



## OBJECTIVES AND MAIN RESULTS

The objectives are the following:

- to develop an asymptotic approximation of the Fisher information (FI) for the estimation of a scalar parameter based on quantized measurements.
- To characterize asymptotically the FI as a function of the quantizer.
- To maximize the FI as a function of the quantization thresholds, which are characterized asymptotically by the quantization interval density.
- To obtain a practical approximation of the optimal quantization thresholds.

The following results are obtained:

- the FI for quantized measurements tends exponentially to the FI for continuous measurements when the number of quantization bits tends to infinity.
- The optimal interval density depends not only on the measurement density but also on the derivative of the score function for the estimation problem.
- In the location estimation problem the performance gap between uniform and nonuniform quantization is small.
- The practical implementation of the optimal asymptotic quantizer may depend on the true parameter, indicating that an adaptive approach may be necessary.

## CONTEXT

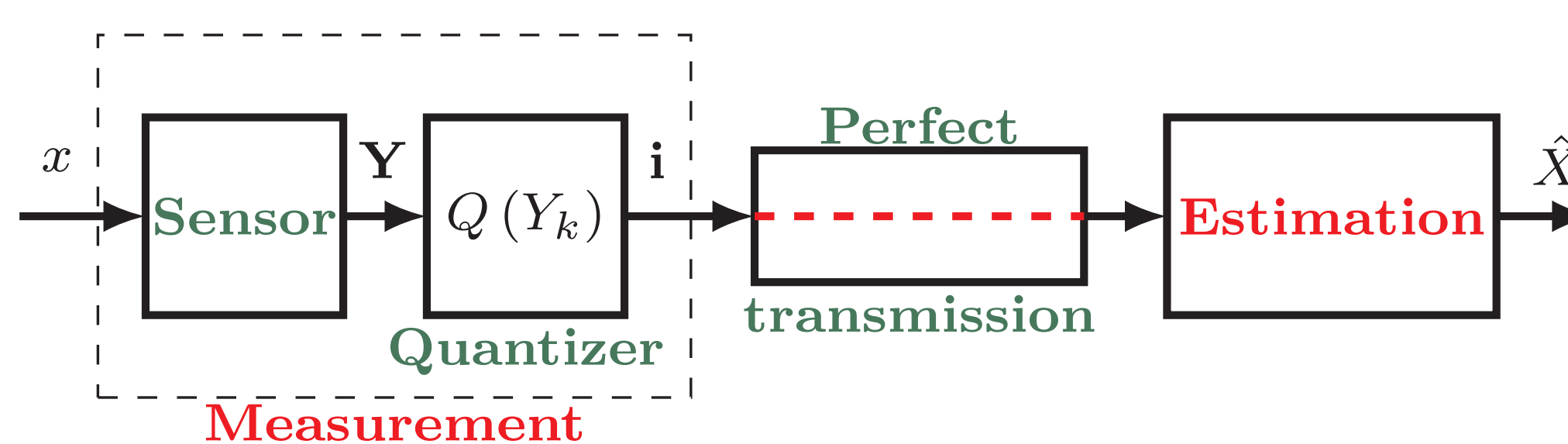
Remote estimation



under the following constraints:

- energy constraints.
- Bandwidth constraints.
- Complexity constraints.

Possible solution – Quantization:



## ESTIMATION PROBLEM

**Estimate**  $x \in \mathbb{R}$ , parameter of the CDF  $F(y; x)$  based on i.i.d.  $Y_k$

$$\mathbf{Y} = [Y_1 \ Y_2 \ \dots \ Y_N]^T.$$

Due to the constraints,  $\mathbf{Y}$  is quantized

$$\mathbf{i} = [i_1 \ i_2 \ \dots \ i_N]^T = [Q(Y_1) \ Q(Y_2) \ \dots \ Q(Y_N)]^T,$$

$$Q(Y) = i, \quad \text{if } Y \in \Delta_i = [\tau_{i-1}, \tau_i),$$

where  $i \in \{1, \dots, N_I\}$ ,  $N_I$  – number of quantization intervals ( $\Delta_i$ ) and  $\tau_i$  – quantizer thresholds.

**Bound and asymptotic performance (CRB)**

$$\text{Var} [\hat{X}] \geq \text{CRB}_q = \frac{1}{N I_q}, \quad I_q = \mathbb{E} [S_q^2],$$

$$S_q = \frac{\partial \log \mathbb{P}(i; x)}{\partial x}, \quad \mathbb{P}(i; x) = F(\tau_i; x) - F(\tau_{i-1}; x).$$

