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# Counting function for interior transmission eigenvalues

Luc Robbiano\*

October 23, 2013

## Abstract

In this paper we give results on the counting function associated with the interior transmission eigenvalues. For a complex refraction index we estimate of the counting function by  $Ct^n$ . In the case where the refraction index is positive we give an equivalent of the counting function.

## Keywords

Interior transmission eigenvalues; Weyl law;

AMS 2010 subject classification: 35P10; 35P20; 35J57.

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## 1 Introduction

In this paper we give an estimate to the counting function associated with the interior transmission eigenvalues. We recall the problem. Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ . Let  $n(x)$  be a smooth function defined in  $\overline{\Omega}$ , called the refraction index. We say that  $k \neq 0$  is a interior transmission eigenvalue if there exists  $(w, v) \neq (0, 0)$  such that

$$\begin{cases} \Delta w + k^2 n(x)w = 0 \text{ in } \Omega, \\ \Delta v + k^2 v = 0 \text{ in } \Omega, \\ w = v \text{ on } \partial\Omega, \\ \partial_\nu w = \partial_\nu v \text{ on } \partial\Omega, \end{cases} \quad (1)$$

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where  $\partial_\nu$  is the exterior normal derivative to  $\partial\Omega$ . We consider here the function  $n(x)$  complex valued. In physical models, we have  $n(x) = n_1(x) + in_2(x)/k$  where  $n_j$  are real valued. Taking  $u = w - v$  and  $\tilde{v} = k^2v$ , we obtain the following equivalent system if  $k \neq 0$ ,

$$\begin{cases} (\Delta + k^2(1+m))u + mv = 0 \text{ in } \Omega, \\ (\Delta + k^2)v = 0 \text{ in } \Omega, \\ u = \partial_\nu u = 0 \text{ on } \partial\Omega, \end{cases} \quad (2)$$

where, for simplicity, we have replaced  $\tilde{v}$  by  $v$  and  $n$  by  $1+m$ .

When  $k \in \mathbb{R}$ , this problem is related with scattering problem. We can find a precise result in Colton and Kress [8, Theorem 8.9] first proved by Colton, Kirsch and Päiväranta [7] and in a survey by Cakoni and Haddar [5].

As the problem is not self-adjoint even for  $n(x)$  real valued, usual tools used in self-adjoint cases cannot be applied, in particular, even for operator with compact resolvent, the existence of  $k$  is not always true.

A lot of results was obtained this last years using several methods. When  $n(x)$  is real, Päiväranta and Sylvester [20] proved that there exist interior transmission eigenvalues; Cakoni, Gintides, and Haddar [4] proved that the set of  $k_j^2$  is infinite and discrete. For  $n(x)$  complex valued Sylvester [23] proved that this set is discrete finite or infinite. In [22] we proved that there exist a infinite number of complex eigenvalues and the associated generalized eigenspaces span a dense space in  $L^2(\Omega) \oplus L^2(\Omega)$ .

Other related problems are studied in literature, problem with cavities studied by Cakoni, Çayören, and Colton [2], Cakoni, Colton, and Haddar [3], problem for operators of order  $m > 2$  by Hitrik, Krupchyk, Ola, and Päiväranta [11, 12].

Lakshitanov and Vainberg [15, 16, 17] studied the counting function for problems with different boundary conditions. For Problem (1), in [18] they obtain a lower estimate of the counting function for real interior transmission eigenvalues.

For counting function in [22] we gave some non optimal estimate. This estimate was improved by Dimassi and Petkov [9]. Their estimate have the same size than the one found below in Theorem 8 except for a factor  $3\sqrt{3}$ . For constant  $m$  there is a recent result given by Pham and Stefanov [21] where they give an equivalent of counting function in this case.

The main results of this paper are Theorem 7 and Theorem 8. In Theorem 7 we prove than the counting function satisfies an estimate in  $Ct^n$  and we prove that if we denote by  $\lambda_j$  the eigenvalues of the problem

$$\sum_{j \in \mathbb{N}} \frac{1}{\lambda_j^p - z^p} = (2\pi)^{-n} |z|^{-p+n/2} \int_{\Omega} \left( ((1+m(x))^{-p} |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1} \right) d\xi dx + o(|z|^{-p+n/2}),$$

when  $|z|$  goes to  $\infty$  and  $z$  in a line outside a domain related with the range of  $n(x)$ .

In the case where  $n(x) > 0$  is real valued or if  $n_1(x) > 0$  in the case  $n(x) = n_1(x) + n_2(x)/k$  this estimate allows, applying a tauberian theorem to give an equivalent of the counting function. We find that  $N(t) \sim \alpha t^n$  where the precise value of  $\alpha$  is given in Theorem 8. These results are also proven by Faierman [10] in a preprint. The methods used are very close that the one used here but he assumes that  $n(x) \neq 1$  every where. This condition excludes the case of cavity. Here we assume only that  $n(x) \neq 1$  in a neighborhood of the boundary.

## 2 Notations and background

Let  $\Omega$  be a  $\mathcal{C}^\infty$  bounded domain in  $\mathbb{R}^n$ . Let  $n(x) \in \mathcal{C}^\infty(\overline{\Omega})$  be complex valued. We set  $m(x) = n(x) - 1$ . We consider also the case where  $n(x) = n_1(x) + in_2(x)/k$  where  $n_j(x)$  are real valued and  $k$  the spectral parameter. This case is different of the previous one but can be treated similarly. We assume that for all  $x \in \overline{\Omega}$ ,  $n(x) \neq 0$ , or  $n_1(x) \neq 0$  or equivalently  $m(x) \neq -1$ . We assume that there exists a neighborhood  $W$  of  $\partial\Omega$  such that for  $x \in \overline{W}$ ,  $n(x) \neq 1$  or  $n_1(x) \neq 1$  or equivalently  $m(x) \neq 0$ . Actually if  $n(x) \neq 1$  for all  $x \in \partial\Omega$ , such a neighborhood  $W$  exists.

We denote by  $C_e$  the cone in  $\mathbb{C}$  defined by

$$C_e = \{z \in \mathbb{C}, \exists x \in \overline{\Omega}, \exists \lambda \geq 0, \text{ such that } z = \lambda(1 + \overline{m(x)})\}. \quad (3)$$

In the case where  $n(x) = n_1(x) + in_2(x)/k$ ,  $C_e = [0, \infty)$  if  $n_1(x) > 0$  for all  $x \in \overline{\Omega}$ , and  $C_e = [-\infty, 0]$  if  $n_1(x) < 0$  for all  $x \in \overline{\Omega}$ .

Here we give some notations useful for the statement of the results. We use the notations and the results proven in [22], except some change of sign.

Let  $z \in \mathbb{C}$ , we denote by  $B_z(u, v) = (f, g)$  the mapping defined from  $H_0^2(\Omega) \oplus \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\}$  to  $L^2(\Omega) \oplus L^2(\Omega)$  by

$$\begin{cases} \left(\frac{-1}{1+m}\Delta - z\right)u - \frac{m}{1+m}v = f \text{ in } \Omega \\ (-\Delta - z)v = g \text{ in } \Omega \end{cases} \quad (4)$$

In the case where  $n = n_1 + in_2/k$  we must change the definition of  $B_z$ . We define  $m_1(x) = n_1(x) - 1$  and  $m_2(x) = n_2(x)$ . The mapping  $\hat{B}_k(u, v) = (f, g)$  is given by

$$\begin{cases} \left(\frac{-1}{1+m_1}\Delta - k^2 - ik\frac{m_2}{1+m_1}\right)u + \left(\frac{-m_1}{1+m_1} - \frac{im_2}{k(1+m_1)}\right)v = f \text{ in } \Omega \\ (-\Delta - k^2)v = g \text{ in } \Omega \end{cases} \quad (5)$$

Remark that the principal symbol of  $\hat{B}_k$  is the same than the one of  $B_z$  if we set  $z = k^2$ . Under an assumption on  $z$ ,  $B_z$  is invertible for some  $z$ .

**Theorem 1.** *Assume  $C_e \neq \mathbb{C}$ , then there exists  $z \in \mathbb{C}$  such that  $B_z$  is a bijective map from  $H_0^2(\Omega) \oplus \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\}$  to  $L^2(\Omega) \oplus L^2(\Omega)$ .*

In the case  $n(x) = n_1(x) + n_2(x)/k$ , here  $C_e \neq \mathbb{C}$  and we have the same result.

**Theorem 2.** *There exists  $k \in \mathbb{C}$  such that  $\hat{B}_k$  is bijective from  $H_0^2(\Omega) \oplus \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\}$  to  $L^2(\Omega) \oplus L^2(\Omega)$ .*

If for  $z \in \mathbb{C}$  the solution  $B_z(u, v) = (f, g)$  exists we denote by  $R_z(f, g) = (u, v)$ . In case where  $n(x) = n_1(x) + in_2(x)/k$  if for  $k \in \mathbb{C}$  the solution of  $\hat{B}_k(u, v) = (f, g)$  exists we denote by  $\hat{R}_k(f, g) = (u, v)$ .

