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GAUSSIAN TYPE LOWER BOUNDS FOR THE DENSITY OF SOLUTIONS OF SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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Abstract. In this paper we obtain Gaussian type lower bounds for the density of solutions to stochastic differential equations (sde's) driven by a fractional Brownian motion with Hurst parameter H. In the one dimensional case with additive noise, our study encompasses all parameters $H \in (0,1)$, while the other cases are restricted to the case $H > 1/2$. We rely on a mix of pathwise methods for stochastic differential equations and stochastic analysis tools.

Contents

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1. Introduction

Let $B = (B^1, \dots, B^d)$ be a d dimensional fractional Brownian motion (fBm in the sequel) defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Hurst parameter $H \in (0,1)$. Recall that this means that B is a centered Gaussian process indexed by $\mathbb{R}_+$, whose coordinate processes are independent and their covariance structure is defined by

$$R(t,s) := \mathbf{E}\left[B_s^j B_t^j\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad \text{for } s,t \in [0,1] \text{ and } j = 1, \dots, d. \tag{1}$$

This implies that the variance of an increment is given by

$$\mathbf{E}\left[\left(B_t^j - B_s^j\right)^2\right] = |t-s|^{2H}, \quad \text{for} \quad s,t \in \mathbb{R}_+. \tag{2}$$

In particular, this process is γ-Hölder continuous a.s. for any $\gamma < H$ and is an H-self similar process. This converts fBm into a very natural generalization of Brownian motion and explains the fact that it is used in applications [14, 21, 23].

We are concerned here with the following class of stochastic differential equations (sde's) in $\mathbb{R}^m$ driven by B on the time interval $[0,1]$:

$$X_t = a + \int_0^t V_0(X_s)ds + \sum_{i=1}^d \int_0^t V_i(X_s)dB_s^i, \tag{3}$$

where $a \in \mathbb{R}^m$ is a generic initial condition and $\{V_i;\, 0 \le i \le d\}$ is a collection of smooth and bounded vector fields of $\mathbb{R}^m$. Though equation (3) can be solved thanks to rough paths methods in the general case $H > 1/4$, $d \ge 1$, we shall consider in the sequel three situations which can be handled without recurring to this kind of technique:

(1) The one-dimensional case with additive noise, which can be treated via simple ODE techniques.
(2) The one-dimensional situation, namely $m = d = 1$, where the equation can be solved thanks to a Doss-Sussman type methodology as mentioned in [15].
(3) The case of a Hurst exponent $H > 1/2$, for which Young integration methods are available (see e.g [11, 19, 25]).

Hence, we always understand the solution to equation (3) according to the settings mentioned above. We shall see however that rough path type arguments shall be involved in some of our proofs.

The process defined as the solution of (3) is obviously worth studying, and a natural step in this direction is to analyze the density of the random variable X_t for a fixed $t > 0$. To this respect, the following results are available in our cases of interest:

(1) For $m = d = 1$, the existence of density for $\mathcal{L}(X_t)$ has been examined in [15].
(2) Whenever $H > 1/2$ and in a multidimensional setting, the existence of density is established in [20], while smoothness under elliptic assumptions is handled in [12].

Let us also mention that for a multidimensional equation (3) and $H \in (1/4, 1/2)$, rough paths techniques also enable the study of densities of the solution. We refer to [6, 7] for existence and [5] for smoothness results for $\mathcal{L}(X_t)$. However, the only Gaussian type estimate for the density we are aware of is the one contained in [3], which relies heavily on a skew-symmetric assumption for the vector fields $V_1, \dots, V_d$.

The current article is thus dedicated to give Gaussian type lower bounds for the density of X_t. More specifically, we work under the following assumptions on the coefficients of equation (3):

Hypothesis 1.1. *The coefficients $V_0, \dots, V_d$ of equation (3) satisfy the following conditions:*

(1) *If $m = d = 1$, then V_0, $V_1 \in \mathcal{C}_b^3$ and we also assume $\lambda \leq |V_1| \leq \Lambda$.*
(2) *In the multidimensional case, the vector fields $V_0, \dots, V_d$ belong to the space $\mathcal{C}_b^\infty$ of smooth functions bounded together with all their derivatives. Furthermore, if $V(x)$ denotes the matrix $(V_1(x), \dots, V_d(x)) \in \mathbb{R}^{m\times d}$ for all $x \in \mathbb{R}^m$, then we assume the following uniform elliptic condition:*

$$\lambda\, Id_m \leq V(x)V^*(x) \leq \Lambda\, Id_m, \quad \text{for all} \quad x \in \mathbb{R}^m, \tag{4}$$

where the inequalities are understood in the matrix sense and where λ, Λ are two given strictly positive constants which are independent of x.

With these hypotheses in hand, our main goal is to prove the following result:

Theorem 1.2. *Consider equation (3), under the following three specific situations:*

I. *$m = d = 1$, $H \in (0, 1)$, $V_0 \in \mathcal{C}_b^1$ and the noise is additive (i.e., V_1 is a non vanishing real constant).*
II. *$m = d = 1$, $H \in (1/2, 1)$ and Hypothesis 1.1(1) is satisfied for V_0, V_1.*
III. *Arbitrary m, $d \in \mathbb{N}$, $H \in (1/2, 1)$ and $V_0 \dots V_d$ satisfying Hypothesis 1.1 (2).*

Then, the solution X_t of equation (3) possesses a density $p_t(x)$ such that for every $x \in \mathbb{R}^m$ and $t \in (0, 1]$ we have:

$$p_t(x) \geq \frac{c_1}{t^{mH}} \exp\left(-\frac{c_2\,|x-a|^2}{t^{2H}}\right), \tag{5}$$

for some constants c_1, c_2 only depending on d, m and $V_0, \dots, V_d$.

As mentioned above, this is (to the best of our knowledge) the first Gaussian lower bounds obtained for equations driven by fBm in a general setting. It should also be mentioned that the lower bound (5) can be complemented by a similar upper bound contained in [4].

Let us say a few words about the methodology we rely on in order to obtain our lower bound (5). Generally speaking it is based on Malliavin calculus tools, but the three results mentioned in Theorem 1.2 are proved in different ways:

(1) In the one dimensional additive case, we invoke a recent formula for densities introduced in [16] which yields an easy way to estimate p_t in the case of additive stochastic equations. We thus include this study for didactical purposes, and also because we obtain (slightly non optimal) Gaussian upper and lower bounds with elegant methods.

(2) The one dimensional case with multiplicative noise is based on the Doss-Sussmann's transform and Girsanov type arguments. It is rather easy to implement and yields results when the criterion of [16] can not be applied.

(3) As far as the general case is concerned, it will be basically handled thanks to the decomposition of random variables strategy introduced in [2, 13]. However, let us point out two important differences between the fBm and the diffusion case:

(i) In the case of the sde (3) without drift coefficient V_0, the first step of the method implemented (for a fixed $t \in (0,1]$) in [2, 13] amounts to introduce a partition $\{t_j;\, 0 \le j \le n\}$ such that $t_0 = 0$ and $t_n = t$, with n large enough, and then split X_t into small contributions of the form

$$X_{t_{j+1}} - X_{t_j} = \sum_{i=1}^{d} V_i(X_{t_j}) \left[B^i_{t_{j+1}} - B^i_{t_j} \right] + \sum_{i=1}^{d} \int_0^t \left[V_i(X_s) - V_i(X_{t_j}) \right] dB^i_s. \tag{6}$$

Then a main conditionally Gaussian contribution $V_i(X_{t_j})[B^i_{t_{j+1}} - B^i_{t_j}]$ is identified in the right hand side of equation (6), while the other terms are a small remainder in the Malliavin calculus sense in comparison with the first. Roughly speaking, the Gaussian lower bound (5) is then obtained by adding those main contributions and proving that the remainder does not significantly modify the estimate. However, let us highlight the fact that this general scheme does not fit to the fractional Brownian motion setting.

Indeed, due to the fBm dependence structure, the main contributions to the variance of X_t in the current situation come from the cross terms $\mathbf{E}[(B^i_{t_{j+1}} - B^i_{t_j})(B^i_{t_{k+1}} - B^i_{t_k})]$ for $j \neq k$. We have thus decided to express equation (3) as an anticipative Stratonovich type equation with respect to the Wiener process induced by B. This is known to be an inefficient way to solve the original equation, but turns out to be very useful in order to analyze the law of X_t. We shall detail this strategy at Section 5.1.

(ii) In the case of an equation driven by usual Brownian motion, the Malliavin-Sobolev norms involved in the computations give deterministic contributions after conditioning, due to the independence of increments of the Wiener process. This is not true anymore in the fBm case, and we thus need to add a proper localization to the arguments of [2, 13].

Our article is structured as follows: Section 2 is devoted to recall some useful facts on fractional Brownian motion and stochastic differential equations. We handle the one dimensional case with additive noise at Section 3 and the one dimensional case with multiplicative noise in Section 4 with different methodologies. Finally, the bulk of our article focuses on the general multidimensional case contained in Section 5. Some auxiliary results used in Section 5 dealing with stochastic derivatives are given in an Appendix.

Notations: Throughout this paper, unless otherwise specified we use $|\cdot|$ for Euclidean norms and $\|\cdot\|_{L^p}$ for the $L^p(\Omega)$ norm with respect to the underlying probability measure $\mathbf{P}$. For a random variable X, $\mathcal{L}(X)$ denotes its law and for a σ-field $\mathcal{F}$, $X \in \mathcal{F}$ denotes the fact that X is $\mathcal{F}$-measurable.

Consider a finite-dimensional vector space V and a subset $U \subset \mathbb{R}^d$. The space of V-valued Hölder continuous functions defined on U, with k-derivatives which are γ- Hölder continuous with $\gamma \in (0,1)$, will be denoted by $\mathcal{C}^{k+\gamma}(U;V)$, or just $\mathcal{C}^\gamma$ when $U = [0,1]$. For a function $g \in \mathcal{C}^\gamma(V)$ and $0 \le s < t \le 1$, we shall consider the semi-norms

$$\|g\|_{s,t,\gamma} = \sup_{s \le u < v \le t} \frac{|g_v - g_u|_V}{|v-u|^\gamma},$$

The semi-norm $\|g\|_{0,1,\gamma}$ will simply be denoted by $\|g\|_\gamma$. Similarly, for an open set U, $\mathcal{C}^1_b(U;V)$ denotes the space of bounded continuously differentiable functions with bounded first derivative. For $x, y \in \mathbb{R}^m$, we set $\mathbf{1}_{\{y \ge x\}} := \prod_{k=1}^m \mathbf{1}_{\{y_k \ge x_k\}}$. Vectors $x \in \mathbb{R}^m$ denote column vectors, their j-th component is denoted by x^j and the transpose of x is denoted by x^*. The identity matrix of order $m \times m$ is denoted by Id_m.

Finally, let us mention that generic constants will be denoted by c, c_H, c_V, etc. independently of their actual value which may change from one line to the next. This rule will also apply for the constants M and M' which will appear as localization parameters, with the following additional convention: each time a localization constant appears, it increases its value by the addition of a fixed universal constant from the previous value. For a detailed explanation, see (15).

2. Stochastic calculus for fractional Brownian motion

This section is devoted to give the basic elements of stochastic calculus with respect to B which allow to understand the remainder of the paper. For some fixed $H \in (0,1)$, we consider $(\Omega, \mathcal{F}, \mathbf{P})$ the canonical probability space associated with the fractional Brownian motion (in short fBm) with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0,1])$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbf{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \dots, B_t^d),\ t \in [0,1]\}$ is a fractional Brownian motion with Hurst parameter H. In this context, let us recall that B is a d-dimensional centered Gaussian process, whose covariance structure is induced by equation (2).

2.1. Malliavin calculus tools.

Gaussian techniques are obviously essential in the analysis of fBm driven differential equations like (3), and we proceed here to introduce some of them (see Chapter 5 in [17] for further details).

2.1.1. *Wiener space associated to fBm.* Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0,1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ under the distance defined by through the scalar product:

$$\langle (\mathbf{1}_{[0,t_1]}, \cdots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \cdots, \mathbf{1}_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^{d} R(t_i, s_i).$$

The space $\mathcal{H}$ is isometric to the reproducing kernel Hilbert space associated to R.

Furthermore, if $(e_1, \dots, e_d)$ designates the canonical basis of $\mathbb{R}^d$, one constructs an isometry $K^* \equiv K^*_{H,1} : \mathcal{H} \to L^2([0,1]; \mathbb{R}^d)$ such that $K^*(\mathbf{1}_{[0,t]}\, e_i) = \mathbf{1}_{[0,t]}\ K_H(t, \cdot)\, e_i$, where the kernel $K = K_H$ is given by

$$\begin{aligned} K(t,s) &= c_H\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du, \quad H > \frac{1}{2}, \\ K(t,s) &= c_{H,1} \left(\frac{s}{t}\right)^{1/2-H} (t-s)^{H-1/2} + c_{H,2}\, s^{1/2-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}}\, du, \quad H < \frac{1}{2}, \end{aligned} \tag{7}$$

for $0 \le s \le t$ and some explicit universal constants c_H, $c_{H,1}, c_{H,2}$. With a slight abuse of notation we will denote the associated integral operator by $Kf(x) = \int_0^x f(s) K(x,s) ds$. Note that we have that $R(s,t) = \int_0^{s \wedge t} K(t,r) K(s,r)\, dr$. Moreover, let us observe that $K^* \equiv K_1^*$ can be represented in the following form: for $H \in (1/2, 1)$ we have

$$[K^* \varphi]_t = \int_t^1 \varphi_r\, \partial_r K(r,t)\, dr$$

while for $H \in (0,1/2)$ it holds that

$$[K^*\varphi]_t = K(1,t)\,\varphi_t + \int_t^1 (\varphi_r - \varphi_t)\;\partial_r K(r,t)\,dr.$$

When $H \in (1/2,1)$ it can be shown that $L^{1/H}([0,1],\mathbb{R}^d) \subset \mathcal{H}$, and when $H \in (0,1/2)$ one has $\mathcal{C}^\gamma \subset \mathcal{H} \subset L^2([0,1])$ for all $\gamma > \frac{1}{2} - H$. We shall also use the following representations of the inner product in $\mathcal{H}$:

(i) For $H \in (1/2,1)$ and $\phi,\psi \in \mathcal{H}$ we have

$$\langle K^*\phi, K^*\psi\rangle_{L^2([0,1])} = \langle \phi,\psi\rangle_{\mathcal{H}} = c_H \int_0^1\int_0^1 |s-t|^{2H-2}\,\langle \phi_s,\psi_t\rangle_{\mathbb{R}^d}\,dsdt\;. \tag{8}$$

(ii) For $H \in (0,1/2)$, consider any family of partitions $\pi = (t_j)$ of $[0,1]$, and set $Q_{jk} = \sum_{i=1}^d \mathbf{E}[\Delta_j^i(B)\Delta_k^i(B)]$ with $\Delta_j^i(B) = B^i_{t_j} - B^i_{t_{j-1}}$. Then, for $\phi,\psi \in \mathcal{H}$ we have

$$\langle \phi,\psi\rangle_{\mathcal{H}} = \lim_{|\pi|\to 0} \sum_{j,k} \langle \phi_{t_{j-1}}, \psi_{t_{k-1}}\rangle_{\mathbb{R}^d}\, Q_{jk}. \tag{9}$$

Let us also recall that there exists a d-dimensional Wiener process W defined on $(\Omega,\mathcal{F},\mathbf{P})$ such that B can be expressed as

$$B_t = \int_0^t K(t,r)\,dW_r, \quad t \in [0,T]. \tag{10}$$

This formula will be referred to as Volterra's representation of fBm. Formula (10) has various important implications. For example, it is readily checked that $\mathcal{F}_t \equiv \sigma\{B_s;\, 0 \le s \le t\} = \sigma\{W_s;\, 0 \le s \le t\}$. This filtration will appear in the sequel.

