

# THE RIGHT ANGLE TO LOOK AT ORTHOGONAL SETS

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ABSTRACT. If  $X$  and  $Y$  are orthogonal hyperdefinable sets such that  $X$  is simple, then any group  $G$  interpretable in  $X \cup Y$  has a normal hyperdefinable  $X$ -internal subgroup  $N$  such that  $G/N$  is  $Y$ -internal;  $N$  is unique up to commensurability. In order to make sense of this statement, local simplicity theory for hyperdefinable sets is developed.

## INTRODUCTION

Two definable sets  $X$  and  $Y$  in some structure are said to be *orthogonal* if every definable subset of  $X \times Y$  is a finite union of *rectangles*, i.e. of subsets of the form  $U \times V$  with  $U \subseteq X$  and  $V \subseteq Y$  definable. It easily follows that if  $X$  and  $Y$  are orthogonal groups, every definable subgroup of  $X \times Y$  is of the form  $U \times V$  with  $U \leq X$  and  $V \leq Y$  subgroups. However, the situation is considerably more complicated for a group  $G$  definable, or more generally interpretable, in  $X \cup Y$ , as it need not be the sum of a group interpretable in  $X$  and a group interpretable in  $Y$ . In fact, an example by Berarducci and Mamino [2, Example 1.2] shows that  $G$  need not have *any* subgroup interpretable in either  $X$  or  $Y$ . However, they prove [2, Theorem 7.1] that if  $X$  is superstable of finite and definable Lascar rank, then any group  $G$  interpretable in  $X \cup Y$  has a normal subgroup  $N$  interpretable in  $X$ , such that  $G/N$  is interpretable in  $Y$ .

In this paper we shall generalize their result to the case where  $X$  is merely simple. In this context, definability has to be replaced by type-definability, as even for a definable group the tools of simplicity theory in general only yield type-definable subgroups. In fact, we even have to study *hyperdefinable* groups, since the quotient  $G/N$ , for  $N$

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type-definable, will be of that form. We therefore put ourselves in the hyperdefinable context and assume right from the start that our orthogonal sets  $X$  and  $Y$  are merely hyperdefinable. To this end, we shall include a quick development of local simplicity theory for hyperdefinable sets in section 4.

Another problem is that of parameters. The usual hypothesis would be that of *stable embedding*, i.e. that every hyperdefinable subset of  $X$  is hyperdefinable with parameters in  $X$ . We shall circumvent this issue by only ever considering parameters from  $X \cup Y$ , as orthogonality automatically yields stable embeddedness of  $X$  and of  $Y$  in  $X \cup Y$ .

We shall work in a big  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous monster model  $\mathfrak{M}$ , where  $\kappa$  is bigger than any cardinality we wish to consider. We shall not usually distinguish between elements and tuples.

## 1. ORTHOGONALITY

**Definition 1.1.** A set  $X$  is *hyperdefinable* over some parameters  $A$  if it is of the form  $Y/E$ , where  $Y$  is a type-definable set in (at most) countably many variables and  $E$  a type-definable equivalence relation on  $Y$ , both with parameters in  $A$ . For an element  $y \in Y$  we denote the class of  $y$  modulo  $E$  by  $y_E$ . If  $y = (y_0, \dots, y_n)$  is a tuple, we put  $y_E = ((y_0)_E, \dots, (y_n)_E)$ ; similarly, we write  $y_E z$  if  $y_i E z_i$  for all  $i \leq n$ .

For a tuple  $y_E$  in  $X$  and some parameters  $B \supseteq A$ , the type  $\text{tp}(y_E/B)$  is given by all partial types over  $B$  of the form

$$\exists z \in Y [z E y \wedge \varphi(z)]$$

true of  $y$ . It is easy to see that (in the monster model) two tuples in  $X$  have the same type over  $B$  if and only if they are conjugate by an automorphism fixing  $B$ . We similarly define the type over a hyperimaginary set containing  $A$ .

Note that we can also consider the type of a hyperimaginary over  $B$  as a partial real type over  $B$ . If  $B$  is hyperimaginary, this corresponds to a partial type over a representative of  $B$ , and any two such types (for different representatives) are equivalent. We shall say that a partial type  $\pi(y)$  is a (partial)  $X$ -type if  $\pi(y) \vdash y_E \in X$  and  $\pi(y)$  is  $E$ -invariant.

If  $a$  is hyperimaginary, a *representative* for  $a$  is any real (or imaginary) tuple  $\bar{a}$  with  $a \in \text{dcl}^{\text{heq}}(\bar{a})$ .

From now on, all tuples and parameter sets are hyperimaginary, unless stated otherwise.

**Definition 1.2.** Let  $X, Y$  be  $A$ -hyperdefinable sets in some structure  $\mathfrak{M}$ . We say that  $X$  and  $Y$  are *orthogonal* over  $A$ , denoted  $X \perp_A Y$ , if for any tuples  $a \in X$  and  $b \in Y$ , the partial type  $\text{tp}(a/A) \cup \text{tp}(b/A)$  determines  $\text{tp}(ab/A)$ .

**Example 1.3.** If  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are two structures and  $\mathfrak{N} = \mathfrak{M}_1 \times \mathfrak{M}_2$  with a predicate  $X$  for  $\mathfrak{M}_1$  and a predicate  $Y$  for  $\mathfrak{M}_2$ , then  $X$  and  $Y$  are orthogonal in  $\mathfrak{N}$ .

**Remark 1.4.** If  $X$  and  $Y$  are orthogonal type-definable sets and  $Z \subseteq X \times Y$  is relatively definable, then  $Z$  is a finite union of rectangles  $A_i \times B_i$ , where  $A_i \subseteq X$  and  $B_i \subseteq Y$  are relatively definable.

