$, $p_{i,i-1} = \mu_i/(\lambda_i + \mu_i)$ for $i \geq 1$. Since λ_n/μ_n vanishes as $n \rightarrow +\infty$, one can choose n_0 large enough so that

$$p := \sup_{n \geq n_0} \lambda_n/(\lambda_n + \mu_n) < 1/2.$$

Moreover, for $n \geq n_0$, H_n is stochastically dominated by T , the hitting time of $n-1$ by a simple random walk starting at n , with probability transitions $(1-p, p)$. Thus, $\sup_{n \geq n_0} \mathbb{E}_\infty(H_n^2) \leq \mathbb{E}_\infty(T^2) < \infty$ because $p < 1/2$ and the sequences $(\mathbb{E}_\infty(H_n))_n$ and $(\mathbb{E}_\infty(H_n^2))_n$ are bounded. It entails that the right hand side of (34) is bounded and there is $C > 0$ such that

$$\mathbb{E}_\infty(H_n^2) \leq C \frac{\lambda_n}{\mu_n}, \quad n \geq 1. \quad (35)$$

Finally, using that $l = 0$, $\mathbb{E}_\infty\left(\sum_{n \geq 1} \left(\frac{H_n}{n}\right)^2\right) \leq C \sum_{n \geq 1} \frac{1}{n^2} \frac{\lambda_n}{\mu_n} < \infty$. In particular, H_n/n almost surely goes to 0 as $n \rightarrow +\infty$ and we get the expected convergence. \square

Proof of Theorem 4.1(ii). From Theorem 3.1(ii), we know that under \mathbb{P}_∞ ,

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow[n \rightarrow +\infty]{(d)} Z = \sum_{k \geq 0} \alpha (1-\alpha)^k Z_k$$

with $\alpha \in (0, 1]$ and where $(Z_k)_k$ is a sequence of i.i.d. random variables whose Laplace transform satisfies (14). From this equation, one deduces $\mathbb{E}_\infty[Z_0] = 1$, which implies $\mathbb{E}_\infty(Z) = 1 < \infty$. In particular, $\mathbb{P}_\infty(Z < +\infty) = 1$. Furthermore, using again (14),

$$\mathbb{P}_\infty(Z = 0) \leq \mathbb{P}_\infty(Z_0 = 0) = \lim_{a \rightarrow +\infty} G(a) = 0.$$

Hence, for any $\epsilon > 0$, there exist $0 < A \leq B$ such that $\mathbb{P}_\infty(Z \in [A, B]) \geq 1 - \epsilon/2$ and for n large enough,

$$\mathbb{P}_\infty(A \leq T_n/\mathbb{E}_\infty(T_n) \leq B) \geq 1 - \epsilon. \quad (36)$$

Moreover, according to (19), for $N \geq 0$,

$$\frac{\mathbb{E}_\infty[T_{n+N}]}{\mathbb{E}_\infty(T_n)} \xrightarrow[n \rightarrow +\infty]{} (1-\alpha)^N.$$

Since $0 < \alpha \leq 1$, there exists N_0 such that for all $n, N \geq N_0$,

$$\mathbb{E}_\infty[T_{n+N}]/\mathbb{E}_\infty(T_n) \leq \min(1/(2B), A/2). \quad (37)$$

By the definition (29) of the function v , we have

$$\mathbb{E}_\infty(T_{v(t)}) \leq t < \mathbb{E}_\infty(T_{v(t)-1}).$$

It implies that for any $t > 0$ and $N \geq 1$,

$$\mathbb{P}_\infty\left(T_{v(t)+N} \leq \frac{t}{2}\right) \geq \mathbb{P}_\infty\left(T_{v(t)+N} \leq \frac{\mathbb{E}_\infty(T_{v(t)})}{2}\right) = \mathbb{P}_\infty\left(\frac{T_{v(t)+N}}{\mathbb{E}_\infty(T_{v(t)+N})} \leq \frac{\mathbb{E}_\infty(T_{v(t)})}{2\mathbb{E}_\infty(T_{v(t)+N})}\right).$$

Hence, using (36) and (37), there is $N \geq 1$ such that for t small enough

$$\mathbb{P}_\infty(T_{v(t)+N} \leq t/2) \geq \mathbb{P}_\infty\left(\frac{T_{v(t)+N}}{\mathbb{E}_\infty(T_{v(t)+N})} \leq B\right) \geq 1 - \epsilon.$$

We similarly get that for t small enough

$$\mathbb{P}_\infty(T_{v(t)-N} \geq 2t) \geq \mathbb{P}_\infty\left(\frac{T_{v(t)-N}}{\mathbb{E}_\infty(T_{v(t)-N})} \geq A\right) \geq 1 - \epsilon.$$

Then, we have

$$\mathbb{P}_\infty(T_{v(t)+N} \leq t/2, T_{v(t)-N} \geq 2t) \geq 1 - 2\epsilon.$$

It means that $\mathbb{P}_\infty(X(t) \in [v(t) - N, v(t) + N]) \geq 1 - 2\epsilon$ and ensures that $X(t)/v(t)$ tends to 1 in probability as $t \rightarrow 0$. \square

4.2 Central limit theorem

We have proved that X satisfies a strong law of large numbers when $l = 0$ and $(\mu_n)_n$ regularly varies. Under a little stronger assumption, we are now giving a central limit theorem (C.L.T.).

Theorem 4.2. *If $\lim_{n \rightarrow +\infty} \frac{\lambda_n}{\mu_n} = 0$, $\sum_{n \geq 1} \frac{1}{n} \frac{\lambda_n}{\mu_n} < +\infty$ and $(\mu_n)_n$ regularly varies with index $\rho > 1$, then*

$$\sqrt{(2\rho - 1)v(t)} \left(\frac{X(t)}{v(t)} - 1 \right) \xrightarrow[t \rightarrow 0]{(d)} N, \quad (38)$$

where N follows a standard normal distribution.