2.1.2. *Malliavin calculus for B.* Isometry arguments allow to define the Wiener integral $B(h) = \int_0^1 \langle h_s, dB_s\rangle$ for any element $h \in \mathcal{H}$, such that it satisfies $\mathbf{E}[B(h_1)\;B(h_2)] = \langle h_1,\, h_2\rangle_{\mathcal{H}}$ for any $h_1,h_2 \in \mathcal{H}$. A $\mathcal{F}$-measurable real valued random variable F is then said to be cylindrical if it can be written, for a given $n \ge 1$, as

$$F = f\left(B(h^1),\dots,B(h^n)\right) = f\Big(\int_0^1 \langle h_s^1, dB_s\rangle, \dots, \int_0^1 \langle h_s^n, dB_s\rangle\Big)\;,$$

where $h^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The set of cylindrical random variables is denoted by $\mathcal{S}$.

The Malliavin derivative with respect to B is defined as follows: for $F \in \mathcal{S}$, the derivative of F is the $\mathbb{R}^d$ valued stochastic process $(\mathbf{D}_t F)_{0\le t\le 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^n h_t^i\,\frac{\partial f}{\partial x_i}\left(B(h^1),\dots,B(h^n)\right).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$\mathbf{D}^k_{t_1,\dots,t_k} F = \mathbf{D}_{t_1}\dots\mathbf{D}_{t_k} F.$$

For any $p \geq 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $L^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\boldsymbol{D}^{k,p}$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbf{E}\left[F^p\right] + \sum_{j=1}^{k} \mathbf{E}\left[\left\| \mathbf{D}^j F \right\|^p_{\mathcal{H}^{\otimes j}} \right] \right)^{\frac{1}{p}},$$

for $k \geq 0$ and $p \geq 1$. In particular, $\|F\|_{0,p} \equiv \|F\|_p = (\mathbf{E}\left[F^p\right])^{1/p}$. The dual operator of $\mathbf{D}$ is denoted by $\boldsymbol{\delta}$, which corresponds to the Skorohod integral with respect to the fBm B on the interval $[0,1]$. The set of smooth integrands is defined as $\boldsymbol{D}^\infty = \cap_{k,p\geq 1} \boldsymbol{D}^{k,p}(\mathcal{H})$, the set of smooth processes is $\boldsymbol{L}^{k,p}(\mathcal{H})$ and the Malliavin covariance matrix of F is denoted by $\boldsymbol{\Gamma}_F$.

As mentioned in the introduction, our lower bound (5) will be obtained by considering equation (3) as an equation driven by the underlying Wiener process W defined in (10), meaning that we shall also use stochastic analysis estimates with respect to W. We refer to Chapter 1 in [17] for this classical setting, and just mention here a few notations: we denote by D the differentiation operator with respect to W and by δ the corresponding dual operator (Skorohod integral). The respective norms in the Sobolev spaces $D^{k,p}(L^2([0,1]))$ are denoted by $\|\cdot\|_{k,p}$ and the space of smooth integrands by $L^{k,p}$. The following simple relation between $\mathbf{D}$ and D is then shown in [17], Proposition 5.2.1 :

Proposition 2.1. *Let $D^{1,2}$ is the Malliavin-Sobolev space corresponding to the Wiener process W. Then $\boldsymbol{D}^{1,2} = (K^*)^{-1} D^{1,2}$ and for any $F \in D^{1,2}$ we have $\mathrm{D}F = K^* \mathbf{D}F$ whenever both members of the relation are well defined.*

In fact the above proposition says that the derivatives $\boldsymbol{D}$ and D are somewhat interchangeable. Indeed, using formula (5.14) in [17] which gives an explicit formula for $(K^*)^{-1}$ one obtains such a property. In particular, we will use that for $F \in \mathcal{F}_t$ with $F \in D^{k,p}$ and for $u = (u_1, \ldots, u_k) \in [0,1]^k$ we have

$$|\mathrm{D}^k_{u_1,\ldots,u_k} F| \leq \text{ess sup}_{u_i \leq r_i;\ i=1,\ldots,k} |\mathbf{D}^k_{r_1,\ldots,r_k} F| K(t,u_1) \ldots K(t,u_k). \tag{11}$$

For the proof of (11) and other needed properties see Appendix 6.

Some of our computations in Section 5 will rely on some conditional Malliavin calculus arguments, for which some definitions need to be recalled. First, for a given $t \in [0,1]$ and $F \in L^2(\Omega)$, we shorten notations and write

$$\mathbf{E}_t[F] := \mathbf{E}[F|\,\mathcal{F}_t],$$

and also set $\mathbf{P}_t$ for the respective conditional probability and $Cov_t(G)$ for the conditional covariance matrix of a Gaussian vector G. We shall only use conditional Malliavin calculus with respect to the underlying Wiener process W, for which we recall the following definitions: For a random variable F and $t \in [0,1]$, let $\|F\|_{k,p,t}$ and $\Gamma_{F,t}$ be the quantities defined (for $k \geq 0$, $p > 0$) by:

$$\|F\|_{k,p,t} = \left(\mathbf{E}_t\left[F^p\right] + \sum_{j=1}^{k} \mathbf{E}_t \left[\left\| D^j F \right\|^p_{(L^2_t)^{\otimes j}} \right] \right)^{\frac{1}{p}}, \text{ and } \boldsymbol{\Gamma}_{F,t} = \left(\langle \mathbf{D}F^i, \mathbf{D}F^j \rangle_{L^2_t} \right)_{1 \leq i,\, j \leq d}, \tag{12}$$

where we have set $L^2_t \equiv L^2([t,1])$.

With this notation in hand, we give a conditional version of the integration by parts formula with respect to the Wiener process W borrowed from [17, Proposition 2.1.4].

Proposition 2.2. *Fix $n \geq 1$. Let F, Z_s, $G \in (D^\infty)^d$ be three random vectors where $Z_s \in \mathcal{F}_s$-measurable and $(\det_{\Gamma_{F+Z_s}})^{-1}$ has finite moments of all orders. Let $g \in \mathcal{C}_p^\infty(\mathbb{R}^d)$. Then, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \{1, \dots, d\}^n$, there exists a r.v. $H_\alpha^s(F, G) \in \cap_{p\geq 1} \cap_{m\geq 0} D^{m,p}$ such that*

$$\mathbf{E}\left[(\partial_\alpha g)(F + Z_s)\, G | \mathcal{F}_s\right] = \mathbf{E}\left[g(F + Z_s)\, H_\alpha^s(F, G) | \mathcal{F}_s\right], \tag{13}$$

where $H_\alpha^s(F, G)$ is recursively defined by

$$H_{(i)}^s(F, G) = \sum_{j=1}^d \delta_s\left(G\left(\Gamma_{F,s}^{-1}\right)_{ij} \mathbf{D}F^j\right), \quad H_\alpha^s(F, G) = H_{(\alpha_n)}^s(F, H_{(\alpha_1,\dots,\alpha_{n-1})}^s(F, G)).$$

Here δ_s denotes the Skorohod integral with respect to the Wiener process W on the interval $[s, 1]$. Furthermore, the following norm estimates with $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}$ hold true:

$$\|H_\alpha^s(F, G)\|_{p,s} \leq c \|\det(\Gamma_{F,s})^{-1}\|_{2^{n-1}q_1,s}^n \|F\|_{n+2,2^n q_2,s}^{2(dn+1)} \|G\|_{n,q_3,s}.$$

We will also resort to a localized version of the above bounds. Namely, we introduce a family of functions $\Phi_{M,\epsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ indexed by $M, \epsilon > 0$, which are regularizations of $\mathbf{1}_{\{x\leq M\}}$. Specifically, we define a function $\phi_\epsilon = \epsilon^{-1}\phi : \mathbb{R} \to \mathbb{R}$ with

$$\phi(x) := c_\phi \exp\left(-\frac{1}{1 - x^2}\right) \mathbf{1}_{\{|x|<1\}},$$

where c_ϕ is a normalization constant chosen in order to have $\int_{\mathbb{R}} \phi(x)\, dx = 1$. Then, we define

$$\Phi_{M,\epsilon}(y) := 1 - \int_{-\infty}^y \phi_\epsilon\left(x - M\right) dx. \tag{14}$$

It is then readily checked that $\Phi_{M,\epsilon}(z) = 0$ for $|z| > M + \epsilon$, $\Phi_M(z) = 1$ on $[0, M - \epsilon]$ and $\Phi_{M,\epsilon} \in C_b^\infty$. We will use the above localization function in two situations: one for $M >> 1$, $\epsilon = 1$ and in that case we simplify the notation using $\Phi_M \equiv \Phi_{M,1}$. In a second case M will not be a large quantity and therefore we will have to choose ϵ accordingly.

Consider now $Z \in D^\infty$. Under the same conditions as for Proposition 2.2 we get a conditional integration by parts formula of the form (13) localized by Z, with the following modification on the estimation of the norms of H_α^s:

$$\|H_\alpha^s(F, G\,\Phi_M(Z))\|_{p,s} \leq c \|\det(\Gamma_{F,s})^{-1}\Phi_{M'}(Z)\|_{p_3,s}^{k_3} \|F\,\Phi_{M'}(Z)\|_{k_2,p_2,s}^{k_4} \|G\,\Phi_{M'}(Z)\|_{k_1,p_1,s}, \tag{15}$$

for some appropriate positive integers $k_1, p_1, k_2, p_2, k_3, p_3, k_4$, and where we recall our convention on increasing constants $M' > M$. Notice that (15) is valid for localizations of the form $\Phi_{M,\epsilon}(Z)$ as well.

2.2. **Differential equations driven by fBm.** Recall that X is the solution of (3), and that our working assumptions are summarized in Hypothesis 1.1. We have distinguished 3 situations:

(1) The one dimensional additive case, for which equation (3) can be reduced to an ordinary differential equation by considering the process $Z = X - B$.

(2) The one dimensional multiplicative case, handled thanks to the Doss-Sussman transform (see e.g [15]).
(3) The multidimensional case with $H \in (1/2, 1)$, solved in a pathwise way by interpreting stochastic integrals as generalized Riemann-Stieljes type integrals.

In this section we give a brief account on the known results in the last situation.

In the case $H \in (1/2, 1)$, (3) is solved thanks to a fixed point argument, after interpreting the stochastic integral in the (pathwise) Young sense (see e.g. [11]). Let us recall that Young's integral can be defined in the following way:

Proposition 2.3. *Let $f \in \mathcal{C}^\gamma$, $g \in \mathcal{C}^\kappa$ with $\gamma + \kappa > 1$, and $0 \le s \le t \le 1$. Then the integral $\int_s^t g_\xi \, df_\xi$ is well-defined as a Riemann-Stieltjes integral. Moreover, the following estimation is fulfilled:*

$$\left| \int_s^t g_\xi \, df_\xi \right| \le C \|f\|_\gamma \|g\|_\kappa |t-s|^\gamma,$$

where the constant C only depends on γ and κ.

With this definition in mind and under Hypothesis 1.1, we can solve (3). Specifically, it is proven in [19] that equation (3) driven by B admits a unique γ-Hölder continuous solution X, for any $\frac{1}{2} < \gamma < H$. Moreover, the following moments bounds are shown in [12]:

Proposition 2.4. *Let $H \in (1/2, 1)$ and assume that $V_0, \dots, V_d$ satisfy Hypothesis 1.1. Then for $t \in [0, 1]$ and $\frac{1}{2} < \gamma < H$, we have*

$$\|X\|_{0,t,\infty} \le |a| + c_V \|B\|_{0,t,\gamma}^{1/\gamma}. \tag{16}$$

Moreover $X_t \in \boldsymbol{D}^\infty$ and for all $n \ge 1$, $i_1, \dots, i_n \in \{1, \dots, d\}^n$ and $0 \le s < t \le 1$ the following bound holds true:

$$\sup_{s \le u, r_1 \dots r_n \le t} |\mathbf{D}_{r_1 \dots r_n}^{i_1 \dots i_n} X_u| \le C_{V,n} \exp\left(c_{V,n} \|B\|_{s,t,\gamma}^{1/\gamma} \right). \tag{17}$$

We remark here that due to Proposition 5.3 (iii), the good definition of the supremum in (17) can be justified. Furthermore, a bound for γ-Hölder norms with $\frac{1}{2} < \gamma < H$ is provided by [8, Eq.(10.15)] for X together with its Malliavin derivatives:

Proposition 2.5. *Under the same assumptions as for Proposition 2.4 we have*

$$\|X\|_{s,t,\gamma} \le c_{1,V} \left(\|B\|_{s,t,\gamma} \vee \|B\|_{s,t,\gamma}^{\frac{1}{\gamma}} \right), \quad \|\mathbf{D}_{r_1 \dots r_n}^{i_1 \dots i_n} X_u\|_{s,t,\gamma} \le c_{2,V,n} \exp\left(c_{3,V,n} \|B\|_{s,t,\gamma}^{\frac{1}{\gamma}} \right).$$

3. One dimensional additive case

This section is devoted to prove our main Theorem 1.2 in the particular case $m = d = 1$ with additive noise. In this context, one can take advantage of the results obtained by Nourdin and Viens in [16] in order to derive Gaussian type upper and lower bounds for p_t. Let us then first recall what those results are.

3.1. **General bounds on densities of one-dimensional random variables.** Recall that we denote the Malliavin-Sobolev spaces with respect to the fBm B by $\boldsymbol{D}^{k,p}$, and consider a real-valued centered random variable $F \in \boldsymbol{D}^{1,2}$. We define a function g on $\mathbb{R}$ by:

$$g(z) := \mathbf{E}\left[\left\langle \mathbf{D}F, -\mathbf{D}\mathbf{L}^{-1}F\right\rangle_{\mathcal{H}} | F = z\right],$$

where the operator $\mathbf{L}$ is the Ornstein-Uhlenbeck operator associated to the fBm B (see [17] for further details), which can be defined using the chaos expansion by the formula $\mathbf{L} = -\sum_{n=0}^{\infty} n\mathbf{J}_n$. Based on the function g, the following simple criterion for Gaussian type bounds has been obtained in [16]:

Proposition 3.1. *Let $F \in \boldsymbol{D}^{1,2}$ with $\mathbf{E}[F] = 0$. If there exist c_1, $c_2 > 0$ such that*

$$c_1 \leq g(F) \leq c_2, \quad \mathbf{P} - a.s,$$

then the law of F has a density ρ satisfying, for almost all $z \in \mathbb{R}$,

$$\frac{\mathbf{E}[|F|]}{2c_2} \exp\left(-\frac{z^2}{2c_1}\right) \leq \rho(z) \leq \frac{\mathbf{E}[|F|]}{2c_1} \exp\left(-\frac{z^2}{2c_2}\right).$$

Interestingly enough, [16, Proposition 3.7] also gives an alternative formula for $g(F)$ which is suitable for computational purposes. Indeed, if we write $\mathbf{D}F = \Phi_F(B)$, where $\Phi_F : \mathbb{R}^{\mathcal{H}} \to \mathcal{H}$ is a measurable mapping, then the following relation holds true:

$$g(F) = \int_0^{\infty} e^{-\theta}\, \mathbf{E}\left[\left\langle \Phi_F(B), \Phi_F(e^{-\theta}B + \sqrt{1-e^{-2\theta}}B')\right\rangle_{\mathcal{H}} \Big| F\right] d\theta, \tag{18}$$

where B' stands for an independent copy of B, and is such that B and B' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbf{P} \times \mathbf{P}')$. Here we abuse the notation by letting $\mathbf{E}$ be the mathematical expectation with respect to $\mathbf{P} \times \mathbf{P}'$, while $\mathbf{E}'$ is the mathematical expectation with respect to $\mathbf{P}'$ only. One can thus recast relation (18) as

$$g(F) = \int_0^{\infty} \mathbf{E}\left[\mathbf{E}'\left[\left\langle \mathbf{D}F, \mathbf{D}F^{\theta}\right\rangle_{\mathcal{H}}\right] \Big| F\right] d\theta, \tag{19}$$

where, for any random variable X defined in $(\Omega, \mathcal{F}, \mathbf{P})$, X^{θ} denotes the following shifted random variable in $\Omega \times \Omega'$:

$$X^{\theta}(\omega, \omega') = X\left(e^{-\theta}\omega + \sqrt{1-e^{-2\theta}}\omega'\right), \quad \omega \in \Omega,\, \omega' \in \Omega'.$$

3.2. **Main result in the additive one-dimensional case.** Before stating our result let us point out that we assume through this subsection $V_1 \equiv \sigma$. That is, X is the solution of

$$X_t = x + \int_0^t V_0(X_s)ds + \sigma\, B_t, \qquad t \in [0,1] \tag{20}$$

where $\sigma > 0$ is a strictly positive constant, V_0 satisfies $\|V_0'\|_{\infty} \leq M$ for some constant $M > 0$ and B is a fBm with $H \in (0,1)$. Under this setting, we are able to get the following bounds:

Theorem 3.2. *Assume that V_0 satisfies that $\|V_0'\|_{\infty} \leq M$, for some constant $M > 0$, $\sigma > 0$ and $H \in (0,1)$. Then, for all $t \in (0,1]$, X_t possesses a density p_t and there exist some constants c_1 and c_2 depending only on M and H such that for all $z \in \mathbb{R}$*

$$\frac{\mathbf{E}[|X_t - m|]}{c_1\sigma^2 t^{2H}} \exp\left(-\frac{(z-m)^2}{c_2\sigma^2 t^{2H}}\right) \leq p_t(z) \leq \frac{\mathbf{E}[|X_t - m|]}{c_2\sigma^2 t^{2H}} \exp\left(-\frac{(z-m)^2}{c_1\sigma^2 t^{2H}}\right), \tag{21}$$

where $m := \mathbf{E}[X_t]$.