*Proof:* For any  $z = (x, y) \in Z$  we have that  $\text{tp}(x) \cup \text{tp}(y) \vdash (x, y) \in Z$ . By compactness there are relatively definable subsets  $A_z \subseteq X$  in  $\text{tp}(x)$  and  $B_z \subseteq Y$  in  $\text{tp}(y)$  with  $A_z \times B_z \subseteq Z$ . Again by compactness, finitely many of these rectangles suffice to cover  $Z$ .  $\square$

For the rest of this section,  $X$  and  $Y$  will be orthogonal  $\emptyset$ -hyperdefinable sets.

**Proposition 1.5.** *If  $X' \subseteq X^{\text{heq}}$  and  $Y' \subseteq Y^{\text{heq}}$  are hyperdefinable over some parameters  $A \subseteq X^{\text{heq}} \cup Y^{\text{heq}}$ , then  $X' \perp_A Y'$ .*

*Proof:* Suppose  $A = (a, b)$  with  $a \in X^{\text{heq}}$  and  $b \in Y^{\text{heq}}$ , and consider tuples  $a' \in X'$  and  $b' \in Y'$ . Choose representatives  $\bar{a}, \bar{a}' \in X$  of  $a, a'$  and  $\bar{b}, \bar{b}' \in Y$  of  $b, b'$ . Then  $\text{tp}(\bar{a}\bar{a}') \cup \text{tp}(\bar{b}\bar{b}') \vdash \text{tp}(\bar{a}\bar{a}'\bar{b}\bar{b}')$ .

Now if  $a'' \equiv_A a'$  and  $b'' \equiv_A b'$ , we can find  $A$ -conjugates  $\tilde{a}\tilde{a}''$  of  $\bar{a}\bar{a}'$  and  $\tilde{b}\tilde{b}''$  of  $\bar{b}\bar{b}'$  such that  $a''\tilde{a}\tilde{a}'' \equiv_A a'\bar{a}\bar{a}'$  and  $b''\tilde{b}\tilde{b}'' \equiv_A b'\bar{b}\bar{b}'$ . By orthogonality of  $X$  and  $Y$ , we obtain  $\bar{a}''\tilde{a}\tilde{b}''\tilde{b} \equiv \bar{a}'\bar{a}\bar{b}\bar{b}'$ , whence  $a''ab''b \equiv a'ab'b$ , and thus  $a''b'' \equiv_A a'b'$ .  $\square$

Note that this in particular shows that orthogonality is preserved under adding parameters from  $X^{\text{heq}} \cup Y^{\text{heq}}$ .

**Proposition 1.6.**  *$X$  is stably embedded in  $X \cup Y$ : For tuples  $a \in X^{\text{heq}}$  and  $b \in Y^{\text{heq}}$ , every  $ab$ -hyperdefinable subset  $X'$  of  $X^{\text{heq}}$  is hyperdefinable over  $a$ .*

*Proof:* If  $\Phi(x, a, b)$  hyperdefines  $X$  and  $\Psi(y) = \text{tp}(b)$ , put

$$\Phi'(x, a) = \exists y [\Psi(y) \wedge \Phi(x, a, y)].$$

Clearly  $\Phi(x, a, b) \vdash \Phi'(x, a)$ . Conversely, suppose  $a' \models \Phi'(x, a)$ , and choose  $b' \models \Psi$  with  $a' \models \Phi(x, a, b')$ . By orthogonality  $a'ab \equiv a'ab'$ , whence  $a' \models \Phi(x, a, b)$ , and  $\Phi'(x, a)$  hyperdefines  $X'$ .  $\square$

We put  $\text{dcl}_X^{\text{heq}}(A) = \text{dcl}^{\text{heq}}(A) \cap X^{\text{heq}}$  and  $\text{bdd}_X(A) = \text{bdd}(A) \cap X^{\text{heq}}$ .

**Corollary 1.7.** *Suppose  $a \in X^{\text{heq}}$  and  $b \in Y^{\text{heq}}$ . Then*

$$\text{dcl}_X^{\text{heq}}(a, b) = \text{dcl}_X^{\text{heq}}(a) \quad \text{and} \quad \text{bdd}_X(a, b) = \text{bdd}_X(a).$$

*Proof:* Immediate from Proposition 1.6.  $\square$

## 2. WEAK ELIMINATION OF HYPERIMAGINARIES

In this section,  $X$  and  $Y$  will be  $\emptyset$ -hyperdefinable sets.

**Definition 2.1.** Let  $Z$  be  $\emptyset$ -hyperdefinable. We say that  $Z$  has *weak elimination of hyperimaginaries* with respect to  $X^{\text{heq}}$  and  $Y^{\text{heq}}$  if for every  $z \in (X \cup Y)^{\text{heq}}$  there is some  $x \in X^{\text{heq}}$  and  $y \in Y^{\text{heq}}$  with  $xy \in \text{bdd}(z)$  and  $z \in \text{dcl}^{\text{heq}}(xy)$ .

**Theorem 2.2.** *If  $X$  and  $Y$  are orthogonal, then  $X \cup Y$  has weak elimination of hyperimaginaries with respect to  $X^{\text{heq}}$  and  $Y^{\text{heq}}$ .*

*Proof:* Consider  $z \in (X \cup Y)^{\text{heq}}$ , say  $z = (z_X, z_Y)_E$  for some tuples  $z_X \in X$ ,  $z_Y \in Y$  and type-definable equivalence relation  $E$ . For  $x \equiv z_X$  consider the hyperdefinable equivalence relation  $E_x$  on  $\text{tp}(z_Y)$  given by

$$yE_x y' \iff (x, y)E(x, y').$$

Then  $E_x$  is  $\emptyset$ -hyperdefinable by Proposition 1.6, and we may replace  $z_Y$  by  $z_Y/E_x$ . Similarly, we may replace  $z_X$  by  $z_X/E_y$ . We claim that now  $E$  has bounded classes on  $\text{tp}(z_X) \times \text{tp}(z_Y)$ . If not, there is a non-constant indiscernible sequence  $(x_i, y_i : i < \omega)$  in some  $E$ -class. By orthogonality, for  $i < j$

$$\text{tp}(x_i, x_j) \cup \text{tp}(y_i, y_j) \vdash \text{tp}((x_i, y_i), (x_j, y_j)).$$

But  $\text{tp}(y_i, y_j) = \text{tp}(y_i, y_k)$  for  $i < j < k$ , whence

$$\text{tp}((x_i, y_i), (x_j, y_j)) = \text{tp}((x_i, y_i), (x_j, y_k))$$

and  $(x_j, y_k)E(x_i, y_i)E(x_j, y_j)$  holds. By triviality of  $E_x$  we get  $y_j = y_k$ , whence  $x_j = x_k$  by triviality of  $E_y$ , a contradiction.