Proof. We first prove that under the assumptions of Theorem 4.2, T_n satisfies the C.L.T. stated in Theorem 3.4. We have already shown at the beginning of the proof of Theorem 4.1(i) that assumptions (3), (4) and (5) hold if $l = 0$ and $(\mu_n)_n$ regularly varies. It then remains to check that (24) and (25) are also satisfied. From Lemma 3.2, $\text{Var}_\infty(\tau_n) \sim 1/\mu_{n+1}^2$ as $n \rightarrow +\infty$, which implies that $(\text{Var}_\infty(\tau_n))_n$ regularly varies with index -2ρ . Then the fact that $\text{Var}_\infty(T_n) = \sum_{i=n+1}^{\infty} \text{Var}_\infty(\tau_i)$ and Lemma A.4 ensure that $\text{Var}_\infty(T_n)$ regularly varies with index $1 - 2\rho$ and we get

$$\text{Var}_\infty(T_n) \sim \frac{1}{(1 - 2\rho)\mu_{n+1}^2}. \quad (39)$$

Therefore we have $\frac{\text{Var}_\infty(\tau_n)}{\text{Var}_\infty(T_n)} \underset{n \rightarrow +\infty}{\sim} \frac{1 - 2\rho}{n}$, which entails (24). By the triangular inequality and the binomial theorem, we have

$$\mathbb{E}_\infty\left(|\tau_n - \mathbb{E}_\infty(\tau_n)|^3\right) \leq \mathbb{E}_\infty(\tau_n^3) + 3m_n \mathbb{E}_\infty(\tau_n^2) + 4m_n^3.$$

Thanks to Lemma 3.2, all the terms of the r.h.s. are of order of magnitude $1/\mu_{n+1}^3$ as $n \rightarrow +\infty$. Thus, using again Lemma A.4 and (39), there is a positive constant C' such that

$$\frac{\sum_{k \geq n} \mathbb{E}_\infty\left(|\tau_k - \mathbb{E}_\infty(\tau_k)|^3\right)}{\text{Var}_\infty(T_n)^{3/2}} \leq C' \frac{2\rho - 1}{(3\rho - 1)\sqrt{n}}$$

and vanishes as $n \rightarrow +\infty$. So (25) holds and T_n satisfies the following C.L.T.

$$\tilde{Z}_n := \frac{T_n - \mathbb{E}_\infty(T_n)}{\sqrt{\text{Var}_\infty(T_n)}} \xrightarrow[n \rightarrow +\infty]{(d)} N. \quad (40)$$

We now prove that X satisfies the C.L.T. (38). To do so, we first establish a C.L.T. for the process Y where we recall from the proof of Theorem 4.1(i) that Y denotes the a.s. non-increasing process defined as $Y(t) = n$ if $t \in [T_n, T_{n-1})$. This process is more tractable than X and we will derive the C.L.T. for X from that of Y .

As Y is non-increasing, we can follow the proof of C.L.T for renewal processes (as suggested by Aldous for Kingman's coalescent, cf. [1]). More precisely, for any $t > 0, x \in \mathbb{R}$, we have

$$\mathbb{P}_\infty\left(\sqrt{v(t)}\left(\frac{Y(t)}{v(t)} - 1\right) \geq x\right) = \mathbb{P}_\infty(Y(t) \geq s_x(t)) = \mathbb{P}_\infty(T_{s_x(t)} \geq t)$$

where we denote $s_x(t) := \lfloor v(t) + x\sqrt{v(t)} \rfloor$ ($\lfloor \cdot \rfloor$ is the floor function). We then have

$$\mathbb{P}_\infty\left(\sqrt{v(t)}\left(\frac{Y(t)}{v(t)} - 1\right) \geq x\right) = \mathbb{P}_\infty\left(\tilde{Z}_{s_x(t)} \geq \frac{t - \mathbb{E}_\infty(T_{s_x(t)})}{\sqrt{\text{Var}_\infty(T_{s_x(t)})}}\right).$$

Using (40), we just need to prove that

$$\frac{t - \mathbb{E}_\infty(T_{s_x(t)})}{\sqrt{\text{Var}_\infty(T_{s_x(t)})}} \xrightarrow[t \rightarrow 0]{} x\sqrt{2\rho - 1} \quad (41)$$

to obtain the expected C.L.T. for Y . From the definition (29) of the function v , we have

$$\mathbb{E}_\infty(T_{v(t)}) - \mathbb{E}_\infty(T_{s_x(t)}) \leq t - \mathbb{E}_\infty(T_{s_x(t)}) \leq \mathbb{E}_\infty(T_{v(t)-1}) - \mathbb{E}_\infty(T_{s_x(t)}). \quad (42)$$

Let us first deal with the l.h.s. and write

$$\mathbb{E}_\infty(T_n) - \mathbb{E}_\infty(T_{\lfloor n+x\sqrt{n} \rfloor}) = \sum_{k=n}^{\lfloor n+x\sqrt{n} \rfloor - 1} m_k \underset{n \rightarrow +\infty}{\sim} x\sqrt{n}m_n.$$

Indeed,

$$\begin{aligned} \left| \sum_{k=n}^{\lfloor n+x\sqrt{n} \rfloor - 1} m_k - (\lfloor n+x\sqrt{n} \rfloor - n)m_n \right| &\leq m_n \sum_{k=n}^{\lfloor n+x\sqrt{n} \rfloor - 1} \left| \frac{m_k}{m_n} - 1 \right| \\ &\leq x\sqrt{n}m_n \sup_{u \in [0, x]} \left| \frac{m_{\lfloor n+u\sqrt{n} \rfloor}}{m_n} - 1 \right|, \end{aligned}$$

and the second part of Lemma A.5 ensures that the latter supremum vanishes as $n \rightarrow +\infty$ since m_n regularly varies.

Then, by successively applying Lemma A.5 with $f(y) = y, g(y) = \lfloor y + u\sqrt{y} \rfloor$ and $h(n) = \text{Var}_\infty(T_n)$, Lemma 3.2, Lemma A.4(i) and (39), we have the equivalences

$$\frac{\mathbb{E}_\infty(T_n) - \mathbb{E}_\infty(T_{\lfloor n+x\sqrt{n} \rfloor})}{\sqrt{\text{Var}_\infty(T_{\lfloor n+x\sqrt{n} \rfloor})}} \underset{n \rightarrow +\infty}{\sim} \frac{x\sqrt{n}m_n}{\sqrt{\text{Var}_\infty(T_n)}} \underset{n \rightarrow +\infty}{\sim} x\sqrt{2\rho - 1}.$$