Remark 3.3. The advantage of the Nourdin-Viens method of estimating densities is that upper and lower bounds are obtained with similar proofs. The drawback is the restriction to one dimensional additive situations. Also notice that the exponents in equation (21) are optimal if one can prove that $\mathbf{E}[|X_t - m|] \asymp \sigma\, t^H$. This easy step is left to the reader for the sake of conciseness.

Strategy of the proof. Obviously, we shall mainly rely on Proposition 3.1. We thus define $F = X_t - \mathbf{E}[X_t]$, where X_t is the solution of (20). We get a centered random variable, and we shall prove that there exists two constants $0 < K_1 < K_2$ such that

$$K_1\sigma^2 t^{2H} \leq g(F) \leq K_2\sigma^2 t^{2H}. \tag{22}$$

Notice first that in the present case, it is easily seen that for any $t > 0$ we have $X_t \in \mathbb{D}^{1,2}$ (this is a particular case of [20]). Furthermore, the Malliavin derivative of X_t satisfies the following equation for $r \leq t$:

$$\mathbf{D}_r X_t = \int_r^t V_0'(X_s)\mathbf{D}_r X_s ds + \sigma.$$

This equation can be solved explicitly, and we obtain

$$\mathbf{D}_r X_t = \sigma e^{\int_r^t V_0'(X_s)ds}. \tag{23}$$

We shall now bound $g(F)$ according to this explicit expression, and we separate the cases $H \in (1/2, 1)$ and $H \in (0, 1/2)$. Notice that the Brownian case, i.e. $H = 1/2$, is well known and it is thus omitted here for the sake of conciseness. □

3.3. **Case $\mathbf{H > \frac{1}{2}}$.** Recall that we wish to prove (22), for which we can use the explicit expression of $\mathbf{D}_r X_t$ obtained in (23). Furthermore, owing to expression (8) for the inner product in $\mathcal{H}$ we can write $g(F)$ as

$$\begin{aligned} g(F) &= c_H \int_0^\infty \mathbf{E}\left[\mathbf{E}'\left(\int_0^t\int_0^t \mathbf{D}_u X_t\, \mathbf{D}_v X_t^\theta |u-v|^{2H-2} du dv\right) \Big| F\right] d\theta \\ &= c_H\sigma^2 \int_0^\infty e^{-\theta}\mathbf{E}\left[\mathbf{E}'\left(\int_0^t\int_0^t e^{\int_u^t V_0'(X_s)ds} e^{\int_v^t V_0'(X_s^\theta)ds} |u-v|^{2H-2} du dv\right) \Big| F\right] d\theta. \end{aligned} \tag{24}$$

The lower and upper bounds follow from applying

$$e^{-M} \leq e^{\int_u^t V_0'(X_s^\theta)ds} \leq e^M, \text{ for any } 0 \leq u \leq t \leq 1.$$

3.4. **Case $0 < \mathbf{H} < \frac{1}{2}$.** As in the case $H > \frac{1}{2}$, we prove (22). We thus go back to equation (19) and we observe that we can reduce the problem to the existence of two constants $0 < c_1 < c_2$ such that

$$c_1 t^{2H} \leq \left\langle \mathbf{D}X_t, \mathbf{D}X_t^{\theta} \right\rangle_{\mathcal{H}} \leq c_2 t^{2H}. \tag{25}$$

The proof of these inequalities will rely on the following quadratic programming lemma, which is a slight variation of [5, Lemma 6.2]:

Lemma 3.4. *Let $Q \in \mathbb{R}^n \otimes \mathbb{R}^n$ be a strictly positive symmetric matrix such that $\sum_{j=1}^n Q_{ij} \geq 0$ for all $i = 1, \dots, n$. For two positive constants a and b, consider the sets $\mathcal{A} = [a, \infty)^n$ and $\mathcal{B} = [b, \infty)^n$. Then*

$$\inf \left\{ x^* Q \tilde{x};\, \tilde{x} \in \mathcal{A},\, x \in \mathcal{B} \right\} = ab \sum_{i,j=1}^n Q_{ij}.$$

Proof. Set $\mathbf{a} = a\,\mathbf{1} \in \mathbb{R}^n$ and $\mathbf{b} = b\,\mathbf{1} \in \mathbb{R}^n$. The Lagrangian of our quadratic programming problem is a function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ defined as

$$L(x, \tilde{x}, \lambda_1, \lambda_2) = x^* Q \tilde{x} - \lambda_1^* \left(x - \mathbf{b} \right) - \lambda_2^* \left(\tilde{x} - \mathbf{a} \right).$$

It is readily checked that $\nabla_x L(x, \tilde{x}, \lambda_1, \lambda_2) = Q\tilde{x} - \lambda_1$ and $\nabla_{\tilde{x}} L(x, \tilde{x}, \lambda_1, \lambda_2) = Qx - \lambda_2$, which vanishes for $x = Q^{-1}\lambda_2$ and $\tilde{x} = Q^{-1}\lambda_1$. Therefore,

$$\begin{aligned} \inf \left\{ L(x, \tilde{x}, \lambda_1, \lambda_2);\, x,\, \tilde{x} \in \mathbb{R}^n \right\} &= L\left(Q^{-1}\lambda_2,\, Q^{-1}\lambda_1,\, \lambda_1,\, \lambda_2 \right) \\ &= -\lambda_1^* Q^{-1} \lambda_2 + \lambda_1^* \mathbf{b} + \lambda_2^* \mathbf{a} =: G(\lambda_1,\, \lambda_2). \end{aligned}$$

We have thus obtained a dual problem of the form

$$\max \left\{ G(\lambda_1,\, \lambda_2); \lambda_1,\, \lambda_2 \in \mathbb{R}^n_+ \right\}. \tag{26}$$

Let us now solve Problem (26). We first maximize G without positivity constraints on λ_1 and λ_2: we get $\nabla_{\lambda_1} G(\lambda_1,\, \lambda_2) = -Q^{-1}\lambda_2 + \mathbf{b}$ and $\nabla_{\lambda_2} G(\lambda_1,\, \lambda_2) = -\lambda_1^* Q^{-1} + \mathbf{a}$, which vanishes for $\lambda_1^\circ = Q\mathbf{a}$ and $\lambda_2^\circ = Q\mathbf{b}$. Observe now that our assumption $\sum_{j=1}^n Q_{ij} \geq 0$ for all $i = 1, \dots, n$ implies $\lambda_1^\circ,\, \lambda_2^\circ \geq 0$, so that λ_1° and λ_2° are feasible for the dual problem. Hence

$$\max \left\{ G(\lambda_1,\, \lambda_2); \lambda_1,\, \lambda_2 \in \mathbb{R}^n_+ \right\} = G(\lambda_1^\circ,\, \lambda_2^\circ) = ab \sum_{i,j=1}^n Q_{ij},$$

which finishes the proof. □

Importantly enough, Lemma 3.4 can be applied in order to get a lower bound on $\mathcal{H}$ norms:

Proposition 3.5. *Let B be a 1-dimensional fBm on $[0, \tau]$, let $\mathcal{H} \equiv \mathcal{H}_\tau$ be the associated reproducing kernel Hilbert space and $f, \tilde{f} \in \mathcal{H}$ such that $f_u \geq b$ and $\tilde{f}_u \geq a$ for any $u \in [0, \tau]$. Then $\langle f, \tilde{f} \rangle_{\mathcal{H}} \geq a\, b\, \tau^{2H}$.*

Proof. Recall that, owing to relation (9), we have $\langle f, \tilde{f} \rangle_{\mathcal{H}} = \lim_{|\pi| \to 0} I_\pi(f, \tilde{f})$, where π stands for a generic partition $\{0 = t_0 < \cdots < t_n = \tau\}$ and

$$I_\pi(f, \tilde{f}) = \sum_{i,j=1}^n f_{t_{i-1}}\, Q_{ij}\, \tilde{f}_{t_{j-1}}, \quad \text{with} \quad Q_{ij} = \mathbf{E}[\Delta_i(B) \Delta_j(B)],$$

where we recall that $\Delta_i(B) = B_{t_i} - B_{t_{i-1}}$. We assume for the moment that Q satisfies the hypothesis of Lemma 3.4, and we get

$$I_\pi(f, \tilde{f}) \geq ab \sum_{i,j=1}^{n} Q_{ij} = ab \sum_{i,j=1}^{n} \mathbf{E}\left[\Delta_i(B)\Delta_j(B)\right] = ab\, \mathbf{E}\left[B_\tau^2\right] = ab\, \tau^{2H},$$

which is our claim.

Let us now prove that Q satisfies the hypothesis of Lemma 3.4. First, the strict positivity of Q stems from the local non determinism of B (see e.g [24]). Indeed, for $u \in \mathbb{R}^n$ we have

$$u^* Q u = \mathrm{Var}\left(\sum_{j=0}^{n-1} u_j\, \Delta_j(B)\right) \geq c_n \sum_{j=1}^{n} u_j^2\, |t_j - t_{j-1}|^{2H},$$

where the lower bound is the definition of local nondeterminism. Thus $u^* Q u > 0$ as long as $u \neq 0$.

Let us now check that for a fixed i we have $\sum_{j=1}^{n} Q_{ij} \geq 0$. To this aim, write

$$\sum_{j=1}^{n} Q_{ij} = \mathbf{E}\left[\Delta_i(B) B_\tau\right] = \int_{t_i}^{t_{i+1}} \partial_u R(\tau, u)\, du.$$

Going back to expression (1), it is now easily seen that for $u < \tau$ we have

$$\partial_u R(\tau, u) = H\left(u^{2H-1} + (\tau - u)^{2H-1}\right) > 0,$$

which completes the proof. □

We can now go back to the proof of relation (25), which is divided again in two steps:
Step 1: Lower bound. Thanks to relation (23) and since $\|V_0'\|_\infty \leq M$, we have that

$$\sigma e^{-tM} \leq \mathbf{D}_r X_t.$$

Thus we just have to apply Proposition 3.5 to the Malliavin derivative in order to obtain

$$\left\langle \mathbf{D}X_t, \mathbf{D}X_t^\theta \right\rangle_{\mathcal{H}} \geq \sigma^2 t^{2H} e^{-2M}, \tag{27}$$

which is our desired lower bound.

Step 2: Upper bound. In order to obtain an upper bound for $g(F)$ we will use the representation of $\mathcal{H}$ through fractional derivatives. Indeed, apply first Cauchy-Schwarz inequality in order to get

$$\left\langle \mathbf{D}X_t, \mathbf{D}X_t^\theta \right\rangle_{\mathcal{H}} \leq \|\mathbf{D}X_t\|_{\mathcal{H}}\, \|\mathbf{D}X_t^\theta\|_{\mathcal{H}}. \tag{28}$$

We then invoke Lemma 6.1 to bound $\|\mathbf{D}X_t^\theta\|_{\mathcal{H}}$. This boils down to estimate

$$a = \sup_{r \in [0,t]} |\mathbf{D}_r X_t^\theta|, \quad \text{and} \quad b = \sup_{r,v \in [0,t]} \frac{\mathbf{D}_r X_t^\theta - \mathbf{D}_v X_t^\theta}{(v-r)^\gamma},$$

with $1/2 - H < \gamma < 1/2$.

Now starting from expression (23) and owing to the fact that V_0' is uniformly bounded by M, we trivially get $a \leq \sigma\, e^M$. As far as b is concerned, we write

$$\left|\mathbf{D}_r X_t^\theta - \mathbf{D}_v X_t^\theta\right| \leq \sigma\, e^{\int_v^t V_0'(X_s^\theta) ds} \left|1 - e^{\int_u^v V_0'(X_s^\theta) ds}\right| \leq \sigma\, M\, e^{2M}\, (v - r).$$

We thus end up with the inequalities

$$a \leq \sigma\, e^M, \quad \text{and} \quad b \leq \sigma\, M\, e^{2M} t^{1-\gamma}.$$

We now apply Lemma 6.1 with constants a and b and we obtain

$$\|\mathbf{D}X_t\|_{\mathcal{H}} \leq c_H \left(\sigma\, e^M t^H + \sigma\, M\, e^{2M} t^{1+H}\right) \leq c_H\, \sigma\, e^M t^{2H},$$

and hence

$$\left\langle \mathbf{D}X_t, \mathbf{D}X_t^{\theta} \right\rangle_{\mathcal{H}} \leq c_H \sigma^2\, M^2\, e^{4M}\, t^{2H}.$$

Finally, putting together the last bound and (27), we get (22) in the case $H \in (0, 1/2)$, which finishes the proof of Theorem 3.2.

4. One dimensional non-vanishing diffusion coefficient case

We turn now to the case $m = d = 1$, $H \in (\frac{1}{2}, 1)$ for a non constant elliptic coefficient σ. Observe that this special case is treated in a separate section because: (i) The Gaussian bound is obtained with weaker conditions on the coefficients than in the multidimensional case. (ii) The proof is shorter due to specific one-dimensional techniques based on Doss-Sussman's transform and Girsanov's theorem. This is detailed below.

4.1. **Doss-Sussmann transformation.** The idea of the method is to first consider a one dimensional equation of Stratonovich type without drift and then apply Girsanov's theorem for fBm in order to obtain a characterization of the density.

In order to carry out this plan, we start by using an independent copy of $(\Omega, \mathcal{F}, \mathbf{P})$ called $(\Omega', \mathcal{F}', \mathbf{P}')$ supporting a fBm denoted by B'. On $(\Omega', \mathcal{F}', \mathbf{P}')$, let Y be the unique solution to:

$$Y_t = a + \int_0^t V_1(Y_s) \circ dB'_s, \tag{29}$$

where $V_1 \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})$, $V_1 \neq 0$ and $H \in (\frac{1}{2}, 1)$. We also call W' the underlying Wiener process appearing in the Volterra type representation (10) for B'. We now recall here some details from Doss and Sussmann's classical computations adapted to our fBm context.