Thus  $(z_X, z_Y) \in \text{bdd}(z)$ , and trivially  $z \in \text{dcl}^{\text{heq}}(z_X, z_Y)$ .  $\square$

We shall put  $\text{bdd}_{XY}(A) = \text{bdd}(A) \cap (X^{\text{heq}} \cup Y^{\text{heq}})$ .

**Corollary 2.3.** *If  $X \perp Y$ , then  $\text{bdd}_{XY}(A)$  and  $\text{bdd}(A) \cap (X \cup Y)^{\text{heq}}$  are interdefinable for any  $A \subset X \cup Y$ . Moreover, for  $a \in (X \cup Y)^{\text{heq}}$  we have  $\text{tp}(a/\text{bdd}_{XY}(A)) \vdash \text{tp}(a/\text{bdd}(A))$ .*

*Proof:* The first assertion is immediate from weak elimination of hyperimaginaries. For the second assertion, note that equality of type over  $\text{bdd}(A)$  is a bounded type-definable equivalence relation, and the class of  $a$  is definable over  $\text{bdd}_{XY}(A)$ .  $\square$

### 3. INTERNALITY AND ANALYSABILITY

Again,  $X$  and  $Y$  will be  $\emptyset$ -hyperdefinable sets.

**Definition 3.1.** We say that  $X$  is (almost)  $Y$ -internal if there is some parameter set  $A$  such that for every  $a \in X$  there is a tuple  $b \in Y$  with  $a \in \text{dcl}^{\text{heq}}(Ab)$  (or  $a \in \text{bdd}(Ab)$ , respectively).<sup>1</sup>

If the parameters  $A$  can be chosen in some set  $Z$ , we say that  $X$  is  $Y$ -internal *within*  $Z$ .

We say that  $X$  is  $Y$ -analysable (within  $Z$ ) if there is a sequence  $(a_i : i < \alpha)$  such that  $\text{tp}(a_i/a_j : j < i)$  is  $Y$ -internal (within  $Z$ ) for every  $i < \alpha$ , and  $a \in \text{bdd}(a_i : i < \alpha)$ .

**Proposition 3.2.** *Suppose that  $X$  and  $Y$  are orthogonal. If an  $\emptyset$ -hyperdefinable set  $X'$  is  $X$ -analysable within  $X \cup Y$ , then  $X'$  is almost  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ ; if  $X'$  is  $X$ -internal within  $X \cup Y$ , then  $X'$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ .*

*Proof:* We first show the assertion for (almost)  $X$ -internal  $X'$ . So suppose  $\bar{a} \in X$  and  $\bar{b} \in Y$  are such that for every  $x \in X'$  there is a tuple  $a \in X$  with  $x \in \text{bdd}(\bar{a}\bar{b}a)$ . Let  $\bar{x}$  be the set of  $\bar{a}\bar{b}a$ -conjugates of  $x$ , and  $E$  the type-definable equivalence relation on  $\text{tp}(\bar{a}\bar{b}a)$  given by

$$(\bar{a}'\bar{b}'a')E(\bar{a}''\bar{b}''a'') \Leftrightarrow \exists \bar{x}' \bar{x}'\bar{a}'\bar{b}'a' \equiv \bar{x}'\bar{a}''\bar{b}''a'' \equiv \bar{x}\bar{a}\bar{b}a.$$

Clearly,  $\bar{x}$  is interdefinable with  $(\bar{a}\bar{b}a)_E$ . By weak elimination of hyperimaginaries, there is  $\tilde{a} \in X^{\text{heq}}$  and  $\tilde{b} \in Y^{\text{heq}}$  with  $\tilde{a}\tilde{b} \in \text{bdd}(\bar{x})$  and  $\bar{x} \in \text{dcl}^{\text{heq}}(\tilde{a}\tilde{b})$ . Note that if  $X'$  is  $X$ -internal and  $x \in \text{dcl}^{\text{heq}}(\bar{a}\bar{b}a)$ , then  $x = \bar{x}$ .

Suppose  $\tilde{b} \notin \text{bdd}_Y(\emptyset)$ . Then there is an  $\emptyset$ -conjugate  $\tilde{b}'$  of  $\tilde{b}$  outside  $\text{bdd}(\bar{a}\tilde{b})$ ; if  $\sigma$  is an automorphism mapping  $\tilde{b}'$  to  $\tilde{b}$ , put  $\bar{a}'\bar{b}' = \sigma(\bar{a}\tilde{b})$ .

<sup>1</sup>In simplicity theory, this is called *finite generation*; for *internality* we would require for every  $a \in X$  the existence of some  $A \perp a$  and tuple  $b \in Y$  with  $a \in \text{dcl}^{\text{heq}}(Ab)$ .

Then  $\tilde{b} \notin \text{bdd}(\bar{a}'\bar{b}')$ . On the other hand, since  $\bar{a}'\bar{b}' \equiv \bar{a}\bar{b}$  and  $\bar{x} \subset X'$ , there is  $a' \in X$  with  $x' \in \text{bdd}(\bar{a}'\bar{b}'a')$  for all  $x' \in \bar{x}$ . Then in particular  $\tilde{b} \in \text{bdd}(\bar{a}'\bar{b}'a')$ , whence  $\tilde{b} \in \text{bdd}(\bar{b}')$  by Corollary 1.7, a contradiction.