Following the same steps for the r.h.s of (42) ensures that (41) holds. We end the proof by deducing (38) from the C.L.T. satisfied by Y . Indeed, since

$$\sqrt{v(t)} \left(\frac{X(t)}{v(t)} - 1 \right) = \sqrt{v(t)} \left(\frac{Y(t)}{v(t)} - 1 \right) + \frac{X(t) - Y(t)}{\sqrt{v(t)}},$$

it is now sufficient to show that the second term of the latter goes to 0 in probability as $t \rightarrow 0$. Keeping the same notation as in the proof of Theorem 4.1(i), from (32), we almost surely have

$$0 \leq \frac{X(t) - Y(t)}{\sqrt{v(t)}} \leq \frac{H_{Y(t)}}{\sqrt{Y(t)}} \frac{\sqrt{Y(t)}}{\sqrt{v(t)}}. \quad (43)$$

From (35), there is C such that

$$\mathbb{E}_\infty \left(\sum_{n \geq 1} \left(\frac{H_n}{\sqrt{n}} \right)^2 \right) \leq C \sum_{n \geq 1} \frac{1}{n} \frac{\lambda_n}{\mu_n}.$$

Since this series converges by hypothesis, H_n/\sqrt{n} almost surely goes to 0 as $n \rightarrow +\infty$. Hence, since we also have $Y(t) \sim v(t)$ as $t \rightarrow 0$ with probability 1, the r.h.s. of (43) vanishes as $t \rightarrow 0$, which ends up the proof. \square

5 Tail distribution at 0 of the extinction time

In the following result, we focus on the probability that the extinction of the process X occurs for small times.

Theorem 5.1. (i) *If for every $n \geq 0$, $\lambda_n = 0$ (pure death case) and if (μ_n) regularly varies at $+\infty$ with index $\rho > 1$, then*

$$t \longmapsto -\log \mathbb{P}_\infty(T_0 \leq t)$$

regularly varies at 0 with index $1/(1 - \rho)$.

(ii) *If $\lim_{n \rightarrow +\infty} \frac{\lambda_n}{\mu_n} = 0$ and $(\mu_n)_n$ regularly varies with index $\rho > 1$,*

$$\frac{\log(-\log \mathbb{P}_\infty(T_0 \leq t))}{\log t} \xrightarrow{t \rightarrow 0} \frac{1}{1 - \rho}.$$

In the pure death case (i), the time of extinction is the sum of independent exponential random variables. That allows us to get an explicit expression of its Laplace exponent and the result comes from a Tauberian theorem (Lemma A.6). It's a key point where we needed the regular variation of $(\mu_n)_n$. To prove (ii), we first use the speed of coming down from infinity obtained in Theorem 4.1 on time interval $[0, t/2]$. Then we compare the trajectory with the pure death case via a coupling argument and conclude thanks to (i).

Proof of Theorem 5.1(i). Let ϕ denote the Laplace transform of T_0

$$\phi(a) := \mathbb{E}_\infty [\exp(-aT_0)], \quad a > 0.$$

Let us prove that $-\log \phi$ regularly varies with index $1/\rho$ at $+\infty$. Lemma A.6 will thus imply the result.

In the pure death case, the times τ_i are independent exponential random variables with respective parameters μ_{i+1} . Then, for $a > 0$,

$$\phi(a) = \prod_{i \geq 0} \mathbb{E}[\exp(-a\tau_i)] = \prod_{i \geq 1} \frac{1}{1 + a/\mu_i}.$$

Using that $\log(1 + a) = a(1 + r(a))$ with r satisfying $\lim_{a \rightarrow 0} r(a) = 0$, we write

$$-\log \phi(a) = \sum_{i: \mu_i \leq a \log a} \log(1 + a/\mu_i) + (1 + R(a)) \sum_{i: \mu_i \geq a \log a} a/\mu_i,$$

where

$$0 \leq R(a) \leq \sup_{y \in [0, 1/\log a]} r(y) \xrightarrow{a \rightarrow +\infty} 0.$$

Moreover, since $(\mu_i)_i$ regularly varies with index ρ , we know (cf. [6, Thm 1.5.3]) that the increasing sequence $(\inf_{n \geq i} \mu_n)_i$ regularly varies with index ρ . Therefore, according to Proposition A.3, the application

$$a \mapsto i_a := \min\{i \in \mathbb{N} : \inf_{n \geq i} \mu_n \geq a \log a\}$$

is regularly varying at ∞ with index $1/\rho$. Then,

$$h_1(a) := \sum_{i: \mu_i \geq a \log a} \frac{a}{\mu_i} = a \sum_{i \geq i_a} \frac{1}{\mu_i}$$

is regularly varying at ∞ with index $1 + (1 - \rho)/\rho = \rho$ since $\sum_{i \geq i_a} 1/\mu_i$ is the composition of the two functions $a \mapsto i_a$ and $n \mapsto \sum_{i \geq n} 1/\mu_i$, which both regularly vary with respective indices $1/\rho$ and $1 - \rho$.

Furthermore, for $a \geq 1$, we have

$$h_2(a) = \sum_{i \leq i_a} \log(1 + a/\mu_i) = \sum_{i \leq i_a} \left(\log a + \log \left(\frac{1}{a} + \frac{1}{\mu_i} \right) \right) = i_a \log a + \sum_{i \leq i_a} \log \left(\frac{1}{a} + \frac{1}{\mu_i} \right).$$

The second term in the r.h.s. is less than $\sum_{i \geq 1} \log(1 + 1/\mu_i)$ which is finite since $(\mu_i)_i$ regularly varies with index $\rho > 1$. Hence, h_2 regularly varies at $+\infty$ as i_a , that is, with index $1/\rho$.

Putting all the pieces together, $\log \phi = h_1 + h_2(1 + R)$ is a negative function which regularly varies with index $1/\rho$ at $+\infty$. Then, according to Lemma A.6, the function

$$t \mapsto -\log \mathbb{P}_\infty(T_0 \leq t)$$

regularly varies at 0 with index $1/(1 - \rho)$, which concludes the proof. \square

Proof of Theorem 5.1(ii). We now suppose that $l = 0$ and that $(\mu_n)_n$ regularly varies with index $\rho > 1$. Let $(E_i)_{i \geq 1}$ be a sequence of independent exponential random variables with respective parameters $\lambda_i + \mu_i$. First, to go from $+\infty$ to 0, the process X has to reach each integer level. That gives the upper bound

$$\mathbb{P}_\infty(T_0 \leq t) \leq \mathbb{P}_\infty\left(\sum_{i=1}^{\infty} E_i \leq t\right), \quad t > 0. \quad (44)$$

Moreover, for $n \geq 1$ and $t \geq 0$, we have

$$\mathbb{P}_n(T_0 \leq t) \geq \mathbb{P}_n(X \downarrow) \mathbb{P}_n(T_0 \leq t | X \downarrow) = \prod_{i=1}^n \frac{\mu_i}{\lambda_i + \mu_i} \mathbb{P}_n\left(\sum_{i=1}^n E_i \leq t\right),$$

where $X \downarrow := \{\text{The process } X \text{ is non-increasing}\}$. Indeed, conditionally on $X \downarrow$, $(X(t) : t \geq 0)$ is a pure death process with death rates $\mu_n = \lambda_n + \mu_n$.

