



HAL
open science

The right angle to look at orthogonal sets

Frank Olaf Wagner

► **To cite this version:**

| Frank Olaf Wagner. The right angle to look at orthogonal sets. 2014. hal-00876033v2

HAL Id: hal-00876033

<https://hal.science/hal-00876033v2>

Preprint submitted on 3 Dec 2014 (v2), last revised 6 Jul 2016 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE RIGHT ANGLE TO LOOK AT ORTHOGONAL SETS

FRANK O. WAGNER

ABSTRACT. If X and Y are orthogonal hyperdefinable sets such that X is simple, then any group G interpretable in $X \cup Y$ has a normal hyperdefinable X -internal subgroup N such that G/N is Y -internal; N is unique up to commensurability. In order to make sense of this statement, local simplicity theory for hyperdefinable sets is developed.

INTRODUCTION

Two definable sets X and Y in some structure are said to be *orthogonal* if every definable subset of $X \times Y$ is a finite union of *rectangles*, i.e. of subsets of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ definable. It easily follows that if X and Y are orthogonal groups, every definable subgroup of $X \times Y$ is of the form $U \times V$ with $U \leq X$ and $V \leq Y$ subgroups. However, the situation is considerably more complicated for a group G definable, or more generally interpretable, in $X \cup Y$, as it need not be the sum of a group interpretable in X and a group interpretable in Y . In fact, an example by Berarducci and Mamino [2, Example 1.2] shows that G need not have *any* subgroup interpretable in either X or Y . However, they prove [2, Theorem 7.1] that if X is superstable of finite and definable Lascar rank, then any group G interpretable in $X \cup Y$ has a normal subgroup N interpretable in X , such that G/N is interpretable in Y .

In this paper we shall generalize their result to the case where X is merely simple. In this context, definability has to be replaced by type-definability, as even for a definable group the tools of simplicity theory in general only yield type-definable subgroups. In fact, we even have to study *hyperdefinable* groups, since the quotient G/N , for N

2000 *Mathematics Subject Classification.* 03C45.

Key words and phrases. orthogonality; simplicity; internality.

Partially supported by ANR-09-BLAN-0047 Modig and ANR-13-BS01-0006 ValCoMo.

type-definable, will be of that form. We therefore put ourselves in the hyperdefinable context and assume right from the start that our orthogonal sets X and Y are merely hyperdefinable. To this end, we shall include a quick development of local simplicity theory for hyperdefinable sets in section 4.

Another problem is that of parameters. The usual hypothesis would be that of *stable embedding*, i.e. that every hyperdefinable subset of X is hyperdefinable with parameters in X . We shall circumvent this issue by only ever considering parameters from $X \cup Y$, as orthogonality automatically yields stable embeddedness of X and of Y in $X \cup Y$.

We shall work in a big κ -saturated and strongly κ -homogeneous monster model \mathfrak{M} , where κ is bigger than any cardinality we wish to consider. We shall not usually distinguish between elements and tuples.

1. ORTHOGONALITY

Definition 1.1. A set X is *hyperdefinable* over some parameters A if it is of the form Y/E , where Y is a type-definable set in (at most) countably many variables and E a type-definable equivalence relation on Y , both with parameters in A . For an element $y \in Y$ we denote the class of y modulo E by y_E . If $y = (y_0, \dots, y_n)$ is a tuple, we put $y_E = ((y_0)_E, \dots, (y_n)_E)$; similarly, we write $y_E z$ if $y_i E z_i$ for all $i \leq n$.

For a tuple y_E in X and some parameters $B \supseteq A$, the type $\text{tp}(y_E/B)$ is given by all partial types over B of the form

$$\exists z \in Y [z E y \wedge \varphi(z)]$$

true of y . It is easy to see that (in the monster model) two tuples in X have the same type over B if and only if they are conjugate by an automorphism fixing B . We similarly define the type over a hyperimaginary set containing A .

Note that we can also consider the type of a hyperimaginary over B as a partial real type over B . If B is hyperimaginary, this corresponds to a partial type over a representative of B , and any two such types (for different representatives) are equivalent. We shall say that a partial type $\pi(y)$ is a (partial) X -type if $\pi(y) \vdash y_E \in X$ and $\pi(y)$ is E -invariant.

If a is hyperimaginary, a *representative* for a is any real (or imaginary) tuple \bar{a} with $a \in \text{dcl}^{\text{heq}}(\bar{a})$.

From now on, all tuples and parameter sets are hyperimaginary, unless stated otherwise.

Definition 1.2. Let X, Y be A -hyperdefinable sets in some structure \mathfrak{M} . We say that X and Y are *orthogonal* over A , denoted $X \perp_A Y$, if for any tuples $a \in X$ and $b \in Y$, the partial type $\text{tp}(a/A) \cup \text{tp}(b/A)$ determines $\text{tp}(ab/A)$.

Example 1.3. If \mathfrak{M}_1 and \mathfrak{M}_2 are two structures and $\mathfrak{N} = \mathfrak{M}_1 \times \mathfrak{M}_2$ with a predicate X for \mathfrak{M}_1 and a predicate Y for \mathfrak{M}_2 , then X and Y are orthogonal in \mathfrak{N} .