## ASYMPTOTIC FI

**Rewriting the FI**

$$S_c = \frac{\partial \log f(y; x)}{\partial x}, \quad I_c = \mathbb{E} [S_c^2], \quad L = \mathbb{E} [(S_c - S_q)^2] = \sum_{i=1}^{N_I} \int_{\Delta_i} \left[ \frac{\partial \log f(y; x)}{\partial x} - \frac{\partial \log \mathbb{P}(i; x)}{\partial x} \right]^2 f(y; x) dy.$$

$$I_q = I_c - \mathbb{E} [(S_c - S_q)^2].$$

**Quantization loss**

**Asymptotic approximation** – Evaluation of  $L$  when  $N_I \rightarrow +\infty$ ,  $\Delta_i \rightarrow 0$

$$\text{Loss} - \begin{cases} L = \sum_{i=1}^{N_I} \left\{ (S_{c,i}^{(y)})^2 f_i \frac{\Delta_i^3}{12} + o(\Delta_i^3) \right\}, \\ L = \sum_{i=1}^{N_I} \left\{ (S_{c,i}^{(y)})^2 f_i \frac{\Delta_i}{12 N_I^2 \lambda_i^2} + o\left(\frac{1}{N_I^2}\right) \Delta_i \right\}, \end{cases} \text{ where } \lambda_i = \frac{1}{N_I \Delta_i} \text{ is the interval density.}$$

$$\text{Asymp. loss} - \lim_{N_I \rightarrow \infty} N_I^2 L = \frac{1}{12} \int \frac{\left(\frac{\partial S_c(y; x)}{\partial y}\right)^2 f(y; x)}{\lambda^2(y)} dy \quad \text{FI} - I_q \approx I_c - \frac{1}{12 N_I^2} \int \frac{\left(\frac{\partial S_c(y; x)}{\partial y}\right)^2 f(y; x)}{\lambda^2(y)} dy.$$

$$\text{Optimal density} - \lambda^*(y) \propto \left(\frac{\partial S_c(y; x)}{\partial y}\right)^{\frac{2}{3}} f^{\frac{1}{3}}(y; x) \quad \text{FI} - I_q^* \approx I_c - \frac{2^{-2N_B}}{12} \left[ \int \left(\frac{\partial S_c(y; x)}{\partial y}\right)^{\frac{2}{3}} f^{\frac{1}{3}}(y; x) dy \right]^3.$$

## LOCATION ESTIMATION

**Gaussian** –  $f(y; x) = \frac{1}{\delta \sqrt{\pi}} e^{-\left(\frac{y-x}{\delta}\right)^2}$ ,

**Cauchy** –  $f(y; x) = \frac{1}{\delta \pi} \frac{1}{1 + \left(\frac{y-x}{\delta}\right)^2}$ ,

$$\lambda^* = \frac{1}{\delta \sqrt{3\pi}} e^{-\left(\frac{y-x}{\sqrt{3}\delta}\right)^2}, \quad I_q^* \approx \frac{2}{\delta^2} [1 - \pi \sqrt{3} 2^{-(2N_B+1)}]. \quad \lambda^* = \frac{1}{\delta B\left(\frac{1}{2}; \frac{5}{6}\right)} \frac{\left[1 - \left(\frac{y-x}{\delta}\right)^2\right]^{\frac{2}{3}}}{\left[1 + \left(\frac{y-x}{\delta}\right)^2\right]^{\frac{5}{3}}}, \quad I_q^* \approx \frac{1}{2\delta^2} \left[1 - \frac{B\left(\frac{1}{2}; \frac{5}{6}\right)^3 2^{-2N_B+1}}{3\pi}\right].$$

$N_B$	Optimal	Uniform	Companding approx.	Optimal	Uniform	Companding approx.
1	1.27323954 <sup>†</sup>	–	1.27323954	0.40528473 <sup>†</sup>	–	0.40528473
2	1.76503630 <sup>†</sup>	1.76503630	1.75128300	0.43433896 <sup>†</sup>	0.43433896	0.40528473
3	1.93090199 <sup>†</sup>	1.92837814	1.92740111	0.48474865 <sup>†</sup>	0.45600797	0.47893785
4	1.97874454 <sup>*</sup>	1.97841622	1.98038526	0.49533850 <sup>*</sup>	0.48136612	0.49504170
5	1.99468613 <sup>*</sup>	1.99353005	1.99489906	0.49883463 <sup>*</sup>	0.49204506	0.49879785
6	1.99867153 <sup>*</sup>	1.99807736	1.99869886	0.49970866 <sup>*</sup>	0.49656712	0.49970408
7	1.99966788 <sup>*</sup>	1.99943563	1.99967136	0.49992716 <sup>*</sup>	0.49851056	0.49992659
8	1.99991697 <sup>*</sup>	1.99983649	1.99991741	0.49998179 <sup>*</sup>	0.49935225	0.49998172

## ADAPTIVE SOLUTION FOR PLACING THE THRESHOLDS

Threshold approx. –  $\tau_i^* = F_\lambda^{-1}\left(\frac{i}{N_I}\right)$  – depends on  $x$ .

Solution – adaptive approach – use  $\hat{X}_k$  as  $x$ .

$$\hat{X}_k = \hat{X}_{k-1} + \frac{1}{k I_q} \eta(i_k), \quad \eta(i) = \frac{f(\tau_{i-1}^*; x) - f(\tau_i^*; x)}{F(\tau_i^*; x) - F(\tau_{i-1}^*; x)}.$$