For $\eta \in (0, 1)$ and $t > 0$, we denote $n_t := v(t)(1 + \eta)$. Hence, for $t > 0$, by applying the Markov property, we get the lower bound

$$\begin{aligned} \mathbb{P}_\infty(T_0 \leq t) &\geq \mathbb{P}_\infty(X(t/2) \leq n_{t/2}) \mathbb{P}_{n_{t/2}}(T_0 \leq t/2) \\ &\geq \mathbb{P}_\infty(X(t/2) \leq n_{t/2}) \prod_{i=1}^{n_{t/2}} \frac{\mu_i}{\lambda_i + \mu_i} \mathbb{P}_\infty\left(\sum_{i=1}^{n_{t/2}} E_i \leq t/2\right). \end{aligned} \quad (45)$$

Putting together (44) and (45), at a logarithmic scale, we obtain

$$\begin{aligned} -\log \mathbb{P}_\infty\left(\sum_{i=1}^{\infty} E_i \leq t\right) &\leq -\log \mathbb{P}_\infty(T_0 \leq t) \leq -\log \mathbb{P}_\infty\left(\sum_{i=1}^{n_{t/2}} E_i \leq t/2\right) \\ &\quad + \sum_{i=1}^{n_{t/2}} \log\left(1 + \frac{\lambda_i}{\mu_i}\right) - \log \mathbb{P}_\infty(X_{t/2} \leq n_{t/2}). \end{aligned} \quad (46)$$

We know from the pure death case that

$$t \mapsto \log \mathbb{P}_\infty\left(\sum_{i=1}^{n_{t/2}} E_i \leq t/2\right) \text{ and } t \mapsto \log \mathbb{P}_\infty\left(\sum_{i=1}^{\infty} E_i \leq t\right)$$

both regularly vary at 0 with common index $1/(1-\rho)$. Moreover, according to Theorem 4.1(i),

$$\log \mathbb{P}_\infty(X(t/2) \leq n_{t/2}) = \log \mathbb{P}_\infty\left(\frac{X(t/2)}{v(t/2)} \leq 1 + \eta\right) \xrightarrow{t \rightarrow 0} 0.$$

Let us deal with the remaining term of (46), namely $\sum_{i=1}^{n_{t/2}} \log(1 + \lambda_i/\mu_i)$. Since $l = 0$, $(\lambda_n/\mu_n)_n$ is bounded and $\sum_{i=1}^{n_{t/2}} \log(1 + \lambda_i/\mu_i) \leq C n_{t/2}$ for some $C > 0$. Then, $\sum_{i=1}^{n_{t/2}} \log(1 + \lambda_i/\mu_i)$ is upper bounded by a regularly varying function with index $1/(1-\rho)$ since $n(t/2) = v(t/2)(1+\eta)$.

Plugging our four estimates into (46) ensures that there exist two slowly varying functions \underline{l} and \bar{l} such that

$$t^{1/(1-\rho)} \underline{l}(t) \leq -\log \mathbb{P}_\infty(T_0 \leq t) \leq t^{1/(1-\rho)} \bar{l}(t).$$

So for $t < 1$

$$\frac{\log \underline{l}(t)}{\log t} \geq \frac{\log(-\log \mathbb{P}_\infty(T_0 \leq t))}{\log t} - \frac{1}{1-\rho} \geq \frac{\log \bar{l}(t)}{\log t}.$$

Adding that the right and left hand sides tend to zero according to Lemma A.2(ii), the proof is complete. \square

6 Application to inhomogeneous birth and death processes

Thanks to the previous results, we can estimate the probability of extinction for birth and death processes. An estimation of the probability of extinction before a small time comes from Theorem 5.1 and an estimation to be extincted after a large time t can be obtained from the exponential moments obtained in Proposition 2.3 by Markov inequality.

As an application, we can now state some asymptotic results for population dynamics in varying environment. We consider a time inhomogeneous birth and death process $(Z_t, t \geq 0)$, associated with a varying environment. The birth and death rates at time t are respectively $\lambda_k(t)$ and $\mu_k(t)$ when the population size is equal to k . We say that an environment is favorable if the process persists with positive probability in this environment.

In case of random environment and under uniform assumptions on the birth and death rates, which make that either all the environments are favorable or all the environments are non favorable, extinction criteria are known, see e.g. Theorem 3.2 in [18] and Theorem 3.3 in [9]. Here we consider a case where favorable and non favorable environments can be mixed, successively in time. The assumption below focuses on time intervals of unfavorable environments.

Assumption A. There exists a sequence of successive and disjoint time intervals $([a_i, a_i + t_i], i \in \mathbb{N})$ such that for each $t \in \cup_{i \geq 0} [a_i, a_i + t_i)$,

$$\lambda_k(t) \leq \lambda_k, \quad \mu_k(t) \geq \mu_k,$$

where $\lambda_k/\mu_k \rightarrow 0$ as $k \rightarrow +\infty$ and $(\mu_k)_k$ regularly varies with index $\rho > 1$.

Assumption A means that on the successive intervals $[a_i, a_i + t_i)$, the environment is unfavorable and strongly increases the sub-criticality of the process. The intervals $[a_i, a_i + t_i)$ can be seen as competition phases. We will show that under Assumption A and even if the population is very large at time a_i , the process can go down to extinction during the time $[a_i, a_i + t_i)$, even for some t_i tending to zero and whatever happens during the phases $]a_i + t_i, a_{i+1}]$. Nevertheless, the durations t_i cannot go to zero too fast and we provide a quantitative criterion. Such a framework is motivated from ecology by the fact that favorable environments can alternate with unfavorable environments. It may be due to variations of the climate conditions and the resources available, which can affect both the natality, the mortality and the competition. One particular motivating example is a case with linear birth rates and $\rho > 1$ corresponding to a polynomial competition term ($\rho = 2$ yields the logistic competition). See [17] for discussions about the value of ρ .