Remark 1.4. If X and Y are orthogonal type-definable sets and $Z \subseteq X \times Y$ is relatively definable, then Z is a finite union of rectangles $A_i \times B_i$, where $A_i \subseteq X$ and $B_i \subseteq Y$ are relatively definable.

Proof: For any $z = (x, y) \in Z$ we have that $\text{tp}(x) \cup \text{tp}(y) \vdash (x, y) \in Z$. By compactness there are relatively definable subsets $A_z \subseteq X$ in $\text{tp}(x)$ and $B_z \subseteq Y$ in $\text{tp}(y)$ with $A_z \times B_z \subseteq Z$. Again by compactness, finitely many of these rectangles suffice to cover Z . \square

For the rest of this section, X and Y will be orthogonal \emptyset -hyperdefinable sets.

Proposition 1.5. *If $X' \subseteq X^{\text{heq}}$ and $Y' \subseteq Y^{\text{heq}}$ are hyperdefinable over some parameters $A \subseteq X^{\text{heq}} \cup Y^{\text{heq}}$, then $X' \perp_A Y'$.*

Proof: Suppose $A = (a, b)$ with $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, and consider tuples $a' \in X'$ and $b' \in Y'$. Choose representatives $\bar{a}, \bar{a}' \in X$ of a, a' and $\bar{b}, \bar{b}' \in Y$ of b, b' . Then $\text{tp}(\bar{a}\bar{a}') \cup \text{tp}(\bar{b}\bar{b}') \vdash \text{tp}(\bar{a}\bar{a}'\bar{b}\bar{b}')$.

Now if $a'' \equiv_A a'$ and $b'' \equiv_A b'$, we can find A -conjugates $\tilde{a}\tilde{a}''$ of $\bar{a}\bar{a}'$ and $\tilde{b}\tilde{b}''$ of $\bar{b}\bar{b}'$ such that $a''\tilde{a}\tilde{a}'' \equiv_A a'\bar{a}\bar{a}'$ and $b''\tilde{b}\tilde{b}'' \equiv_A b'\bar{b}\bar{b}'$. By orthogonality of X and Y , we obtain $\bar{a}''\tilde{a}\tilde{b}''\tilde{b} \equiv \bar{a}'\bar{a}\bar{b}\bar{b}'$, whence $a''\tilde{a}\tilde{b}''\tilde{b} \equiv a'\bar{a}\bar{b}\bar{b}'$, and thus $a''b'' \equiv_A a'b'$. \square

Note that this in particular shows that orthogonality is preserved under adding parameters from $X^{\text{heq}} \cup Y^{\text{heq}}$.

Proposition 1.6. *X is stably embedded in $X \cup Y$: For tuples $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, every ab -hyperdefinable subset X' of X^{heq} is hyperdefinable over a .*

Proof: If $\Phi(x, a, b)$ hyperdefines X and $\Psi(y) = \text{tp}(b)$, put

$$\Phi'(x, a) = \exists y [\Psi(y) \wedge \Phi(x, a, y)].$$

Clearly $\Phi(x, a, b) \vdash \Phi'(x, a)$. Conversely, suppose $a' \models \Phi'(x, a)$, and choose $b' \models \Psi$ with $a' \models \Phi(x, a, b')$. By orthogonality $a'ab \equiv a'ab'$, whence $a' \models \Phi(x, a, b)$, and $\Phi'(x, a)$ hyperdefines X' . \square

We put $\text{dcl}_X^{\text{heq}}(A) = \text{dcl}^{\text{heq}}(A) \cap X^{\text{heq}}$ and $\text{bdd}_X(A) = \text{bdd}(A) \cap X^{\text{heq}}$.

Corollary 1.7. *Suppose $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$. Then*

$$\text{dcl}_X^{\text{heq}}(a, b) = \text{dcl}_X^{\text{heq}}(a) \quad \text{and} \quad \text{bdd}_X(a, b) = \text{bdd}_X(a).$$

Proof: Immediate from Proposition 1.6. \square

2. WEAK ELIMINATION OF HYPERIMAGINARIES

In this section, X and Y will be \emptyset -hyperdefinable sets.

Definition 2.1. Let Z be \emptyset -hyperdefinable. We say that Z has *weak elimination of hyperimaginaries* with respect to X^{heq} and Y^{heq} if for every $z \in (X \cup Y)^{\text{heq}}$ there is some $x \in X^{\text{heq}}$ and $y \in Y^{\text{heq}}$ with $xy \in \text{bdd}(z)$ and $z \in \text{dcl}^{\text{heq}}(xy)$.

