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THE RIGHT ANGLE TO LOOK AT ORTHOGONAL SETS

FRANK O. WAGNER

ABSTRACT. If X and Y are orthogonal hyperdefinable sets such that X is simple, then any group G interpretable in $X \cup Y$ has a normal hyperdefinable X -internal subgroup N such that G/N is Y -internal; N is unique up to commensurability. In order to make sense of this statement, local simplicity theory for hyperdefinable sets is developed.

INTRODUCTION

Two definable sets X and Y in some structure are said to be *orthogonal* if every definable subset of $X \times Y$ is a finite union of *rectangles*, i.e. of subsets of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ definable. It easily follows that if X and Y are orthogonal groups, every definable subgroup of $X \times Y$ is of the form $U \times V$ with $U \leq X$ and $V \leq Y$ subgroups. However, the situation is considerably more complicated for a group G definable, or more generally interpretable, in $X \cup Y$, as it need not be the sum of a group interpretable in X and a group interpretable in Y . In fact, an example by Berarducci and Mamino [2, Example 1.2] shows that G need not have *any* subgroup interpretable in either X or Y . However, they prove [2, Theorem 7.1] that if X is superstable of finite and definable Lascar rank, then any group G interpretable in $X \cup Y$ has a normal subgroup N interpretable in X , such that G/N is interpretable in Y .

In this paper we shall generalize their result to the case where X is merely simple. In this context, definability has to be replaced by type-definability, as even for a definable group the tools of simplicity theory in general only yield type-definable subgroups. In fact, we even have to study *hyperdefinable* groups, since the quotient G/N , for N type-definable, will be of that form. We therefore put ourselves in the

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hyperdefinable context and assume right from the start that our orthogonal sets X and Y are merely hyperdefinable. To this end, we shall include a quick development of local simplicity theory for hyperdefinable sets in section 4.

Another problem is that of parameters. The usual hypothesis would be that of *stable embedding*, i.e. that every hyperdefinable subset of X is hyperdefinable with parameters in X . We shall circumvent this issue by only ever considering parameters from $X \cup Y$, as orthogonality automatically yields stable embeddedness of X and of Y in $X \cup Y$.

We shall work in a big κ -saturated and strongly κ -homogeneous monster model \mathfrak{M} , where κ is bigger than any cardinality we wish to consider. We shall not usually distinguish between elements and tuples.

1. ORTHOGONALITY

Definition 1.1. A set X is *hyperdefinable* over some parameters A if it is of the form Y/E , where Y is a type-definable set in (at most) countably many variables and E a type-definable equivalence relation on Y , both with parameters in A . For an element $y \in Y$ we denote the class of y modulo E by y_E . If $y = (y_0, \dots, y_n)$ is a tuple, we put $y_E = ((y_0)_E, \dots, (y_n)_E)$; similarly, we write $y_E z$ if $y_i E z_i$ for all $i \leq n$.

For a tuple y_E in X and some parameters $B \supseteq A$, the type $\text{tp}(y_E/B)$ is given by all formulas over B of the form

$$\exists z \in Y [z E y \wedge \varphi(z)]$$

true of y . It is easy to see that (in the monster model) two tuples in X have the same type over B if and only if they are conjugate by an automorphism fixing B . We similarly define the type over a hyperimaginary set containing A .

Note that we can also consider the type of a hyperimaginary over B as a partial real type over B . If B is hyperimaginary, this corresponds to a partial type over a representative of B , and any two such types (for different representatives) are equivalent. We shall say that a partial type $\pi(x)$ is a (partial) X -type if $\pi(x) \vdash x \in Y$ and $\pi(x)$ is E -invariant.

If a is hyperimaginary, a *representative* for a is any real (or imaginary) tuple \bar{a} with $a \in \text{dcl}^{\text{heq}}(\bar{a})$.

From now on, all tuples and parameter sets are hyperimaginary, unless stated otherwise.

Definition 1.2. Let X, Y be A -hyperdefinable sets in some structure \mathfrak{M} . We say that X and Y are *orthogonal* over A , denoted $X \perp_A Y$, if for any tuples $a \in X$ and $b \in Y$, the partial type $\text{tp}(a/A) \cup \text{tp}(b/A)$ determines $\text{tp}(ab/A)$.

Example 1.3. If \mathfrak{M}_1 and \mathfrak{M}_2 are two structures and $\mathfrak{N} = \mathfrak{M}_1 \times \mathfrak{M}_2$ with a predicate X for \mathfrak{M} and a predicate Y for \mathfrak{N} , then X and Y are orthogonal in \mathfrak{N} .

Remark 1.4. If X and Y are orthogonal type-definable sets and $Z \subseteq X \times Y$ is relatively definable, then Z is a finite union of rectangles $A_i \times B_i$, where $A_i \subseteq X$ and $B_i \subseteq Y$ are relatively definable.