Let us also remark that the environment may also model the effect of some predation, when the dynamics of the predator does not depend on the number of preys (generalist predator), see e.g. [8]. As a last motivation, we mention the use of inhomogeneous birth and death processes in epidemiology, see for example [19, 3] in the linear case. The two forthcoming results give the minimal duration of the competition phases which leads to a.s. extinction.

Proposition 6.1. *Under Assumption A, if there exists $\beta < \rho - 1$ such that, for i large enough and some constant $c > 0$,*

$$t_i \geq c/\log^\beta i, \tag{47}$$

then for every $n \in \mathbb{N}$, the process Z becomes extinct in finite time \mathbb{P}_n -a.s.

Proof. We denote by T_0 the absorbing time of a process with birth rate λ_k and death rate μ_k and $\tilde{\mathbb{P}}_\infty$ its law issued from infinity (which is well defined since $l = 0$ and $(\mu_k)_k$ regularly varies with index $\rho > 1$). By the Markov property and a monotonicity argument, a simple induction yields for every $n \geq 0$, $k \geq 1$,

$$\mathbb{P}_n(Z_{a_k+t_k} > 0) = \mathbb{E}_n \left(\mathbb{P}_n(Z_{a_k+t_k} > 0 \mid (Z_s, s \leq a_k)) \mathbf{1}_{\{Z_{a_{k-1}+t_{k-1}} > 0\}} \right) \leq \prod_{i \leq k} \tilde{\mathbb{P}}_\infty(T_0 > t_i),$$

so that

$$\log \mathbb{P}_n(\forall t > 0 : Z_t > 0) \leq \sum_{i \geq 0} \log \tilde{\mathbb{P}}_\infty(T_0 > t_i) = \sum_{i \geq 0} \log(1 - \tilde{\mathbb{P}}_\infty(T_0 \leq t_i)).$$

We know from Theorem 5.1(ii) that for every $\eta > 0$ and for t small enough,

$$\tilde{\mathbb{P}}_\infty(T_0 \leq t) \geq \exp \left(-t^{-\left(\frac{1}{\rho-1} + \eta\right)} \right).$$

Since $\tilde{\mathbb{P}}_\infty(T_0 \leq t_i) \geq \tilde{\mathbb{P}}_\infty(T_0 \leq c/\log^\beta i)$, we get that for any $\eta > 0$ and i large enough,

$$\tilde{\mathbb{P}}_\infty(T_0 \leq t_i) \geq \exp \left(-\left(\frac{\log^\beta i}{c} \right)^{\frac{1}{\rho-1} + \eta} \right).$$

By hypothesis, there is η small enough such that $\beta\left(\frac{1}{\rho-1} + \eta\right) < 1$. Therefore, $\left(\frac{\log^\beta i}{c}\right)^{\frac{1}{\rho-1} + \eta} < \log i$ for i large enough and

$$\sum_{i \geq 0} \exp \left(-\left(\frac{\log^\beta i}{c} \right)^{\frac{1}{\rho-1} + \eta} \right) = \infty.$$

Finally, $\mathbb{P}_n(\forall t > 0 : Z_t > 0) = 0$, which ends up the proof. \square

The result of Proposition 6.1 is completed by the following example where (47) fails and where the process survives with positive probability. Let us define the birth and death process Z as follows. We assume that $a_0 = 0$ and that $\lambda_k(t) = \lambda > 0$ for any k (for simplicity). Moreover, for every $t \in \cup_{i \geq 0} [a_i, a_i + t_i)$, $\mu_k(t) = \mu_k$ regularly varies with index $\rho > 1$ and for every $t \in \cup_{i \geq 0} [a_i + t_i, a_{i+1})$, $\mu_k(t) = 0$.

We assume that $t_i \leq c/\log^\beta i$ with $\beta > \rho - 1$. Let $\eta > 0$ be such that $\beta\left(\frac{1}{\rho-1} - \eta\right) > 1$. We now prove that the sequence $(a_i)_{i \geq 1}$ can be chosen such that the process Z survives with positive probability.

Let $\epsilon_i \in (0, 1)$ such that $\prod_{i \geq 0} (1 - \epsilon_i) > 0$. Let $x_0 = 1$ and for each $i \geq 1$, choose $x_i \in \mathbb{N}$ such that

$$\mathbb{P}_{x_i}(T_0 > t_i) \geq 1 - (1 - \epsilon_i)\mathbb{P}_\infty(T_0 > t_i),$$

so that for i large enough, by Theorem 5.1(ii),

$$\mathbb{P}_{x_i}(T_0 > t_i) \geq 1 - \epsilon_i \exp \left(-t_i^{-\left(\frac{1}{\rho-1} - \eta\right)} \right).$$

Then,

$$\prod_{i \geq 0} \mathbb{P}_{x_i}(T_0 > t_i) \geq \prod_{i \geq 0} \left(1 - (1 - \epsilon_i) \exp \left(-t_i^{-\left(\frac{1}{\rho-1} - \eta\right)} \right) \right) > 0.$$

Observing that the process Z is a pure birth process during the time intervals $[a_i + t_i, a_{i+1})$, one can choose the times a_i ($i \geq 1$) such that for every $i \geq 0$, $\mathbb{P}_1(Z_{a_{i+1}-a_i-t_i} \geq x_{i+1}) \geq (1 - \epsilon_i)$. By Markov property for every $n \geq 1$, it yields

$$\mathbb{P}_n(\forall t \geq 0 : Z_t > 0) \geq \prod_{i \geq 0} \mathbb{P}_{x_i}(T_0 > t_i) \mathbb{P}_1(Z_{a_{i+1}-a_i-t_i} \geq x_{i+1}) > 0$$

and ends up the proof.

A Regular varying functions

In this section, we give several results that deal with regularly varying functions. The interested reader can see [6] for more details.

Definition A.1. (i) A function $g : [0, +\infty) \rightarrow (0, +\infty)$ has regular variations at $L \in \{0, +\infty\}$ if there exists ρ such that for all $a > 0$,

$$\lim_{x \rightarrow L} \frac{g(ax)}{g(x)} = a^\rho.$$

(ii) A sequence of real non-zero numbers $(u_n)_{n \geq 0}$ regularly varies if there exists ρ such that for all $a > 0$

$$\lim_{n \rightarrow +\infty} \frac{u_{[an]}}{u_n} = a^\rho$$

where $[\cdot]$ is the floor function.