Now assume that  $x \in X'$  and  $(x_i : i < \alpha)$  is an  $X$ -analysis of  $x$  within  $X \cup Y$ . We show inductively on  $i$  that  $\text{tp}(x_j : j < i)$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ . So suppose  $\text{tp}(x_j : j < k)$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$  for all  $k < i$ . If  $i$  is limit, then clearly  $\text{tp}(x_j : j < i)$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ . If  $i = k + 1$ , then  $\text{tp}(x_k/x_j : j < k)$  is  $X$ -internal within  $\text{bdd}_Y(x_j : j < k)$ , and there is  $a \in X$  with

$$x_k \in \text{dcl}^{\text{heq}}(a, \text{bdd}_Y(x_j : j < k), x_j : j < k).$$

Or, by  $X$ -internality of  $\text{tp}(x_j : j < k)$  within  $\text{bdd}_Y(\emptyset)$  there is  $a' \in X$  with  $(x_j : j < k) \in \text{dcl}^{\text{heq}}(a', \text{bdd}_Y(\emptyset))$ . Then

$$\text{bdd}_Y(x_j : j < k) \subseteq \text{bdd}_Y(a', \text{bdd}_Y(\emptyset)) = \text{bdd}_Y(\emptyset)$$

by Corollary 1.7, and  $x_k \in \text{bdd}(a, a', \text{bdd}_Y(\emptyset))$ . So  $\text{tp}(x_j : j < i)$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ , and  $\text{tp}(x)$  is almost  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ .  $\square$

**Corollary 3.3.** *Suppose that  $X$  and  $Y$  are orthogonal. Let  $X'$  and  $Y'$  be  $\emptyset$ -hyperdefinable. If  $X'$  is  $X$ -internal within  $X \cup Y$  and  $Y'$  is  $Y$ -internal within  $X \cup Y$ , then  $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$ .*

*Proof:* This is immediate from Proposition 3.2 and 1.5.  $\square$

**Corollary 3.4.** *Suppose that  $X$  and  $Y$  are orthogonal. If an  $(X \cup Y)$ -hyperdefinable set  $Z$  is almost  $X$ - and almost  $Y$ -internal within  $X \cup Y$ , then it is bounded.*

*Proof:* We may assume that  $Z$  is hyperdefinable over  $\emptyset$ . Let  $z \in Z$ . Since  $Z$  is almost  $X$ -internal, there is  $b \in \text{bdd}_Y(\emptyset)$  and  $x \in X$  such that  $z \in \text{bdd}(xb)$ . Let  $\bar{z}$  be the set of  $xb$ -conjugates of  $z$ . Then  $\text{tp}(\bar{z})$  is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ , and still almost  $Y$ -internal within  $X \cup Y$ . So there is  $a \in \text{bdd}_X(\emptyset)$  and  $y \in Y$  such that  $\bar{z} \in \text{bdd}(ya)$ . Let  $\bar{\bar{z}}$  be the set of  $ya$ -conjugates of  $\bar{z}$ . Then  $\text{tp}(\bar{\bar{z}})$  is  $Y$ -internal within  $\text{bdd}_X(\emptyset)$ , and still  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ . Hence  $\text{tp}(\bar{\bar{z}}) \perp_{\text{bdd}_{XY}(\emptyset)} \text{tp}(\bar{\bar{z}})$ , so  $\text{tp}(\bar{\bar{z}})$  is bounded. Thus  $Z$  is bounded as well.  $\square$

**Corollary 3.5.** *Suppose that  $X$  and  $Y$  are orthogonal. If  $Z \subseteq (X \cup Y)^{\text{heq}}$  is  $\emptyset$ -hyperdefinable and almost  $X$ -internal within  $X \cup Y$ , then it is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$ .*

*Proof:* By weak elimination of hyperimaginaries there is  $x \in X^{\text{heq}}$  and  $y \in Y^{\text{heq}}$  with  $z \in \text{dcl}^{\text{heq}}(xy)$  and  $xy \in \text{bdd}(z)$ . So  $\text{tp}(y)$  is  $Y$ -internal and almost  $X$ -internal, whence bounded.  $\square$

## 4. LOCAL SIMPLICITY

**Definition 4.1.** Let  $A \subseteq B$ , and  $\pi(x, B)$  be a partial type over  $B$ . We say that  $\pi(x, B)$  does not divide over  $A$  if for any indiscernible sequence  $(B_i : i < \omega)$  in  $\text{tp}(B/A)$  the partial type

$$\bigcup_{i < \omega} \pi(x, B_i)$$

is consistent. Clearly,  $\text{tp}(a/B)$  divides over  $A$  if and only if  $\text{tp}(a_0/B)$  does so for some finite subtuple  $a_0 \subseteq a$ .

**Example 4.2.** If  $\text{tp}(a) \perp \text{tp}(b)$ , then  $\text{tp}(a/b)$  does not divide over  $\emptyset$ .

We now define the appropriate version of local rank. We follow Ben Yaacov's terminology [1, Definition 1.4], more general than [5, Definition 4.3.5].

**Definition 4.3.** Let  $\pi(x)$ ,  $\Phi(x, y)$  and  $\Psi(y_1, \dots, y_k)$  be partial types in (at most) countably many variables.

- (1)  $\Psi$  is a *k-inconsistency witness* for  $\Phi$  if

$$\models \forall y_1 \dots y_k \neg \exists x [\Psi(y_1, \dots, y_k) \wedge \bigwedge_{i=1}^k \Phi(x, y_i)].$$

- (2) The *local  $(\Phi, \Psi)$ -rank*  $D(\cdot, \Phi, \Psi)$  is defined on partial types in  $x$  as follows:

- $D(\pi(x), \Phi, \Psi) \geq 0$  if  $\pi(x)$  is consistent.
- $D(\pi(x), \Phi, \Psi) \geq n + 1$  if there is a sequence  $(a_i : i < \omega)$  such that  $\models \Psi(\bar{a})$  for any  $k$ -tuple  $\bar{a} \subset (a_i : i < \omega)$ , and  $D(\pi(x) \wedge \Phi(x, a_i), \Phi, \Psi) \geq n$  for all  $i < \omega$ .