Theorem 2.2. *If X and Y are orthogonal, then $X \cup Y$ has weak elimination of hyperimaginaries with respect to X^{heq} and Y^{heq} .*

Proof: Consider $z \in (X \cup Y)^{\text{heq}}$, say $z = (z_X, z_Y)_E$ for some tuples $z_X \in X$, $z_Y \in Y$ and type-definable equivalence relation E . For $x \equiv z_X$ consider the hyperdefinable equivalence relation E_x on $\text{tp}(z_Y)$ given by

$$yE_x y' \iff (x, y)E(x, y').$$

Then E_x is \emptyset -hyperdefinable by Proposition 1.6, and we may replace z_Y by z_Y/E_x . Similarly, we may replace z_X by z_X/E_y . We claim that now E has bounded classes on $\text{tp}(z_X) \times \text{tp}(z_Y)$. If not, there is a non-constant indiscernible sequence $(x_i, y_i : i < \omega)$ in some E -class. By orthogonality, for $i < j$

$$\text{tp}(x_i, x_j) \cup \text{tp}(y_i, y_j) \vdash \text{tp}((x_i, y_i), (x_j, y_j)).$$

But $\text{tp}(y_i, y_j) = \text{tp}(y_i, y_k)$ for $i < j < k$, whence

$$\text{tp}((x_i, y_i), (x_j, y_j)) = \text{tp}((x_i, y_i), (x_j, y_k))$$

and $(x_j, y_k)E(x_i, y_i)E(x_j, y_j)$ holds. By triviality of E_x we get $y_j = y_k$, whence $x_j = x_k$ by triviality of E_y , a contradiction.

Thus $(z_X, z_Y) \in \text{bdd}(z)$, and trivially $z \in \text{dcl}^{\text{heq}}(z_X, z_Y)$. \square

We shall put $\text{bdd}_{XY}(A) = \text{bdd}(A) \cap (X^{\text{heq}} \cup Y^{\text{heq}})$.

Corollary 2.3. *If $X \perp Y$, then $\text{bdd}_{XY}(A)$ and $\text{bdd}(A) \cap (X \cup Y)^{\text{heq}}$ are interdefinable for any $A \subset X \cup Y$. Moreover, for $a \in (X \cup Y)^{\text{heq}}$ we have $\text{tp}(a/\text{bdd}_{XY}(A)) \vdash \text{tp}(a/\text{bdd}(A))$.*

Proof: The first assertion is immediate from weak elimination of hyperimaginaries. For the second assertion, note that equality of type over $\text{bdd}(A)$ is a bounded type-definable equivalence relation, and the class of a is definable over $\text{bdd}_{XY}(A)$. \square

3. INTERNALITY AND ANALYSABILITY

Again, X and Y will be \emptyset -hyperdefinable sets.

Definition 3.1. We say that X is (almost) Y -internal if there is some parameter set A such that for every $a \in X$ there is a tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Ab)$ (or $a \in \text{bdd}(Ab)$, respectively).¹

If the parameters A can be chosen in some set Z , we say that X is Y -internal *within* Z .

We say that X is Y -analysable (within Z) if there is a sequence $(a_i : i < \alpha)$ such that $\text{tp}(a_i/a_j : j < i)$ is Y -internal (within Z) for every $i < \alpha$, and $a \in \text{bdd}(a_i : i < \alpha)$.

Proposition 3.2. *Suppose that X and Y are orthogonal. If an \emptyset -hyperdefinable set X' is X -analysable within $X \cup Y$, then X' is almost X -internal within $\text{bdd}_Y(\emptyset)$; if X' is X -internal within $X \cup Y$, then X' is X -internal within $\text{bdd}_Y(\emptyset)$.*

Proof: We first show the assertion for (almost) X -internal X' . So suppose $\bar{a} \in X$ and $\bar{b} \in Y$ are such that for every $x \in X'$ there is a tuple $a \in X$ with $x \in \text{bdd}(\bar{a}\bar{b}a)$. Let \bar{x} be the set of $\bar{a}\bar{b}a$ -conjugates of x , and E the type-definable equivalence relation on $\text{tp}(\bar{a}\bar{b}a)$ given by

$$(\bar{a}'\bar{b}'a')E(\bar{a}''\bar{b}''a'') \Leftrightarrow \exists \bar{x}' \bar{x}'\bar{a}'\bar{b}'a' \equiv \bar{x}'\bar{a}''\bar{b}''a'' \equiv \bar{x}\bar{a}\bar{b}a.$$

Clearly, \bar{x} is interdefinable with $(\bar{a}\bar{b}a)_E$. By weak elimination of hyperimaginaries, there is $\tilde{a} \in X^{\text{heq}}$ and $\tilde{b} \in Y^{\text{heq}}$ with $\tilde{a}\tilde{b} \in \text{bdd}(\bar{x})$ and $\bar{x} \in \text{dcl}^{\text{heq}}(\tilde{a}\tilde{b})$. Note that if X' is X -internal and $x \in \text{dcl}^{\text{heq}}(\bar{a}\bar{b}a)$, then $x = \bar{x}$.

Suppose $\tilde{b} \notin \text{bdd}_Y(\emptyset)$. Then there is an \emptyset -conjugate \tilde{b}' of \tilde{b} outside $\text{bdd}(\bar{a}\tilde{b})$; if σ is an automorphism mapping \tilde{b}' to \tilde{b} , put $\bar{a}'\bar{b}' = \sigma(\bar{a}\tilde{b})$.

¹In simplicity theory, this is called *finite generation*; for *internality* we would require for every $a \in X$ the existence of some $A \downarrow a$ and tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Ab)$.