In both cases, the real number ρ is called the index and when $\rho = 0$, one says that variations are slow.

According to [6, Thm. 1.9.5], $(u_n)_n$ regularly varies if and only if the function $x \mapsto u_{[cx]}$ regularly varies at $+\infty$. Then, all the following results that we state for regularly varying functions also hold for regularly varying sequences.

Regularly varying functions can be compared with power functions as it is recorded in the following proposition.

Lemma A.2. Let g be a slowly varying function at $L \in \{0, +\infty\}$, with index ρ .

(i) For all $\varepsilon > 0$, $\lim_{x \rightarrow L} g(x)/(x^{\rho-\varepsilon}) = L$ and $\lim_{x \rightarrow L} g(x)/(x^{\rho+\varepsilon}) = 1/L$, where we use the convention $1/(+\infty) = 0$ and $1/0 = +\infty$.

(ii) $\lim_{x \rightarrow L} \log g(x)/\log x = \rho$.

Proof. We prove the first point when $L = +\infty$. By the definition of a regularly varying function, for $a > 0$ and x large enough, $g(ax) \geq g(x)a^\rho/2$. Then, for $\varepsilon > 0$,

$$\frac{g(ax)}{(ax)^{\rho-\varepsilon}} \geq a^\varepsilon \frac{g(x)}{2x^{\rho-\varepsilon}}$$

and by letting $a \rightarrow +\infty$, we obtain $\liminf_{n \rightarrow +\infty} g(x)/(x^{\rho-\varepsilon}) = +\infty$.

The second point stems from the first point. Indeed, if for instance $L = +\infty$, for all $\varepsilon > 0$ and t large enough, we have $t^{\rho-\varepsilon} \leq g(t) \leq t^{\rho+\varepsilon}$ and at a logarithmic scale, we have the result. The proof is the same if $L = 0$. \square

In the two following results, one sees that the class of regularly varying functions is stable by inversion and summation.

Lemma A.3. [6, Thm 1.5.12] *Let g be a regularly varying function at $+\infty$ with index $\alpha > 0$. Then, the generalized inverse*

$$g^{-1}(t) := \inf\{s \geq 0; g(s) \geq t\}, \quad t \geq 0$$

is well-defined, regularly varies at $+\infty$ with index $1/\alpha$ and as $t \rightarrow +\infty$,

$$g(g^{-1}(t)) \sim g^{-1}(g(t)) \sim t.$$

The same result holds if g regularly varies at 0 with a negative index and if $g^{-1}(t) = \inf\{s \geq 0; g(s) \leq t\}$.

Lemma A.4. *Let g be a function that regularly varies at $+\infty$ with index $\rho < -1$. Then $R(n) = \sum_{k \geq n} g(k)$ regularly varies with index $\rho + 1$ and*

$$\sum_{k \geq n} g(k) \underset{n \rightarrow +\infty}{\sim} -\frac{ng(n)}{\rho + 1}.$$

Proof. We only prove the first point since the proof of the second one uses similar arguments. First, since $\rho < -1$, according to Lemma A.2, $\sum_{k \geq 0} g(k)$ and $\int_0^{+\infty} g(x)dx$ are both convergent. Moreover, thanks to [6, Thm 1.5.3], any regularly varying function with a negative index is equivalent to a non-increasing function. Then, without loss of generality, one can suppose that g is non-increasing. If $I_n := \int_n^{+\infty} g(x)dx$, since g is now non-increasing, a classical comparison between series and integrals entails that $1 - \frac{g(n)}{I_n} \leq \frac{R_n}{I_n} \leq 1$. Using that g regularly varies, according to [6, Thm 1.5.11],

$$\lim_{n \rightarrow +\infty} \frac{ng(n)}{I_n} = -(\rho + 1). \quad (48)$$

Hence, from the last two displays, we get $I_n \sim R_n$ as $n \rightarrow +\infty$. We also see from (48) that I regularly varies at $+\infty$ with index $1 + \rho$. Since I and R are equivalent, R also regularly varies with the same index. \square

We end this section by giving two results that involve regularly varying functions. The second one is a Tauberian theorem, which is a key result in the proof of Theorem 5.1(i).

Lemma A.5. *Let $x_0 \in [0, +\infty]$ and let f and g be two positive functions such that*

$$f(x) \underset{x \rightarrow x_0}{\rightarrow} L \in \{0, +\infty\}, \quad \frac{f(x)}{g(x)} \underset{x \rightarrow x_0}{\rightarrow} 1.$$

If h regularly varies at L , then

$$\frac{h(f(x))}{h(g(x))} \underset{x \rightarrow x_0}{\rightarrow} 1.$$

Moreover, if $f(x) = f(x, t) = g(x)(1 + t\varepsilon(x))$ with $\lim_{x \rightarrow x_0} \varepsilon(x) = 0$, the previous convergence holds uniformly in t in any compact subset of \mathbb{R} .

Proof. We only prove the case $L = 0$. Let fix $\varepsilon > 0$. By definition of a regularly varying function, there exist $\eta, \eta' > 0$ such that for every $a \in [1 - \eta, 1 + \eta]$ and $y \in (0, \eta')$,

$$1 - \varepsilon \leq \frac{h(ay)}{h(y)} \leq 1 + \varepsilon.$$

Furthermore, for x close enough to x_0 , we have $g(x) \leq \eta'$ and $(1 - \eta) \leq f(x)/g(x) \leq (1 + \eta)$, so that

$$\left| \frac{h\left(g(x) \cdot \frac{f(x)}{g(x)}\right)}{h(g(x))} - 1 \right| \leq \varepsilon,$$

which ends up the first part of the proof. The second part follows in the same way since $1 + t\varepsilon(x)$ goes to 1 uniformly in t in any compact set. \square

Lemma A.6. [6, Thm. 4.12.9] *Let ν be a positive measure on $(0, +\infty)$ whose Laplace transform*

$$\phi(a) := \int_0^\infty e^{-ax} d\nu(x)$$

converges for all $a > 0$. Let $\rho < 1$. Then, $a \mapsto -\log \phi(a)$ regularly varies at $+\infty$ with index ρ if and only if $x \mapsto -\log \nu(0, x]$ regularly varies at 0 with index $\rho/(\rho - 1)$.

B Proof of Technical results

We first prove the equivalence of (5) and $S < \infty$ under the assumptions (3) and (4).