**Theorem 3.** *Assume  $C_e \neq \mathbb{C}$ , there exists  $z \in \mathbb{C}$  such that the resolvent  $R_z$  from  $\overline{H}^2(\Omega) \oplus L^2(\Omega)$  to itself is compact.*

*In particular, applying the Riesz theory, the spectrum is finite or is a discrete countable set. If  $\lambda \neq 0$  is in the spectrum,  $\lambda$  is an eigenvalue associated with a finite dimensional generalized eigenspace.*

**Theorem 4.** *There exists  $k \in \mathbb{C}$  such that the resolvent  $\hat{R}_k$  from  $\overline{H}^2(\Omega) \oplus L^2(\Omega)$  to itself is compact.*

*In particular, we can apply the Riesz theory, the spectrum is finite or a discrete countable set. If  $\lambda \neq 0$  is in the spectrum,  $\lambda$  is an eigenvalue associated with a finite dimensional generalized eigenspace.*

**Remark 1.** Actually if  $z_0 \notin C_e \cup [0, \infty)$  for all  $\lambda > 0$  large enough we can take  $z = \lambda z_0$  in Theorems 1 and 3.

If  $k_0^2 \notin C_e \cup [0, \infty)$  for all  $\lambda > 0$  large enough we can take  $k = \lambda k_0$  in the Theorems 2 and 4. Here we estimate the resolvent in the exterior of a conic neighborhood of  $C_e \cup [0, \infty)$ . In particular if  $n_1(x) > 0$ , the eigenvalues  $k^2$  are in all small conic neighborhood of  $(0, +\infty)$ , except for a finite number of eigenvalues.

In general for a non self-adjoint problem, we cannot claim that the spectrum is non empty. In the following theorem, with a stronger assumption on  $C_e$ , we can prove that the spectrum is non empty.

We say that  $C_e$  is contained in a sector with angle less than  $\theta$  if there exist  $\theta_1 < \theta_2$ , such that  $C_e \subset \{z \in \mathbb{C}, z = 0 \text{ or } \frac{z}{|z|} = e^{i\varphi}, \text{ where } \theta_1 \leq \varphi \leq \theta_2\}$ , and  $\theta_2 - \theta_1 \leq \theta$ .

**Theorem 5.** *Assume that  $C_e$  is contained in a sector with angle less than  $\theta$  with  $\theta < 2\pi/p$  where  $4p > n$  and  $\theta < \pi/2$ . Then there exists  $z$  such that the spectrum of  $R_z$  is infinite and the space spanned by the generalized eigenspaces is dense in  $H_0^2(\Omega) \oplus \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\}$ .*

**Theorem 6.** *There exists  $k$  such that the spectrum of  $\hat{R}_k$  is infinite and the space spanned by the generalized eigenspaces is dense in  $H_0^2(\Omega) \oplus \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\}$ .*

**Remark 2.** These results are based on the theory given in Agmon [1] and using the spectral results on Hilbert-Schmidt operators. In this theory we deduce that the spectrum is infinite from the proof that the generalized eigenspaces form a dense subspace in the closure of the range of  $R_z$  [resp.  $\hat{R}_k$ ]. In [22] we proved that  $R_z^p$  [resp.  $\hat{R}_k^p$ ] is a Hilbert-Schmidt operator if  $4p > n$ . We can deduce the spectral decomposition of  $R_z$  [resp.  $\hat{R}_k$ ] from that of  $R_z^p$  [resp.  $\hat{R}_k^p$ ].

Let  $z_j$  be the elements of the spectrum of  $R_z$  [resp.  $\hat{R}_k$ ] and  $E_j$  the generalized associated eigenspace. We denote by  $N(t) = \sum_{|z_j|^{-1} \leq t^2} \dim E_j$ .

If  $z_j$  is an eigenvalue of  $R_z$ ,  $\lambda_j = -z + 1/z_j$  is an eigenvalue of  $B_0$  and we have  $N(t) \sim \#\{j, |\lambda_j| \leq t^2\}$ , where  $\lambda_j$  are counted with multiplicity.

### 3 Results

We denote by  $\omega_j$  for  $j = 1, \dots, p$ , the roots of  $z^p = 1$ .

**Theorem 7.** *We assume as in Theorem 5 that  $\theta < 2\pi/p$  and  $\theta < \pi/2$  where  $p$  satisfies  $2p > n$  and  $4p > 4 + n$ . Then, there exists  $C > 0$  such that  $N(t) \leq Ct^n$ .*

Moreover let  $\mu \in \mathbb{C}$  such that  $|\mu| = 1$  and we assume that  $\omega_j \mu \notin C_e \cup (0, +\infty)$  for  $j = 1, \dots, p$ . We denote by  $a(x) = (1 + m(x))^{-1}$  or  $a(x) = (1 + m_1(x))^{-1}$  if  $n(x) = n_1(x) + k^{-1}n_2(x)$ . We fix  $z_0$  such that the resolvent  $R_{z_0}$  exist and let  $\mu_j$  such that  $1/\mu_j$  are the eigenvalues of  $R_{z_0}$  counted with multiplicity. Then we have

$$\sum_{j \in \mathbb{N}} \frac{1}{\mu_j^p - z^p} = (2\pi)^{-n} |z|^{-p+n/2} \int_{\Omega} \int ((a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1}) d\xi dx + o(|z|^{-p+n/2}), \quad (6)$$

when  $z = r\mu$  and  $r$  goes to  $\infty$ .

**Remark 3.** The first part of the theorem improve [22, Theorem 7] where we found the estimate  $N(t) \leq Ct^{n+4}$ .

**Theorem 8.** *We assume  $n_1(x) = 1 + m_1(x) > 0$  for all  $x \in \bar{\Omega}$  when  $n(x) = n_1(x) + n_2/k$ . Then*

$$N(t) = \alpha t^n + o(t^n) \text{ where } \alpha = (2\pi)^{-n} \text{Vol}(B_1) \int_{\Omega} ((1 + m_1(x))^{n/2} + 1) dx.$$

**Remark 4.** By a more precise study in a neighborhood of the boundary we can obtain a result with a smaller remainder in (6) but this estimate does not allow to prove a result better on the counting function. Malliavin [19] was proved a tauberian theorem with a sharp remainder but this requires an estimate on  $\sum_{j \in \mathbb{N}} \frac{1}{\lambda_j^p - z^p}$  in a complex domain except in a parabola neighborhood of  $(0, \infty)$ . Here the estimate are proved in a complex domain except in a conic neighborhood of  $(0, \infty)$ . It is maybe possible to improve this result following Hitrik, Krupchyk, Ola and Päiväranta [13] where they prove that the eigenvalues are in a parabolic neighborhood of  $(0, \infty)$ .

### 4 Proof of Theorem 7

The proof is based on Lemmas 4.1 and 4.2 below. We introduce some notations.

We set  $S = R_{z_0}$  and  $T = S^p$  where  $p$  satisfies the assumption of Theorem 7.

We set  $T_\lambda = T(I - \lambda T)^{-1}$ .

As  $T_\lambda$  is a matrix of operators we denote

$$T_\lambda = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

We denote by

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad (7)$$

the kernel of  $T_{z^p}$ .

**Lemma 4.1.** *Under the assumption of Theorem 7, there exists  $C > 0$  such that  $N(t) \leq Ct^n$ .*

*Moreover let  $V$  a conical neighborhood of  $C_e \cup [0, \infty)$ , there exists  $R > 0$  such all  $z \in \mathbb{C}$ , satisfying  $|z| \geq R$  and  $\omega_j z \notin \bar{V}$ , we have*

$$\int_{\Omega} K_{11}(x, x) dx + \int_{\Omega} K_{22}(x, x) dx = \sum_{j \in \mathbb{N}} \frac{1}{\lambda_j^p - z^p}. \quad (8)$$

**Lemma 4.2.** *With the notation of Theorem 7 we have  $|z|^{p-n/2} (\int_{\Omega} K_{11}(x, x) dx + \int_{\Omega} K_{22}(x, x) dx)$  goes to  $(2\pi)^{-n} \int_{\Omega} \int ((a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1}) d\xi dx$  when  $|z|$  goes to  $\infty$ .*

Clearly Lemmas 4.1 and 4.2 imply Theorem 7.