Then $\tilde{b} \notin \text{bdd}(\bar{a}'\bar{b}')$. On the other hand, since $\bar{a}'\bar{b}' \equiv \bar{a}\bar{b}$ and $\bar{x} \subset X'$, there is $a' \in X$ with $x' \in \text{bdd}(\bar{a}'\bar{b}'a')$ for all $x' \in \bar{x}$. Then in particular $\tilde{b} \in \text{bdd}(\bar{a}'\bar{b}'a')$, whence $\tilde{b} \in \text{bdd}(\bar{b}')$ by Corollary 1.7, a contradiction.

Now assume that $x \in X'$ and $(x_i : i < \alpha)$ is an X -analysis of x within $X \cup Y$. We show inductively on i that $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. So suppose $\text{tp}(x_j : j < k)$ is X -internal within $\text{bdd}_Y(\emptyset)$ for all $k < i$. If i is limit, then clearly $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. If $i = k + 1$, then $\text{tp}(x_k/x_j : j < k)$ is X -internal within $\text{bdd}_Y(x_j : j < k)$, and there is $a \in X$ with

$$x_k \in \text{dcl}^{\text{heq}}(a, \text{bdd}_Y(x_j : j < k), x_j : j < k).$$

Or, by X -internality of $\text{tp}(x_j : j < k)$ within $\text{bdd}_Y(\emptyset)$ there is $a' \in X$ with $(x_j : j < k) \in \text{dcl}^{\text{heq}}(a', \text{bdd}_Y(\emptyset))$. Then

$$\text{bdd}_Y(x_j : j < k) \subseteq \text{bdd}_Y(a', \text{bdd}_Y(\emptyset)) = \text{bdd}_Y(\emptyset)$$

by Corollary 1.7, and $x_k \in \text{bdd}(a, a', \text{bdd}_Y(\emptyset))$. So $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$, and $\text{tp}(x)$ is almost X -internal within $\text{bdd}_Y(\emptyset)$. \square

Corollary 3.3. *Suppose that X and Y are orthogonal. Let X' and Y' be \emptyset -hyperdefinable. If X' is X -internal within $X \cup Y$ and Y' is Y -internal within $X \cup Y$, then $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$.*

Proof: This is immediate from Proposition 3.2 and 1.5. \square

Corollary 3.4. *Suppose that X and Y are orthogonal. If an $(X \cup Y)$ -hyperdefinable set Z is almost X - and almost Y -internal within $X \cup Y$, then it is bounded.*

Proof: We may assume that Z is hyperdefinable over \emptyset . Let $z \in Z$. Since Z is almost X -internal, there is $b \in \text{bdd}_Y(\emptyset)$ and $x \in X$ such that $z \in \text{bdd}(xb)$. Let \bar{z} be the set of xb -conjugates of z . Then $\text{tp}(\bar{z})$ is X -internal within $\text{bdd}_Y(\emptyset)$, and still almost Y -internal within $X \cup Y$. So there is $a \in \text{bdd}_X(\emptyset)$ and $y \in Y$ such that $\bar{z} \in \text{bdd}(ya)$. Let $\bar{\bar{z}}$ be the set of ya -conjugates of \bar{z} . Then $\text{tp}(\bar{\bar{z}})$ is Y -internal within $\text{bdd}_X(\emptyset)$, and still X -internal within $\text{bdd}_Y(\emptyset)$. Hence $\text{tp}(\bar{\bar{z}}) \perp_{\text{bdd}_{XY}(\emptyset)} \text{tp}(\bar{\bar{z}})$, so $\text{tp}(\bar{\bar{z}})$ is bounded. Thus Z is bounded as well. \square

Corollary 3.5. *Suppose that X and Y are orthogonal. If $Z \subseteq (X \cup Y)^{\text{heq}}$ is \emptyset -hyperdefinable and almost X -internal within $X \cup Y$, then it is X -internal within $\text{bdd}_Y(\emptyset)$.*

Proof: By weak elimination of hyperimaginaries there is $x \in X^{\text{heq}}$ and $y \in Y^{\text{heq}}$ with $z \in \text{dcl}^{\text{heq}}(xy)$ and $xy \in \text{bdd}(z)$. So $\text{tp}(y)$ is Y -internal and almost X -internal, whence bounded. \square

4. LOCAL SIMPLICITY

Definition 4.1. Let $A \subseteq B$, and $\pi(x, B)$ be a partial type over B . We say that $\pi(x, B)$ does not divide over A if for any indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ the partial type

$$\bigcup_{i < \omega} \pi(x, B_i)$$

is consistent. Clearly, $\text{tp}(a/B)$ divides over A if and only if $\text{tp}(a_0/B)$ does so for some finite subtuple $a_0 \subseteq a$.

Example 4.2. If $\text{tp}(a) \perp \text{tp}(b)$, then $\text{tp}(a/b)$ does not divide over \emptyset .

We now define the appropriate version of local rank. We follow Ben Yaacov's terminology [1, Definition 1.4], more general than [5, Definition 4.3.5].

Definition 4.3. Let $\pi(x)$, $\Phi(x, y)$ and $\Psi(y_1, \dots, y_k)$ be partial types in (at most) countably many variables.