Proof: For any $z = (x, y) \in Z$ we have that $\text{tp}(x) \cup \text{tp}(y) \vdash (x, y) \in Z$. By compactness there are relatively definable subsets $A_z \subseteq X$ in $\text{tp}(x)$ and $B_z \subseteq Y$ in $\text{tp}(y)$ with $A_z \times B_z \subseteq Z$. Again by compactness, finitely many of these rectangles suffice to cover Z . \square

For the rest of this section, X and Y will be orthogonal \emptyset -hyperdefinable sets.

Proposition 1.5. *If $X' \subseteq X^{\text{heq}}$ and $Y' \subseteq Y^{\text{heq}}$ are hyperdefinable over some parameters $A \subseteq X^{\text{heq}} \cup Y^{\text{heq}}$, then $X' \perp_A Y'$.*

Proof: Suppose $A = (a, b)$ with $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, and consider tuples $a' \in X'$ and $b' \in Y'$. Choose representatives $\bar{a}, \bar{a}' \in X$ and $\bar{b}, \bar{b}' \in Y$. Then $\text{tp}(\bar{a}\bar{a}') \cup \text{tp}(\bar{b}\bar{b}') \vdash \text{tp}(\bar{a}\bar{a}'\bar{b}\bar{b}')$.

Now if $a'' \equiv_A a'$ and $b'' \equiv_A b'$, we can find A -conjugates $\tilde{a}\tilde{a}''$ of $\bar{a}\bar{a}'$ and $\tilde{b}\tilde{b}''$ of $\bar{b}\bar{b}'$ such that $a''\tilde{a}\tilde{a}'' \equiv_A a'\bar{a}\bar{a}'$ and $b''\tilde{b}\tilde{b}'' \equiv_A b'\bar{b}\bar{b}'$. By orthogonality of X and Y , we obtain $\tilde{a}''\tilde{a}\tilde{b}''\tilde{b} \equiv \bar{a}'\bar{a}\bar{b}\bar{b}'$, whence $a''ab''b \equiv a'ab'b$, and thus $a''b'' \equiv_A a'b'$. \square

Note that this in particular shows that orthogonality is preserved under adding parameters from $X^{\text{heq}} \cup Y^{\text{heq}}$.

Proposition 1.6. *X is stably embedded in $X \cup Y$: For tuples $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, every ab -hyperdefinable subset X' of X^{heq} is hyperdefinable over a .*

Proof: If $\Phi(x, a, b)$ hyperdefines X and $\Psi(Y) = \text{tp}(b)$, put

$$\Phi'(x, a) = \exists Y [\Psi(Y) \wedge \Phi(x, a, Y)].$$

Clearly $\Phi(x, a, b) \vdash \Phi'(x, a)$. Conversely, suppose $a' \models \Phi'(x, a)$, and choose $b' \models \Psi$ with $a' \models \Phi(x, Aa, b')$. By orthogonality $a'ab \equiv a'ab'$, whence $a' \models \Phi(x, a, b)$, and $\Phi'(x, a)$ hyperdefines X' . \square

We put $\text{dcl}_X^{\text{heq}}(A) = \text{dcl}^{\text{heq}}(A) \cap X^{\text{heq}}$ and $\text{bdd}_X(A) = \text{bdd}(A) \cap X^{\text{heq}}$.

Corollary 1.7. *Suppose $a \in X^{heq}$ and $b \in Y^{heq}$. Then*

$$\text{dcl}_X^{heq}(a, b) = \text{dcl}_X^{heq}(a) \quad \text{and} \quad \text{bdd}_X(a, b) = \text{bdd}_X(a).$$

Proof: Immediate from Proposition 1.6. □

2. WEAK ELIMINATION OF HYPERIMAGINARIES

In this section, X and Y will be \emptyset -hyperdefinable sets.

Definition 2.1. Let Z be \emptyset -hyperdefinable. We say that Z has *weak elimination of hyperimaginaries* with respect to X^{heq} and Y^{heq} if for every $z \in (X \cup Y)^{heq}$ there is some $x \in X^{heq}$ and $y \in Y^{heq}$ with $xy \in \text{bdd}(z)$ and $z \in \text{dcl}^{heq}(xy)$.

Theorem 2.2. *If X and Y are orthogonal, then $X \cup Y$ has weak elimination of hyperimaginaries with respect to X^{heq} and Y^{heq} .*

Proof: Consider $z \in (X \cup Y)^{heq}$, say $z = (z_X, z_Y)_E$ for some tuples $z_X \in X$, $z_Y \in Y$ and type-definable equivalence relation E . For $x \equiv z_X$ consider the hyperdefinable equivalence relation E_x on $\text{tp}(z_Y)$ given by

$$yE_x y' \iff (x, y)E(x, y').$$

Then E_x is \emptyset -hyperdefinable by Proposition 1.6, and we may replace z_Y by z_Y/E_x . Similarly, we may replace z_X by z_X/E_y . We claim that now E has bounded classes on $\text{tp}(z_X) \times \text{tp}(z_Y)$. If not, there is a non-constant indiscernible sequence $(x_i, y_i : i < \omega)$ in some E -class. By orthogonality, for $i < j$

$$\text{tp}(x_i, x_j) \cup \text{tp}(y_i, y_j) \vdash \text{tp}((x_i, y_i), (x_j, y_j)).$$