#### 4.1 Proof of Lemma 4.1

We recall that  $\omega_j$  for  $j = 1, \dots, p$ , are the roots of  $z^p = 1$ , we have

$$(1 - z^p S^p) = \prod_{j=1}^p (1 - \omega_j z S). \quad (9)$$

The operator  $(1 - z^p S^p)$  is invertible if and only if  $(1 - \omega_j z S)$  is invertible for all  $j$ . Thus we have

$$T_{z^p} = S^p \prod_{j=1}^p (1 - \omega_j z S)^{-1} = \prod_{j=1}^p S_{\omega_j z}.$$

If we denote by

$$S_z = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

We recall that  $S = R_{z_0}$  and

$$S_z = (R_{z_0})_z = R_{z_0} (I - z R_{z_0})^{-1} = (R_{z_0}^{-1} - z)^{-1} = (B_{z_0} - z)^{-1} = (B_0 - z_0 - z)^{-1} = R_{z_0+z}, \quad (10)$$

if the resolvent  $R_{z_0+z}$  exists. In what follows,  $z_0$  is fixed thus  $|z| \sim |z + z_0|$  for large  $|z|$ .

We introduce some notation for Sobolev spaces.

We denote the semi-classical  $H^s$  norm by  $\|w\|_{H_{sc}^s}^2 = \int (1 + h^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$ . Let  $w$  be a distribution on  $\Omega$ , we denote by  $\|w\|_{\overline{H}_{sc}^s(\Omega)} = \inf\{\|\beta\|_{H_{sc}^s}, \text{ where } \beta|_{\Omega} = w\}$ . We recall that we denote  $D = -ih\partial$ , and if  $s$  is an integer the quantity  $\sum_{|\alpha| \leq s} \|D^\alpha w\|_{L^2(\Omega)}^2$  is equivalent to  $\|w\|_{\overline{H}_{sc}^s(\Omega)}^2$  uniformly with respect to  $h$ . When  $h = 1$  we denote the space by  $\overline{H}^s(\Omega)$ .

We apply the results of [22]. The estimates below are given by [22, theorem 10] for  $k \geq 1$ . The estimate on  $S_{12}$  and  $S_{22}$  for  $k = 0$  are also given by [22, theorem 10]. For  $k = 0$ , [22, Proposition 2.3] gives the estimate on  $S_{21}$  and [22, Lemma 2.1] gives the estimate on  $S_{11}$ . The relation between  $z$  and  $h$  is  $zh^2 = \mu$ , in particular  $h^2|z| = 1$ .

$$\begin{aligned} S_{11} &: \overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+2}(\Omega) \text{ with } \|S_{11}\|_{\overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+2}(\Omega)} \leq C|z|^{-1} \\ S_{12} &: \overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+4}(\Omega) \text{ with } \|S_{12}\|_{\overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+4}(\Omega)} \leq C|z|^{-2} \\ S_{21} &: \overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k}(\Omega) \text{ with } \|S_{21}\|_{\overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k}(\Omega)} \leq C \\ S_{22} &: \overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+2}(\Omega) \text{ with } \|S_{22}\|_{\overline{H}_{sc}^{2k}(\Omega) \rightarrow \overline{H}_{sc}^{2k+2}(\Omega)} \leq C|z|^{-1} \end{aligned} \quad (11)$$

We denote by

$$\Lambda_z = \begin{pmatrix} \sqrt{|z|} & 0 \\ 0 & 1/\sqrt{|z|} \end{pmatrix}$$

We remark that

$$\Lambda_z A \Lambda_z^{-1} = \begin{pmatrix} A_{11} & |z|A_{12} \\ (1/|z|)A_{21} & A_{22} \end{pmatrix}, \text{ where } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (12)$$

We deduce from (11) that  $\Lambda_z S_{\omega_j z} \Lambda_z^{-1} : L^2(\Omega) \oplus L^2(\Omega) \rightarrow \overline{H}_{sc}^2(\Omega) \oplus L^2(\Omega)$  with an operator norm less than  $C|z|^{-1}$  and  $\Lambda_z S_{\omega_j z} \Lambda_z^{-1} : \overline{H}_{sc}^{2k+2}(\Omega) \oplus \overline{H}_{sc}^{2k}(\Omega) \rightarrow H_{sc}^{2k+4}(\Omega) \oplus \overline{H}_{sc}^{2k+2}(\Omega)$  with an operator norm less than  $C|z|^{-1}$ .

As

$$\Lambda_z T_{z^p} \Lambda_z^{-1} = \prod_{j=1}^p \Lambda_z S_{\omega_j z} \Lambda_z^{-1}, \quad (13)$$

we deduce that

$$\Lambda_z T_{z^p} \Lambda_z^{-1} : L^2(\Omega) \oplus L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p}(\Omega) \oplus \overline{H}_{sc}^{2p-2}(\Omega), \quad (14)$$

with an operator norm less than  $C|z|^{-p}$ .

We can prove that  $N(t) \leq Ct^n$ . First we weaken (14) to consider  $\Lambda_z T_{z^p} \Lambda_z^{-1}$  as a map between  $L^2(\Omega) \oplus L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p-2}(\Omega) \oplus \overline{H}_{sc}^{2p-2}(\Omega)$ , with an operator norm less than  $C|z|^{-p}$ .

As  $\|v\|_{H^s} \leq h^{-2s} \|v\|_{H_{sc}^s}$ , we obtain

$$\begin{aligned} \|\Lambda_z T_{z^p} \Lambda_z^{-1}\|_{L^2(\Omega) \oplus L^2(\Omega) \rightarrow \overline{H}^{2p-2}(\Omega) \oplus \overline{H}^{2p-2}(\Omega)} &\leq C|z|^{-1}, \\ \|\Lambda_z T_{z^p} \Lambda_z^{-1}\|_{L^2(\Omega) \oplus L^2(\Omega) \rightarrow L^2(\Omega) \oplus L^2(\Omega)} &\leq C|z|^{-p}. \end{aligned} \quad (15)$$

We can apply the theorem 13.5 in Agmon [1], that is, if  $m > n/2$  we have

$$\| \|T\| \| \leq C \|T\|_m^{n/(2m)} \|T\|_0^{1-n/(2m)},$$

where  $\| \|T\| \|$  is the Hilbert-Schmidt norm and  $\|T\|_m$  is the operator norm of the map  $L^2(\Omega) \oplus L^2(\Omega) \rightarrow \overline{H}^m(\Omega) \oplus \overline{H}^m(\Omega)$ . We apply this estimate with  $m = 2p - 2 > n/2$  and we have

$$\| \|\Lambda_z T_{z^p} \Lambda_z^{-1}\| \| \lesssim |z|^{-\frac{n}{4(p-1)}} |z|^{-(1-\frac{n}{4(p-1)})p} = |z|^{-p+n/4}.$$

We can follow the proof of Theorem 7 in [22]. If we denote by  $\mu_j$  complex numbers such that  $\mu_j^{-1}$  are eigenvalue of  $S$  counted with multiplicity, then  $\frac{1}{\mu_j^p - z^p}$  are the eigenvalues of  $T_{z^p}$  and thus the eigenvalues of  $\Lambda_z T_{z^p} \Lambda_z^{-1}$ . We obtain

$$\sum_j \frac{1}{|\mu_j^p - z^p|^2} \leq \| \|\Lambda_z T_{z^p} \Lambda_z^{-1}\| \|^2 \leq C|z|^{-2p+n/2}.$$

Let  $\mu \in \mathbb{C}$  such that  $|\mu| = 1$ , and  $\omega_j \mu \notin C_e \cup (0, \infty)$  for all  $j = 1, \dots, p$ . We take  $z = t^2 \mu$ . If  $|\mu_j| \leq t^2$ , we have  $|\mu_j^p - z^p| \leq 2t^{2p}$ . Then we have

$$\sum_{|\mu_j| \leq t^2} \frac{1}{4t^{4p}} \leq \sum_j \frac{1}{|\mu_j^p - z^p|^2} \leq \| \|T_{z^p}\| \|^2 \leq Ct^{-4p+n}.$$

Then we obtain  $N(t) \leq Ct^n$ .

