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► To cite this version:

| Sylvain Corlay. Properties of the Ornstein-Uhlenbeck bridge. 2013. hal-00875342v4

HAL Id: hal-00875342

<https://hal.science/hal-00875342v4>

Preprint submitted on 22 Jan 2014

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Properties of the Ornstein-Uhlenbeck bridge

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January 21, 2014

Abstract

This paper presents a study of the properties of the Ornstein-Uhlenbeck bridge, specifically, we derive its Karhunen-Loève expansion for any value of the initial variance and mean-reversion parameter (or mean-repulsion if negative). We also show that its canonical decomposition can be obtained using some techniques related to generalized bridges. Finally, we present an application to the optimal functional quantization of the Ornstein-Uhlenbeck bridge.

Keywords: Ornstein-Uhlenbeck bridge, canonical decomposition, Karhunen-Loève, filtration enlargement, functional quantization

Introduction

Let θ and μ be two real numbers and $\sigma > 0$. The Ornstein-Uhlenbeck process of long-term mean μ , mean-reversion parameter θ and volatility σ is defined as the solution of the S.D.E.

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad (1)$$

where W is a standard Brownian motion and $X_0 \stackrel{\mathcal{L}}{\sim} \mathcal{N}(x_0, \sigma_0^2)$ is independent of W . We have

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(u-t)} dW_u.$$

For a finite horizon $T > 0$, let us derive the covariance function of the Ornstein-Uhlenbeck bridge. For $\theta \neq 0$, we have

$$\begin{aligned} \mathbb{E}[(X_t - \mathbb{E}[X_t|X_T])(X_s - \mathbb{E}[X_s|X_T])] &= \frac{1}{2\theta} e^{-\theta(s+t)} ((2\theta\sigma_0^2 - \sigma^2) + \sigma^2 e^{2\theta s \wedge t}) \\ &\quad - \frac{1}{2\theta} e^{-\theta T} \frac{(\sigma^2 e^{\theta t} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta t})(\sigma^2 e^{\theta s} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta s})}{\sigma^2 e^{\theta T} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta T}}, \end{aligned}$$

and for $\theta = 0$, we have $\mathbb{E}[(X_t - \mathbb{E}[X_t|X_T])(X_s - \mathbb{E}[X_s|X_T])] = \sigma_0^2 + \sigma^2 s \wedge t - \frac{(\sigma_0^2 + \sigma^2 t)(\sigma_0^2 + \sigma^2 s)}{\sigma_0^2 + \sigma^2 T}$.

While the Karhunen-Loève expansion of Brownian motion and the Brownian bridge are known, no closed-form expression is available in the general case of a Gaussian process with a continuous covariance function such as fractional Brownian motion. Asymptotics on the rates of decay of the eigenvalues can sometime be obtained as they are related with the mean-regularity of the process, the small-ball probabilities and the rate of decay of the quantization error. However, closed-form expressions are very useful for a variety of numerical applications such as the use of functional quantization for cubature. Examples of Gaussian processes for which a closed-form expression of the Karhunen-Loève expansion exists are available in [2] and [7]. The case of the Ornstein-Uhlenbeck process is derived in [5].

The paper is organized as follows: Section 1 covers background on the Ornstein-Uhlenbeck bridge and its canonical decomposition. Using a recent result on generalized bridges, we retrieve the canonical decomposition already derived by Barczy and Kern in [3]. In Section 2, we present the derivation of the Karhunen-Loève expansion of the Ornstein-Uhlenbeck bridge. In Section 3 we derive the optimal functional quantization of Ornstein-Uhlenbeck bridges of various parameters.

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1 Canonical decompositions in the enlarged filtration

1.1 Backgrounds on generalized bridges

Let $(X_t)_{t \in [0, T]}$ be a continuous centered Gaussian semimartingale starting from 0 on $(\Omega, \mathcal{A}, \mathbb{P})$ and \mathcal{F}^X its natural filtration. Fernique's theorem [8] ensures that $\int_0^T \mathbb{E}[X_t^2] dt < +\infty$.

As Alili in [1], we are interested in the conditioning with respect to a finite family $\bar{Z}_T := (Z_T^i)_{i \in I}$ of Gaussian random variables, which are the terminal values of processes of the form $Z_t^i = \int_0^t f_i(s) dX_s$, $i \in I$, for some finite set of bounded measurable functions $\bar{f} = (f_i)_{i \in I}$. A *generalized bridge* for $(X_t)_{t \in [0, T]}$ corresponding to \bar{f} with end-point $\bar{z} = (z_i)_{i \in I}$ is a process $(X_t^{\bar{f}, \bar{z}})_{t \in [0, T]}$ with distribution $X^{\bar{f}, \bar{z}} \stackrel{\mathcal{L}}{\sim} \mathcal{L}(X | Z_T^i = z_i, i \in I)$.