But $\text{tp}(y_i, y_j) = \text{tp}(y_i, y_k)$ for $i < j < k$, whence

$$\text{tp}((x_i, y_i), (x_j, y_j)) = \text{tp}((x_i, y_i), (x_j, y_k))$$

and $(x_j, y_k)E(x_i, y_i)E(x_j, y_j)$ holds. By triviality of E_x we get $y_j = y_k$, whence $x_j = x_k$ by triviality of E_y , a contradiction.

Thus $(z_X, z_Y) \in \text{bdd}(z)$, and trivially $z \in \text{dcl}^{heq}(z_X, z_Y)$. □

We shall put $\text{bdd}_{XY}(A) = \text{bdd}(A) \cap (X^{heq} \cup Y^{heq})$.

Corollary 2.3. *If $X \perp Y$, then $\text{bdd}_{XY}(A)$ and $\text{bdd}(A) \cap (X \cup Y)^{heq}$ are interdefinable for any $A \subset X \cup Y$. Moreover, for $a \in (X \cup Y)^{heq}$ we have $\text{tp}(a/\text{bdd}_{XY}(A)) \vdash \text{tp}(a/\text{bdd}(A))$.*

Proof: The first assertion is immediate from weak elimination of hyperimaginaries. For the second assertion, note that equality of type over $\text{bdd}(A)$ is a bounded type-definable equivalence relation, and the class of a is definable over $\text{bdd}_{XY}(A)$. \square

3. INTERNALITY AND ANALYSABILITY

Again, X and Y will be \emptyset -hyperdefinable sets.

Definition 3.1. We say that X is (almost) Y -internal if there is some parameter set A such that for every $a \in X$ there is a tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Ab)$ (or $a \in \text{bdd}(Ab)$, respectively).¹

If the parameters A can be chosen in some set Z , we say that X is Y -internal *within* Z .

We say that X is Y -analysable (within Z) if there is a sequence $(a_i : i < \alpha)$ such that $\text{tp}(a_i/a_j : j < i)$ is Y -internal (within Z) for every $i < \alpha$, and $a \in \text{bdd}(a_i : i < \alpha)$.

Proposition 3.2. *Suppose that X and Y are orthogonal. If an \emptyset -hyperdefinable set X' is X -analysable within $X \cup Y$, then X' is almost X -internal within $\text{bdd}_Y(\emptyset)$; if X' is X -internal within $X \cup Y$, then X' is X -internal within $\text{bdd}_Y(\emptyset)$.*

Proof: We first show the assertion for (almost) X -internal X' . So suppose $\bar{a} \in X$ and $\bar{b} \in Y$ is a tuple such that for every $x \in X'$ there is $a \in X$ with $x \in \text{bdd}(\bar{a}\bar{b}a)$. Let \bar{x} be the set of $\bar{a}\bar{b}a$ -conjugates of x , and E the type-definable equivalence relation on $\text{tp}(\bar{a}\bar{b}a)$ given by

$$(\bar{a}'\bar{b}'a')E(\bar{a}''\bar{b}''a'') \Leftrightarrow \exists \bar{x}' \bar{x}'\bar{a}'\bar{b}'a' \equiv \bar{x}'\bar{a}''\bar{b}''a'' \equiv \bar{x}\bar{a}\bar{b}a.$$

Clearly, \bar{x} is interdefinable with $(\bar{a}\bar{b}a)_E$. By weak elimination of hyperimaginaries, there is $\tilde{a} \in X^{\text{heq}}$ and $\tilde{b} \in Y^{\text{heq}}$ with $\tilde{a}\tilde{b} \in \text{bdd}(\bar{x})$ and $\bar{x} \in \text{dcl}^{\text{heq}}(\tilde{a}\tilde{b})$. Note that if X' is X -internal and $x \in \text{dcl}^{\text{heq}}(\bar{a}\bar{b}a)$, then $x = \bar{x}$.

Suppose $\tilde{b} \notin \text{bdd}_Y(\emptyset)$. Then there is an \emptyset -conjugate \tilde{b}' of \tilde{b} outside $\text{bdd}(\bar{a}\tilde{b})$; if σ is an automorphism mapping \tilde{b}' to \tilde{b} , put $\bar{a}'\bar{b}' = \sigma(\bar{a}\tilde{b})$. Then $\tilde{b} \notin \text{bdd}(\bar{a}'\bar{b}')$. On the other hand, since $\bar{a}'\bar{b}' \equiv \bar{a}\tilde{b}$ and $\bar{x} \subset X'$, there is $a' \in X$ with $x' \in \text{bdd}(\bar{a}'\bar{b}'a')$ for all $x' \in \bar{x}$. Then in particular $\tilde{b} \in \text{bdd}(\bar{a}'\bar{b}'a')$, whence $\tilde{b} \in \text{bdd}(\bar{b}')$ by Corollary 1.7, a contradiction.