Now we prove Formula (6). Estimate (14) implies that  $\|T_{11}\|_{L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p}(\Omega)} \leq C|z|^{-p}$  and as  $\|v\|_{H^s} \leq h^{-2s} \|v\|_{H_{sc}^s}$ , we have

$$\|T_{11}\|_{L^2(\Omega) \rightarrow \overline{H}^{2p}(\Omega)} \leq C. \quad (16)$$

To estimate the norm of  $T_{22}$ , we shall use that  $S_{12}$  is a mapping from  $L^2(\Omega)$  to  $\overline{H}_{sc}^4(\Omega)$ . Actually if we take  $g \in L^2(\Omega)$ , we have  $\Lambda_z S_{\omega_j z} \Lambda_z^{-1}(0, g) \in \overline{H}_{sc}^4(\Omega) \oplus \overline{H}_{sc}^2(\Omega)$  and  $\|\Lambda_z S_{\omega_j z} \Lambda_z^{-1}(0, g)\|_{\overline{H}_{sc}^4(\Omega) \oplus \overline{H}_{sc}^2(\Omega)} \leq$

$C|z|^{-1}\|g\|_{L^2(\Omega)}$ . We can repeat the previous argument for the  $p-1$  other factors  $\Lambda_z S_{\omega_j z} \Lambda_z^{-1}$  and we obtain that

$$\| \Lambda_z T_{z^p} \Lambda_z^{-1}(0, g) \|_{\overline{H}_{sc}^{2p+2}(\Omega) \oplus \overline{H}_{sc}^{2p}(\Omega)} \leq C|z|^{-p}\|g\|_{L^2(\Omega)}.$$

In particular this means that  $\|T_{22}g\|_{\overline{H}_{sc}^{2p}(\Omega)} \leq C|z|^{-p}\|g\|_{L^2(\Omega)}$ , and

$$\|T_{22}g\|_{\overline{H}^{2p}(\Omega)} \leq C\|g\|_{L^2(\Omega)}. \quad (17)$$

By (15), we have

$$\|T_{11}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} + \|T_{22}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C|z|^{-p}. \quad (18)$$

We can apply the theorem 13.9, Agmon [1]. If  $2p > n$  we have for  $j = 1, 2$ ,  $K_{jj} \in \overline{H}^q(\Omega \times \Omega)$  where  $q = 2p - [n/2] - 1 > n/2$ . In particular the trace  $K_{jj}(x, x)$  is well defined in  $L^2(\Omega)$  and

$$\begin{aligned} (\|K_{jj}(x, x)\|^2 dx)^{1/2} &\leq C \left[ \left( \|T_{jj}\|_{L^2(\Omega) \rightarrow \overline{H}^{2p}(\Omega)} + \|T_{jj}^*\|_{L^2(\Omega) \rightarrow \overline{H}^{2p}(\Omega)} \right)^{n/(2p)} \|T_{jj}\|_{L^2(\Omega) \rightarrow L^2(\Omega)}^{1-n/(2p)} \right. \\ &\quad \left. + \|T_{jj}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \right] \end{aligned} \quad (19)$$

**Remark 5.** The adjoint of  $B_z$  is given by an analogous formula than (4). Indeed we find that the adjoint  $B_z^*(p, q) = (g_1, g_2)$  is given by

$$\begin{aligned} -\Delta((1 + \bar{m})^{-1}p) - \bar{z}p &= g_1 \text{ in } \Omega \\ -\Delta q - \bar{z}q + \bar{m}(1 + \bar{m})^{-1}p &= g_2 \text{ in } \Omega \\ q|_{\partial\Omega} = \partial_\nu q|_{\partial\Omega} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Using the relation between  $R_z$  and  $S_z$  (see (10)), we deduce that the adjoint of  $S_z$  satisfies the same estimate than  $S_z$  given in (11). By (13), the adjoint of  $T_z$  satisfies the same properties than  $T_z$  given in (16), (17) and (18).

For  $j = 1, 2$ , (19) implies, from (16), (17) and (18)

$$\left( \int_{\Omega} |K_{jj}(x, x)|^2 dx \right)^{1/2} \leq C|z|^{-p+(n/2)} \quad (20)$$

We recall that  $\lambda_j^{-1}$  are the eigenvalues of  $S$  counted with multiplicity the eigenvalues of  $T$  are  $\lambda_j^{-p}$ . The indices are such that  $|\lambda_j| \leq |\lambda_{j+1}|$ . As  $N(t) \leq Ct^n$ , this implies that  $|\lambda_j| \geq Cj^{2/n}$  where  $C > 0$ . In particular  $\sum 1/|\lambda_j|^p$  converges if  $2p > n$ .

By Theorem 12.17 in Agmon [1], there exists a constant  $c \in \mathbb{C}$  such that

$$\text{Tr}(TT_{z^p}) = \sum_{j \in \mathbb{N}} \frac{1}{(\mu_j^p - z^p)\mu_j^p} + c. \quad (21)$$

We recall that the trace is defined in the theorem 12.20 in Agmon [1] for an operator  $Q = Q_1 Q_2$  where  $Q_1$  and  $Q_2$  are Hilbert-Schmidt operators. Moreover if  $K$  is the kernel of  $Q$ ,  $K(x, x)$  is definite for almost all  $x$ , we have  $\int_{\Omega} |K(x, x)| dx < \infty$  and

$$\text{Tr}(Q) = \int_{\Omega} K(x, x) dx. \quad (22)$$

We remark as by assumption  $\mu_j^p$  and  $z^p$  are not in the same cone, we have  $|\mu_j^p - z^p| \sim |\mu_j^p| + |z^p|$ . Then

$$\frac{1}{|\mu_j^p - z^p||\mu_j^p|} \leq \frac{C}{|\mu_j^p|} \text{ and } \frac{1}{|\mu_j^p - z^p||\mu_j^p|} \rightarrow 0 \text{ when } |z| \rightarrow +\infty.$$

This implies that  $\sum_{j \in \mathbb{N}} \frac{1}{(\mu_j^p - z^p)\mu_j^p} \rightarrow 0$  when  $|z| \rightarrow +\infty$ .



In [22] we proved that  $\|T_{z^p}\| \leq C|z|^{1-p+n/4}$  as  $1-p+n/4 < 0$  thus  $\|T_{z^p}\|$  goes to 0 as  $|z|$  goes to  $+\infty$ . We have  $|\text{Tr}(TT_{z^p})| \leq \|T\| \|T_{z^p}\|$  goes to 0 as  $|z|$  goes to  $+\infty$ . Then  $c = 0$  in (21). We obtain that

$$\text{Tr}(z^p TT_{z^p}) = \sum_{j \in \mathbb{N}} \left( \frac{1}{\mu_j^p - z^p} - \frac{1}{\mu_j^p} \right).$$

We have  $z^p TT_{z^p} = z^p T^2 (I - z^p T)^{-1} = -T(I - z^p T)(I - z^p T)^{-1} + T(I - z^p T)^{-1} = T_{z^p} - T$ . By Formula (22) the trace of  $z^p TT_{z^p}$  is given by the integral of its kernel and, as the integral of the kernel of  $T_{z^p}$  exists by (20), the integral of trace of kernel of  $z^p TT_{z^p} - T_{z^p}$  does not depend of  $z$ , we obtain

$$\int_{\Omega} K_{11}(x, x) dx + \int_{\Omega} K_{22}(x, x) dx = \sum_{j \in \mathbb{N}} \left( \frac{1}{\mu_j^p - z^p} - \frac{1}{\mu_j^p} \right) + c \quad (23)$$

By (20), for  $j = 1, 2$ ,  $\int_{\Omega} K_{jj}(x, x) dx$  goes to 0 when  $|z|$  goes to 0 and as  $\frac{1}{|\mu_j^p - z^p|} \leq C \frac{1}{|\mu_j^p|}$  and  $\frac{1}{|\mu_j^p - z^p|} \rightarrow 0$  when  $|z| \rightarrow +\infty$ , we obtain  $c = \sum_{j \in \mathbb{N}} \frac{1}{\mu_j^p}$ . Then (23) gives the statement of Lemma 4.1.

## 4.2 Proof of Lemma 4.2

We recall some facts on pseudo-differential operators. Let  $a(x, \xi)$  be in  $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  we say that  $a$  is a symbol of order  $m$  if for all  $\alpha, \beta \in \mathbb{N}^n$ , there exist  $C_{\alpha, \beta} > 0$ , such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|},$$

where  $\langle \xi \rangle^2 = 1 + |\xi|^2$ . In particular a polynomial in  $\xi$  of order  $m$  with coefficients in  $\mathcal{C}^\infty(\mathbb{R}^n)$  with bounded derivatives of all orders, is a symbol of order  $m$ .

With a symbol we can associate an semi-classical operator by the following formula

$$\text{Op}(a)u = a(x, D)u = \frac{1}{(2\pi)^n} \int e^{ix\xi} a(x, h\xi) \hat{u}(\xi) d\xi = \frac{1}{(2h\pi)^n} \int e^{ix\xi/h} a(x, \xi) \hat{u}(\xi/h) d\xi.$$

If  $a(x, \xi)$  is a symbol of order  $m$  (which can depend of  $h$ ), if  $a(x, \xi) = b(x, \xi) + hc(x, \xi)$  where  $b(x, \xi)$  and  $c(x, \xi)$  are symbols of order  $m$ , we call  $b(x, \xi)$  the principal symbol of  $a(x, \xi)$  which is definite modulo  $h$ . This formula makes sense for  $u \in \mathcal{S}(\mathbb{R}^n)$  and we can extend it to  $u \in H^s$  for all  $s$ . For  $a$ , a symbol of order  $m$ , there exists  $C > 0$  such that for all  $u \in H^s$ ,

$$\|a(x, D)u\|_{H^{s-m}} \leq C \|u\|_{H^s}.$$

In the following, when we use pseudo-differential operator we have always cut-off functions supported in  $\Omega$  in each side of the operator. We do not have to consider the action of pseudo-differential operator on  $\overline{H^s}(\Omega)$  space as in [22].