Indeed, as in [15], let us recall that the solution of equation (29) can be expressed as $Y_t = F(B'_t,\, a)$, $t > 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is the flow associated to V_1:

$$\frac{\partial F}{\partial x}(x, y) = V_1(F(x, y)), \qquad F(0, y) = y. \tag{30}$$

We remark that if V_1 is bounded then F satisfies $|F(x, y)| \leq c(1 + |x| + |y|)$.

Next we relate the solution X of equation (3) to the process Y defined by (29). This step is partially borrowed from [18], and we refer to that paper for further details. Indeed, thanks to a Girsanov type transform, the following characterization of the law of the solution to (3) is shown for $m = d = 1$: For any bounded measurable function $U : \mathbb{R} \to \mathbb{R}$, one has

$$\mathbf{E}_{\mathbf{P}}\left[U(X_t)\right] = \mathbf{E}_{\mathbf{P}'}\left[U\left(F(B'_t,\, a)\right)\, \xi\right], \tag{31}$$

where $\xi \equiv \xi_t = \frac{d\mathbf{P}}{d\mathbf{P}'}$ is the random variable defined by

$$\xi = \exp\left(\int_0^t \left[\mathcal{M}_s\, dW'_s - \frac{1}{2}\mathcal{M}_s^2\, ds\right]\right), \tag{32}$$

where we have set $\mathcal{M} = K_H^{-1}(\int_0^{\cdot} V_0 V_1^{-1}(Y_u) du)$.

Notice that in definition (32), the operators K, K^{-1} are respectively defined (with a slight abuse of notation), for $H \geq \frac{1}{2}$ and an appropriate function h, by:

$$[K_H h](s) = I^1_{0^+}(s^{H-\frac{1}{2}}(I^{H-\frac{1}{2}}_{0^+}(s^{\frac{1}{2}-H}h)))(s), \quad \text{and} \quad [K_H^{-1}h](s) = s^{H-\frac{1}{2}}(D^{H-\frac{1}{2}}_{0^+}(s^{\frac{1}{2}-H}h'))(s).$$

We also recall that in the last equation, $I^\alpha_{0^+}$ and $D^\alpha_{0^+}$ denote the fractional integral and fractional derivative, whose expressions are:

$$I^\alpha_{0^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy,$$

and

$$D^\alpha_{0^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{x^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right).$$

Notice that in order for (31) to be satisfied it is required that $\int_0^\cdot V_0 V_1^{-1}(Y_u) du \in I^{H+\frac{1}{2}}_{0+}(L^2[0,T])$. This condition is satisfied due to the γ-Hölderianity of Y for any $\gamma < H$.

Actually one should prove that Novikov's type conditions are satisfied for ξ in order to apply Girsanov's transform and get relation (31). This is achieved in the following lemma:

Lemma 4.1. *Let ξ be the random variable defined by* (32)*. Then*

$$\mathcal{M}_s \leq c_V \, \beta_s, \quad \textit{with} \quad \beta_s := s^{\frac{1}{2}-H} + \|B'\|_{H-\frac{1}{2}}. \tag{33}$$

Furthermore $\mathbf{E}_{\mathbf{P}'}[\xi] = 1$, which justifies the Girsanov identity (31)*. That is, under $\mathbf{P}$, $B = B' + \int_0^\cdot V_0 V_1^{-1}(Y_u) du$ is a H-fBm.*

Proof. According to the expression of K_H^{-1} we have

$$\mathcal{M}_s = \frac{s^{H-\frac{1}{2}}}{\Gamma(H-\frac{1}{2})} \left(\frac{s^{\frac{1}{2}-H} V_0 V_1^{-1}(Y_s)}{s^{H-\frac{1}{2}}} + \left(H-\frac{1}{2}\right) \int_0^s \frac{s^{\frac{1}{2}-H} V_0 V_1^{-1}(Y_s) - u^{\frac{1}{2}-H} V_0 V_1^{-1}(Y_u)}{(s-u)^{H+\frac{1}{2}}} du \right).$$

We now invoke the uniform ellipticity of V_1 and the regularity of V_0 and V_1, which yields

$$\mathcal{M}_s \leq c_V \left(s^{\frac{1}{2}-H} + \int_0^s \frac{|F(B'_s, a) - F(B'_u, a)|}{(s-u)^{H+\frac{1}{2}}} du \right) \leq c_V \beta(s).$$

Now let us have a closer look at the process β: it is readily checked that $\|B'\|_\gamma$ admits quadratic exponential moments for any $\gamma < H$ (see Theorem 3 in [18]). Hence there exists $\lambda > 0$ such that the expected value $\mathbf{E}[\exp(\lambda \int_0^t \beta^2(s) ds)]$ is a finite quantity. Owing to a version of Novikov's condition stated in [9, Theorem 1.1] we deduce that $\mathbf{E}[\xi] = 1$. This concludes the proof. □

4.2. **Main result.** As in the additive case of Section 3, we are able to get both upper and lower Gaussian bounds in a one dimensional context:

Theorem 4.2. *Assume that V_0, $V_1 \in \mathcal{C}_b^1$, $\lambda \leq |V_1| \leq \Lambda$ and $H \in (1/2, 1)$. Then, there exist constants C_1 and C_2 such that for all $t \in (0, 1]$, the solution X_t to equation (29) possesses a density p_t satisfying for all $x \in \mathbb{R}$:*

$$\frac{1}{C_1\sqrt{2\pi t^{2H}}} \exp\left(-C_1 \frac{(x-a)^2}{2t^{2H}}\right) \leq p_t(x) \leq \frac{1}{C_2\sqrt{2\pi t^{2H}}} \exp\left(-C_2 \frac{(x-a)^2}{2t^{2H}}\right). \tag{34}$$

Proof. Step 1: Upper bound. We start from an equivalent of (31) for densities, which is justified by [15] and a duality argument:

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\left[\delta_x\left(F(B'_t,\, a)\right)\, \xi\right], \tag{35}$$

where ξ is the random variable defined in (32). We now integrate by parts in order to get

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\left[\mathbf{1}_{\{F(B'_t,a)\leq x\}}\, H\left(F(B'_t,\, a), \xi\right)\right],$$

with

$$H\left(F(B'_t,\, a), \xi\right) = \delta\left(\frac{\xi\, \mathrm{D}F(B'_t,\, a)}{\|\mathrm{D}F(B'_t,\, a)\|^2_{L^2([0,t])}}\right), \tag{36}$$

where D, δ respectively stand (with a slight abuse of notation) for the Malliavin derivative and divergence operator for the Brownian motion W' under $\mathbf{P}'$. Let us further simplify the expression for the random variable $H(F(B'_t,\, a), \xi)$: setting $K_t(u) \equiv K(t, u)\mathbf{1}_{[0,t]}(u)$, it is readily checked that we have

$$\mathrm{D}_u F(B'_t,\, a) = \partial_x F(B'_t,\, a)\, K_t(u) \quad \text{and} \quad \|\mathrm{D}F(B'_t,\, a)\|^2_{L^2([0,t])} = |\partial_x F(B'_t,\, a)|^2\, t^{2H}.$$

Plugging this information into (36), and defining $Z := \xi\, (\partial_x F(B'_t,\, a))^{-1}$, we end up with

$$H\left(F(B'_t,\, a), \xi\right) = \frac{\delta\left(Z\, K_t\right)}{t^{2H}} = K_1 - K_2,$$

where

$$K_1 = \frac{Z\, B'_t}{t^{2H}}, \quad \text{and} \quad K_2 = \frac{\langle \mathrm{D}Z,\, K_t\rangle_{L^2([0,t])}}{t^{2H}}.$$

We have thus obtained

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\left[\mathbf{1}_{\{F(B'_t,a)\geq x\}}\, K_1\right] - \mathbf{E}_{\mathbf{P}'}\left[\mathbf{1}_{\{F(B'_t,a)\geq x\}}\, K_2\right] =: p_t^1(x) - p_t^2(x), \tag{37}$$

and we shall upper bound these two terms separately.

The term $p_t^1(x)$ can be bounded as follows: for $q_1, q_2, q_3 > 1$ large enough and a parameter $1 < q_4 = 1 + \varepsilon$ with an arbitrary small $\varepsilon > 0$ we have

$$p_t^1(x) \leq \frac{\mathbf{E}^{1/q_1}_{\mathbf{P}'}\left[|B'_t|^{q_1}\right]}{t^{2H}}\, \mathbf{P}'^{1/q_2}\left(F(B'_t,\, a) \geq x\right)\, \mathbf{E}^{1/q_3}_{\mathbf{P}'}\left[|\partial_x F(B'_t,\, a)|^{q_3}\right]\, \mathbf{E}^{1/q_4}_{\mathbf{P}'}\left[\xi^{q_4}\right] \tag{38}$$

We now bound the right hand side of this inequality:

(i) We obviously have $\frac{\mathbf{E}^{1/q_1}_{\mathbf{P}'}\left[|B'_t|^{q_1}\right]}{t^{2H}} \leq ct^{-H}$, since B' is a $\mathbf{P}'$-fBm.

(ii) Let us prove that there exists two positive constants c_1 and c_2 such that

$$c_1(x-a) \leq F^{-1}(x, a) \leq c_2(x-a). \tag{39}$$

Indeed, for a fixed $a \in \mathbb{R}$, F is a continuously differentiable function and $\partial_x F(\cdot, a) \neq 0$ for any $a \in \mathbb{R}$. Then by the inverse function theorem $F^{-1}(\cdot, a)$ exists and it is a continuously differentiable function verifying

$$\partial_x F^{-1}(z, a) = \frac{1}{V_1(F(x, a))} = \frac{1}{V_1(z)}, \quad \text{where} \quad F(x, a) = z.$$

Since F^{-1} is continuous we can now resort to the mean value theorem to get that for any $x_1, x_2 \in \mathbb{R}$ we have

$$F^{-1}(x_2, a) - F^{-1}(x_1, a) = \frac{x_2 - x_1}{V_1(\theta)},$$

for a certain $\theta \in [F^{-1}(x_1, a), F^{-1}(x_2, a)]$. Moreover, it is easily seen that $F^{-1}(x, x) = 0$ for all $x \in \mathbb{R}$, so that $|F^{-1}(x, a)| = \frac{1}{|V_1(\theta)|}|x - a|$. Our claim (39) is now easily proven once we recall that $\lambda \leq |V_1(u)| \leq \Lambda$.

Using (39) and the Gaussian density for B'_t, it is now readily checked that

$$\mathbf{P'}^{1/q_2}\left(F(B'_t,\, a) \geq x\right) \leq c_1 \exp\left(-\frac{c_2(x-a)^2}{t^{2H}}\right).$$

(iii) Equation (30) reveals that $\partial_x F$ is bounded by a constant, so that

$$\mathbf{E}^{1/q_3}_{\mathbf{P}'}\left[|\partial_x F(B'_t,\, a)|^{q_3}\right] \leq c.$$

(iv) Set $S = \int_0^t \mathcal{M}_s\, dW'_s$ and $D = \int_0^t \mathcal{M}_s^2\, ds$, where $\mathcal{M} \equiv K_H^{-1}(\int_0^{\cdot} V_0 V_1^{-1}(Y_u)du)$ as above, and where we recall that $q_4 = 1 + \varepsilon$ with an arbitrarily small $\varepsilon > 0$. It is readily checked that

$$\xi^{q_4} = \exp\left(q_4 S - \frac{q_4}{2} D\right) = \exp\left(q_4 S - \frac{q_4^2}{2} D\right) \exp\left(\frac{q_\varepsilon}{2} D\right),$$

where $q_\varepsilon = q_4^2 - q_4 = \varepsilon(1 + \varepsilon)$. Now observe that the term $\exp(q_4 S - \frac{q_4^2}{2} D)$ is a Girsanov change of measure which corresponds to a shift on B' of the form

$$\hat{B} = B' - q_4 \int_0^{\cdot} V_0 V_1^{-1}(Y_u)du = B - (q_4 - 1)\int_0^{\cdot} V_0 V_1^{-1}(Y_u)du.$$

Calling $\hat{\mathbf{P}}'$ the probability under which $\hat{B}$ is a fractional Brownian motion, we get

$$\mathbf{E}_{\mathbf{P}'}\left[\xi^{q_4}\right] = \mathbf{E}_{\hat{\mathbf{P}}'}\left[\exp\left(\frac{q_\varepsilon}{2} D\right)\right]. \tag{40}$$

Now plug the estimate (33) into (40). This yields

$$\begin{aligned} D &\leq c_V\left(1 + \|B\|^2_{H-\frac{1}{2}}\right) \\ &\leq c_V\left(1 + \|\hat{B} + q_4 \int_0^{\cdot} V_0 V_1^{-1}(Y_u)du\|^2_{H-\frac{1}{2}}\right) \\ &\leq c_V\left(1 + \|\hat{B}\|^2_{H-\frac{1}{2}}\right). \end{aligned}$$

Going back to relation (40) and taking into account the fact that q_ε can be chosen arbitrarily small, we get $\mathbf{E}_{\mathbf{P}'}[\xi^{q_4}] < \infty$.

Gathering all the above estimates into (38), we have thus obtained that

$$p_t^1(x) \le \frac{c_1}{t^H} \exp\left(-\frac{c_2(x-a)^2}{t^{2H}}\right).$$

The same kind of bound can be deduced for $p_t^2(x)$ in (37), which gives our global upper bound in (34).

Step 2: Lower bound. Our strategy to obtain the lower bound in (34) is based on the following decomposition:

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\left[\delta_x(F(B'_t,\, a))(\xi_t - \xi_{c_1 t})\right] + \mathbf{E}_{\mathbf{P}'}\left[\delta_x(F(B'_t,\, a))\xi_{c_1 t}\right] =: \rho_t^1 + \rho_t^2, \tag{41}$$

where c_1 is a constant to be determined later. Observe that the main term will be ρ_t^2, which means that we consider a two point partition of the interval $[0,t]$ and we perform a one-step decomposition of X_t (or Y_t) on $[0, c_1 t]$ and $[c_1 t, t]$, as opposed to the general time interval partition in Section 5.

First, we start studying the main term ρ_t^2: Note that due to (10), we have by Girsanov's Theorem

$$\begin{aligned}\rho_t^2 &= \mathbf{E}_{\mathbf{P}'}\left[\mathbf{E}_{\mathbf{P}'}\left[\delta_x(F(B'_t,\, a))\middle|\mathcal{F}_{c_1 t}\right]\xi_{c_1 t}\right] \\ &= \mathbf{E}_{\mathbf{P}'}\left[\exp\left(-\frac{(F^{-1}(x,a) - \int_0^{c_1 t} K(t,s)dW'_s)^2}{2\int_{c_1 t}^t K(t,s)^2 ds}\right)\frac{\partial_x F^{-1}(x,a)}{\sqrt{2\pi\int_{c_1 t}^t K(t,s)^2 ds}}\xi_{c_1 t}\right] = \mathbf{E}_{\mathbf{P}}[L_{c_1,t}],\end{aligned}$$

where we have set

$$L_{c_1,t} := \exp\left(-\frac{(F^{-1}(x,a) - \int_0^{c_1 t} K(t,s)dW_s + \int_0^{c_1 t} V_0 V_1^{-1}(X_s)ds)^2}{2\int_{c_1 t}^t K(t,s)^2 ds}\right)\frac{\partial_x F^{-1}(x,a)}{\sqrt{2\pi\int_{c_1 t}^t K(t,s)^2 ds}}.$$

In order to determine a lower bound for the above expression, we use the following information:
(i) We have $\partial_x F^{-1}(x,a) \ge \Lambda^{-1}$.
(ii) We apply the inequality $(x+a)^2 \ge \frac{1}{2}x^2 - 2a^2$ to $a^2 := (\int_0^{c_1 t} V_0 V_1^{-1}(X_s)ds)^2 \le c_V t^2$.
(iii) Gaussian convolution identities can be invoked in order to compose the quadratic exponential term defining $L_{c_1,t}$ with the expected value with respect to the Gaussian random variable $\int_0^{c_1 t} K(t,s)dW'_s$.
These ingredients easily entail that

$$\rho_t^2 \ge \frac{c_\varepsilon}{\sqrt{2\pi\hat{\sigma}^2}}\exp\left(-\frac{F^{-1}(x,a)^2}{2\hat{\sigma}^2}\right),$$

for $\hat{\sigma}^2 = 2\int_{c_1 t}^t K(t,s)^2 ds + \int_0^{c_1 t} K(t,s)^2 ds$, and we observe that $\sigma^2 \le \hat{\sigma}^2 \le 2\sigma^2$.