¹In simplicity theory, this is called *finite generation*; for *internality* we would require for every $a \in X$ the existence of some $A \downarrow a$ and tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Ab)$.

Now assume that $x \in X'$ and $(x_i : i < \alpha)$ is an X -analysis of x within $X \cup Y$. We show inductively on i that $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. So suppose $\text{tp}(x_j : j < k)$ is X -internal within $\text{bdd}_Y(\emptyset)$ for all $k < i$. If i is limit, then clearly $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. If $i = k + 1$, then $\text{tp}(x_k/x_j : j < k)$ is X -internal within $\text{bdd}_Y(x_j : j < k)$, and there is $a \in X$ with

$$x_k \in \text{dcl}^{\text{heq}}(a, \text{bdd}_Y(x_j : j < k), x_j : j < k).$$

Or, by X -internality of $\text{tp}(x_j : j < k)$ within $\text{bdd}_Y(\emptyset)$ there is $a' \in X$ with $(x_j : j < k) \in \text{dcl}^{\text{heq}}(a', \text{bdd}_Y(\emptyset))$. Then

$$\text{bdd}_Y(x_j : j < k) \subseteq \text{bdd}_Y(a', \text{bdd}_Y(\emptyset)) = \text{bdd}_Y(\emptyset)$$

by Corollary 1.7, and $x_k \in \text{bdd}(a, a', \text{bdd}_Y(\emptyset))$. So $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$, and $\text{tp}(x)$ is almost X -internal within $\text{bdd}_Y(\emptyset)$. \square

Corollary 3.3. *Suppose that X and Y are orthogonal. Let X' and Y' be \emptyset -hyperdefinable. If X' is X -internal within $X \cup Y$ and Y' is Y -internal within $X \cup Y$, then $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$.*

Proof: This is immediate from Proposition 3.2 and 1.5. \square

Corollary 3.4. *Suppose that X and Y are orthogonal. If an $(X \cup Y)$ -hyperdefinable set Z is almost X - and almost Y -internal within $X \cup Y$, then it is bounded.*

Proof: We may assume that Z is hyperdefinable over \emptyset . Let $z \in Z$. Since Z is almost X -internal, there is $b \in \text{bdd}_Y(\emptyset)$ and $x \in X$ such that $z \in \text{bdd}(xb)$. Let \bar{z} be the set of xb -conjugates of z . Then $\text{tp}(\bar{z})$ is X -internal within $\text{bdd}_Y(\emptyset)$, and still almost Y -internal within $X \cup Y$. So there is $a \in \text{bdd}_X(\emptyset)$ and $y \in Y$ such that $\bar{z} \in \text{bdd}(ya)$. Let $\bar{\bar{z}}$ be the set of ya -conjugates of \bar{z} . Then $\text{tp}(\bar{\bar{z}})$ is Y -internal within $\text{bdd}_X(\emptyset)$, and still X -internal within $\text{bdd}_Y(\emptyset)$. Hence $\text{tp}(\bar{\bar{z}}) \perp_{\text{bdd}_{XY}(\emptyset)} \text{tp}(\bar{\bar{z}})$, so $\text{tp}(\bar{\bar{z}})$ is bounded. Thus Z is bounded as well. \square

Corollary 3.5. *Suppose that X and Y are orthogonal. If $Z \subseteq (X \cup Y)^{\text{heq}}$ is \emptyset -hyperdefinable and almost X -internal within $X \cup Y$, then it is X -internal within $\text{bdd}_Y(\emptyset)$.*

Proof: By weak elimination of hyperimaginaries there is $x \in X^{\text{heq}}$ and $y \in Y^{\text{heq}}$ with $z \in \text{dcl}^{\text{heq}}(xy)$ and $xy \in \text{bdd}(z)$. So $\text{tp}(y)$ is Y -internal and almost X -internal, whence bounded. \square

4. LOCAL SIMPLICITY

Definition 4.1. Let $A \subseteq B$, and $\pi(x, B)$ be a partial type over B . We say that $\pi(x, B)$ does not divide over A if for any indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ the partial type

$$\bigcup_{i < \omega} \pi(x, B_i)$$

is consistent. Clearly, $\text{tp}(a/B)$ divides over A if and only if $\text{tp}(a_0/B)$ does so for some finite subtuple $a_0 \subseteq a$.

Example 4.2. If $\text{tp}(a) \perp \text{tp}(b)$, then $\text{tp}(a/b)$ does not divide over \emptyset .

We now define the appropriate version of local rank. We follow Ben Yaacov's terminology [1, Definition 1.4], more general than [5, Definition 4.3.5].

Definition 4.3. Let $\pi(x)$, $\Phi(x, y)$ and $\Psi(y_1, \dots, y_k)$ be partial types in (at most) countably many variables.