We begin with a description on  $S_z$  in all compact set in  $\Omega$ .

**Lemma 4.3.** *Let  $\mu \in \mathbb{C}$  such that  $|\mu| = 1$  and we assume that  $\mu \notin C_e \cup (0, +\infty)$  for  $j = 1, \dots, p$ . Let  $z \in \mathbb{C}$  such that  $z/\mu \in (0, \infty)$  is large enough. Let  $\theta$  and  $\tilde{\theta}$  be functions in  $\mathcal{C}_0^\infty(\Omega)$  such that  $\theta(x) = 1$  if  $x$  in the support of  $\tilde{\theta}$ , and  $\theta(x) = 1$  if  $x$  in a compact subset of  $\Omega$ . Then we have*

$$\tilde{\theta} \Lambda_z S_z \Lambda_z^{-1} = |z|^{-1} \tilde{\theta} B \theta + |z|^{-1/2} W \theta \Lambda_z S_z \Lambda_z^{-1}, \quad (24)$$

where  $W = \Lambda_z K \Lambda_z^{-1}$  and  $W^*$  are bounded on  $\overline{H^s}(\Omega)$  and the principal symbol of  $B$  is given on the support of  $\tilde{\theta}$  by

$$\begin{pmatrix} (a|\xi|^2 - \mu)^{-1} & (a|\xi|^2 - \mu)^{-1} V (|\xi|^2 - \mu)^{-1} \\ 0 & (|\xi|^2 - \mu)^{-1} \end{pmatrix}.$$

**Proof.** We apply the result proved in [22]. Let us recall the notations and the main results.

We multiply Equations (4) by  $h^2$ , we denote by  $\mu = h^2 z$  where  $\mu$  belongs to a bounded domain of  $\mathbb{C}$ ,  $a = 1/(1+m)$  and  $V = m/(1+m)$ . We change  $(f, g)$  in  $(-f, -g)$ .

We recall the assumption made on  $m$ , we have  $m(x) \neq -1$  for all  $x \in \overline{\Omega}$  and  $m(x) \neq 0$  for  $x$  in a neighborhood of  $\partial\Omega$ .

Thus following (4), we obtain the system

$$\begin{cases} (-ah^2\Delta - \mu)u - h^2Vv = h^2f & \text{in } \Omega \\ (-h^2\Delta - \mu)v = h^2g & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (25)$$

In case where  $n(x) = n_1(x) + n_2(x)/k$  we multiply Equations (5) by  $h^2$ , we denote by  $\mu = -h^2k^2$ ,  $a = 1/(1 + m_1)$ ,  $V = m_1/(1 + m_1) + hm_2/(\nu + \nu m_1)$  where  $\nu = hk$ . We change  $(f, g)$  in  $(-f, -g)$ . Thus following (5) we obtain the system

$$\begin{cases} (-ah^2\Delta + hW_0 - \mu)u - h^2Vv = h^2f & \text{in } \Omega \\ (-h^2\Delta - \mu)v = h^2g & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \quad (26)$$

where  $W_0 = -\nu m_2/(1 + m_2)$ . In particular the principal semi-classical symbol of  $-ah^2\Delta + hW_0 - \mu$  is  $-a|\xi|^2 - \mu$ , the principal semi-classical symbol of  $V$  is  $m_1/(1 + m_1)$ . In what follows only the principal symbols of  $-ah^2\Delta + hW_0 - \mu$  and  $-h^2\Delta - \mu$  must be take account. For simplicity we write the proof for the system (25), the case of system (26) may be treat following the same way.

Now we compute the symbol of the resolvent in  $\Omega$ .

Let  $\phi_0, \phi_1$  and  $\phi_2 \in \mathcal{C}_0^\infty(\Omega)$  where  $\phi_0\phi_1 = \phi_0$  and  $\phi_1\phi_2 = \phi_1$ . We take  $\phi_0$  such that  $\phi_0 = 1$  on the support of  $\tilde{\theta}$ . Let  $Q$  be a parametrix of  $-ah^2\Delta - \mu$  such that  $\phi_0Q\phi_1(-ah^2\Delta - \mu) = \phi_0 - hK$  where  $K$  is of order  $-1$  and the principal symbol of  $Q$  is  $(a|\xi|^2 - \mu)^{-1}$ . As

$$\phi_1(-ah^2\Delta - \mu)\phi_2u = h^2V\phi_1v + h^2\phi_1f \text{ in } \mathbb{R}^n,$$

applying  $\phi_0Q$  to this equation, we obtain

$$\phi_0u = h^2\phi_0Q\phi_1f + h^2\phi_0Q(V\phi_1v) + hK\phi_2u. \quad (27)$$

We apply the same method use above to compute  $v$  (see [22, Section 2.2]). Let  $\phi_3 \in \mathcal{C}_0^\infty(\Omega)$  where  $\phi_2\phi_3 = \phi_2$  and  $\theta = 1$  on the support of  $\phi_3$ . The choices of the  $\phi_j$  are compatible with  $\theta$  and  $\tilde{\theta}$ . We have  $\phi_1\tilde{Q}\phi_2(-h^2\Delta - \mu) = \phi_1 - hK_{-1}$  where  $K_{-1}$  is of order  $-1$  and the principal symbol of  $\tilde{Q}$  is  $(|\xi|^2 - \mu)^{-1}$ .

We apply  $\phi_2$  on the equation on  $v$  in (25), we have

$$\phi_2(-h^2\Delta - \mu)\phi_3v = h^2\phi_2g.$$

Applying the parametrix  $\phi_1\tilde{Q}$ , we have

$$\phi_1v = h^2\phi_1\tilde{Q}\phi_2g + hK_{-1}\phi_3v. \quad (28)$$

With this equation and (27) we obtain

$$\phi_0u = h^2\phi_0Q\phi_1f + h^4\phi_0Q(V\phi_1\tilde{Q}\phi_2g) + hK\phi_2u + h^3\tilde{K}_{-1}\phi_3v, \quad (29)$$

where  $\tilde{K}_{-1}$  is an operator of order  $-1$ .

Let

$$A = \begin{pmatrix} h^2Q\phi_1 & h^4QV\phi_1\tilde{Q}\phi_2 \\ 0 & h^2\phi_1\tilde{Q}\phi_2 \end{pmatrix} \text{ and } K = \begin{pmatrix} K_{11} & h^2K_{12} \\ 0 & K_{22} \end{pmatrix},$$

where  $K_{11} = K\phi_2$ ,  $K_{12} = \tilde{K}_{-1}$  and  $K_{22} = K_{-1}\phi_2$ . In particular  $K_{jk}$  is bounded on  $\overline{H}_{sc}^s(\Omega)$  for all  $s \geq 0$ . Indeed, all the operators contain cut-off thus  $K_{jk}\phi_3u$  is compactly supported in  $\Omega$  if  $u \in \overline{H}_{sc}^s(\Omega)$ .

We recall that  $S = R_{z_0}$  and by (10),  $S_z = R_{z_0+z}$ ,

if the resolvent  $R_{z_0+z}$  exists. In what follows,  $z_0$  is fixed and we have the relation  $z = -\mu/h^2 - z_0$ . In particular  $|z|^{-1/2} \sim h$  for large  $|z|$ . With these relations we have, following (28) and (29),  $S_z(f, g) = (u, v)$ , and

$$\phi_0 S_z = \phi_0 A \phi_3 + |z|^{-1/2} K \phi_3 S_z.$$

Thus we can write

$$\phi_0 \Lambda_z S_z \Lambda_z^{-1} = |z|^{-1} \phi_0 B \phi_3 + |z|^{-1/2} W \phi_3 \Lambda_z S_z \Lambda_z^{-1}, \quad (30)$$

where  $W = \Lambda_z K \Lambda_z^{-1}$  is bounded on  $\overline{H}^s(\Omega)$  and the principal symbol of  $B$  is given on the support of  $\phi_0$  by

$$\begin{pmatrix} (a|\xi|^2 - \mu)^{-1} & (a|\xi|^2 - \mu)^{-1} V (|\xi|^2 - \mu)^{-1} \\ 0 & (|\xi|^2 - \mu)^{-1} \end{pmatrix}.$$

As  $\tilde{\theta}\phi_0 = \tilde{\theta}$  and  $\phi_3\theta = \phi_3$ , (30) gives (24). As  $K$  is a semi-classical pseudo-differential operator,  $W^*$  is also bounded on  $\overline{H}^s(\Omega)$   $\blacksquare$

**Remark 6.** Formula (24) does not give a description on the operator  $S_z$  in  $\Omega$ . It gives only  $S_z$  in all compact in  $\Omega$ . In the proof below we need also estimates on  $S_z$  up the boundary given in (11) to absorb the error terms.