1.1.1 Gaussian semimartingales

We denote by $\nu_{\bar{Z}_T | ((X_t)_{t \in [0, s]})} : \mathcal{B}(\mathbb{R}^I) \times C^0([0, s], \mathbb{R}) \rightarrow \mathbb{R}_+$, the transition kernel corresponding to the conditional distribution $\mathcal{L}(\bar{Z}_T | ((X_t)_{t \in [0, s]}))$. We make the assumption (\mathcal{H}) that for every $s \in [0, T]$, this transition kernel is absolutely continuous with respect to the Lebesgue measure and we denote by $\Pi_{(x_u)_{u \in [0, s]}, T}$ its density. This hypothesis is equivalent to assuming that the conditional covariance matrix $Q(s, T) = \mathbb{E}[(\bar{Z}_T - \mathbb{E}[\bar{Z}_T | (X_u)_{u \in [0, s]}])(\bar{Z}_T - \mathbb{E}[\bar{Z}_T | (X_u)_{u \in [0, s]}])^* | (X_u)_{u \in [0, s]}]$ is invertible.

Theorem 1.1 (Radon-Nikodym derivative). *Under the (\mathcal{H}) hypothesis, for any $s \in [0, T]$, and for $\mathbb{P}_{\bar{Z}_T}$ - almost every $\bar{z} \in \mathbb{R}^I$, $\mathbb{P}[\cdot | \bar{Z}_T = \bar{z}]$ is equivalent to \mathbb{P} on \mathcal{F}_s^X and its Radon-Nikodym density is given by*

$$\frac{d\mathbb{P}[\cdot | \bar{Z}_T = \bar{z}]}{d\mathbb{P}} \Big|_{\mathcal{F}_s^X} = \frac{\Pi_{(X_u)_{u \in [0, s]}, T}(\bar{z})}{\Pi_{0, T}(\bar{z})}.$$

Proposition 1.2 (Generalized bridges as semimartingales). *Let us define the filtration $\mathcal{G}^{X, \bar{f}}$ by $\mathcal{G}_t^{X, \bar{f}} := \sigma(\bar{Z}_T, \mathcal{F}_t^X)$, the enlargement of the filtration \mathcal{F}^X corresponding to the above conditioning. We consider the stochastic process $D_s^{\bar{z}} := \frac{d\mathbb{P}[\cdot | \bar{Z}_T = \bar{z}]}{d\mathbb{P}} \Big|_{\mathcal{F}_s^X} = \frac{\Pi_{(X_t)_{t \in [0, s]}, T}(\bar{z})}{\Pi_{0, T}(\bar{z})}$ for $s \in [0, T]$. Then, under the (\mathcal{H}) hypothesis, and the assumption that $D^{\bar{z}}$ is continuous, X is a continuous $\mathcal{G}^{X, \bar{f}}$ -semimartingale on $[0, T]$.*

Theorem 1.1 and Proposition 1.2 were first established in the Brownian case in [1] and extended to Gaussian semimartingales in [4].

1.1.2 Canonical decomposition

Following the lines of [4], we define $L_t^{\bar{z}} := \int_0^t \frac{d\Pi_{(X_u)_{u \in [0, s]}, T}(\bar{z})}{\Pi_{(X_u)_{u \in [0, s]}, T}(\bar{z})}$. We have

$$d\langle X, L^{\bar{z}} \rangle_s = d\langle X, \mathbb{E}[\bar{Z}_T | (X_u)_{u \in [0, \cdot]}] \rangle_s Q(s, T)^{-1} (\bar{z} - \mathbb{E}[\bar{Z}_T | (X_u)_{u \in [0, s]}])^*.$$

Thus, if $X = V + M$ is the $(\mathcal{F}^X, \mathbb{P})$ -canonical decomposition of X then $M - \langle X, L^{\bar{z}} \rangle$ is a $(\mathcal{G}^{X, \bar{f}}, \mathbb{P}[\cdot | \bar{Z}_T = \bar{z}])$ -martingale, and the canonical decomposition of X in this filtration is given by $X = (V + \langle X, L^{\bar{z}} \rangle) + (M - \langle X, L^{\bar{z}} \rangle)$. In the case where X is a Markov process, the expression for $\langle X, L^{\bar{z}} \rangle$ can be simplified: For every $j \in I$ there exists $g_j \in L^2([0, T])$ such that $\mathbb{E}[Z_T^j | (X_u)_{u \in [0, s]}] = \int_0^s f_j(u) dX_u + g_j(s) X_s$. Hence, if one assumes that functions $(g_j)_{j \in I}$ have finite-variations, which is the case if X is an Ornstein-Uhlenbeck process, then $d\langle X, \mathbb{E}[\bar{Z}_T | (X_u)_{u \in [0, \cdot]}] \rangle_s = (\bar{f}(s) + \bar{g}(s)) d\langle X \rangle_s$, and thus

$$d\langle X, L^{\bar{z}} \rangle_s = \sum_{i \in I} (f_i(s) + g_i(s)) \sum_{j \in I} (Q(s, T)^{-1})_{ij} \left(z_j - \mathbb{E}[Z_T^j | (X_u)_{u \in [0, s]}] \right) d\langle X \rangle_s. \quad (2)$$