Now we estimate the first term ρ_t^1 in (41) and prove that it is upper bounded by a quantity which is smaller than half of the lower bound we have just obtained. For this term we need to use again the integration by parts estimates carried out in (35). In order not to repeat arguments we just mention the main steps: we start by writing

$$\rho_t^1 = \mathbf{E}_{\mathbf{P}'}\left[\delta_x(F(B'_t,\, a))(\xi_t - \xi_{c_1 t})\right] = \mathbf{E}_{\mathbf{P}'}\left[\mathbf{1}_{\{F(B'_t,a)\le x\}} H\left(F(B'_t,\, a), \xi_t - \xi_{c_1 t}\right)\right],$$

and we decompose this expression into $p^1 - p^2$ like in (37), except for the fact that this time Z is replaced by $Z_t := ((\xi_t - \xi_{c_1 t})\, \partial_x F(B'_t,\, a))^{-1}$.

We wish to take advantage of the fact that $\xi_t - \xi_{c_1 t}$ is a small quantity whenever c_1 is close to 1. For this, define the process $\mathcal{M}_{c_1 t,\cdot}$ as $\mathcal{M}_{c_1 t,s} = K_H^{-1}(\int_{c_1 t}^{\cdot} V_0 V_1^{-1}(Y_u) du)$, consider $\theta \in [0,1]$ and define

$$\xi_t(\theta) := \xi_{c_1 t} \exp\left(\theta \int_{c_1 t}^t \mathcal{M}_{c_1 t, s}\, dW'_s - \frac{\theta^2}{2} \int_{c_1 t}^t \mathcal{M}^2_{c_1 t, s}\, ds \right).$$

Then by the mean value theorem, we have

$$\xi_t - \xi_{c_1 t} = \int_0^1 d\theta\, \xi_t(\theta) \left(\int_{c_1 t}^t \mathcal{M}_s\, dW'_s - \theta \int_{c_1 t}^t \mathcal{M}_s^2\, ds \right).$$

Applying Fubini's theorem, one sees that the same estimates as in (38), appear again with the exception that (i) The last term in the decomposition becomes $\mathbf{E}_{\mathbf{P}'}^{1/q_4}\left[(\xi_t(\theta))^{q_4}\right]$ which is handled in the same fashion as before. (ii) There is another term appearing in the decomposition, namely

$$\mathbf{E}_{\mathbf{P}'}^{1/q_5} \left[\left(\int_{c_1 t}^t \mathcal{M}_s\, dW'_s - \theta \int_{c_1 t}^t \mathcal{M}_s^2\, ds \right)^{q_5} \right].$$

Using (33) and usual estimate methods for stochastic integrals, one obtains that the latter term is upper bounded by $c(1 - c_1^{2-2H}) t^{2-2H}$. Therefore taking c_1 sufficiently close to 1 one obtains that this upper bound is smaller than $1/2$ of the lower bound previously obtained. The proof is now complete. □

5. General lower bound

We now wish to obtain Gaussian type lower bounds for the multi-dimensional case of equation (3). However, the computations in this section will be performed on the following simplified version for notational sake (adaptation of our calculations to the drift case are straightforward):

$$X_t = a + \sum_{i=1}^d \int_0^t V_i(X_s) \circ dB_s^i, \tag{42}$$

where $a \in \mathbb{R}^m$ is a generic initial condition, $V_i : \mathbb{R}^m \to \mathbb{R}^m$ $i = 1, \ldots, d$ is a collection of smooth and bounded vectors fields and $B^1, \ldots, B^d$ are d independent fBm's with $H \in (1/2, 1)$. Recall that our goal is then to prove relation (5) in this context. To this aim, we shall assume that Hypothesis 1.1 (especially relation (4)) is satisfied for the remainder of the article.

5.1. **Preliminary considerations.** Let us recall briefly the strategy used in [2, 13] in order to obtain Gaussian lower bounds for solutions of stochastic differential equations. The argument starts with some additional notation: Recall that the natural filtration of B, which is also the natural filtration of the underlying Wiener process W defined by (10), is denoted by $\mathcal{F}_t$. As we have introduced in section 2.1, we write $\mathbf{E}_t$ for the conditional expectation with respect to $\mathcal{F}_t$. Under our working Hypothesis 1.1, let us also mention that the following result is available (see [4, 12] for further details):

Proposition 5.1. *Under Hypothesis 1.1, let X be the unique solution to (42). Then for any $t \in (0,1]$, the random variable X_t is non degenerate in the sense of Definition 2.1.1 in [17]. In particular, the density of X_t admits the representation $p_t(x) = \mathbf{E}[\delta_x(X_t)]$, where δ_x stands for the Dirac measure at point x.*

With this preliminary result in hand, the quantity $\mathbf{E}[\delta_x(X_t)]$ will be analyzed by means of the succesive evaluation of conditional densities of an approximation sequence $\{F_j;\, 0 \le j \le n\}$ such that $X_t = F_n$. We thus consider $p_t(x) = \mathbf{E}[\delta_x(F_n)]$. The discretization procedure is based on a corresponding partition of the time interval as $\pi : 0 = t_0 < \cdots < t_n = t$, and the sequence of random variables F_j which satisfy the relation $F_j \in \mathcal{F}_{t_j}$.

Let us give some hints about the general strategy for the discretization: it is designed to take advantage of conditional Malliavin calculus, which allows to capture the convolution property of Gaussian distributions. We shall thus assume for the moment a structure of the form

$$F_j = F_{j-1} + I_j + R_j, \tag{43}$$

where we recall that $F_{j-1} \in \mathcal{F}_{t_{j-1}}$. In formula (43), the term I_j will stand for a Gaussian random variable (conditionally to $\mathcal{F}_{t_{j-1}}$) and R_j refers to a small remainder term, whose contribution to the density of F_j can be neglected with respect to the one induced by I_j just like in the argument in (41). The local Gaussian bound (5) will be obtained from the density of the sum $\sum_{j=1}^n I_j$. The argument will finish by an application of the Chapman-Kolmogorov formula.

As suggested by equation (6), and setting $\Delta^i_{j+1}(B) := B_{t^i_{j+1}} - B_{t^i_j}$, a natural candidate consists in taking $F_j = X_{t_j}$, which yields

$$I_j = \sum_{i=1}^d V_i(X_{t_j})\, \Delta^i_{j+1}(B), \quad \text{and} \quad R_j = \sum_{i=1}^d \int_0^t \left[V_i(X_s) - V_i(X_{t_j})\right] dB^i_s. \tag{44}$$

However, this simple and natural guess is not suitable for the fBm case. Indeed, the analysis of the variances of I_j induced from the decomposition (44) reveals that a significant amount is generated by the covariances between the increments $\Delta^i_j(B)$. Now, if we write

$$t^{2H} = \mathbf{E}\left[\left(B^i_t\right)^2\right] = \mathbf{E}\left[\left(\sum_{j=1}^n \Delta^i_j(B)\right)^2\right] = \sum_{j,k=1}^n \mathbf{E}\left[\Delta^i_j(B)\Delta^i_k(B)\right], \tag{45}$$

we realize that the diagonal terms in the right hand side expression only accounts for a term of the form $\sum_j |t_j - t_{j-1}|^{2H}$, which vanishes as the mesh of the partition goes to 0 when $H \in (1/2, 1)$. This means that our decomposition (44) will not be able to capture the correct amount of variance contained in X_t, and has to be modified.

There are at least two natural generalizations of the Euler type scheme method described above:

(1) Take into account the off-diagonal terms in (45), and perform a block type analysis.
(2) Express the equation as an equation driven by the Wiener process W defined by relation (10) and take advantage of the independence of the increments of W.

In the current paper we have chosen to follow the second approach above, and thus we first recall how to define equation (42) as a Stratonovich equation with respect to W.

5.2. **Fractional equations as Stratonovich type equations.** In order to handle equation (42) as an equation with respect to W, let us first introduce the following functional space:

Definition 5.2. *Let $|\mathcal{H}|$ be the space of measurable functions $\phi : [0,T] \to \mathbb{R}^n$ such that*

$$\|\phi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^1 \left(\int_0^1 |\phi_r||\phi_u||r-u|^{2H-2} dr \right) du < +\infty.$$

Note that $|\mathcal{H}|$ endowed with the norm $\|\cdot\|_{|\mathcal{H}|}$ is a Banach space of functions, which is also a subspace of $\mathcal{H}$.

In the sequel we also consider random elements with values in $|\mathcal{H}|$. In particular, the norm of ϕ in $\boldsymbol{D}^{1,2}(|\mathcal{H}|)$ is given by

$$\|\phi\|_{\boldsymbol{D}^{1,2}(|\mathcal{H}|)} = \mathbf{E}\left[\|\phi\|^2_{|\mathcal{H}|}\right] + \mathbf{E}\left[\|\mathbf{D}\phi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}\right].$$

For these elements, the following result from [1, Proposition 3] allows to define Stratonovich type integrals (see [17] for a complete definition), which turn out to coincide with Young integrals in our cases of interest. In the sequel, we will use $\int_0^1 u_t \circ dB_t \equiv \sum_{i=1}^d \int_0^1 u_t^i \circ dB_t^i$ to denote Stratonovich integrals.

Proposition 5.3. *Let $u = \{u_t,\, t \in [0,1]\} \in \boldsymbol{D}^{1,2}(|\mathcal{H}|)$, such that*

$$\int_0^1 \int_0^1 |\mathbf{D}_s u_t||t-s|^{2H-2} ds dt < \infty. \tag{46}$$

Then

(i) *The Stratonovich integral $\int_0^1 u_t \circ dB_t$ in the sense of* [17] *exists and we also have*

$$\int_0^1 u_t \circ dB_t = \boldsymbol{\delta}(u) + \alpha_H \int_0^1 \int_0^1 \mathbf{D}_s u_t |t-s|^{2H-2} ds dt. \tag{47}$$

(ii) *Whenever $u \in \mathcal{C}^\gamma$ a.s. with $\gamma > 1/2$ and $H \in (1/2,1)$, the Stratonovich integral $\int_0^1 u_t \circ dB_t$ coincides with the Young integral $\int_0^1 u_t\, dB_t$.*

The next Proposition will allow us to interpret the stochastic integral appearing in (42) as a Stratonovich type integral.

Proposition 5.4. *Let $X = \{X_t,\, t \in [0,1]\}$ be the solution to* (42)*, and assume Hypothesis 1.1. Then the process $X \in \boldsymbol{D}^{1,2}(|\mathcal{H}|)$ and satisfies the equation*

$$X_t = a + \sum_{k=1}^d \int_0^t V_k(X_u) \circ dB_u^k,$$

where the Stratonovich integral can be decomposed as a Skorohod integral plus a trace term as in (47)*.*

Proof. According to Proposition 5.3, we just have to prove that $X \in \boldsymbol{D}^{1,2}(|\mathcal{H}|)$ and satisfies relation (46). We first focus on proving the relation

$$\mathbf{E}\left[\|X\|^2_{|\mathcal{H}|}\right] + \mathbf{E}\left[\|\mathbf{D}X\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}\right] < \infty.$$

In order to see the first part of this inequality, invoke relation (16) and write

$$\begin{aligned}\mathbf{E}\left[\|X\|^2_{|\mathcal{H}|}\right] &= \alpha_H \int_0^1 \int_0^1 \mathbf{E}\left[|X_r||X_s|\right] |r-s|^{2H-2}\, drds \\ &\leq c\,\mathbf{E}\left[\|X\|^2_\infty\right] \int_0^1 \int_0^1 |r-s|^{2H-2}\, drds < c_1.\end{aligned}$$

Along the same lines and owing to (17), it is also readily checked that $\mathbf{E}[\|\mathbf{D}X\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}] < \infty$ and that relation (46) holds true, which ends the proof. Note that due to Proposition 5.3 (ii) and Proposition 2.5, we obtain the other assertions. □

Finally, the following corollary is the key to the the effective decomposition we shall use in order to get our Gaussian lower bound on p_t:

Corollary 5.5. *Under the same assumptions as for Proposition 5.4, the process $K_t^*(V_k(X)) \in dom(\delta)$ and satisfies the equation*

$$X_t = a + \sum_{k=1}^d \int_0^t [K_t^*(V_k(X))]_s \circ dW_s^k = a + \sum_{k=1}^d \int_0^t \left(\int_s^t \partial_u K(u,s) V_k(X_u)\, du \right) \circ dW_s^k, \quad (48)$$

where the anticipative Stratonovich integral with respect to W can be decomposed as a Skorohod integral plus a trace term.

Proof. With Proposition 5.4 in hand, relation (48) is easily deduced from [17, Proposition 5.2.2] and Proposition 2.1. □

5.3. **Discretization procedure.** We now proceed to the decomposition of $F_n := X_t$ as announced in (43), starting from the expression of F_j for $j = 0, \ldots, n$. Indeed, according to expression (48), a natural approximation sequence for X_t based on a partition $0 = t_0 < \ldots < t_n = t$ of $[0,t]$ is the following:

$$F_i = F_{i-1} + I_i + R_i, \quad (49)$$

where, introducing the additional notations

$$\eta_i(u) := \inf(u, t_i), \quad \text{and} \quad g^k_{i,s} := \int_s^t \partial_u K(u,s) V_k(X_{\eta_i(u)})\, du \quad (50)$$

we set (note that $g^k_{i-1,s} \in \mathcal{F}_{t_{i-1}}$)

$$\begin{aligned} F_{i-1} &:= \sum_{k=1}^d \int_0^{t_{i-1}} g^k_{i-1,s} \circ dW_s^k, \\ I_i &:= \sum_{k=1}^d \int_{t_{i-1}}^{t_i} g^k_{i-1,s} \circ dW_s^k = \sum_{k=1}^d V_k(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t,s)\, dW_s^k, \end{aligned} \quad (51)$$

where the last integral above is simply a Wiener integral with respect to W. We also introduce a family of random variables R_i defined by:

$$R_i := \sum_{k=1}^d \int_{t_{i-1}}^{t_i} Q_s^k \circ dW_s^k, \quad (52)$$

where Q is the process defined by

$$Q_s^k := \int_s^t \partial_u K(u,s)[V_k(X_{\eta_i(u)}) - V_k(X_{t_{i-1}})]du. \tag{53}$$

Observe that if V is elliptic and bounded, it is clear from expression (51) that $\sum_i \mathrm{Cov}_{t_{i-1}}(I_i) \asymp t^{2H} Id_m$ up to a constant, independently of the particular values of the t_i's. We shall see however how to choose those values at Condition 5.8.

Finally we introduce some random variables $\Phi_M(\mathrm{N}^i_{\gamma,p}(B))$ for $i = 1, \dots, n$ which allow us to control the supremum norm of the solution of the equation (42) and of their stochastic derivatives. This argument needs to be added in the methodology of [2, 13] and therefore we have to tailor the arguments therein to our situation. The localization random variables are based on the family of functionals $\mathrm{N}^i_{\gamma,p}(B)$ defined by

$$\mathrm{N}^i_{\gamma,p}(B) = \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} \frac{|B_v - B_u|^{2p}}{|v-u|^{2\gamma p+2}}\, dudv,$$

which can be compared to Hölder type norms and have the advantage that they can be differentiated with respect to B. In fact, we can see the aim of introducing this functional in the following proposition which is direct consequence of Garsia-Rodemich-Rumsey's Lemma (see e.g [10]).