(1) Ψ is a *k-inconsistency witness* for Φ if

$$\models \forall y_1 \dots y_n \neg \exists x [\Psi(y_1, \dots, y_n) \wedge \bigwedge_{i=1}^n \Phi(x, y_i)].$$

(2) The *local (Φ, Ψ) -rank* $D(\cdot, \Phi, \Psi)$ is defined on partial types in x as follows:

- $D(\pi(x), \Phi, \Psi) \geq 0$ if $\pi(x)$ is consistent.
- $D(\pi(x), \Phi, \Psi) \geq n + 1$ if there is a sequence $(a_i : i < \omega)$ such that $\models \Psi(\bar{a})$ for any k -tuple $\bar{a} \subset (a_i : i < \omega)$, and $D(\pi(x) \wedge \Phi(x, a_i), \Phi, \Psi) \geq n$ for all $i < \omega$.

If $D(\pi, \Phi, \Psi) \geq n$ for all $n < \omega$, we put $D(\pi, \Phi, \Psi) = \infty$.

Note that in Definition 4.3 (2) we may require $(a_i : i < \omega)$ to be indiscernible. Moreover, $D(\pi(x, a), \Phi, \Psi) \geq n$ is a closed condition on a , and $D(\text{tp}(x/a), \Phi, \Psi) \geq n$ is a closed condition on x over a .

Definition 4.4. Let I be an ordered set. A sequence $I = (a_i : i \in I)$ is *independent over A* , or *A -independent*, if $\text{tp}(a_i/A, a_j : j < i)$ does not divide over A for all $i \in I$. If $A \subseteq B$ and $p \in S(B)$, the sequence $(a_i : i \in I)$ is a *Morley sequence in p over A* if it is B -indiscernible, $a_i \models p$ and $\text{tp}(a_i/B, a_j : j < i)$ does not divide over A for all $i \in I$. If $A = B$, we simply call it a *Morley sequence in p* .

Fact 4.5. [5, Corollary 3.2.5] or [4, Proposition 16.12] *If $\text{tp}(b/cd)$ does not divide over d and $\text{tp}(a/cbd)$ does not divide over bd , then $\text{tp}(ab/cd)$ does not divide over d .*

Theorem 4.6. *Let X be a hyperdefinable set over \emptyset . The following are equivalent:*

- (1) *Symmetry holds on X : For all $a, b, c \in X$, $\text{tp}(a/bc)$ does not divide over b if and only if $\text{tp}(c/ab)$ does not divide over b .*
- (2) *Transitivity holds on X : If $a, b, c, d \in X$, then $\text{tp}(a/bcd)$ does not divide over b if and only if $\text{tp}(a/bc)$ does not divide over b and $\text{tp}(a/bcd)$ does not divide over bc .*
- (3) *Local character holds on X : There is κ such that for all countable $a \in X$ and $A \subset X$ there is $A_0 \subseteq A$ with $|A_0| \leq \kappa$ such that $\text{tp}(a/A)$ does not divide over A_0 . In fact, we can take $\kappa = 2^{|T|}$.*
- (4) *$D(\cdot, \Phi, \Psi) < \omega$ for any partial X -types $\Phi(x, y)$ and inconsistency witness Ψ for Φ .*
- (5) *For any $A \subseteq B \subset X$, a partial X -type $\pi(x, B)$ does not divide over A if and only if there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.*

If any of these conditions is satisfied, then for all $A \subseteq B \subset X$ and $a \in X$ the type $\text{tp}(a/B)$ does not divide over A if and only if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) . Moreover, Extension holds on X : For any partial X -type $\pi(x)$ over B , if π does not divide over A then it has a completion which does not divide over A .

Proof: (1) \Rightarrow (2) Clearly, if $\text{tp}(a/bcd)$ does not divide over b , it does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . Conversely, suppose that $\text{tp}(a/bcd)$ does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . By symmetry, $\text{tp}(d/abc)$ does not divide over bc and $\text{tp}(c/ab)$ does not divide over b . By Fact 4.5 $\text{tp}(cd/ab)$ does not divide over b , so again by symmetry $\text{tp}(a/bcd)$ does not divide over b .

(2) \Rightarrow (4) Suppose there is a partial X -type Φ and an inconsistency witness Ψ for Φ such that $D(x = x, \Phi, \Psi) = \infty$. This means that for all $n < \omega$ there are

$$(b_i, a_i^j : i < n, j < \omega)$$

such that $\models \Psi(\bar{a})$ for all $i < n$ and $\bar{a} \subset (a_i^j : j < \omega)$ of the right length, $b_i \models \bigwedge_{k \leq i} \Phi(x, a_k^0)$, and $(a_i^j : j < \omega)$ is indiscernible over $\{b_k a_k^0 : k < i\}$. By Ramsey's Theorem and compactness we may assume that i is indexed by $\{\pm(1 + \frac{1}{n}) : n > 0\}$ and that the sequence $(b_i a_i^0)_i$ is indiscernible. Put $I = \{b_i : i < -1\}$ and $J = \{b_i : i > 1\}$. Then $\text{tp}(b_1/IJ)$ does not divide over I and $\text{tp}(b_1/IJa_{-1})$ does not divide over IJ , since the former is finitely satisfiable in I and the latter in J . However,

$(a_{-1}^j : j < \omega)$ witnesses that $\Phi(x, a_{-1})$, and hence $\text{tp}(b_1/IJa_{-1})$, divides over I , contradicting transitivity.