If  $D(\pi, \Phi, \Psi) \geq n$  for all  $n < \omega$ , we put  $D(\pi, \Phi, \Psi) = \infty$ .

Note that in Definition 4.3(2) we may require  $(a_i : i < \omega)$  to be indiscernible. Moreover,  $D(\pi(x, a), \Phi, \Psi) \geq n$  is a closed condition on  $a$ , and  $D(\text{tp}(x/a), \Phi, \Psi) \geq n$  is a closed condition on  $x$  over  $a$ .

**Definition 4.4.** Let  $I$  be an ordered set. A sequence  $I = (a_i : i \in I)$  is *independent over  $A$* , or  *$A$ -independent*, if  $\text{tp}(a_i/A, a_j : j < i)$  does not divide over  $A$  for all  $i \in I$ . If  $A \subseteq B$  and  $p \in S(B)$ , the sequence  $(a_i : i \in I)$  is a *Morley sequence in  $p$  over  $A$*  if it is  $B$ -indiscernible,  $a_i \models p$  and  $\text{tp}(a_i/B, a_j : j < i)$  does not divide over  $A$  for all  $i \in I$ . If  $A = B$ , we simply call it a *Morley sequence in  $p$* .

**Fact 4.5.** [5, Corollary 3.2.5] or [4, Proposition 16.12] *If  $\text{tp}(b/cd)$  does not divide over  $d$  and  $\text{tp}(a/cbd)$  does not divide over  $bd$ , then  $\text{tp}(ab/cd)$  does not divide over  $d$ .*

**Theorem 4.6.** *Let  $X$  be a hyperdefinable set over  $\emptyset$ . The following are equivalent:*

- (1) *Symmetry holds on  $X$ : For all  $a, b, c \in X$ ,  $\text{tp}(a/bc)$  does not divide over  $b$  if and only if  $\text{tp}(c/ab)$  does not divide over  $b$ .*
- (2) *Transitivity holds on  $X$ : If  $a, b, c, d \in X$ , then  $\text{tp}(a/bcd)$  does not divide over  $b$  if and only if  $\text{tp}(a/bc)$  does not divide over  $b$  and  $\text{tp}(a/bcd)$  does not divide over  $bc$ .*
- (3) *Local character holds on  $X$ : There is  $\kappa$  such that for all countable  $a \in X$  and  $A \subset X$  there is  $A_0 \subseteq A$  with  $|A_0| \leq \kappa$  such that  $\text{tp}(a/A)$  does not divide over  $A_0$ . In fact, we can take  $\kappa = 2^{|T|}$ .*
- (4)  *$D(\cdot, \Phi, \Psi) < \infty$  for any partial  $X$ -type  $\Phi(x, y)$  and inconsistency witness  $\Psi$  for  $\Phi$ .*
- (5) *For any  $A \subseteq B \subset X$ , a partial  $X$ -type  $\pi(x, B)$  does not divide over  $A$  if and only if there is a Morley sequence  $I$  in  $\text{tp}(B/A)$  such that  $\{\pi(x, B') : B' \in I\}$  is consistent.*

*If any of these conditions is satisfied, then for all  $A \subseteq B \subset X$  and  $a \in X$  the type  $\text{tp}(a/B)$  does not divide over  $A$  if and only if*

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

*for all  $(\Phi, \Psi)$ . Moreover, Extension holds on  $X$ : For any partial  $X$ -type  $\pi(x)$  over  $B$ , if  $\pi$  does not divide over  $A$  then it has a completion which does not divide over  $A$ .*

*Proof:* (1)  $\Rightarrow$  (2) Clearly, if  $\text{tp}(a/bcd)$  does not divide over  $b$ , it does not divide over  $bc$  and  $\text{tp}(a/bc)$  does not divide over  $b$ . Conversely, suppose that  $\text{tp}(a/bcd)$  does not divide over  $bc$  and  $\text{tp}(a/bc)$  does not divide over  $b$ . By symmetry,  $\text{tp}(d/abc)$  does not divide over  $bc$  and  $\text{tp}(c/ab)$  does not divide over  $b$ . By Fact 4.5  $\text{tp}(cd/ab)$  does not divide over  $b$ , so again by symmetry  $\text{tp}(a/bcd)$  does not divide over  $b$ .

(2)  $\Rightarrow$  (4) Suppose there is a partial  $X$ -type  $\Phi$  and an inconsistency witness  $\Psi$  for  $\Phi$  such that  $D(x = x, \Phi, \Psi) = \infty$ . This means that for all  $n < \omega$  there are

$$(b_i, a_i^j : i < n, j < \omega)$$

such that  $\models \Psi(\bar{a})$  for all  $i < n$  and  $\bar{a} \subset (a_i^j : j < \omega)$  of the right length,  $b_i \models \bigwedge_{k \leq i} \Phi(x, a_k^0)$ , and  $(a_i^j : j < \omega)$  is indiscernible over  $\{b_k a_k^0 : k < i\}$ .



By Ramsey's Theorem and compactness we may assume that  $i$  is indexed by  $\{\pm(1 + \frac{1}{n}) : n > 0\}$  and that the sequence  $(b_i a_i^0)_i$  is indiscernible. Put  $I = \{b_i a_i^0 : i < -1\}$  and  $J = \{b_i a_i^0 : i > 1\}$ . Then  $\text{tp}(b_1/IJ)$  does not divide over  $I$  and  $\text{tp}(b_1/IJ a_{-1})$  does not divide over  $IJ$ , since the former is finitely satisfiable in  $I$  and the latter in  $J$ . However,  $(a_{-1}^j : j < \omega)$  witnesses that  $\Phi(x, a_{-1})$ , and hence  $\text{tp}(b_1/IJ a_{-1})$ , divides over  $I$ , contradicting transitivity.