Lemma 4.3 gives the principal symbol of  $S_z$ , the following lemma gives the principal symbol of  $T_{z^p}$ .

**Lemma 4.4.** *Let  $p \in \mathbb{N} \setminus \{0\}$ . Let  $\varphi_0$  and  $\varphi_1$  be functions in  $\mathcal{C}_0^\infty(\Omega)$  such that  $\varphi_0 = 1$  on a neighborhood of  $\text{supp } \varphi_1$ .*

$$\varphi_1 \Lambda_z T_{z^p} \Lambda_z^{-1} \varphi_0 = |z|^{-p} \varphi_1 B_p \varphi_0 + |z|^{-p-1/2} \varphi_1 R_p \varphi_0, \quad (31)$$

where  $\varphi_1 R_p \varphi_0$  satisfies the following property if we denote by  $\varphi_1 R_p \varphi_0 = \begin{pmatrix} R_p^{11} & R_p^{12} \\ R_p^{21} & R_p^{22} \end{pmatrix}$

$$\begin{aligned} R_p^{11} &: L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p}(\Omega) \\ R_p^{12} &: L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p+2}(\Omega) \\ R_p^{21} &: L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p-2}(\Omega) \\ R_p^{22} &: L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p}(\Omega), \end{aligned} \quad (32)$$

where the norm operator are uniformly bounded with respect  $h$ , and the principal symbol of  $B_p$  is

$$\begin{pmatrix} (a^p |\xi|^{2p} - \mu^p)^{-1} & Q_{-2p-2}(x, \xi) \\ 0 & (|\xi|^{2p} - \mu^p)^{-1} \end{pmatrix} \quad (33)$$

Moreover the adjoint of  $R_p^{jq}$  satisfies (32) where the norm operator are uniformly bounded with respect  $h$ .

**Proof.** We argue by induction on  $k$  and for that we must introduce a sequence of cut-off functions. Let  $\chi_k$  and  $\tilde{\chi}_k$  be cut-off functions such that  $\tilde{\chi}_k \tilde{\chi}_{k+1} = \tilde{\chi}_{k+1}$ ,  $\chi_k \tilde{\chi}_k = \tilde{\chi}_k$ ,  $\tilde{\chi}_k \chi_{k+1} = \chi_{k+1}$ . We can assume that  $\tilde{\chi}_0 = 1$  on the support of  $\varphi_0$  and  $\tilde{\chi}_p = 1$  on the support of  $\varphi_1$ . We can apply Formula (24) where  $\tilde{\theta}$  is replaced by  $\tilde{\chi}_k$  and  $\theta$  by  $\chi_k$ . We have

$$\tilde{\chi}_k \Lambda_z S_z \Lambda_z^{-1} = |z|^{-1} \tilde{\chi}_k B \chi_k + |z|^{-1/2} W \chi_k \Lambda_z S_z \Lambda_z^{-1}, \quad (34)$$

where  $W$  and  $W^*$  are bounded on  $\overline{H}^s(\Omega)$ .

We prove by recurrence the following formula

$$\tilde{\chi}_k \prod_{j=1}^k \Lambda_z S_{\omega_j z} \Lambda_z^{-1} \tilde{\chi}_0 = |z|^{-k} \tilde{\chi}_k B_k \tilde{\chi}_0 + |z|^{-k-1/2} R_k \tilde{\chi}_0, \quad (35)$$

where the semi-classical principal symbol of  $B_k$  is given by

$$B_k = \begin{pmatrix} \prod_{j=1}^k (a|\xi|^2 - \omega_j \mu)^{-1} & Q_{-2k-2}(x, \xi) \\ 0 & \prod_{j=1}^k (|\xi|^2 - \omega_j \mu)^{-1} \end{pmatrix}, \quad (36)$$

where  $Q_{-2k-2}(x, \xi)$  is a symbol of order  $-2k - 2$ . The operators  $R_k^{jq}$  and their adjoints satisfy Estimates (32) with  $p = k$ .

For  $k = 1$ , Formula (36) and properties (32) for  $R_1^{jq}$  and their adjoints follow from (12), (11), Remark 5 and (34).

If Formula (35) is true for  $k$  we have

$$\tilde{\chi}_{k+1} \prod_{j=1}^{k+1} \Lambda_z S_{\omega_j z} \Lambda_z^{-1} \tilde{\chi}_0 = L_1 + L_2,$$

where

$$L_1 = \tilde{\chi}_{k+1} \Lambda_z S_{\omega_{k+1} z} \Lambda_z^{-1} \tilde{\chi}_k \prod_{j=1}^k \Lambda_z S_{\omega_j z} \Lambda_z^{-1} \tilde{\chi}_0$$

$$L_2 = \tilde{\chi}_{k+1} \Lambda_z S_{\omega_{k+1} z} \Lambda_z^{-1} (1 - \tilde{\chi}_k) \prod_{j=1}^k \Lambda_z S_{\omega_j z} \Lambda_z^{-1} \tilde{\chi}_0.$$

By (34) for  $k + 1$  and (35) for  $k$ , we have

$$L_1 = \left( |z|^{-1} \tilde{\chi}_{k+1} B \chi_{k+1} + |z|^{-1/2} W \chi_{k+1} \Lambda_z S_z \Lambda_z^{-1} \right) \left( |z|^{-k} \tilde{\chi}_k B_k \tilde{\chi}_0 + |z|^{-k-1/2} R_k \tilde{\chi}_0 \right).$$

The term  $|z|^{-1-k} \tilde{\chi}_{k+1} B \chi_{k+1} \tilde{\chi}_k B_k \tilde{\chi}_0$  gives the first right hand side term of (35), where  $B_{k+1} = B \chi_{k+1} \tilde{\chi}_k B_k$  and the principal symbol is given by Formula (36) on the support of  $\tilde{\chi}_{k+1}$ .

The three other terms have the form of  $R_{k+1}$  and satisfy Estimates (32). Indeed, the power of  $|z|$  is obtained as the operator norm of  $\Lambda_z S_z \Lambda_z^{-1}$  is bounded by  $|z|^{-1}$ . To prove the mapping between the  $H_{sc}^s$ , we denote by  $A_q$  a generic operator of order  $q$  mapping  $H^s$  to  $H^{s-q}$ . We check that

$$\begin{pmatrix} A_{-2} & A_{-4} \\ A_0 & A_{-2} \end{pmatrix} \cdot \begin{pmatrix} A_{-2k} & A_{-2k-2} \\ A_{-2k+2} & A_{-2k} \end{pmatrix} = \begin{pmatrix} A_{-2k-2} & A_{-2k-4} \\ A_{-2k} & A_{-2k-2} \end{pmatrix}. \quad (37)$$

The properties on adjoints follow from the recurrence assumptions on  $R_k$ , the properties on  $W^*$  and Remark 5.

By (34) for  $k + 1$  and as  $\chi_{k+1}(1 - \tilde{\chi}_k) = 0$ , we have

$$L_2 = |z|^{-1/2} W \chi_k \Lambda_z S_z \Lambda_z^{-1} (1 - \tilde{\chi}_k) \prod_{j=1}^k \Lambda_z S_{\omega_j z} \Lambda_z^{-1} \tilde{\chi}_0.$$

By (12) and (11) the operator norm of this term is  $|z|^{-k-1/2}$ . The proof that  $L_2$  satisfies Estimates (32) for  $p = k + 1$  is obtained by (37). The properties on the adjoint of  $L_2$  follow from Remark 5 and the properties on  $W^*$ .

From (36) for  $k = p$ , and as  $z^p - \mu^p = \prod_{j=1}^p (z - \omega_j \mu)$ , we obtain that  $\prod_{j=1}^p (a|\xi|^2 - \omega_j \mu)^{-1} = (a^p |\xi|^{2p} - \mu^p)^{-1}$ . This gives the diagonal terms of the symbol of  $B_p$  in Formula (33). ■

Now we can finish the proof of Lemma 4.2. We take  $\varphi_0$  such that  $\varphi_0(x) = 1$  is  $d(x, \mathbb{R}^n \setminus \Omega) \geq 2\delta$  and  $\varphi_0(x) = 0$  is  $d(x, \mathbb{R}^n \setminus \Omega) \leq \delta$ . We take  $\varphi_1$  such that  $\varphi_1(x) = 1$  is  $d(x, \mathbb{R}^n \setminus \Omega) \geq 4\delta$  and  $\varphi_1(x) = 0$  is  $d(x, \mathbb{R}^n \setminus \Omega) \leq 3\delta$ .