(4) \Rightarrow (3) Assume (4). First, we note that for $A \subseteq B \subset X$, if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) , then $\text{tp}(a/B)$ does not divide over A . This is obvious, as if some A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ witnesses dividing, we can take $\Phi(x, y) = \text{tp}(a, B)$ and $\Psi = \text{tp}(B_1, B_2, \dots, B_n)$ for $n < \omega$ sufficiently large. Then Ψ is an n -inconsistency witness (clearly, we may restrict to countable B), and

$$D(\text{tp}(a/B), \Phi, \Psi) < D(\text{tp}(a/A), \Phi, \Psi).$$

Given $\text{tp}(a/A)$ it is hence enough to take $A_0 \subseteq A$ big enough such that

$$D(\text{tp}(a/A), \Phi, \Psi) = D(\text{tp}(a/A_0), \Phi, \Psi)$$

for all (Φ, Ψ) . There are only $2^{|T|}$ such pairs, so we need at most that many parameters.

(3) \Rightarrow (4) Suppose $D(x = x, \Phi, \Psi) = \infty$. Then for any cardinal κ we can find an indiscernible sequence $(a_i b_i : i \leq \kappa^+)$ as in (2) \Rightarrow (4), such that $b_{\kappa^+} \models \bigwedge_{i \leq \kappa^+} \Phi(x, a_i)$ and $\Phi(x, a_i)$ divides over $\{a_j : j < i\}$ for all $i \leq \kappa^+$. Then $\text{tp}(b_{\kappa^+}/a_i : i < \kappa^+)$ divides over any subset of its domain of cardinality $\leq \kappa$.

(4) \Rightarrow (5). Assume (4). Given $a \in X$ and $A \subseteq B \subset X$, for any pair (Φ, Ψ) and any formula $\varphi(y)$ we can adjoin either $\exists y [xEy \wedge \varphi(y)]$ or $\exists y [xEy \wedge \neg\varphi(y)]$ and preserve $D(\cdot, \Phi, \Psi)$ -rank. By compactness we can thus complete $\text{tp}(a/A)$ to a type p over B of the same $D(\cdot, \Phi, \Psi)$ -rank. In particular, no Φ -instance in p divides over A with Ψ as inconsistency witness. Coding finitely many pairs $(\Phi_i, \Psi_i : i < n)$ in a single one, one obtains an extension such that no Φ_i -instance Ψ_i -divides for any $i < n$; by compactness we can do this for all pairs (Φ, Ψ) simultaneously and obtain an extension which does not divide over A . Taking for B a sufficiently saturated model, a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(A, a_j : j < i)}$ is a Morley sequence in $\text{tp}(a/A)$.

This shows in particular that if $\pi(x, B)$ does not divide over A , then there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.

Conversely, suppose that $\pi(x, B)$ divides over A , as witnessed by an A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ with $\bigcup_{i < \omega} \pi(x, B_i)$ inconsistent. Take any Morley sequence I in $\text{tp}(B/A)$ (if there is none,

we are done). By [5, Corollary 2.2.8] (which is shown there for real tuples, but transfers easily to hyperimaginaries) we may assume that $B_i \hat{\ } I$ is A -conjugate to I for all $i \in I$ and that $(B_i : i < \omega)$ is indiscernible over AI . If $\bar{\pi}(x) = \bigcup_{B' \in I} \pi(x, B')$ were consistent, then $(B_i : i < \omega)$ would witness that

$$D(\bar{\pi}(x) \wedge \pi(x, B_0), \pi(x, y), \Psi) < D(\bar{\pi}(x), \pi(x, y), \Psi)$$

for some inconsistency witness Ψ . But by A -conjugacy the two ranks must be equal, a contradiction.

(5) \Rightarrow (1) Let us first show *Extension*. If $A \subseteq B \subset X$ and $\pi(x, B)$ is a partial X -type which does not divide over A , let $(B_i : i < \alpha)$ be a very long Morley sequence in $\text{tp}(B/A)$. Consider any realization $a \models \bigwedge_{i < \alpha} \pi(x, B_i)$. Since α is large, there is an infinite subset $J \subset \alpha$ such that $p(aAX) = \text{tp}(aAB_i)$ is constant for $i \in J$. Then $p(xAB)$ is a completion of π which does not divide over A , as witnessed by $(B_i : i \in J)$.