(4)  $\Rightarrow$  (3) Assume (4). First, we note that for  $A \subseteq B \subset X$ , if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all  $(\Phi, \Psi)$ , then  $\text{tp}(a/B)$  does not divide over  $A$ . This is obvious, as if some  $A$ -indiscernible sequence  $(B_i : i < \omega)$  in  $\text{tp}(B/A)$  witnesses dividing, we can take  $\Phi(x, y) = \text{tp}(a, B)$  and  $\Psi = \text{tp}(B_1, B_2, \dots, B_n)$  for  $n < \omega$  sufficiently large. Then  $\Psi$  is an  $n$ -inconsistency witness (clearly, we may restrict to countable  $B$ ), and

$$D(\text{tp}(a/B), \Phi, \Psi) < D(\text{tp}(a/A), \Phi, \Psi).$$

Given  $\text{tp}(a/A)$  it is hence enough to take  $A_0 \subseteq A$  big enough such that

$$D(\text{tp}(a/A), \Phi, \Psi) = D(\text{tp}(a/A_0), \Phi, \Psi)$$

for all  $(\Phi, \Psi)$ . There are only  $2^{|T|}$  such pairs, so we need at most that many parameters.

(3)  $\Rightarrow$  (4) Suppose  $D(x = x, \Phi, \Psi) = \infty$ . Then for any cardinal  $\kappa$  we can find an indiscernible sequence  $(a_i b_i : i \leq \kappa^+)$  as in (2)  $\Rightarrow$  (4), such that  $b_{\kappa^+} \models \bigwedge_{i \leq \kappa^+} \Phi(x, a_i)$  and  $\Phi(x, a_i)$  divides over  $\{a_j : j < i\}$  for all  $i \leq \kappa^+$ . Then  $\text{tp}(b_{\kappa^+}/a_i : i < \kappa^+)$  divides over any subset of its domain of cardinality  $\leq \kappa$ .

(4)  $\Rightarrow$  (5). Assume (4). Given  $a \in X$  and  $A \subseteq B \subset X$ , for any pair  $(\Phi, \Psi)$  and any formula  $\varphi(y)$  we can adjoin either  $\exists y [xEy \wedge \varphi(y)]$  or  $\exists y [xEy \wedge \neg \varphi(y)]$  and preserve  $D(\cdot, \Phi, \Psi)$ -rank. By compactness we can thus complete  $\text{tp}(a/A)$  to a type  $p$  over  $B$  of the same  $D(\cdot, \Phi, \Psi)$ -rank. In particular, no  $\Phi$ -instance in  $p$  divides over  $A$  with  $\Psi$  as inconsistency witness. Coding finitely many pairs  $(\Phi_i, \Psi_i : i < n)$  in a single one, one obtains an extension such that no  $\Phi_i$ -instance  $\Psi_i$ -divides for any  $i < n$ ; by compactness we can do this for all pairs  $(\Phi, \Psi)$  simultaneously and obtain an extension which does not divide over  $A$ . Taking for  $B$  a sufficiently saturated model, a sequence  $(a_i : i < \omega) \subset B$  such that  $a_i \models p \upharpoonright_{(A, a_j : j < i)}$  is a Morley sequence in  $\text{tp}(a/A)$ .

This shows in particular that if  $\pi(x, B)$  does not divide over  $A$ , then there is a Morley sequence  $I$  in  $\text{tp}(B/A)$  such that  $\{\pi(x, B') : B' \in I\}$  is consistent.

Conversely, suppose that  $\pi(x, B)$  divides over  $A$ , as witnessed by an  $A$ -indiscernible sequence  $(B_i : i < \omega)$  in  $\text{tp}(B/A)$  with  $\bigcup_{i < \omega} \pi(x, B_i)$  inconsistent. Take any Morley sequence  $I$  in  $\text{tp}(B/A)$ . By [5, Corollary 2.2.8] (which is shown there for real tuples, but transfers easily to hyperimaginaries) we may assume that  $B_i \hat{\ } I$  is  $A$ -conjugate to  $I$  for all  $i \in I$  and that  $(B_i : i < \omega)$  is indiscernible over  $AI$ . If  $\bar{\pi}(x) = \bigcup_{B' \in I} \pi(x, B')$  were consistent, then  $(B_i : i < \omega)$  would witness that

$$D(\bar{\pi}(x) \wedge \pi(x, B_0), \pi(x, y), \Psi) < D(\bar{\pi}(x), \pi(x, y), \Psi)$$

for some inconsistency witness  $\Psi$ . But by  $A$ -conjugacy the two ranks must be equal, a contradiction.

(5)  $\Rightarrow$  (1) Let us first show *Extension*. If  $A \subseteq B \subset X$  and  $\pi(x, B)$  is a partial  $X$ -type which does not divide over  $A$ , let  $(B_i : i < \alpha)$  be a very long Morley sequence in  $\text{tp}(B/A)$ . Consider any realization  $a \models \bigwedge_{i < \alpha} \pi(x, B_i)$ . Since  $\alpha$  is large, there is an infinite subset  $J \subset \alpha$  such that  $p(aAX) = \text{tp}(aAB_i)$  is constant for  $i \in J$ . Then  $p(xAB)$  is a completion of  $\pi$  which does not divide over  $A$ , as witnessed by  $(B_i : i \in J)$ .

Now given  $a, b, c \in X$  such that  $\text{tp}(a/bc)$  does not divide over  $b$ , let  $B \ni bc$  be a sufficiently saturated model, and  $p$  an extension of  $\text{tp}(a/bc)$  to  $B$  which does not divide over  $b$ . Choose a sequence  $(a_i : i < \omega) \subset B$  such that  $a_i \models p \upharpoonright_{(bc, a_j : j < i)}$ . This is a Morley sequence in  $\text{tp}(a/bc)$  over  $b$ . Then  $(a_i : i < \omega)$  is a Morley sequence in  $\text{tp}(a/b)$ , and  $a_i \models \text{tp}(a/bc)$  for all  $i < \omega$ . Hence  $\text{tp}(c/ba)$  does not divide over  $b$ , and symmetry holds.