With the notation of (31) we deduce from (32) that for  $j = 1, 2$  we have

$$\| |z|^{-p-1/2} \varphi_1 R_p^{jj} \varphi_0 \|_{L^2(\Omega) \rightarrow \overline{H}_{sc}^{2p}(\Omega)} \lesssim |z|^{-p-1/2}.$$

This implies as for  $T_{jj}$  in (17) and (18) that

$$\| |z|^{-p-1/2} \varphi_1 R_p^{jj} \varphi_0 \|_{L^2(\Omega) \rightarrow L^2(\Omega)} \lesssim |z|^{-p-1/2} \quad \text{and} \quad \| |z|^{-p-1/2} \varphi_1 R_p^{jj} \varphi_0 \|_{L^2(\Omega) \rightarrow \overline{H}^{2p}(\Omega)} \lesssim |z|^{-1/2}.$$

By Formula (19) applied to the kernel of  $|z|^{-p-1/2} \varphi_1 R_p^{jj} \varphi_0$  denoted by  $K_{jj}^R$  and the properties on  $R_p$  and its adjoint given in Lemma 4.3, we obtain

$$\left( \int_{\Omega} |K_{jj}^R(x, x)|^2 dx \right)^{1/2} \leq C |z|^{-p+(n/2)-1/2}. \quad (38)$$

By the principal symbol of  $B_p$  given in Lemma 4.4 we can compute the integral of the kernel  $K_{jj}(x, x)$ . Denoting by  $b(x, \xi)$  either  $(a^p |\xi|^{2p} - \mu^p)^{-1}$  or  $(|\xi|^{2p} - \mu^p)^{-1}$ , the kernel of a diagonal term is given by  $(2\pi h)^{-n} \int e^{i(x-y)\xi/h} \varphi_1(x) b(x, \xi) \varphi_0(y) d\xi$  and this integral make sense if  $p > n/2$ . Denoting by  $K_{jj}^{B_p}(x, y)$  the diagonal terms of the kernel of  $|z|^{-p} \varphi_1 B_p \varphi_0$ , we obtain

$$\begin{aligned} & \int_{\Omega} K_{11}^{B_p}(x, x) dx + \int_{\Omega} K_{22}^{B_p}(x, x) dx \\ &= (2\pi)^{-n} |z|^{-p+n/2} \int \int \varphi_1(x) \left( (a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1} \right) d\xi dx + O(|z|^{-p+(n+1)/2}), \end{aligned} \quad (39)$$

where the error term  $O(|z|^{(n+1)/2})$  is given by the lower order terms in the symbolic calculus. From (31), (38) and (39), we deduce

$$\begin{aligned} & \int \varphi_0(x) (K_{11}(x, x) + K_{22}(x, x)) dx \\ &= (2\pi)^{-n} |z|^{-p+n/2} \int \int \varphi_1(x) \left( (a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1} \right) d\xi dx + O(|z|^{-p+(n+1)/2}). \end{aligned}$$

Now we can write

$$\Lambda_z T_{z^p} \Lambda_z^{-1} = \varphi_1 \Lambda_z T_{z^p} \Lambda_z^{-1} \varphi_0 + \varphi_1 \Lambda_z T_{z^p} \Lambda_z^{-1} (1 - \varphi_0) + (1 - \varphi_1) \Lambda_z T_{z^p} \Lambda_z^{-1}.$$

If  $K(x, y)$  is the kernel of  $\Lambda_z T_{z^p} \Lambda_z^{-1}$ , the kernel of the left hand side terms are respectively,

$$\varphi_1(x) K(x, y) \varphi_0(y), \quad \varphi_1(x) K(x, y) (1 - \varphi_0(y)) \quad \text{and} \quad (1 - \varphi_1(x)) K(x, y).$$

In particular, from the properties of the supports of  $\varphi_j$ , we have  $\varphi_1(x) K(x, x) (1 - \varphi_0(x)) = 0$ . Let  $F(x) = K_{11}(x, x) + K_{22}(x, x)$ , to show that

$$\left| \int_{\Omega} |z|^{p-n/2} F(x) dx - (2\pi)^{-n} \int \int \left( (a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1} \right) d\xi dx \right| \leq C\delta + C_{\delta} |z|^{-1/2}, \quad (40)$$

We shall prove

$$\begin{aligned} & \left| |z|^{p-n/2} \varphi_1(x) F(x) dx - (2\pi)^{-n} \int \int \left( (a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1} \right) d\xi dx \right| \\ & \leq C\delta + C_{\delta} |z|^{-1/2}, \end{aligned} \quad (41)$$

and

$$|z|^{p-n/2} \left| \int_{\Omega} (1 - \varphi_1(x)) F(x) dx \right| \leq C\delta^{1/2}. \quad (42)$$

Obviously (41) and (42) imply (40), and (40) implies Lemma 4.2. To prove (41), we apply (39). We have

$$\begin{aligned} \int_{\{x, d(x, \mathbb{R}^n \setminus \Omega) \leq 2\delta\}} \int |(a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1}| d\xi dx \\ \leq C \int_{\{x, d(x, \mathbb{R}^n \setminus \Omega) \leq 2\delta\}} \int \langle \xi \rangle^{-2p} d\xi dx \leq C\delta, \end{aligned}$$

indeed  $|\xi|^{2p}$  and  $a^p |\xi|^{2p}$  are not in the same cone as  $\mu^p$  by assumption. Thus the principal term in (41) given by the principal term from (39), can be estimate by  $C\delta$  and the error terms from (39) can be estimate by  $C_{\delta}|z|^{-1/2}$ .

To prove (42), using (20) we obtain

$$\left| \int_{\Omega} (1 - \varphi_1(x)) K_{jj}(x) dx \right| \leq \left( \int_{\Omega} (K_{jj}(x, x))^2 dx \right)^{1/2} \left( \int_{\{x, d(x, \mathbb{R}^n \setminus \Omega) \leq 4\delta\}} dx \right)^{1/2} \leq C|z|^{-p+n/2} \delta^{1/2},$$

which is Estimate (42).

## 5 Proof of Theorem 8

We shall apply the following tauberian Theorem cited in Agmon [1, th. 14.5], The proof is given in Karamata [14].

**Theorem 9** (Tauberian Theorem). *Let  $\sigma(\lambda)$  be a non decreasing function for  $\lambda > 0$ , let  $0 < a < 1$ , let  $\alpha$  be a non-negative number, and suppose that as  $t \rightarrow +\infty$ ,*

$$\int_0^{+\infty} \frac{d\sigma(\lambda)}{\lambda + t} = \alpha t^{a-1} + o(t^{a-1}).$$

Then as  $\lambda \rightarrow +\infty$

$$\sigma(\lambda) = \alpha \frac{\sin \pi a}{\pi a} \lambda^a + o(\lambda^a).$$

Theorem 8 is implied by tauberian theorem and Theorem 7 which gives

$$\begin{aligned} \sum_{j \in \mathbb{N}} \frac{1}{\mu_j^p - z^p} = A|z|^{-p+n/2} + o(|z|^{-p+n/2}), \\ \text{where } A = (2\pi)^{-n} \int_{\Omega} \int ((a^p |\xi|^{2p} - \mu^p)^{-1} + (|\xi|^{2p} - \mu^p)^{-1}) d\xi dx, \text{ with } \mu = z/|z|. \end{aligned} \quad (43)$$

We take  $z = t^{1/p} e^{i\pi/p}$ , where  $t > 0$ , we obtain

$$\sum_{j \in \mathbb{N}} \frac{1}{\mu_j^p + t} = A t^{-1+n/2p} + o(t^{-1+n/2p}). \quad (44)$$

Let  $\mu_j = \delta_j + i\nu_j$  where  $\delta_j$  and  $\nu_j$  are real. Let  $A(t) = \sum_{j \in \mathbb{N}} \frac{1}{\delta_j^p + t}$ .

By assumption (see Remark 1), for all  $\varepsilon > 0$ , there exists  $R_{\varepsilon} > 0$ , such that if  $\delta_j \geq R_{\varepsilon}$  then  $|\nu_j| \leq \varepsilon \delta_j$ .

We have for  $j$  large enough such that  $|\nu_j| \leq \varepsilon \delta_j$ ,

$$\left| \frac{1}{\delta_j^p + t} - \frac{1}{\mu_j^p + t} \right| \leq C\varepsilon \frac{1}{\delta_j^p + t}, \quad (45)$$

where  $C$  depends only on  $p$ .