Now given $a, b, c \in X$ such that $\text{tp}(a/bc)$ does not divide over b , let $B \ni bc$ be a sufficiently saturated model, and p an extension of $\text{tp}(a/bc)$ to B which does not divide over b . Choose a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(bc, a_j : j < i)}$. This is a Morley sequence in $\text{tp}(a/bc)$ over b . Then $(a_i : i < \omega)$ is a Morley sequence in $\text{tp}(a/b)$, and $a_i \models \text{tp}(a/bc)$ for all $i < \omega$. Hence $\text{tp}(c/ba)$ does not divide over b , and symmetry holds.

Finally we show that if (1) – (5) hold and $\text{tp}(a/B)$ does not divide over A for $A \subseteq B \subset X$ and $a \in X$, then $D(\text{tp}(a/A), \Phi, \Psi) \geq n$ implies $D(\text{tp}(a/B), \Phi, \Psi) \geq n$ for all (Φ, Ψ) . For $n = 0$ this is obvious. So suppose $D(\text{tp}(a/A), \Phi, \Psi) \geq n + 1$. Then there is $(d_i : i < \omega)$ indiscernible over A such that $\bar{d} \models \Psi$ for all $\bar{d} \subset (d_i : i < \omega)$ of the right length, and $D(\text{tp}(a/A) \wedge \Phi(x, d_i), \Phi, \Psi) \geq n$ for all $i < \omega$. Let q be a completion of $\text{tp}(a/A) \wedge \Phi(x, d_0)$ with $D(q, \Phi, \Psi) \geq n$. Clearly, we may assume $a \models q$, and that $\text{tp}(d_0/aB)$ does not divide over aA . As $\text{tp}(a/B)$ does not divide over A , by symmetry and transitivity $\text{tp}(ad_0/B)$ does not divide over A , and $\text{tp}(a/d_0B)$ does not divide over d_0A . By induction hypothesis,

$$D(\text{tp}(a/d_0A), \Phi, \Psi) \geq n \quad \text{implies} \quad D(\text{tp}(a/d_0B), \Phi, \Psi) \geq n.$$

As $\text{tp}(d_0/B)$ does not divide over A and $(d_i : i < \omega)$ is A -indiscernible, we may assume that it remains indiscernible over B . But then it witnesses

$$D(\text{tp}(a/B), \Phi, \Psi) \geq D(\text{tp}(a/d_0B), \Phi, \Psi) + 1 \geq n + 1. \quad \square$$

Definition 4.7. An A -hyperdefinable set X is *simple (over A)* if it satisfies any of the conditions of Theorem 4.6 when we adjoin A to the language. If X is simple over A and $a, b, c \in X$, we shall say that a and c are independent over Ab , written $a \perp_{Ab} c$, if $\text{tp}(a/Abc)$ does not divide over Ab .

Note that we only allow tuples and parameters from $A \cup X$. If X is stably embedded, we can of course allow parameters from anywhere.

Remark 4.8. If X is merely hyperdefinable, it may be simple although no definable or even type-definable imaginary set in the ambient structure is simple.

If X is simple, it is now standard to extend the notions of dividing and independence to hyperimaginaries in X^{heq} . Moreover, we can develop basic simplicity theory (canonical bases, the independence theorem, stratified ranks, generic types, stabilizers, see [5]) within X^{heq} , replacing the notion of a model by a boundedly closed subset C of X^{heq} such that X^{heq} -types over C are finitely satisfiable in C .

Proposition 4.9. *Let X and Y be orthogonal \emptyset -hyperdefinable sets such that X is simple over \emptyset . If $A \subset Y$ is a set of parameters, then X is simple over A , and over $\text{bdd}_Y(A)$. In particular, if X' is X -internal within $X \cup Y$, then X' is simple over its parameters of definition.*

Proof: Simplicity over A is obvious from orthogonality; simplicity over $\text{bdd}_Y(A)$ follows. But if X' is hyperdefinable over A and X -internal within $X \cup Y$, it lives in $\text{dcl}^{\text{eq}}(X, \text{bdd}_Y(A))$ by Corollary 3.2, and must be simple as well. \square

5. GROUPS INTERPRETABLE IN ORTHOGONAL SETS

Recall that two hyperdefinable subgroups H_1 and H_2 of some group G are *commensurable* if $H_1 \cap H_2$ has bounded index both in H_1 and in H_2 .

Theorem 5.1. *Suppose X and Y are orthogonal \emptyset -hyperdefinable sets in a structure \mathfrak{M} , and G is an \emptyset -hyperdefinable group in $(X \cup Y)^{\text{heq}}$. If X is simple over \emptyset , there is an \emptyset -hyperdefinable normal X -internal subgroup N of G such that the quotient G/N is Y -internal. N is unique up to commensurability.*

Proof: Let us first show uniqueness: If N' is a second such group, then $N/(N \cap N')$ and $N'/(N \cap N')$ are X -internal and Y -internal, and

hence bounded by orthogonality of X and Y . Thus N and N' are commensurable.