- (1) Ψ is a *k-inconsistency witness* for Φ if

$$\models \forall y_1 \dots y_k \neg \exists x [\Psi(y_1, \dots, y_k) \wedge \bigwedge_{i=1}^k \Phi(x, y_i)].$$

- (2) The *local (Φ, Ψ) -rank* $D(\cdot, \Phi, \Psi)$ is defined on partial types in x as follows:

- $D(\pi(x), \Phi, \Psi) \geq 0$ if $\pi(x)$ is consistent.
- $D(\pi(x), \Phi, \Psi) \geq n + 1$ if there is a sequence $(a_i : i < \omega)$ such that $\models \Psi(\bar{a})$ for any k -tuple $\bar{a} \subset (a_i : i < \omega)$, and $D(\pi(x) \wedge \Phi(x, a_i), \Phi, \Psi) \geq n$ for all $i < \omega$.

If $D(\pi, \Phi, \Psi) \geq n$ for all $n < \omega$, we put $D(\pi, \Phi, \Psi) = \infty$.

Note that in Definition 4.3(2) we may require $(a_i : i < \omega)$ to be indiscernible. Moreover, $D(\pi(x, a), \Phi, \Psi) \geq n$ is a closed condition on a , and $D(\text{tp}(x/a), \Phi, \Psi) \geq n$ is a closed condition on x over a .

Definition 4.4. Let I be an ordered set. A sequence $I = (a_i : i \in I)$ is *independent over A* , or *A -independent*, if $\text{tp}(a_i/A, a_j : j < i)$ does not divide over A for all $i \in I$. If $A \subseteq B$ and $p \in S(B)$, the sequence $(a_i : i \in I)$ is a *Morley sequence in p over A* if it is B -indiscernible, $a_i \models p$ and $\text{tp}(a_i/B, a_j : j < i)$ does not divide over A for all $i \in I$. If $A = B$, we simply call it a *Morley sequence in p* .

Fact 4.5. [5, Corollary 3.2.5] or [4, Proposition 16.12] *If $\text{tp}(b/cd)$ does not divide over d and $\text{tp}(a/cbd)$ does not divide over bd , then $\text{tp}(ab/cd)$ does not divide over d .*

Theorem 4.6. *Let X be a hyperdefinable set over \emptyset . The following are equivalent:*

- (1) *Symmetry holds on X : For all $a, b, c \in X$, $\text{tp}(a/bc)$ does not divide over b if and only if $\text{tp}(c/ab)$ does not divide over b .*
- (2) *Transitivity holds on X : If $a, b, c, d \in X$, then $\text{tp}(a/bcd)$ does not divide over b if and only if $\text{tp}(a/bc)$ does not divide over b and $\text{tp}(a/bcd)$ does not divide over bc .*
- (3) *Local character holds on X : There is κ such that for all countable $a \in X$ and $A \subset X$ there is $A_0 \subseteq A$ with $|A_0| \leq \kappa$ such that $\text{tp}(a/A)$ does not divide over A_0 . In fact, we can take $\kappa = 2^{|T|}$.*
- (4) *$D(\cdot, \Phi, \Psi) < \infty$ for any partial X -type $\Phi(x, y)$ and inconsistency witness Ψ for Φ .*
- (5) *For any $A \subseteq B \subset X$, a partial X -type $\pi(x, B)$ does not divide over A if and only if there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.*

If any of these conditions is satisfied, then for all $A \subseteq B \subset X$ and $a \in X$ the type $\text{tp}(a/B)$ does not divide over A if and only if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) . Moreover, Extension holds on X : For any partial X -type $\pi(x)$ over B , if π does not divide over A then it has a completion which does not divide over A .

Proof: (1) \Rightarrow (2) Clearly, if $\text{tp}(a/bcd)$ does not divide over b , it does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . Conversely, suppose that $\text{tp}(a/bcd)$ does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . By symmetry, $\text{tp}(d/abc)$ does not divide over bc and $\text{tp}(c/ab)$ does not divide over b . By Fact 4.5 $\text{tp}(cd/ab)$ does not divide over b , so again by symmetry $\text{tp}(a/bcd)$ does not divide over b .

(2) \Rightarrow (4) Suppose there is a partial X -type Φ and an inconsistency witness Ψ for Φ such that $D(x = x, \Phi, \Psi) = \infty$. This means that for all $n < \omega$ there are

$$(b_i, a_i^j : i < n, j < \omega)$$

such that $\models \Psi(\bar{a})$ for all $i < n$ and $\bar{a} \subset (a_i^j : j < \omega)$ of the right length, $b_i \models \bigwedge_{k \leq i} \Phi(x, a_k^0)$, and $(a_i^j : j < \omega)$ is indiscernible over $\{b_k a_k^0 : k < i\}$.

By Ramsey's Theorem and compactness we may assume that i is indexed by $\{\pm(1 + \frac{1}{n}) : n > 0\}$ and that the sequence $(b_i a_i^0)_i$ is indiscernible. Put $I = \{b_i a_i^0 : i < -1\}$ and $J = \{b_i a_i^0 : i > 1\}$. Then $\text{tp}(b_1/IJ)$ does not divide over I and $\text{tp}(b_1/IJ a_{-1})$ does not divide over IJ , since the former is finitely satisfiable in I and the latter in J . However, $(a_{-1}^j : j < \omega)$ witnesses that $\Phi(x, a_{-1})$, and hence $\text{tp}(b_1/IJ a_{-1})$, divides over I , contradicting transitivity.