Denoting  $N_\varepsilon$  such that for all  $j \geq N_\varepsilon$ , then  $|\nu_j| \leq \varepsilon \delta_j$ . We have

$$\sum_{j=1}^{N_\varepsilon} \frac{1}{|\mu_j^p + t|} \leq C \sum_{j=1}^{N_\varepsilon} \frac{1}{\delta_j^p + t} \leq CN_\varepsilon/t. \quad (46)$$

We deduce from (44), (45) and (46) that

$$A(t) + O(N_\varepsilon/t) + O(\varepsilon A(t)) = At^{-1+n/2p} + o(t^{-1+n/2p}). \quad (47)$$

For  $\varepsilon$  small enough we deduce there exists  $C > 0$  such that

$$(1/C)t^{-1+n/2p} \leq A(t) \leq Ct^{-1+n/2p}.$$

Using this in (47) we obtain

$$\sum_{j \in \mathbb{N}} \frac{1}{\delta_j^p + t} = At^{-1+n/2p} + o(t^{-1+n/2p}).$$

Denoting  $\sigma(\lambda) = \#\{j \in \mathbb{N}, \delta_j^p \leq \lambda\}$  where the number is counted with multiplicity. We have

$$\sum_{j \in \mathbb{N}} \frac{1}{\delta_j^p + t} = \int_0^{+\infty} \frac{d\sigma(\lambda)}{\lambda + t} = At^{-1+n/2p} + o(t^{-1+n/2p}).$$

By tauberian Theorem we obtain

$$\#\{j \in \mathbb{N}, \delta_j^p \leq \lambda\} = 2pA \frac{\sin(n\pi/(2p))}{n\pi} \lambda^{n/(2p)} + o(\lambda^{n/(2p)}),$$

which is equivalent to

$$\#\{j \in \mathbb{N}, \delta_j \leq t^2\} = 2pA \frac{\sin(n\pi/(2p))}{n\pi} t^n + o(t^n). \quad (48)$$

We recall that  $\mu = z/|z|$  and  $z = t^{1/p}e^{-i\pi/p}$  thus  $\mu = -1$ . To compute  $A$  from (43), we must compute integral as  $\int (b^p|\xi|^{2p} + 1)^{-1}d\xi$ , where  $b > 0$ . We have

$$\begin{aligned} \int (b^p|\xi|^{2p} + 1)^{-1}d\xi &= n\text{Vol}(B_1) \int_0^{+\infty} r^{n-1}(b^p r^{2p} + 1)^{-1}dr \\ &= n(2p)^{-1}b^{-n/2}\text{Vol}(B_1) \int_0^{+\infty} \sigma^{n/(2p)-1}(\sigma + 1)^{-1}d\sigma \\ &= n(2p)^{-1}b^{-n/2}\text{Vol}(B_1)\pi \sin^{-1}(n\pi/(2p)), \end{aligned}$$

where the last integral is computed by residue theorem (see Cartan [6, p 107]).

This gives that

$$2pA \frac{\sin(n\pi/(2p))}{n\pi} = (2\pi)^{-n}\text{Vol}(B_1) \int_{\Omega} (a^{-n/2}(x) + 1)dx = \alpha.$$

Now to prove the statement of Theorem 8, we must prove that  $\alpha = \lim t^{-n} \#\{j \in \mathbb{N}, \delta_j \leq t^2\} = \lim t^{-n} \#\{j \in \mathbb{N}, |\mu_j| \leq t^2\}$  when  $t$  goes to  $\infty$ . Except for a finite number of values,  $\delta_j > 0$  and as  $\delta_j \leq |\mu_j|$  there exists  $C > 0$  such that

$$\#\{j \in \mathbb{N}, |\mu_j| \leq t^2\} \leq \#\{j \in \mathbb{N}, \delta_j \leq t^2\} + C.$$

Thus

$$\limsup_{t \rightarrow \infty} t^{-n} \#\{j \in \mathbb{N}, |\mu_j| \leq t^2\} \leq \alpha. \quad (49)$$

For all  $\varepsilon > 0$ , there exists  $J_\varepsilon$  such that for all  $j \geq J_\varepsilon$ ,  $|\nu_j| \leq \varepsilon \delta_j$  then  $|\mu_j| \leq (1 + \varepsilon)\delta_j$ . Thus there exist  $C_\varepsilon > 0$  such that

$$\#\{j \in \mathbb{N}, \delta_j \leq t^2\} \leq \#\{j \in \mathbb{N}, (1 + \varepsilon)^{-1}|\mu_j| \leq t^2\} + C_\varepsilon,$$

which is equivalent to

$$\#\{j \in \mathbb{N}, \delta_j \leq (1 + \varepsilon)^{-1}t^2\} \leq \#\{j \in \mathbb{N}, |\mu_j| \leq t^2\} + C_\varepsilon.$$

We obtain

$$(1 + \varepsilon)^{-n/2}\alpha \leq \liminf_{t \rightarrow \infty} t^{-n}\#\{j \in \mathbb{N}, |\mu_j| \leq t^2\}.$$

As this estimate is true for all  $\varepsilon > 0$  and from (49) we have

$$\alpha \leq \liminf_{t \rightarrow \infty} t^{-n}\#\{j \in \mathbb{N}, |\mu_j| \leq t^2\} \leq \limsup_{t \rightarrow \infty} t^{-n}\#\{j \in \mathbb{N}, |\mu_j| \leq t^2\} \leq \alpha.$$

This is the statement of Theorem 8.

## References

- [1] Agmon Shmuel, Lectures on elliptic boundary value problems. Van Nostrand Mathematical Studies, No. 2, 1965.
- [2] Cakoni Fioralba, Çayören Mehmet, Colton David, Transmission eigenvalues and the nondestructive testing of dielectrics. *Inverse Problems* **24** (2008) 065016.
- [3] Cakoni Fioralba, Colton David, Haddar Housseem, The interior transmission problem for regions with cavities. *SIAM J. Math. Anal.* **42** (2010) 145–162.
- [4] Cakoni Fioralba, Gintides Drossos, Haddar Housseem, The existence of an infinite discrete set of transmission eigenvalues. *SIAM J. Math. Anal.* **42** (2010) 237–255.
- [5] Cakoni Fioralba, Haddar Housseem, Transmission Eigenvalues in Inverse Scattering Theory. <http://hal.inria.fr/hal-00741615>
- [6] Cartan Henri , Théorie élémentaire des fonctions analytiques. Hermann.
- [7] Colton David, Kirsch Andreas, Päivärinta Lassi, Far-field patterns for acoustic waves in an inhomogeneous medium. *SIAM J. Math. Anal.* **20** (1989) 1472–1483.
- [8] Colton David, Kress Rainer, Inverse acoustic and electromagnetic scattering theory. *Applied Mathematical Sciences*, **93**, Springer-Verlag, Berlin, 1992.
- [9] Dimassi Mouez , Petkov Vesselin, Upper bound for the counting function of interior transmission eigenvalues, arxiv: 1308.2594
- [10] Faierman Melvin, Transmission eigenvalues for parameter-elliptic boundary problems, preprint.
- [11] Hitrik Michael, Krupchyk Katsiaryna, Ola Petri, Päivärinta Lassi. Transmission eigenvalues for operators with constant coefficients. *SIAM J. Math. Anal.* **42** (2010) 2965–2986.
- [12] Hitrik Michael, Krupchyk Katsiaryna, Ola Petri, Päivärinta Lassi, Transmission eigenvalues for elliptic operators. *SIAM J. Math. Anal.* **43** (2011) 2630–2639.
- [13] Hitrik Michael, Krupchyk Katsiaryna, Ola Petri, Päivärinta Lassi, The interior transmission problem and bounds on transmission eigenvalues, arxiv: arXiv:1009.5640v1
- [14] Karamata, J., Neuer Beweis und Verallgemeinerung einiger Tauberian-Sätze. *Math. Z.* **33** (1931) 294–299



- [15] Lakshtanov Evgeny, Vainberg Boris, Ellipticity in the Interior Transmission Problem in Anisotropic Media. *SIAM J. Math. Anal.* **44** (2012) 1165–1174.
- [16] Lakshtanov Evgeny, Vainberg Boris, Remarks on interior transmission eigenvalues, Weyl formula and branching billiards. *J. Phys. A: Math. Theor* **45** 125202.
- [17] Lakshtanov Evgeny, Vainberg Boris, Bounds on positive interior transmission eigenvalues. arxiv: 1206.3782v2.
- [18] Lakshtanov Evgeny, Vainberg Boris, Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem. arxiv: arXiv:1212.6785
- [19] Malliavin, Paul, Un théorème taubérien relié aux estimations de valeurs propres. Séminaire Jean Leray (1962-1963) 224–231 [http://www.numdam.org/item?id=SJL\\_1962-1963\\_\\_\\_\\_224\\_0](http://www.numdam.org/item?id=SJL_1962-1963____224_0)
- [20] Päivärinta Lassi, Sylvester John, Transmission eigenvalues. *SIAM J. Math. Anal.* **40** (2008) 738–753.
- [21] Pham Ha, Stefanov Plamen, Weyl asymptotics of the transmission eigenvalues for a constant index of refraction, arxiv:1309.3616
- [22] Robbiano Luc, Spectral analysis on interior transmission eigenvalues, <http://hal.archives-ouvertes.fr/hal-00634867/fr/>
- [23] Sylvester John, Discreteness of Transmission Eigenvalues via Upper Triangular Compact Operators. *SIAM J. Math. Anal.* **44** (2012) 341–354.