By Theorem 2.2 every element $g \in G$ is of the form $(g_X, g_Y)_E$ for some $g_X \in X^{heq}$ and $g_Y \in Y^{heq}$, both bounded over g , and some type-definable equivalence relation E with bounded classes, depending on $\text{tp}(g)$. Hence $\text{tp}(g/g_Y)$ is X -internal and $\text{tp}(g/g_X)$ is Y -internal. Now if $h = (h_X, h_Y)_E$ and $gh = ((gh)_X, (gh)_Y)_E$, then

$$(gh)_X \in \text{bdd}(g_X, g_Y, h_X, h_Y),$$

whence $(gh)_X \in \text{bdd}(g_X, h_X)$ by Corollary 1.7. Similarly $(gh)_Y \in \text{bdd}(g_Y, h_Y)$.

Now X is simple, as is $\text{tp}(g, h/g_Y, h_Y)$ for any $g, h \in G$. Hence we can consider $g, h \in G$ such that $g \perp_{g_Y, h_Y} h$. Then for any stratified local rank D

$$\begin{aligned} (\dagger) \quad D(gh/(gh)_Y) &\geq D(gh/(gh)_Y, g_Y, h_Y, g) = D(h/g_Y, h_Y, g) \\ &= D(h/g_Y, h_Y) = D(h/h_Y), \end{aligned}$$

as also $h \perp_{h_Y} g_Y$ by orthogonality (Example 4.2). Similarly

$$D(gh/(gh)_Y) \geq D(g/g_Y).$$

Now suppose G is a subset of $(X^m \times Y^n)/E$. Then if $g = (\bar{x}_g, \bar{y}_g)_E$ we have $g \perp_{g_Y} \bar{y}_g$, and $g_Y \in \text{bdd}(\bar{y}_g)$, whence $D(g/g_Y) = D(g/\bar{y}_g)$. By compactness, there is a G -type $p((\bar{x}, \bar{y})_E)$ implying that $D((\bar{x}, \bar{y})_E/\bar{y})$ is maximal for all local stratified ranks. But if $g, h \models p$ with $g \perp_{g_Y, h_Y} h$, then we must have equality in (\dagger) , which implies that g, h and gh are pairwise independent over $g_Y, h_Y, (gh)_Y$. Put

$$S = \text{stab}(g/g_Y, h_Y, (gh)_Y).$$

By [3, Lemme 1.2 and Remarque 1.3] the element g is generic in the coset Sg , and this coset is hyperdefinable over $\text{bdd}(g_Y, h_Y, (gh)_Y)$. Moreover, S is X -internal since $\text{tp}(g/g_Y)$ is.

Suppose that S is not commensurable with S^h for some $h \in G$. Then SS^h is still X -internal, with

$$D(SS^h) \geq D(S) = D(p)$$

for every stratified local rank D , and for at least one such rank D_0 we have $D_0(SS^h) > D_0(p)$. Choose $g' \in SS^h$ with $D_0(g'/h) = D_0(SS^h)$. By pre-multiplying with a generic element of S and post-multiplying with a generic element of S^h , the inequality (\dagger) implies that we may assume $D(g'/h) \geq D(p)$ for every stratified local rank D . However,

$\text{tp}(g'/h)$ is X -internal, so $g'_Y \in \text{bdd}(g')$ implies $g'_Y \in \text{bdd}(h)$ by Lemma 1.7 and Proposition 3.2. Thus

$$D(g'/g'_Y) \geq D(g'/h) \geq D(p) \quad \text{and} \quad D_0(g'/g'_Y) \geq D_0(g'/h) > D_0(p),$$

contradicting our choice of p . Hence S is commensurable with all its conjugates, and the locally connected component N of S is normal in G [5, Definition 4.5.15 and Corollary 4.5.16]. In fact the same proof, with NZ instead of SS^h , shows that if Z is an X -internal hyperdefinable subset of G , then Z is covered by boundedly many cosets of N . In particular for any $g' \in G$ the type $\text{tp}(g'/g'_Y)$ is covered by boundedly many cosets of N . But then $g'N \in \text{bdd}(g'_Y)$, and G/N is almost Y -internal, whence Y -internal by Corollary 3.5. \square

Question 5.2. If X and Y are orthogonal type-definable sets and G is a relatively definable group in $(X \cup Y)^{eq}$, can we find a relatively definable normal X -internal subgroup N such that G/N is Y -internal?

Question 5.3. What can we say if neither X nor Y is simple? Is it true that in every hyperdefinable subgroup of $((X \cup Y)^{heq})$ there is a maximal normal hyperdefinable X -internal subgroup N_X , a maximal normal hyperdefinable Y -internal subgroup N_Y , an X -internal hyperdefinable local group G_X , a Y -internal hyperdefinable local group G_Y and a hyperdefinable locally bounded equivalence relation E on $G_X \times G_Y$ such that $G/(N_X N_Y)$ is isogenous, or even isomorphic, to $(G_X \times G_Y)/E$?

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