Proposition 5.6. *Let $H > \frac{1}{2}$ and p such that $0 < \gamma < H - \frac{1}{2p}$, then we have $\|B\|_{t_{i-1},t_i,\gamma} \leq c_{\gamma,p}[\mathrm{N}^i_{\gamma,p}(B)]^{1/2p}$.*

The next step is to study the conditional densities of the approximation sequence F_i. To this aim, one has to control various terms for which the localization technique of Malliavin Calculus turns out to be useful. Specifically, recall that we have introduced families of functions $\Phi_M, \Phi_{M,\varepsilon}$ given by expression (14). In the sequel we localize our expectations using functionals of the type $\Phi_M(\mathrm{N}^i_{\gamma,p}(B))$ and $\Phi_{c_i,\epsilon}(\sum_{j=1}^d \int_{t_{i-1}}^{t_i} |\mathrm{D}^j_r R_i|^2 dr)$ for some constants c_i, ϵ of the form:

$$c_i := \frac{\lambda}{4}\int_{t_{i-1}}^{t_i} K(t,s)^2 ds > 0, \quad \text{and} \quad \epsilon_i := \frac{c_i}{2} > 0. \tag{54}$$

Furthermore, in order to ease notations, notice that we will simply write:

$$\Phi_M \equiv \Phi_M(\mathrm{N}^i_{\gamma,p}(B)), \quad \text{and} \quad \Phi_{c_i,\epsilon_i} \equiv \Phi_{c_i,\epsilon_i}\left(\sum_{j=1}^d \int_{t_{i-1}}^{t_i} |\mathrm{D}^j_r R_i|^2 dr\right). \tag{55}$$

With this additional notation in hand, we can proceed to the first step of our approximation scheme: since F_i is $\mathcal{F}_{t_{i-1}}$ conditionally non-degenerate and the localizations Φ_M and $\Phi_{c_i,\epsilon_i} \in D^\infty$, we can write

$$\mathbf{E}_{t_{i-1}}[\delta_x(F_i)] = \mathbf{E}_{t_{i-1}}[\delta_x(F_i)\,\Phi_M\,\Phi_{c_i,\epsilon_i}] + \mathbf{E}_{t_{i-1}}[\delta_x(F_i)\,(1 - \Phi_M\,\Phi_{c_i,\epsilon_i})],$$

and due to the non-negativity of the second term, we have

$$\mathbf{E}_{t_{i-1}}[\delta_x(F_i)] \geq \mathbf{E}_{t_{i-1}}[\delta_x(F_i)\,\Phi_M\Phi_{c_i,\epsilon_i}].$$

Recalling that $F_i = F_{i-1} + I_i + R_i$, we then obtain the following decomposition:

$$\mathbf{E}_{t_{i-1}}\left[\delta_x(F_i)\Phi_M\Phi_{c_i,\epsilon_i}\right] = J_{1,i} + J_{2,i} + J_{3,i}, \tag{56}$$

where

$$J_{1,i} = \mathbf{E}_{t_{i-1}}[\delta_x(F_{i-1} + I_i)], \qquad J_{2,i} = \mathbf{E}_{t_{i-1}}[\delta_x(F_{i-1} + I_i)(\Phi_M \Phi_{c_i,\epsilon_i} - 1)], \tag{57}$$

and

$$J_{3,i} = \sum_{j=1}^{m} \mathbf{E}_{t_{i-1}} \left[\Phi_M \Phi_{c_i,\epsilon_i} \int_0^1 \partial_{x_j} \delta_x(F_{i-1} + I_i + \rho R_i)\, R_i^j \, d\rho \right]. \tag{58}$$

Our aim is now to prove that in this decomposition $J_{1,i}$ should yield the main contribution, while $J_{2,i}$ is small because of the quantity $(\Phi_M \Phi_{c_i,\epsilon_i} - 1)$ whenever M and n are large enough and $J_{3,i}$ is small due to the presence of the difference between $X_{t_i} - X_{t_{i-1}}$ in R_i. We shall implement this strategy in the next subsections.

5.4. **Upper and lower bounds on $\mathbf{J_{1,i}}$.** The main information which will be used about $J_{1,i}$ is the following:

Proposition 5.7. *Let $J_{1,i}$ be defined by (57). Then under Hypothesis 1.1 we have*

$$J_{1,i} = \mathbf{E}_{t_{i-1}}[\delta_x(F_{i-1} + I_i)] = \frac{\exp\left(-\frac{1}{2}(x - F_{i-1})^* \, \Sigma_{i-1}^{-1} (x - F_{i-1})\right)}{(2\pi)^{m/2} \, |\Sigma_{i-1}|^{1/2}}, \tag{59}$$

where Σ_{i-1} is a deterministic (conditionally to $\mathcal{F}_{t_{i-1}}$) matrix such that

$$\lambda \left(\int_{t_{i-1}}^{t_i} K^2(t,u)\, du \right) Id_m \leq \Sigma_{i-1} \leq \Lambda \left(\int_{t_{i-1}}^{t_i} K^2(t,u)\, du \right) Id_m,$$

and where the two strictly positive constants λ, Λ satisfy (4).

Proof. The fact that I_{i-1} is conditionally Gaussian is clear from expression (51), and this immediately yields our claim (59). Furthermore,

$$\begin{aligned}
\Sigma_{i-1} &:= \mathrm{Cov}_{t_{i-1}}(I_i) = \mathbf{E}_{t_{i-1}}[I_i \, I_i^*] \\
&= \mathbf{E}_{t_{i-1}} \left[\left(\sum_{k=1}^{d} V_k(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t,u)\, dW_u^k \right) \left(\sum_{l=1}^{d} V_l^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t,u)\, dW_u^l \right) \right] \\
&= \sum_{k=1}^{d} V_k(X_{t_{i-1}}) V_k^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K^2(t,u)\, du,
\end{aligned}$$

which finishes the proof of our second claim thanks to Hypothesis 1.1. □

The previous proposition induces a natural choice for the partition (t_i) in terms of the kernel K:

Condition 5.8. *We choose the partition $0 = t_0 < \ldots < t_n = t$ of $[0,t]$ such that we have $\int_{t_{i-1}}^{t_i} K^2(t,u)\, du = \frac{t^{2H}}{n} =: \sigma_n^2$ for all $i = 1, \ldots, n$.*

With this choice in hand, let us note the following properties for further use:

Lemma 5.9. *Let $t_0, \ldots, t_n$ be the partition of $[0,t]$ defined by Condition 5.8. Then*

(i) *The partition is constructed in a unique way.*

(ii) *We have $0 \leq t_i - t_{i-1} \leq c_H \, n^{-1/(2H)}$ for all $i = 1, \ldots, n$.*

(iii) *The parameters c_i defined at (54) are all equal to $\frac{\lambda t^{2H}}{4n}$.*

Proof. Our first claim stems from the fact that $\int_0^t K^2(t,u)\,du = t^{2H}$ and $v \mapsto \int_v^\tau K^2(t,u)\,du$ is a strictly decreasing function for all $0 \le v \le \tau \le t$.

In order to prove our item (ii) recall expression (7), from which we easily deduce the bound

$$K(t,s) \ge c_H (t-s)^{H-1/2}. \tag{60}$$

Consider now a fixed point $\tau \in (0,t]$ and $0 \le v \equiv v_\tau < \tau \le t$ such that $\int_v^\tau K^2(t,u)\,du = \frac{t^{2H}}{n}$. Thanks to the bound (60) we have $v_\tau \ge w_\tau$ where $w_\tau \equiv w$ is defined by

$$c_H \int_w^\tau (t-u)^{2H-1}\,du = \frac{t^{2H}}{n} \quad \Longleftrightarrow \quad c_H\left[(t-w)^{2H} - (t-\tau)^{2H}\right] = \frac{t^{2H}}{n}.$$

In addition, since $2H > 1$ we have $(t-w)^{2H} - (t-\tau)^{2H} \ge (\tau - w)^{2H}$ for $w < \tau < t$, which means that $w_\tau \ge x_\tau$ where x_τ is defined by the equation $(\tau - x)^{2H} = \frac{c_H t^{2H}}{n}$. The latter equation can be solved explicitly as $x_\tau = \tau - \frac{c_H t}{n^{1/(2H)}}$, and summarizing our last considerations we end up with the relation

$$\tau - v_\tau \le \frac{c_H t}{n^{1/(2H)}},$$

which easily yields our assertion (ii). The proof of (iii) is straightforward. □

Now we state the following Corollary to Proposition 5.7, whose immediate proof is left to the reader:

Corollary 5.10. *Let $J_{1,i}$ be defined by* (57)*. Then under Hypothesis 1.1 and Condition 5.8 we have for $\sigma_n^2 = \frac{t^{2H}}{n}$*

$$J_{1,i} \ge \frac{1}{(2\pi)^{m/2}(\Lambda\sigma_n^2)^{m/2}}\, \exp\left(-\frac{|x - F_{i-1}|^2}{2\lambda\,\sigma_n^2}\right), \tag{61}$$

Summarizing the considerations of this section, we have obtained that the main contribution to $\mathbf{E}_{t_{i-1}}[\delta_x(F_i)]$, $J_{1,i}$, is of the order given by (61). Most of our work is now devoted to prove that the contributions of $J_{2,i}$ and $J_{3,i}$ are smaller than a fraction of (61) if M, n are conveniently chosen.

5.5. **Upper bounds for $\mathbf{J_{2,i}}$.** We start the control of $J_{2,i}$ by stating a bound in terms of the localization we have chosen:

Proposition 5.11. *Let $J_{2,i}$ be the quantity defined by* (57)*. Then there exists positive constants $c_{\lambda,\Lambda}$, k_1, k_2 and p_1 independent of n such that:*

$$|J_{2,i}| \le c_{\lambda,\Lambda}\, \left(\sigma_n^2\right)^{-k_2} L_n^i(k_1,p_1), \quad \textit{where} \quad L_n^i(k_1,p_1) \equiv \|1 - \Phi_M \Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}},$$

with $\sigma_n^2 = \frac{t^{2H}}{n}$, and where we recall that the norms $\|\cdot\|_{k,p,t}$ have been introduced at equation (12) *and the random variables $\Phi_M, \Phi_{c_i,\epsilon_i}$ at equation* (55)*.*

Proof. Our strategy hinges on the conditional integration by parts formula we have introduced in Proposition 2.2 , which gives for some constants k_i, p_i, $i = 1, ..., 4$,

$$\begin{aligned} |J_{2,i}| &= \left|\mathbf{E}_{t_{i-1}}\left[\mathbf{1}_{\{F_{i-1}+I_i>x\}} H^{t_{i-1}}_{(1,\dots,m)}(I_i, 1 - \Phi_M\Phi_{c_i,\epsilon_i})\right]\right| \\ &\le c_{1,q} \|\det(\Gamma_{I_i,t_{i-1}})^{-1}\|^{k_3}_{p_3,t_{i-1}} \|I_i\|^{k_4}_{k_2,p_2,t_{i-1}} \|1 - \Phi_M\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}. \end{aligned} \tag{62}$$

Here, we have used that $\mathbf{1}_{\{F_{i-1}+I_i>x\}} \leq 1$.

In order to bound the right hand side of (62) we start by computing the Malliavin derivatives of I_i. Recall that due to (51), we have for $j = 1, \ldots, d$, $\alpha > 1$ and $r, r_1, \ldots, r_\alpha > t_{i-1}$ that

$$\mathrm{D}^j_r I_i = V_j(X_{t_{i-1}}) K(t,r)\, \mathbf{1}_{[t_{i-1},t_i]}(r), \quad \text{and} \quad \mathrm{D}^\alpha_{r_1 \ldots r_\alpha} I_i = 0.$$

As far as $\Gamma_{I_i,t_{i-1}}$ is concerned, it is a conditionally deterministic quantity such that for i, $j = 1, \ldots, d$, we can write

$$\begin{aligned}\Gamma_{I_i,t_{i-1}} &= \sum_{j=1}^d \left\langle \mathrm{D}^j I_i, \mathrm{D}^j I_i^* \right\rangle_{L^2([t_{i-1},t_i])} \\ &= \sum_{j=1}^d V_j(X_{t_{i-1}}) V_j^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K^2(t,s) ds = \sigma_n^2\, V(X_{t_{i-1}}) V^*(X_{t_{i-1}}).\end{aligned}$$

Using the ellipticity condition of Hypothesis 1.1(2) for V, we thus obtain that

$$0 \leq \Gamma^{-1}_{I_i,t_{i-1}} \leq \frac{1}{\lambda \sigma_n^2} Id_m.$$

Therefore $\|I_i\|^{k_4}_{k_2,p_2,t_{i-1}} \leq C\, (\sigma_n^2 \Lambda)^{\frac{k_4}{2}}$ and

$$\|\det(\Gamma_{I_i,t_{i-1}})^{-1}\|^{k_3}_{p_3,t_{i-1}} \leq \left(\frac{1}{\lambda \sigma_n^2}\right)^{m k_3}.$$

Substituting these inequalities in (62), our proof is now finished. □

From the above Proposition 5.11, we see that in order to get a convenient bound for $J_{2,i}$ we need to study the random variable $\|1 - \Phi_M \Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$. A suitable information for us will be the following bound:

Proposition 5.12. *Let $L^i_n(k_1,p_1) = \|1 - \Phi_M \Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$ be the random variable defined at Proposition 5.11. Then for any p (recall that $\Phi_M \equiv \Phi_M(\mathrm{N}^i_{\gamma,p}(B))$) large enough so that $2p(1-\frac{\gamma}{H}) - k_1 - \frac{2}{H} > 0$ with $H - \frac{1}{2} < \gamma < H - \frac{1}{2p}$, the following holds true: For any $\eta > 0$ there exists $C_{M,\eta,p} > 0$ such that*

$$\mathbf{E}[L^i_n(k_1,p_1)] \leq C_{M,\eta,p}\, n^{-\eta}. \tag{63}$$

Proof. We sketch the proof for $p_1 = 2$. The general case follows similarly. We start noting that it is enough to find a proper bound for $\|1-\Phi_M\|_{k_1,p_1,t_{i-1}}$ and $\|1-\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$ separately. We start with the first one. Using Chebyshev inequality we have, for any $k_2 \geq 1$ and $\gamma \in (0, H-\frac{1}{2p})$

$$\mathbf{E}\left[|1-\Phi_M|^2\right] \leq \mathbf{P}(\mathrm{N}^i_{\gamma,p}(B) > M-1) \leq \frac{\mathbf{E}\left[|\mathrm{N}^i_{\gamma,p}(B)|^{k_2}\right]}{(M-1)^{k_2}}. \tag{64}$$

Evaluate now $\mathbf{E}[|\mathrm{N}^i_{\gamma,p}(B)|^{k_2}]$. A simple application of Jensen's inequality yields:

$$\mathbf{E}\left[|\mathrm{N}^i_{\gamma,p}(B))|^{k_2}\right] = \mathbf{E}\left[\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\frac{|B_v-B_u|^{2p}}{|v-u|^{2p\gamma+2}}dudv\right)^{k_2}\right]$$
$$\leq c\,|t_i-t_{i-1}|^{2(k_2-1)}\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\frac{\mathbf{E}\left[|B_v-B_u|^{2pk_2}\right]}{|v-u|^{(2p\gamma+2)k_2}}dudv\right) \leq c_{k_2,p,\gamma,H}|t_i-t_{i-1}|^{2k_2p(H-\gamma)}. \quad (65)$$

Recalling from that Lemma 5.9(ii) that $|t_i-t_{i-1}| \leq c_H\, n^{-1/(2H)}$ and taking k_2 large enough, we obtain that $\mathbf{E}[|\mathrm{N}^i_{\gamma,p}(B))|^{k_2}] \leq c_{M,\eta}\, n^{-\eta}$ for an arbitrary large η. This is consistent with relation (63).