Lemma B.1. *The series*

$$S = \sum_{i \geq 1} \pi_i + \sum_{n \geq 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{\mu_n} \quad (49)$$

have the same behavior.

Proof. First, according to (3), as $n \rightarrow +\infty$, $\lambda_n/\mu_n \rightarrow l < 1$ and the first term of the r.h.s. of (49) converges. It remains to study the convergence of the series $\sum_{n \geq 1} A_n$ where for $n \geq 1$,

$$A_n := \sum_{j \geq 1} \frac{\lambda_{n+1} \cdots \lambda_{n+j-1}}{\mu_{n+1} \cdots \mu_{n+j}}.$$

We have $A_n \geq 1/\mu_{n+1}$ since it is the first term of the sum. Moreover, according to assumption (3), for n large enough, we have $\lambda_n/\mu_n \leq l' := (1 + l)/2$ and

$$A_n \leq \sum_{j \geq 1} l'^{j-1} \frac{1}{\mu_{n+j}} \leq \frac{1}{1 - l'} \frac{M}{\mu_{n+1}}$$

where $M = \sup\{\mu_n/\mu_{j+n} : j, n \geq 1\}$ is finite thanks to assumption (4). Putting all pieces together, for n large enough, we have

$$\frac{1}{\mu_{n+1}} \leq A_n \leq \frac{1}{1 - l'} \frac{M}{\mu_{n+1}}$$

and the series $\sum_n \frac{1}{\mu_n}$ and $\sum_n A_n$ have the same behaviors. \square

Now we consider a bounded sequence $(u_n)_{n \geq 1}$ satisfying for every $n \geq 1$

$$u_n = a_n u_{n+1} + b_n, \quad \text{where } a_n \geq 0, \quad \limsup_{n \rightarrow +\infty} a_n < 1, \quad b_n > 0. \quad (50)$$

We prove the following result used in particular in our work to control the moments of τ_n .

Lemma B.2. *Assuming (50), then for every $n \geq 1$,*

$$u_n = \sum_{i \geq n} b_i \prod_{j=n}^{i-1} a_j \quad (\text{where by convention } \prod_{j=n}^{n-1} a_j = 1). \quad (51)$$

Moreover, if $\sup\{b_{n+k}/b_n : k, n \geq 1\} < \infty$, there exists $C > 0$ such that for every $n \geq 1$

$$b_n \leq u_n \leq C b_n \quad \text{and thus} \quad \sup_{k, n \geq 1} \frac{u_{n+k}}{u_n} < \infty. \quad (52)$$

If in addition we assume that $a_n \xrightarrow[n \rightarrow +\infty]{} 0$, then $u_n \underset{n \rightarrow +\infty}{\sim} b_n$.

Proof. By the recurrence property (50), for all $N \geq n \geq 1$,

$$u_n = \sum_{i=n}^N b_i \prod_{j=n}^{i-1} a_j + u_{N+1} \prod_{j=n}^N a_j. \quad (53)$$

Since $\limsup a_n < 1$ and $(u_n)_n$ is bounded, the second term of the r.h.s. of (53) vanishes as $N \rightarrow +\infty$ and (51) is proved.

We now suppose $K := \sup\{b_{n+k}/b_n : k, n \geq 1\} < +\infty$. Moreover $\limsup_n a_n < 1$ ensures that there exists $\varepsilon > 0$ and n_0 such that for $n \geq n_0$, $a_n \leq 1 - \varepsilon$. So writing (51) as

$$u_n = b_n \left(1 + \sum_{i \geq 1} \frac{b_{i+n}}{b_n} \prod_{j=n}^{i+n-1} a_j \right), \quad (54)$$

Then, for n large enough,

$$b_n \leq u_n \leq b_n \left(1 + K \sum_{i \geq 1} (1 - \varepsilon)^{i-1} \right),$$

which ends the proof of (52). If we also have $\lim_{n \rightarrow +\infty} a_n = 0$, we obtain $u_n \sim b_n$ as $n \rightarrow +\infty$ from (54) thanks to the dominated convergence theorem. \square

C Examples

In this paragraph, we give examples that fulfill the Assumptions (3), (4) and (5). They illustrate the law of large numbers and central limit theorems of this paper. Special attention is paid to the examples motivated by population dynamics, such as Example 1. For these motivations and convenience, we assume here that the birth rate satisfies $\lambda_n \leq C.n$ for some

constant $C > 0$ and every $n \in \mathbb{N}$. It captures the linear branching rate and allows for example to take into account cooperation for small populations, as Allee effect. We are also focusing on the case $l = 0$, which means that the death rate prevails for large population owing to the competition. Let us note from Lemma 3.2 that such assumptions ensure that $\mathbb{E}_\infty(\tau_n) \sim 1/\mu_{n+1}$ as $n \rightarrow +\infty$.

Example 1. We assume that $\mu_n = n^\rho \log^\gamma n$ with either $\rho > 1$ or $\rho = 1$ and $\gamma > 1$. This death rate regularly varies with index ρ , so that the almost sure convergence of Theorem 3.1 holds. Thus, T_n satisfies a strong law of large numbers with speed

$$\mathbb{E}_\infty(T_n) \underset{n \rightarrow +\infty}{\sim} \sum_{k \geq n+1} \frac{1}{k^\rho \log^\gamma k} \underset{n \rightarrow +\infty}{\sim} \begin{cases} \frac{1}{(\rho-1)n^{\rho-1} \log^\gamma n} & \text{if } \rho > 1 \\ \frac{1}{(\gamma-1) \log^{\gamma-1} n} & \text{if } \rho = 1 \end{cases}.$$

Moreover, since $\text{Var}_\infty(T_n) \sim 1/[(2\rho-1)n^{2\rho-1} \log^{2\gamma} n]$ as $n \rightarrow +\infty$, according to Theorem 3.4, T_n satisfies the C.L.T.

$$\begin{aligned} \frac{\sqrt{2\rho-1}}{\rho-1} \sqrt{n} \{(\rho-1)n^{\rho-1} \log^\gamma n T_n - 1\} &\underset{n \rightarrow +\infty}{\xrightarrow{(d)}} N \quad (\rho > 1) \\ \frac{\sqrt{n} \log n}{(\gamma-1)} \{(\gamma-1) \log^{\gamma-1} n T_n - 1\} &\underset{n \rightarrow +\infty}{\xrightarrow{(d)}} N \quad (\rho = 1). \end{aligned}$$

Concerning the asymptotic behavior of $X(t)$ as $t \rightarrow 0$, when $\rho > 1$, v regularly varies at 0 with index $1/(1-\rho)$ and is generally not explicit. However, if $\gamma = 0$, that is, if $\mu_n = n^\rho$, we have $v(t) \sim ((\rho-1)t)^{1/(1-\rho)}$ as $t \rightarrow 0$.