(4) \Rightarrow (3) Assume (4). First, we note that for $A \subseteq B \subset X$, if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) , then $\text{tp}(a/B)$ does not divide over A . This is obvious, as if some A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ witnesses dividing, we can take $\Phi(x, y) = \text{tp}(a, B)$ and $\Psi = \text{tp}(B_1, B_2, \dots, B_n)$ for $n < \omega$ sufficiently large. Then Ψ is an n -inconsistency witness (clearly, we may restrict to countable B), and

$$D(\text{tp}(a/B), \Phi, \Psi) < D(\text{tp}(a/A), \Phi, \Psi).$$

Given $\text{tp}(a/A)$ it is hence enough to take $A_0 \subseteq A$ big enough such that

$$D(\text{tp}(a/A), \Phi, \Psi) = D(\text{tp}(a/A_0), \Phi, \Psi)$$

for all (Φ, Ψ) . There are only $2^{|T|}$ such pairs, so we need at most that many parameters.

(3) \Rightarrow (4) Suppose $D(x = x, \Phi, \Psi) = \infty$. Then for any cardinal κ we can find an indiscernible sequence $(a_i b_i : i \leq \kappa^+)$ as in (2) \Rightarrow (4), such that $b_{\kappa^+} \models \bigwedge_{i \leq \kappa^+} \Phi(x, a_i)$ and $\Phi(x, a_i)$ divides over $\{a_j : j < i\}$ for all $i \leq \kappa^+$. Then $\text{tp}(b_{\kappa^+}/a_i : i < \kappa^+)$ divides over any subset of its domain of cardinality $\leq \kappa$.

(4) \Rightarrow (5). Assume (4). Given $a \in X$ and $A \subseteq B \subset X$, for any pair (Φ, Ψ) and any formula $\varphi(y)$ we can adjoin either $\exists y [xEy \wedge \varphi(y)]$ or $\exists y [xEy \wedge \neg \varphi(y)]$ and preserve $D(\cdot, \Phi, \Psi)$ -rank. By compactness we can thus complete $\text{tp}(a/A)$ to a type p over B of the same $D(\cdot, \Phi, \Psi)$ -rank. In particular, no Φ -instance in p divides over A with Ψ as inconsistency witness. Coding finitely many pairs $(\Phi_i, \Psi_i : i < n)$ in a single one, one obtains an extension such that no Φ_i -instance Ψ_i -divides for any $i < n$; by compactness we can do this for all pairs (Φ, Ψ) simultaneously and obtain an extension which does not divide over A . Taking for B a sufficiently saturated model, a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(A, a_j : j < i)}$ is a Morley sequence in $\text{tp}(a/A)$.

This shows in particular that if $\pi(x, B)$ does not divide over A , then there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.

Conversely, suppose that $\pi(x, B)$ divides over A , as witnessed by an A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ with $\bigcup_{i < \omega} \pi(x, B_i)$ inconsistent. Take any Morley sequence I in $\text{tp}(B/A)$. By [5, Corollary 2.2.8] (which is shown there for real tuples, but transfers easily to hyperimaginaries) we may assume that $B_i \hat{\ } I$ is A -conjugate to I for all $i \in I$ and that $(B_i : i < \omega)$ is indiscernible over AI . If $\bar{\pi}(x) = \bigcup_{B' \in I} \pi(x, B')$ were consistent, then $(B_i : i < \omega)$ would witness that

$$D(\bar{\pi}(x) \wedge \pi(x, B_0), \pi(x, y), \Psi) < D(\bar{\pi}(x), \pi(x, y), \Psi)$$

for some inconsistency witness Ψ . But by A -conjugacy the two ranks must be equal, a contradiction.

(5) \Rightarrow (1) Let us first show *Extension*. If $A \subseteq B \subset X$ and $\pi(x, B)$ is a partial X -type which does not divide over A , let $(B_i : i < \alpha)$ be a very long Morley sequence in $\text{tp}(B/A)$. Consider any realization $a \models \bigwedge_{i < \alpha} \pi(x, B_i)$. Since α is large, there is an infinite subset $J \subset \alpha$ such that $p(aAX) = \text{tp}(aAB_i)$ is constant for $i \in J$. Then $p(xAB)$ is a completion of π which does not divide over A , as witnessed by $(B_i : i \in J)$.

Now given $a, b, c \in X$ such that $\text{tp}(a/bc)$ does not divide over b , let $B \ni bc$ be a sufficiently saturated model, and p an extension of $\text{tp}(a/bc)$ to B which does not divide over b . Choose a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(bc, a_j : j < i)}$. This is a Morley sequence in $\text{tp}(a/bc)$ over b . Then $(a_i : i < \omega)$ is a Morley sequence in $\text{tp}(a/b)$, and $a_i \models \text{tp}(a/bc)$ for all $i < \omega$. Hence $\text{tp}(c/ba)$ does not divide over b , and symmetry holds.