In order to achieve the first claim, we also have to get a bound for the derivatives of Φ_M. In fact, we shall only detail the calculations for the first derivative, since estimates for the higher order derivatives are similar. Now we resort to the explicit expression of the Malliavin derivative of Φ_M given in [22, Remark 3.2], which can be computed as:

$$|\mathbf{D}^j_s\Phi_M| \leq |\Phi'_M(\mathrm{N}^i_{\gamma,p}(B))|\,\mu_s,$$

where $s\in[t_{i-1},t_i]$ and the process μ is defined by

$$\mu_s = 4p\int_{t_{i-1}}^{s}\int_{s}^{t_i}\frac{|B_\xi-B_\eta|^{2p-1}}{|\xi-\eta|^{2\gamma p+2}}d\xi\,d\eta\,\mathbf{1}_{[t_{i-1},t_i]}(s).$$

We thus get

$$\begin{aligned}\|\mathrm{D}\Phi_M\|^2_{L^2([t_{i-1},t_i])} &= \|\mathbf{D}\Phi_M\|^2_{\mathcal{H}([t_{i-1},t_i])}\\ &\leq c_H\,|\Phi'_M(\mathrm{N}^i_{\gamma,p}(B))|^2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\mu_u|u-v|^{2H-2}\mu_v\,dudv.\end{aligned}$$

Furthermore, taking into account the fact that p is large enough, we obtain that for any exponent $\kappa>0$ the following hold true:

$$\begin{aligned}\mathbf{E}[|\mu_s|^\kappa] &\leq (2p)^\kappa(t_i-t_{i-1})^{2(\kappa-1)}\int_{t_{i-1}}^{s}\int_{s}^{t_i}\frac{\mathbf{E}\left[|B_\xi-B_\eta|^{(2p-1)\kappa}\right]}{|\xi-\eta|^{(2\gamma p+2)\kappa}}d\xi d\eta\\ &\leq c_{p,\kappa,\gamma,H}\,|t_i-t_{i-1}|^{2p\kappa(H-\gamma)-\kappa H}.\end{aligned}$$

Applying this inequality with $\kappa=4$, recalling from (14) that $\|\Phi'_M\|_\infty = \|\phi\|_\infty \leq c_\phi e^{-1}$, and plugging (64), this yields

$$\begin{aligned}\mathbf{E}\left[\|\mathrm{D}\Phi_M\|^2_{L^2[t_{i-1},t_i]}\right] &\leq c_H\|\Phi'_M\|^2_\infty\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}|u-v|^{2H-2}\mathbf{E}[\mu_u\mu_v\mathbf{1}_{\{\mathrm{N}^i_{\gamma,p}(B)>M-1\}}]dudv\\ &\leq c_{H,p}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}|u-v|^{2H-2}\,\mathbf{E}^{\frac14}[\mu_u^4]\,\mathbf{E}^{\frac14}[\mu_v^4]\,\frac{\mathbf{E}\left[|\mathrm{N}^i_{\gamma,p}(B)|^{k_3}\right]}{(M-1)^{k_3}}\,dudv,\end{aligned}$$

for an arbitrary large k_3. Proceeding as in (65), for an arbitrary large k_4 we end up with:

$$\mathbf{E}\left[\|\mathrm{D}\Phi_M\|^2_{L^2[t_{i-1},t_i]}\right] \leq c_{H,p,M}|t_i-t_{i-1}|^{2p(k_4+2)(H-\gamma)},$$

which corresponds again to our assertion (63) since $|t_i-t_{i-1}|\leq c_H\,n^{-1/(2H)}$.

Finally, let us mention that along the same lines, we get a bound of the following form for the ℓ-th derivative:

$$\begin{aligned}\mathbf{E}\left[\|\mathrm{D}^{j_1,\dots,j_\ell}\Phi_M\|^2_{L^2([t_{i-1},t_i]^\ell)}\right] &\leq c_{H,p}\max_{j=1,\dots,\ell}\|\Phi_M^{(j)}\|_\infty|t_i-t_{i-1}|^{2p(2+k_5)(H-\gamma)}\\ &\leq c_{H,p,M}\,|t_i-t_{i-1}|^{2p(2+k_5)(H-\gamma)},\end{aligned}$$

for an arbitrary large k_5. This yields our claim (63), invoking again the upper bound $|t_i - t_{i-1}| \leq c_H\, n^{-1/(2H)}$.

The calculation for $\|1-\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$ is similar and we skip details for sake of conciseness. It is based on the fact that for any $k_6>0$, Chebyshev's inequality and Lemma 6.4 (postponed to the Appendix) imply that:

$$\begin{aligned}&\mathbf{P}\left(\sum_{j=1}^d\int_{t_{i-1}}^{t_i}|\mathrm{D}^j_rR^i|^2dr>\frac{\lambda}{8}\int_{t_{i-1}}^{t_i}K(t,s)^2ds\right)\\ &\leq\left(\frac{\lambda}{8}\int_{t_{i-1}}^{t_i}K(t,s)^2ds\right)^{-k_6}\mathbf{E}\left[\left(\sum_{j=1}^d\int_{t_{i-1}}^{t_i}|\mathrm{D}^j_rR^i|^2dr\right)^{k_6}\right]\\ &\leq c\left(\lambda\sigma_n^2\right)^{-k_6}(t_i-t_{i-1})^{(2\gamma+1)k_6}\leq cn^{(1-\frac{2\gamma+1}{2H})k_6}.\end{aligned}$$

Here we have used the result in Lemma 5.9 (ii) once more and the fact that $\gamma > H-\frac{1}{2}$. □

5.6. **Upper bounds for $\mathbf{J_{3,i}}$.** We now turn to the main technical issue in our computations, namely the bound on $J_{3,i}$. Our aim is thus to prove the following proposition:

Proposition 5.13. *Let $J_{3,i}$ be the quantity defined by* (58). *Then there exist $c>0$ and $k>0$ such that for any $H-\frac{1}{2}<\gamma<H$.*

$$|J_{3,i}|\leq\frac{c_{M,V,m}(t_i-t_{i-1})^\gamma}{(\sigma_n^2)^{m/2}}\leq\frac{c_{M,V,m}}{n^{\gamma/2H}\,(\sigma_n^2)^{m/2}}. \tag{66}$$

Proof. We start from expression (58) and normalize $I_i+\rho R_i$ in the following way: we just set $I_i+\rho R_i=\sigma_n\,\mathcal{U}_i$, where $\mathcal{U}_i=\sigma_n^{-1}(I_i+\rho R_i)$. We thus have

$$J_{3,i}=\sum_{j=1}^m\mathbf{E}_{t_{i-1}}\left[\Phi_M\Phi_{c_i,\epsilon_i}\int_0^1\partial_{x_j}\delta_x(F_{i-1}+\sigma_n\,\mathcal{U}_i)\,R_i^j\,d\rho\right].$$

Along the same lines as in (62), the integration by parts formula (15) now yields

$$J_{3,i}=\sigma_n^{-(m+1)}\sum_{j=1}^m\int_0^1\mathbf{E}_{t_{i-1}}\left[\mathbf{1}_{\{I_i+\rho R_i>x-F_{i-1}\}}H^{t_{i-1}}_{(j,1,\dots,m)}(\mathcal{U}_i,R_i^j\Phi_M\Phi_{c_i,\epsilon_i})\right]d\rho.$$

Hence the following bound holds true (see [17, p. 102]):

$$|J_{3,i}|\leq c_{1,q}\,\sigma_n^{-(m+1)}\,A_1\int_0^1A_2(\rho)\,A_3(\rho)\,d\rho,$$

where the quantities A_1, $A_2(\rho)$, $A_3(\rho)$ are respectively defined by

$$A_1=\max_{j=1,..,m}\|R_i^j\,\Phi_{M'}\|_{k_1,p_1,t_{i-1}},\quad A_2(\rho)=\|\det(\Gamma^{-1}_{\mathcal{U}_i,t_{i-1}})\Phi_{M'}\Phi_{c_i,\epsilon_i}\|^{k_3}_{p_3,t_{i-1}}$$

and

$$A_3(\rho) = \|\mathcal{U}_i\, \Phi_{M'}\|^{k_4}_{k_2,p_2,t_{i-1}},$$

and where we also recall that R_i^j is defined by (52). Then the first inequality in (66) follows from Lemmas 6.4, 6.5 and 6.6 which have been postponed to the Appendix, and by choosing γ such that $H - \frac{1}{2} < \gamma$. In order to go from the first inequality in (66) to the second one, we simply apply Lemma 5.9. □

5.7. **Lower bound.** Let us first summarize the considerations of the previous section: starting from decomposition (56) and applying Corollary 5.10, Propositions 5.11, 5.12 and 5.13 and the forthcoming relation (76), we have obtained the following facts: The inequality $\mathbf{E}_{t_{i-1}}[\delta_x(F_i)] \geq J_{1,i} + J_{2,i} + J_{3,i}$ holds true, and thus

$$\mathbf{E}_{t_{i-1}}[\delta_x(F_i)] \geq \frac{1}{(2\pi)^{m/2}(\Lambda\sigma_n^2)^{m/2}} \exp\left(-\frac{|x - F_{i-1}|^2}{2\lambda\,\sigma_n^2}\right) - c_{\lambda,\Lambda}\,\left(\sigma_n^2\right)^{-k_2} L_n^i(k_1,p_1) - \frac{c_{M,V,m}}{n^{\gamma/2H}\,(\sigma_n^2)^{m/2}}, \quad (67)$$

with the additional information $\mathbf{E}[L_n^i(k_1,p_1)] \leq C_{M,\eta}\, n^{-\eta}$ for an arbitrarily large exponent η.

We are now ready to prove the main theorem of this article:

Proof of Theorem 1.2. With equation (67) in hand, we shall follow the strategy designed in [2, 13]: Fix $x-a$ throughout the proof and define the balls $B_i = B(y_i,\, c_1\sigma_n)$ for $i = 1, \ldots, n$ where $y_i = a + \frac{i}{n}(x-a)$. We also define below an additional sequence $\{x_i;\, i = 1\ldots, n\}$, such that $x_i \in B_i$ and $x_n = x$. The constant c_1 will be fixed later on.

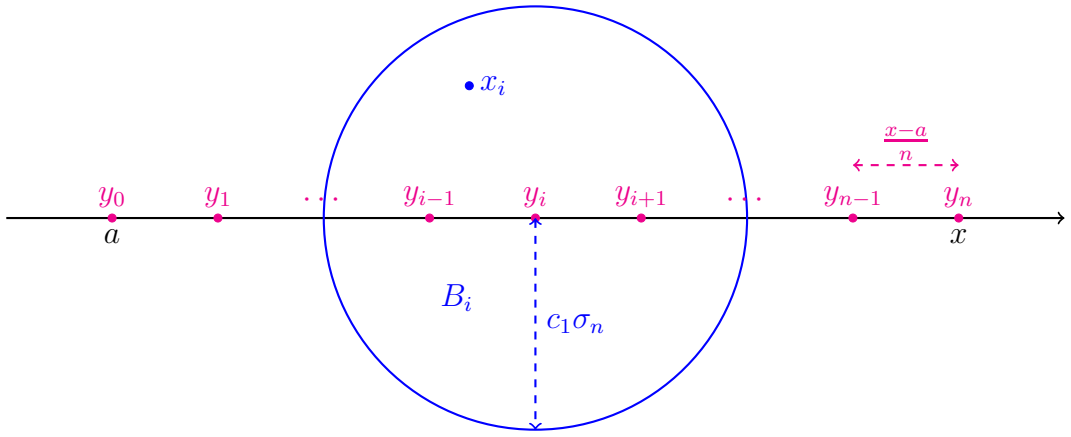

FIGURE 1. Space partition for the lower bound, with sequence $y_1, \ldots, y_n$ and x_i.

We shall now proceed in a backward recursive way on the index i. For instance in order to go from n to $n-1$, we resort to (67) in order to write:

$$\mathbf{E}\left[\delta_x(F_n)\right] = \mathbf{E}[\mathbf{E}_{t_{n-1}}\left[\delta_x(F_n)\right]] \geq \frac{c_{V,m}}{\sigma_n^m}\,\mathbf{E}\left[\exp\left(-\frac{|x - F_{n-1}|^2}{2\lambda\sigma_n^2}\right) - c_{M,V,m}\, n^{-\kappa}\right],$$

for a certain strictly positive constant κ. Hence

$$\begin{aligned}\mathbf{E}\left[\delta_x(F_n)\right] &\geq \frac{c_{V,m}}{\sigma_n^m}\int_{\mathbb{R}}\mathbf{E}\left[\left(\exp\left(-\frac{|x-F_{n-1}|^2}{2\lambda\sigma_n^2}\right)-c_{M,V,m}\,n^{-\kappa}\right)\delta_{x_{n-1}}(F_{n-1})\right]dx_{n-1}\\ &\geq \frac{c_{V,m}}{\sigma_n^m}\int_{B_{n-1}}\mathbf{E}\left[\left(\exp\left(-\frac{|x-F_{n-1}|^2}{2\lambda\sigma_n^2}\right)-c_{M,V,m}\,n^{-\kappa}\right)\delta_{x_{n-1}}(F_{n-1})\right]dx_{n-1}.\end{aligned}$$

We now observe the following: if we wish the term $\delta_{x_{n-1}}(F_{n-1})$ to give a non null contribution, the relations

$$x_{n-1}\in B(y_{n-1},c_1\sigma_n),\quad x-y_{n-1}=\frac{x-a}{n},\quad \sigma_n=\frac{t^H}{n^{1/2}},\quad |F_{n-1}-x_{n-1}|\leq c_1\sigma_n$$

must be satisfied. Moreover, from these conditions, it is easily seen that $|x-F_{n-1}|\leq 4c_1\sigma_n$ whenever $n\geq\frac{|x-a|^2}{c_1t^{2H}}$. We thus define a constant $c_2\geq\frac{1}{4c_1}$ such that

$$n=\frac{c_2\,|x-a|^2}{t^{2H}}. \tag{68}$$

Then if we take c_1 such that $\exp\left(-\frac{8c_1^2}{\lambda}\right)\geq\frac{1}{2}$ and n such that $c_{M,V,m}\,n^{-\kappa}\leq 1/4$, we obtain

$$\mathbf{E}\left[\delta_x(F_n)\right]\geq\frac{c_{V,m}}{4\sigma_n^m}\int_{B_{n-1}}\mathbf{E}\left[\delta_{x_{n-1}}(F_{n-1})\right]dx_{n-1}.$$

These arguments can now be iterated backward from $i=n-1$ to 1, and the reader can easily check that the only additional required condition is the compatibility relation $y_{i+1}-y_i\leq c_1\sigma_n$ (this will be verified below). Denoting by α_m the volume of a unit ball in $\mathbb{R}^m$ (namely $\alpha_m=\pi^{m+2}/\Gamma(\frac{m}{2}+1)$), we end up with

$$\begin{aligned}\mathbf{E}\left[\delta_x(F_n)\right] &\geq \left(\frac{c_{V,m}}{4\,\sigma_n^m}\right)^n|B(0,c_1\sigma_n)|^{n-1}=\left(\frac{c_{V,m}}{4}\right)^n\left(\frac{n^{1/2}}{t^H}\right)^{nm}\left(\frac{c_1t^H}{n^{1/2}}\right)^{m(n-1)}\alpha_m^{n-1} \qquad (69)\\ &=\left(\frac{c_{V,m}}{4}\right)^n(c_1^m\alpha_m)^{n-1}\left(\frac{n^{1/2}}{t^H}\right)^m=\frac{1}{\alpha_m\,(c_1t^H)^m}\exp\left(n\ln\left(\frac{c_{V,m}c_1^m\alpha_m}{4}\right)+\frac{m}{2}\,\ln(n)\right).\end{aligned}$$

Once here, we are reduced to tune our parameters according to the following constraints:
(i) Recalling (68), we have that if c_1 is taken small enough so that $\rho\equiv-\ln(c_{V,m}c_1^m\alpha_m/4)>0$ and (as alluded to above) such that $\exp(-8c_1^2/\lambda)\geq\frac{1}{2}$ and $n\ln(\rho)+m\ln(n)\geq 0$ for all $n\in\mathbb{N}$, we get

$$\exp\left(n\ln\left(\frac{c_{V,m}c_1^m\alpha_m}{4}\right)\right)=\exp\left(-\frac{\rho c_2\,\|x-a\|^2}{t^{2H}}\right).$$

We remark here that the values of c_1, c_2 and $c_{M,V,m}$ are fixed independently of n. It is now easily seen that our bound (69) is of the form (5).
(ii) We now choose the constant c_2 in (68) so that the compatibility relation $y_{i+1}-y_i\leq c_1\sigma_n$ is satisfied. Towards this aim, recall that

$$|y_{i+1}-y_i|=\frac{|x-a|}{n}=\frac{|x-a|}{n^{1/2}}\frac{1}{n^{1/2}},$$

and since $n = c_2 \frac{|x-a|^2}{t^{2H}}$, we get

$$|y_{i+1} - y_i| = \frac{|x-a|}{n^{1/2}} c_2^{-1/2} \frac{t^H}{|x-a|} = c_2^{-1/2} \sigma_n.$$

It is thus sufficient to take $c_2^{-1/2} \leq c_1 \wedge (2c_1^{1/2})$, which also satisfies that $n \geq \frac{|x-a|^2}{4c_1 t^{2H}}$. This completes our proof. □

6. Appendix: Some properties of stochastic derivatives

We start this technical section with a general bound on the space $\mathcal{H}$ related to fractional Brownian motion.