Example 2. Let us illustrate the regimes (ii) of Theorems 3.1 and 4.1.

If $\mu_n = (n!)^\gamma$ with $\gamma > 0$, $\mathbb{E}_\infty(T_n) \sim ((n+1)!)^{-\gamma}$. Hence, $\lim_{n \rightarrow +\infty} \mathbb{E}_\infty(\tau_n) / \mathbb{E}_\infty(T_n) = 1$ and Theorem 3.1(ii) yields

$$((n+1)!)^\gamma T_n \underset{n \rightarrow +\infty}{\xrightarrow{(d)}} E$$

where E is exponential with parameter 1.

If $\mu_n = e^{-\beta n}$ with $\beta > 0$, $\mathbb{E}_\infty(T_n) \sim e^{-\beta(n+1)}/(1-e^{-\beta})$. Thus, the conditions of Theorem 3.1(ii) hold true with $\alpha = 1 - e^{-\beta}$ and

$$e^{\beta(n+1)} T_n \underset{n \rightarrow +\infty}{\xrightarrow{(d)}} \sum_{k \geq 0} e^{-\beta k} E_k$$

where the E_k 's are i.i.d. exponential with parameter 1. In that case, we can explicit the speed v of Theorem 4.1 and we get $X(t) \sim -(\log t)/\beta$ as $t \rightarrow 0$, in probability.

Example 3. In Theorem 3.1, we did not consider the case where $r_n = \mathbb{E}_\infty(\tau_n) / \mathbb{E}_\infty(T_n)$ does not converge. Then one can (only) state analogous results along the convergent subsequences. For instance, if $\mu_{2n} = \mu_{2n+1} = 3^{-2n}$, we have

$$r_{2n} \xrightarrow{n \rightarrow +\infty} \frac{4}{9} \quad \text{and} \quad r_{2n+1} \xrightarrow{n \rightarrow +\infty} \frac{4}{5}.$$

Theorem 3.1(ii) still holds in that case, but for the two subsequences $(T_{2n}/\mathbb{E}_\infty(T_{2n}))_n$ and $(T_{2n+1}/\mathbb{E}_\infty(T_{2n+1}))_n$, which converge in distribution to different limits.

One can also find examples where $0 = \liminf r_n < \limsup r_n$. Then, $(T_n)_n$ has two subsequences satisfying the two regimes (i) and (ii) of Theorem 3.1.

Example 4. In this last example, we exhibit a sequence of death rates $(\mu_n)_n$ such that the law of large numbers of Theorem 3.1(i) holds in probability but not almost surely.

For that purpose, we set $\mu_n = \exp(n/\log n) \log n$. One can check that $l = 0$ and

$$\mathbb{E}_\infty(\tau_n) \sim 1/\mu_{n+1}, \quad \mathbb{E}_\infty(T_n) \sim S(n+1) := \sum_{k \geq n+1} 1/\mu_k$$

as $n \rightarrow +\infty$. Moreover, as μ_n is non-decreasing,

$$\int_n^\infty \frac{e^{-x/\log(x)}}{\log x} dx \leq S(n) \leq \int_n^\infty \frac{e^{-x/\log(x)}}{\log x} dx + \frac{e^{-n/\log(n)}}{\log n}.$$

and

$$\int_n^\infty \frac{e^{-x/\log(x)}}{\log x} dx \underset{n \rightarrow +\infty}{\sim} \int_n^\infty \left(\frac{1}{\log x} + \frac{1}{(\log x)^2} \right) e^{-x/\log(x)} dx = e^{-n/\log(n)}.$$

Combining the two last displays and recalling $r_n = \mathbb{E}_\infty(\tau_n)/\mathbb{E}_\infty(T_n)$, we have

$$S(n) \sim \exp(-n/\log n), \quad r_n \sim 1/\log n, \quad r_n \rightarrow 0, \quad \sum_n r_n^2 = \infty,$$

so that $T_n/S(n+1)$ goes to 1 in probability but the almost sure convergence is not guaranteed. Indeed, let us assume now that $V_n := T_n/S(n+1)$ does converge a.s. toward 1 and find a contradiction. We have

$$V_{n+1} - V_n = V_{n+1} \left(1 - \frac{S(n+2)}{S(n+1)} \right) - \frac{\tau_n}{S(n+1)}. \quad (55)$$

By hypothesis, the left hand side of the latter a.s. vanishes as $n \rightarrow +\infty$. Moreover, simple computations leads to $S(n+1)/S(n) \rightarrow 1$ and the first term in the r.h.s. of the last display a.s. goes to 0 since our assumption implies that V_n is bounded a.s. Hence, putting all pieces together, the term $\tau_n/S(n+1)$ of (55) has to go to 0 a.s.

To get a contradiction thanks to Borel-Cantelli's Lemma, it suffices to prove that for ε small enough,

$$\sum_{n \geq 0} \mathbb{P}_\infty(\tau_n/S(n+1) > \varepsilon) = \infty,$$

recalling that the r.v. τ_n are independent. By a coupling argument, τ_n stochastically dominates $\hat{\tau}_n$ the hitting time of n by a pure death process with death rates $(\mu_k)_k$ and starting at $n+1$. In other words, τ_n is stochastically larger than the exponential r.v. $\hat{\tau}_n$ with parameter μ_{n+1} . Then, $\mathbb{P}_\infty(\tau_n/S(n+1) > \varepsilon) \geq \exp(-\varepsilon \mu_{n+1} S(n+1))$. Thanks to previous computations, $\mu_n S(n) \sim \log n$ as $n \rightarrow +\infty$. Then, there is $C > 0$ such that

$$\mathbb{P}_\infty(\tau_n/S(n+1) > \varepsilon) \geq e^{-\varepsilon C \log n} = \frac{1}{n^{C\varepsilon}},$$

which completes the proof because $\sum_{n \geq 0} \mathbb{P}(\tau_{n+1}/S(n) > \varepsilon)$ is infinite as soon as ε is small enough.

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