Finally we show that if (1) – (5) hold and  $\text{tp}(a/B)$  does not divide over  $A$  for  $A \subseteq B \subset X$  and  $a \in X$ , then  $D(\text{tp}(a/A), \Phi, \Psi) \geq n$  implies  $D(\text{tp}(a/B), \Phi, \Psi) \geq n$  for all  $(\Phi, \Psi)$ . For  $n = 0$  this is obvious. So suppose  $D(\text{tp}(a/A), \Phi, \Psi) \geq n + 1$ . Then there is  $(d_i : i < \omega)$  indiscernible over  $A$  such that  $\bar{d} \models \Psi$  for all  $\bar{d} \subset (d_i : i < \omega)$  of the right length, and  $D(\text{tp}(a/A) \wedge \Phi(x, d_i), \Phi, \Psi) \geq n$  for all  $i < \omega$ . Let  $q$  be a completion of  $\text{tp}(a/A) \wedge \Phi(x, d_0)$  with  $D(q, \Phi, \Psi) \geq n$ . Clearly, we may assume  $a \models q$ , and that  $\text{tp}(d_0/aB)$  does not divide over  $aA$ . As  $\text{tp}(a/B)$  does not divide over  $A$ , by symmetry and transitivity  $\text{tp}(ad_0/B)$  does not divide over  $A$ , and  $\text{tp}(a/d_0B)$  does not divide over  $d_0A$ . By induction hypothesis,

$$D(\text{tp}(a/d_0A), \Phi, \Psi) \geq n \quad \text{implies} \quad D(\text{tp}(a/d_0B), \Phi, \Psi) \geq n.$$

As  $\text{tp}(d_0/B)$  does not divide over  $A$  and  $(d_i : i < \omega)$  is  $A$ -indiscernible, we may assume that it remains indiscernible over  $B$ . But then it witnesses

$$D(\text{tp}(a/B), \Phi, \Psi) \geq D(\text{tp}(a/d_0B), \Phi, \Psi) + 1 \geq n + 1. \quad \square$$

**Definition 4.7.** An  $A$ -hyperdefinable set  $X$  is *simple (over  $A$ )* if it satisfies any of the conditions of Theorem 4.6 when we adjoin  $A$  to the language. If  $X$  is simple over  $A$  and  $a, b, c \in X$ , we shall say that  $a$  and  $c$  are independent over  $Ab$ , written  $a \perp_{Ab} c$ , if  $\text{tp}(a/Abc)$  does not divide over  $Ab$ .

Note that we only allow tuples and parameters from  $A \cup X$ . If  $X$  is stably embedded, we can of course allow parameters from anywhere.

**Remark 4.8.** If  $X$  is merely hyperdefinable, it may be simple although no definable or even type-definable imaginary set in the ambient structure is simple.

If  $X$  is simple, it is now standard to extend the notions of dividing and independence to hyperimaginaries in  $X^{\text{heq}}$ . Moreover, we can develop basic simplicity theory (canonical bases, the independence theorem, stratified ranks, generic types, stabilizers, see [5]) within  $X^{\text{heq}}$ , replacing the notion of a model by a boundedly closed subset  $C$  of  $X^{\text{heq}}$  such that  $X^{\text{heq}}$ -types over  $C$  are finitely satisfiable in  $C$ .

**Proposition 4.9.** *Let  $X$  and  $Y$  be orthogonal  $\emptyset$ -hyperdefinable sets such that  $X$  is simple over  $\emptyset$ . If  $A \subset Y$  is a set of parameters, then  $X$  is simple over  $A$ , and over  $\text{bdd}_Y(A)$ . In particular, let  $Z$  be a set hyperdefinable over some parameters  $A \subset X \cup Y$ . If  $Z$  is  $X$ -internal within  $X \cup Y$ , then  $Z$  is simple over  $A$ ; if  $Z \subseteq (X \cup Y)^{\text{heq}}$  is almost  $X$ -internal, then  $Z$  is simple as well.*

*Proof:* Simplicity over  $A$  is obvious from orthogonality; simplicity over  $\text{bdd}_Y(A)$  follows. Now if  $Z$  is  $X$ -internal within  $X \cup Y$ , it lives in  $\text{dcl}^{\text{eq}}(X, \text{bdd}_Y(A))$  by Corollary 3.2, and must be simple as well; if  $Z \subseteq (X \cup Y)^{\text{heq}}$  is almost  $X$ -internal within  $X \cup Y$ , it is  $X$ -internal within  $\text{bdd}_Y(\emptyset)$  by Corollary 3.5.  $\square$

## 5. GROUPS INTERPRETABLE IN ORTHOGONAL SETS

Recall that two hyperdefinable subgroups  $H_1$  and  $H_2$  of some group  $G$  are *commensurable* if  $H_1 \cap H_2$  has bounded index both in  $H_1$  and in  $H_2$ .

**Theorem 5.1.** *Suppose  $X$  and  $Y$  are orthogonal  $\emptyset$ -hyperdefinable sets in a structure  $\mathfrak{M}$ , and  $G$  is an  $\emptyset$ -hyperdefinable group in  $(X \cup Y)^{heq}$ . If  $X$  is simple over  $\emptyset$ , there is an  $\emptyset$ -hyperdefinable normal  $X$ -internal subgroup  $N$  of  $G$  such that the quotient  $G/N$  is  $Y$ -internal.  $N$  is unique up to commensurability.*

*Proof:* Let us first show uniqueness: If  $N'$  is a second such group, then  $N/(N \cap N')$  and  $N'/(N \cap N')$  are  $X$ -internal and  $Y$ -internal, and hence bounded by orthogonality of  $X$  and  $Y$ . Thus  $N$  and  $N'$  are commensurable.