Lemma 6.1. *Let $H \in (0, 1/2)$, $t \in (0, 1]$ and consider the space $\mathcal{H}$ defined on $[0, t]$ as in Section 2.1. Let f be an element of $\mathcal{C}^\gamma([0, t])$ for $1/2 - H < \gamma < 1/2$, with $\|f\|_\infty \leq a$ and $\|f\|_{0,t,\gamma} \leq b$. Then*

$$\|f\|_{\mathcal{H}} \leq c_H \left(a\, t^H + b\, t^{\gamma+H} \right).$$

Proof. For a function g defined on $[0, t]$, recall that its fractional derivative is given by

$$D_{t^-}^{1/2-H} g_u = \frac{g_u}{(t-u)^{1/2-H}} + \int_u^t \frac{g_u - g_v}{(v-u)^{3/2-H}}\, dv. \tag{70}$$

Consider now $f \in \mathcal{C}^\gamma([0, t])$ satisfying the conditions above, and set $g_u = u^{-(1/2-H)} f_u$. According to [17, formula 5.31], we have

$$\|f\|_{\mathcal{H}}^2 \leq c_H \int_0^t s^{1-2H} \left| D_{t^-}^{1/2-H} g_s \right|^2 ds. \tag{71}$$

We now proceed to estimate the right hand side of relation (71).

Indeed, plugging definition (70) into (71), it is readily checked that

$$\|f\|_{\mathcal{H}}^2 \leq c_H \left(\int_0^t A_s^2\, ds + \int_0^t B_s^2\, ds \right), \quad \text{with} \quad A_s = \frac{f_s}{(t-s)^{1/2-H}}, \quad B_s = \int_s^t \frac{f_s - \psi_v\, f_v}{(v-s)^{3/2-H}}\, dv,$$

where we have set $\psi_v = (s/v)^{1/2-H}$. It is then easily seen that $\int_0^t A_s^2\, ds \leq c_H\, a^2\, t^{2H}$. In order to bound B, notice that the function ψ is well defined on $[s, t]$ and satisfies $\psi_s = 1$, $\psi_v \leq 1$ and $|\psi_v'| \leq v^{-1}$.

$$|f_s - \psi_v\, f_v| \leq |f_s - f_v|\, \psi_v + |f_s||1 - \psi_v| \leq b\, (v-s)^\gamma + a|1 - \psi_v|^\gamma \leq \left(b + \frac{a}{s^\gamma} \right) (v-s)^\gamma.$$

Dividing this inequality by $(v-s)^{3/2-H}$, recalling that $\gamma \leq 1/2$ and integrating over $[s, t]$, we get

$$|B_s| \leq c_H \left(b + \frac{a}{s^\gamma} \right) (t-s)^{\gamma-(1/2-H)},$$

which entails that

$$\int_0^t B_s^2\, ds \leq c_H \left(a^2\, t^{2H} + b^2\, t^{2(\gamma+H)} \right).$$

Gathering our bounds on $\int_0^t A_s^2\, ds$ and $\int_0^t B_s^2\, ds$, our proof is now finished.

□

Let us now state a bound on Malliavin derivatives.

Proof of relation (11). We focus on the first derivative case, the other ones being handled in a similar fashion. We will thus prove that

$$|\mathrm{D}_u F| \leq \text{ess sup}_{u\leq r} |\mathbf{D}_r F| K(t,u).$$

Indeed, according to Proposition 2.1, we have that for $F \in \mathcal{F}_t$

$$|\mathrm{D}_u F| = |[K_t^* \mathbf{D}F]_u| = |\int_u^t \mathbf{D}_r F \partial_r K(r,u) dr| \leq \text{ess sup}_{u\leq r\leq t} |\mathbf{D}_r F| K(t,u),$$

which is exactly our claim. □

We now turn to the bounds on the process Q featuring in the definition of our remainders R_i (see decomposition (49) of X_t):

Lemma 6.2. *Let X be the solution to* (42)*, let η_i be the function defined by* (50) *and Q the process given by* (53)*. If $r_1, s \in (t_{i-1}, t_i)$ then the following bounds hold true:*

$$\left|Q_s^k\right| \leq c_V K(t,s) |t_i - t_{i-1}|^\gamma Z_0^i \tag{72}$$

$$\left|\mathrm{D}^l_{r_1} Q_s^k\right| \leq c_V K(t,s) K(t,r_1) Z_1^i, \tag{73}$$

for $\mathcal{F}_1$-measurable random variables Z_0^i, Z_1^i defined by $Z_0^i = \|B\|_{t_{i-1},t,\gamma} \vee \|B\|^\gamma_{t_{i-1},t,\gamma}$ and

$$Z_1^i = \sup\left\{|\mathbf{D}^l_{r_1}(X_v - X_{t_{i-1}})|,\ t_{i-1} \leq r_1 \leq v \leq t_i\right\}, \tag{74}$$

admitting moments of all orders. In general, we can extend these results to Malliavin derivatives of arbitrary order $\ell \geq 1$ in the following way: for $r_1, s \in (t_{i-1}, t_i)$ and $r_2, \ldots, r_\ell < t_i$ we have

$$|\mathrm{D}^{j_1 \ldots j_\ell}_{r_1 \ldots r_\ell} Q_s^k| \leq c_V\, K(t,s)\, Z_\ell^i \prod_{j=1}^n K(t,\, r_j), \tag{75}$$

for $Z_\ell^i \equiv \sup\left\{|\mathbf{D}^{j_1 \ldots j_\ell}_{r_1 \ldots r_\ell}(X_v - X_{t_{i-1}})|,\ t_{i-1} \leq r_i \leq v \leq t_i, i = 1, \ldots, n\right\}$, which is a $\mathcal{F}_1$-measurable random variable with moments of all orders.

Proof. The bound (72) is an easy consequence of (53), Proposition 2.5 and the fact that $\partial_u K(u,s) \geq 0$. Moreover, observe that whenever $r_1 > t_{i-1}$ we have $\mathrm{D}_{r_1} V_k(X_{t_{i-1}}) = 0$. Hence, using Proposition 2.1 we get:

$$\begin{aligned}
|\mathrm{D}^l_{r_1} Q_s^k| &= \left|\int_{s\vee r_1}^t \partial_u K(u,s)\, \mathrm{D}^l_{r_1} V_k(X_{\eta_i(u)}) du\right| = \left|\int_{s\vee r_1}^t \partial_u K(u,s) [K_t^* \mathbf{D}^l_\cdot V_k(X_{\eta_i(u)})]_{r_1} du\right| \\
&= \left|\int_{s\vee r_1}^t \partial_u K(u,s) \left(\int_{r_1}^t \mathbf{D}^l_{r_2} V_k(X_{\eta_i(u)}) \partial_{r_2} K(r_2, r_1) dr_2\right) du\right|.
\end{aligned}$$

It is thus readily checked that

$$|\mathrm{D}^l_{r_1} Q_s^k| \leq c_V Z_1^i \left|\int_{s\vee r_1}^t \partial_u K(u,s) K(t,r_1) du\right| \leq c_V Z_1^i K(t,s) K(t,r_1).$$

The general result (75) is now obtained by means of an induction argument and resorting to the same techniques as in the case of the first order derivative (namely $\ell = 1$). □

Remark 6.3. Note that due to the definition (74) of Z_l^i and Proposition 2.5 which controls the derivatives of X using the Hölder norms of B, the random variables Z verify:

$$|Z_j^i| \leq C_V \exp\left(C_V \|B\|_{t_{i-1},t_i,\gamma}^{\frac{1}{\gamma}}\right),$$

for any $\gamma \in (\frac{1}{2}, H)$. Hence, applying Proposition 5.6 we obtain

$$|Z_j^i| \leq C_V \exp\left(C_{V,\gamma}\left(\mathrm{N}_{\gamma,p}^i(B)\right)^{\frac{1}{2\gamma p}}\right),$$

for any p such that $0 < \gamma < H - \frac{1}{2p}$. This relation yields in particular that $Z_j^i \in \cap_{q\geq 1} L^q(\Omega)$. Furthermore, once we localize by the random variables Φ_M or $\Phi_{M'}$, we end up with:

$$\max_{0\leq l\leq k}\left(Z_l^i\,\Phi_{M'}\right) \leq c_{M,V,m}, \quad \text{with} \quad c_{M,V,m} = c_{V,m}\exp(c_{V,m}(M')^{\frac{1}{2\gamma p}}). \tag{76}$$

In the next proposition, we give norm estimates for the remainder terms R_i needed in the upper bound for $J_{3,i}$.

Lemma 6.4. *In the setting of Proposition 5.4 and Corollary 5.5, with the definition* (52) *and* (55)*, the following estimate is valid:*

$$\|R_i^j \Phi_{M'}\|_{k_1,p_1,t_{i-1}} \leq c_{V,M}\,(t_i - t_{i-1})^\gamma\,\sigma_n. \tag{77}$$

Proof. This result obviously involves the control of many derivative terms. For the sake of conciseness, we only sketch the bound for $\mathrm{D}R_i$. Now recall that

$$R_i = \sum_{k=1}^d \int_{t_{i-1}}^{t_i} Q_s^k \circ dW_s^k.$$

We now apply a small variant of [17, Proposition 1.3.8] to Stratonovich integrals, which states that for $r \in [t_{i-1}, t_i]$ we have

$$\mathrm{D}_r^j R_i = Q_r^j + \sum_{k=1}^d \int_{t_{i-1}}^{t_i} \mathrm{D}_r^j Q_s^k \circ dW_s^k. \tag{78}$$

Let us now evaluate the $L^2[t_{i-1}, t_i]$ norm of $\mathrm{D}_r^j R_i$. The main contribution for this norm comes from the term Q in the right hand side of (78), for which we obtain, according to (72),

$$\begin{aligned}\int_{t_{i-1}}^{t_i}\left(Q_r^j\right)^2 dr &\leq c_V\,|t_i - t_{i-1}|^{2\gamma}\left(Z_0^i\right)^2 \int_{t_{i-1}}^{t_i} K^2(t,r)\,dr \\ &= c_V\,\left(Z_0^i\right)^2\,|t_i - t_{i-1}|^{2\gamma}\,\sigma_n^2,\end{aligned}$$

and thus

$$\mathbf{E}_{t_{i-1}}^{1/p_1}\left[\|Q\|_{L^2([t_{i-1},t_i])}^{p_1}\Phi_{M'}\right] \leq c_V\,|t_i - t_{i-1}|^\gamma\,\sigma_n\,\mathbf{E}_{t_{i-1}}^{p_1}\left[\left(Z_0^i\right)^{p_1}\,\Phi_{M'}\right] \leq c_{V,M}\,|t_i - t_{i-1}|^\gamma\,\sigma_n,$$

which is consistent with our claim (77).

Let us give another example of term which has to be analyzed in order to bound the norm of $\mathrm{D}_r^j R_i$: the term A defined as

$$A := \mathbf{E}_{t_{i-1}}^{1/p_1}\left[\left(\int_{t_{i-1}}^{t_i} dr \int_{t_{i-1}}^{t_i} ds\,\left[\mathrm{D}_r^j Q_s^k\right]^2\right)^{\frac{p_1}{2}}\,\Phi_{M'}\right].$$

Along the same lines as above, using (72), we find

$$A \leq c_{M,V} \int_{t_{i-1}}^{t_i} ds K^2(t,s) \int_{t_{i-1}}^{t_i} dr K^2(t,r) = c_{M,V} \sigma_n^4,$$

which is a remainder term with respect to (77). Notice that many other higher order terms have to be evaluated in order to complete the proof. We omit these cumbersome but routine developments for sake of conciseness. □

We now turn to the bound on $A_2(\rho)$:

Lemma 6.5. *Recall that $A_2(\rho)$ is defined as $A_2(\rho) = \| \det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1} \Phi_{M'} \Phi_{c_i,\epsilon_i} \|_{p_3,t_{i-1}}^{k_3}$. Then this quantity is uniformly bounded in n, ρ and $\omega \in \Omega$.*

Proof. First remark that using Proposition 4 in [2], we have that

$$\det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1} \Phi_{c_i,\epsilon_i} \leq \sigma_n^{2m} \left(\frac{1}{2} \lambda \int_{t_{i-1}}^{t_i} K^2(t,s) ds - \sum_{j=1}^{d} \int_{t_{i-1}}^{t_i} |\mathrm{D}_r^j R^i|^2 dr \right)^{-m} \Phi_{c_i,\epsilon_i}.$$

Moreover, we have localized $\sum_{j=1}^{d} \int_{t_{i-1}}^{t_i} |\mathrm{D}_r^j R^i|^2 dr$ by Φ_{c_i,ϵ_i} with $c_i = \frac{\lambda \sigma_n^2}{8}$. Thus we end up with

$$\det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1} \Phi_{c_i,\epsilon_i} \leq \sigma_n^2 \left(\frac{\lambda}{4} \int_{t_{i-1}}^{t_i} K^2(t,s) ds \right)^{-1},$$

from which the result follows. □

The estimates for $A_3(\rho)$ are obtained in a similar fashion. In fact, we have:

Lemma 6.6. *The same conclusion as in Lemma 6.5 holds true for the quantity $A_3(\rho) = \| \mathcal{U}_i \, \Phi_{M'} \|_{k_2,p_2,t_{i-1}}^{k_4}$.*

Proof. With respect to Lemma 6.4, we only need to consider additionally the bound for

$$\| I_i \Phi_{M'} \|_{k_2,p_2,t_{i-1}} \leq c \| I_i \|_{k_2,p_3,t_{i-1}} \| \Phi_{M'} \|_{k_2,p_4,t_{i-1}}.$$

The above follows from Hölder's inequality. Therefore the result follows from straightforward calculations for I_i as in the proof of Proposition 5.11. □

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