Finally we show that if (1) – (5) hold and $\text{tp}(a/B)$ does not divide over A for $A \subseteq B \subset X$ and $a \in X$, then $D(\text{tp}(a/A), \Phi, \Psi) \geq n$ implies $D(\text{tp}(a/B), \Phi, \Psi) \geq n$ for all (Φ, Ψ) . For $n = 0$ this is obvious. So suppose $D(\text{tp}(a/A), \Phi, \Psi) \geq n + 1$. Then there is $(d_i : i < \omega)$ indiscernible over A such that $\bar{d} \models \Psi$ for all $\bar{d} \subset (d_i : i < \omega)$ of the right length, and $D(\text{tp}(a/A) \wedge \Phi(x, d_i), \Phi, \Psi) \geq n$ for all $i < \omega$. Let q be a completion of $\text{tp}(a/A) \wedge \Phi(x, d_0)$ with $D(q, \Phi, \Psi) \geq n$. Clearly, we may assume $a \models q$, and that $\text{tp}(d_0/aB)$ does not divide over aA . As $\text{tp}(a/B)$ does not divide over A , by symmetry and transitivity $\text{tp}(ad_0/B)$ does not divide over A , and $\text{tp}(a/d_0B)$ does not divide over d_0A . By induction hypothesis,

$$D(\text{tp}(a/d_0A), \Phi, \Psi) \geq n \quad \text{implies} \quad D(\text{tp}(a/d_0B), \Phi, \Psi) \geq n.$$

As $\text{tp}(d_0/B)$ does not divide over A and $(d_i : i < \omega)$ is A -indiscernible, we may assume that it remains indiscernible over B . But then it witnesses

$$D(\text{tp}(a/B), \Phi, \Psi) \geq D(\text{tp}(a/d_0B), \Phi, \Psi) + 1 \geq n + 1. \quad \square$$

Definition 4.7. An A -hyperdefinable set X is *simple (over A)* if it satisfies any of the conditions of Theorem 4.6 when we adjoin A to the language. If X is simple over A and $a, b, c \in X$, we shall say that a and c are independent over Ab , written $a \perp_{Ab} c$, if $\text{tp}(a/Abc)$ does not divide over Ab .

Note that we only allow tuples and parameters from $A \cup X$. If X is stably embedded, we can of course allow parameters from anywhere.

Remark 4.8. If X is merely hyperdefinable, it may be simple although no definable or even type-definable imaginary set in the ambient structure is simple.

If X is simple, it is now standard to extend the notions of dividing and independence to hyperimaginaries in X^{heq} . Moreover, we can develop basic simplicity theory (canonical bases, the independence theorem, stratified ranks, generic types, stabilizers, see [5]) within X^{heq} , replacing the notion of a model by a boundedly closed subset C of X^{heq} such that X^{heq} -types over C are finitely satisfiable in C .

Proposition 4.9. *Let X and Y be orthogonal \emptyset -hyperdefinable sets such that X is simple over \emptyset . If $A \subset Y$ is a set of parameters, then X is simple over A , and over $\text{bdd}_Y(A)$. In particular, let Z be a set hyperdefinable over some parameters $A \subset X \cup Y$. If Z is X -internal within $X \cup Y$, then Z is simple over A ; if $Z \subseteq (X \cup Y)^{\text{heq}}$ is almost X -internal, then Z is simple as well.*

Proof: Simplicity over A is obvious from orthogonality; simplicity over $\text{bdd}_Y(A)$ follows. Now if Z is X -internal within $X \cup Y$, it lives in $\text{dcl}^{\text{eq}}(X, \text{bdd}_Y(A))$ by Corollary 3.2, and must be simple as well; if $Z \subseteq (X \cup Y)^{\text{heq}}$ is almost X -internal within $X \cup Y$, it is X -internal within $\text{bdd}_Y(\emptyset)$ by Corollary 3.5. \square

5. GROUPS INTERPRETABLE IN ORTHOGONAL SETS

Recall that two hyperdefinable subgroups H_1 and H_2 of some group G are *commensurable* if $H_1 \cap H_2$ has bounded index both in H_1 and in H_2 .

Theorem 5.1. *Suppose X and Y are orthogonal \emptyset -hyperdefinable sets in a structure \mathfrak{M} , and G is an \emptyset -hyperdefinable group in $(X \cup Y)^{heq}$. If X is simple over \emptyset , there is an \emptyset -hyperdefinable normal X -internal subgroup N of G such that the quotient G/N is Y -internal. N is unique up to commensurability.*

Proof: Let us first show uniqueness: If N' is a second such group, then $N/(N \cap N')$ and $N'/(N \cap N')$ are X -internal and Y -internal, and hence bounded by orthogonality of X and Y . Thus N and N' are commensurable.

By Theorem 2.2 every element $g \in G$ is of the form $(g_X, g_Y)_E$ for some $g_X \in X^{heq}$ and $g_Y \in Y^{heq}$, both bounded over g , and some type-definable equivalence relation E with bounded classes, depending on $\text{tp}(g)$. Hence $\text{tp}(g/g_Y)$ is X -internal and $\text{tp}(g/g_X)$ is Y -internal. Now if $h = (h_X, h_Y)_E$ and $gh = ((gh)_X, (gh)_Y)_E$, then

$$(gh)_X \in \text{bdd}(g_X, g_Y, h_X, h_Y),$$

whence $(gh)_X \in \text{bdd}(g_X, h_X)$ by Corollary 1.7. Similarly $(gh)_Y \in \text{bdd}(g_Y, h_Y)$.