By Theorem 2.2 every element  $g \in G$  is of the form  $(g_X, g_Y)_E$  for some  $g_X \in X^{heq}$  and  $g_Y \in Y^{heq}$ , both bounded over  $g$ , and some type-definable equivalence relation  $E$  with bounded classes, depending on  $\text{tp}(g)$ . Hence  $\text{tp}(g/g_Y)$  is  $X$ -internal and  $\text{tp}(g/g_X)$  is  $Y$ -internal. Now if  $h = (h_X, h_Y)_E$  and  $gh = ((gh)_X, (gh)_Y)_E$ , then

$$(gh)_X \in \text{bdd}(g_X, g_Y, h_X, h_Y),$$

whence  $(gh)_X \in \text{bdd}(g_X, h_X)$  by Corollary 1.7. Similarly  $(gh)_Y \in \text{bdd}(g_Y, h_Y)$ .

Now  $X$  is simple, as is  $\text{tp}(g, h/g_Y, h_Y)$  for any  $g, h \in G$ . Hence we can consider  $g, h \in G$  such that  $g \perp_{g_Y, h_Y} h$ . Then for any stratified local rank  $D$

$$\begin{aligned} (\dagger) \quad D(gh/(gh)_Y) &\geq D(gh/(gh)_Y, g_Y, h_Y, g) = D(h/g_Y, h_Y, g) \\ &= D(h/g_Y, h_Y) = D(h/h_Y), \end{aligned}$$

as also  $h \perp_{h_Y} g_Y$  by orthogonality (Example 4.2). Similarly

$$D(gh/(gh)_Y) \geq D(g/g_Y).$$

Now suppose  $G$  is a subset of  $(X^m \times Y^n)/E$ , where  $m, n$  are at most countable. Then if  $g = (\bar{x}_g, \bar{y}_g)_E$  we have  $g \perp_{g_Y} \bar{y}_g$ , and  $g_Y \in \text{bdd}(\bar{y}_g)$ , whence  $D(g/g_Y) = D(g/\bar{y}_g)$ . By compactness, there is a  $G$ -type  $p((\bar{x}, \bar{y})_E)$  implying that  $D((\bar{x}, \bar{y})_E/\bar{y})$  is maximal for all local stratified ranks. But if  $g, h \models p$  with  $g \perp_{g_Y, h_Y} h$ , then we must have equality in  $(\dagger)$ , which implies that  $g, h$  and  $gh$  are pairwise independent over  $g_Y, h_Y, (gh)_Y$ . Put

$$S = \text{stab}(g/g_Y, h_Y, (gh)_Y).$$

By [3, Lemme 1.2 and Remarque 1.3] the element  $g$  is generic in the coset  $Sg$ , and this coset is hyperdefinable over  $\text{bdd}(g_Y, h_Y, (gh)_Y)$ . Moreover,  $S$  is  $X$ -internal since  $\text{tp}(g/g_Y)$  is.

Suppose that  $S$  is not commensurable with  $S^h$  for some  $h \in G$ . Then  $SS^h$  is still  $X$ -internal, with

$$D(SS^h) \geq D(S) = D(p)$$

for every stratified local rank  $D$ , and for at least one such rank  $D_0$  we have  $D_0(SS^h) > D_0(p)$ . Choose  $g' \in SS^h$  with  $D_0(g'/h) = D_0(SS^h)$ . By pre-multiplying with a generic element of  $S$  and post-multiplying with a generic element of  $S^h$ , the inequality (†) implies that we may assume  $D(g'/h) \geq D(p)$  for every stratified local rank  $D$ . However,  $\text{tp}(g'/h)$  is  $X$ -internal, so  $g'_Y \in \text{bdd}(g')$  implies  $g'_Y \in \text{bdd}(h)$  by Lemma 1.7 and Proposition 3.2. Thus

$$D(g'/g'_Y) \geq D(g'/h) \geq D(p) \quad \text{and} \quad D_0(g'/g'_Y) \geq D_0(g'/h) > D_0(p),$$

contradicting our choice of  $p$ . Hence  $S$  is commensurable with all its conjugates, and the locally connected component  $N$  of  $S$  is normal in  $G$  [5, Definition 4.5.15 and Corollary 4.5.16]. In fact the same proof, with  $NZ$  instead of  $SS^h$ , shows that if  $Z$  is an  $X$ -internal hyperdefinable subset of  $G$ , then  $Z$  is covered by boundedly many cosets of  $N$ . In particular for any  $g' \in G$  the type  $\text{tp}(g'/g'_Y)$  is covered by boundedly many cosets of  $N$ . But then  $g'N \in \text{bdd}(g'_Y)$ , and  $G/N$  is almost  $Y$ -internal, whence  $Y$ -internal by Corollary 3.5.  $\square$

**Question 5.2.** If  $X$  and  $Y$  are orthogonal type-definable sets and  $G$  is a relatively definable group in  $(X \cup Y)^{eq}$ , can we find a relatively definable normal  $X$ -internal subgroup  $N$  such that  $G/N$  is  $Y$ -internal?

**Question 5.3.** What can we say if neither  $X$  nor  $Y$  is simple? Is it true that in every hyperdefinable subgroup of  $((X \cup Y)^{heq})$  there is a maximal normal hyperdefinable  $X$ -internal subgroup  $N_X$ , a maximal normal hyperdefinable  $Y$ -internal subgroup  $N_Y$ , an  $X$ -internal hyperdefinable local group  $G_X$ , a  $Y$ -internal hyperdefinable local group  $G_Y$  and a hyperdefinable locally bounded equivalence relation  $E$  on  $G_X \times G_Y$  such that  $G/(N_X N_Y)$  is isogenous, or even isomorphic, to  $(G_X \times G_Y)/E$ ?

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