Now X is simple, as is $\text{tp}(g, h/g_Y, h_Y)$ for any $g, h \in G$. Hence we can consider $g, h \in G$ such that $g \perp_{g_Y, h_Y} h$. Then for any stratified local rank D

$$\begin{aligned} (\dagger) \quad D(gh/(gh)_Y) &\geq D(gh/(gh)_Y, g_Y, h_Y, g) = D(h/g_Y, h_Y, g) \\ &= D(h/g_Y, h_Y) = D(h/h_Y), \end{aligned}$$

as also $h \perp_{h_Y} g_Y$ by orthogonality (Example 4.2). Similarly

$$D(gh/(gh)_Y) \geq D(g/g_Y).$$

Now suppose G is a subset of $(X^m \times Y^n)/E$, where m, n are at most countable. Then if $g = (\bar{x}_g, \bar{y}_g)_E$ we have $g \perp_{g_Y} \bar{y}_g$, and $g_Y \in \text{bdd}(\bar{y}_g)$, whence $D(g/g_Y) = D(g/\bar{y}_g)$. By compactness, there is a G -type $p((\bar{x}, \bar{y})_E)$ implying that $D((\bar{x}, \bar{y})_E/\bar{y})$ is maximal for all local stratified ranks. But if $g, h \models p$ with $g \perp_{g_Y, h_Y} h$, then we must have equality in (\dagger) , which implies that g, h and gh are pairwise independent over $g_Y, h_Y, (gh)_Y$. Put

$$S = \text{stab}(g/g_Y, h_Y, (gh)_Y).$$

By [3, Lemme 1.2 and Remarque 1.3] the element g is generic in the coset Sg , and this coset is hyperdefinable over $\text{bdd}(g_Y, h_Y, (gh)_Y)$. Moreover, S is X -internal since $\text{tp}(g/g_Y)$ is.

Suppose that S is not commensurable with S^h for some $h \in G$. Then SS^h is still X -internal, with

$$D(SS^h) \geq D(S) = D(p)$$

for every stratified local rank D , and for at least one such rank D_0 we have $D_0(SS^h) > D_0(p)$. Choose $g' \in SS^h$ with $D_0(g'/h) = D_0(SS^h)$. By pre-multiplying with a generic element of S and post-multiplying with a generic element of S^h , the inequality (†) implies that we may assume $D(g'/h) \geq D(p)$ for every stratified local rank D . However, $\text{tp}(g'/h)$ is X -internal, so $g'_Y \in \text{bdd}(g')$ implies $g'_Y \in \text{bdd}(h)$ by Lemma 1.7 and Proposition 3.2. Thus

$$D(g'/g'_Y) \geq D(g'/h) \geq D(p) \quad \text{and} \quad D_0(g'/g'_Y) \geq D_0(g'/h) > D_0(p),$$

contradicting our choice of p . Hence S is commensurable with all its conjugates, and the locally connected component N of S is normal in G [5, Definition 4.5.15 and Corollary 4.5.16]. In fact the same proof, with NZ instead of SS^h , shows that if Z is an X -internal hyperdefinable subset of G , then Z is covered by boundedly many cosets of N . In particular for any $g' \in G$ the type $\text{tp}(g'/g'_Y)$ is covered by boundedly many cosets of N . But then $g'N \in \text{bdd}(g'_Y)$, and G/N is almost Y -internal, whence Y -internal by Corollary 3.5. \square

Question 5.2. If X and Y are orthogonal type-definable sets and G is a relatively definable group in $(X \cup Y)^{eq}$, can we find a relatively definable normal X -internal subgroup N such that G/N is Y -internal?

Question 5.3. What can we say if neither X nor Y is simple? Is it true that in every hyperdefinable subgroup of $((X \cup Y)^{heq})$ there is a maximal normal hyperdefinable X -internal subgroup N_X , a maximal normal hyperdefinable Y -internal subgroup N_Y , an X -internal hyperdefinable local group G_X , a Y -internal hyperdefinable local group G_Y and a hyperdefinable locally bounded equivalence relation E on $G_X \times G_Y$ such that $G/(N_X N_Y)$ is isogenous, or even isomorphic, to $(G_X \times G_Y)/E$?

REFERENCES

- [1] Itai Ben Yaacov. *Simplicity in compact abstract theories*. J. Math. Logic 3(2):163–191, 2003.
- [2] Alessandro Berarducci and Marcello Mamino. *Groups definable in two orthogonal sorts*. Preprint. arXiv:1304.1380.
- [3] Thomas Blossier, Amador Martín Pizarro and Frank O. Wagner. *A la recherche du tore perdu*. Submitted.
<http://hal.archives-ouvertes.fr/hal-00758982>
- [4] Enrique Casanovas. *Simple Theories and Hyperimaginaries*. Lecture Notes in Logic 39. Cambridge University Press, 2011.

- [5] Frank O. Wagner. *Simple Theories*. Mathematics and Its Applications 503. Kluwer Academic Publishers, 2000.

UNIVERSITÉ DE LYON; CNRS; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN UMR5208, 43 BD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: `wagner@math.univ-lyon1.fr`