1.2 Centered Ornstein-Uhlenbeck bridge starting from 0

We perform the conditioning of a centered Ornstein-Uhlenbeck process X starting from 0 (meaning that $\mu = x_0 = \sigma_0 = 0$) by $X_T = z$. With the same notation as the previous section, we have $\mathbb{E}[X_T | (X_u)_{u \in [0, s]}] = X_s e^{-\theta(T-s)}$ and thus $d\langle X, E[Z_T | (X_u)_{u \in [0, \cdot]}] \rangle_s = e^{-\theta(T-s)} d\langle X \rangle_s = \sigma^2 e^{-\theta(T-s)} ds$. In this case, Equation (2) simplifies to

$$d\langle X, L^z \rangle_s = e^{-\theta(T-s)} \sigma^2 Q(s, T)^{-1} \left(\bar{z} - X_s e^{-\theta(T-s)} \right) ds. \quad (3)$$

Moreover, $Q(s, T) = \mathbb{E} \left[\left(\int_s^T \sigma e^{\theta(u-T)} dW_u \right)^2 \right] = \sigma^2 e^{-2\theta T} \int_s^T e^{2\theta u} du$, and thus

$$Q(s, T)^{-1} = \begin{cases} \frac{2\theta}{\sigma^2} e^{-2\theta T} \frac{1}{e^{2\theta T} - e^{2\theta s}} & \text{if } \theta \neq 0, \\ \frac{1}{\sigma^2(T-s)} & \text{if } \theta = 0. \end{cases}$$

If $\theta \neq 0$, plugging this into (3) yields $d\langle X, L^z \rangle_s = 2\theta \frac{ze^{\theta(T+s)} - X_s e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}} ds$, and we finally obtain the canonical decomposition of Barczy and Kern [3],

$$dX_t = -\theta \coth(\theta(T-t)) X_t dt + \frac{\theta z}{\sinh(\theta(T-t))} dt + \overbrace{(\theta X_t dt + dX_t + d\langle X, L^z \rangle_t)}^{(\mathcal{G}^W, \mathbb{P}[\cdot | Z_T = z])\text{-martingale}}. \quad (4)$$

By Lévy's characterization of Brownian motion, the martingale part is of the form $\sigma d\widetilde{W}_t$ where \widetilde{W} is a $(\mathcal{G}^W, \mathbb{P}[\cdot | Z_T = z])$ -Brownian motion. If $\theta = 0$, we retrieve the Brownian bridge $dX_t = \frac{z - X_t}{T-t} dt + \sigma d\widetilde{W}_t$.

1.3 The case of a non-deterministic starting point ($\sigma_0^2 \neq 0$)

The conditional distribution of X_0 knowing X_T , $\mathcal{N}(\mathbb{E}[X_0 | X_T], \text{Var}(X_0 | X_T))$ is completely determined by the covariance and expectation given in the introduction. We have $X_t = (X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})) + \widetilde{X}_t$, where $\widetilde{X}_t = \int_0^t \sigma e^{\theta(u-t)} dW_u$ is a centered Ornstein-Uhlenbeck process starting from 0. \widetilde{X}_t only depends on X_T through its dependence on $\widetilde{X}_T = X_T - X_0 e^{-\theta T} - \mu(1 - e^{-\theta T})$. Thus, plugging $z = X_T - X_0 e^{-\theta T} - \mu(1 - e^{-\theta T})$ into (4) gives the $(\mathcal{G}^W, \mathbb{P}[\cdot | X_T])$ -canonical decomposition of \widetilde{X} .

2 Karhunen-Loève expansion

In this section, we derive the Karhunen-Loève expansion of the Ornstein-Uhlenbeck bridge of any initial variance or mean-reversion parameter. The method of derivation is the same as the one used for the Ornstein-Uhlenbeck process in [5].

The covariance operator of the Ornstein-Uhlenbeck bridge is defined by $T^{OB} f(s) := \int_0^T c(s, t) f(t) dt$, where $c(s, t)$ is the covariance function $\mathbb{E}[(X_t - \mathbb{E}[X_t | X_T])(X_s - \mathbb{E}[X_s | X_T])]$ given in the introduction.

Proposition 2.1. *If $f \in C([0, T])$ and $g := T^{OB} f$, then g satisfies the boundary value problem*

$$g'' - \theta^2 g = -\sigma^2 f, \quad g(T) = 0, \quad \sigma_0^2 g'(0) = (\sigma^2 - \theta \sigma_0^2) g(0) \quad (5)$$

Conversely, if $f \in C([0, T])$ and $g \in C^2([0, T])$ satisfy these three properties, then $g = T^{OB} f$.

This is proved by differentiating twice under the integral signs and evaluating at $t = 0$ and $t = T$.

As a consequence, the eigensystem $T^{OB} f = \lambda f$ amounts to solving $\lambda g'' + (\sigma^2 - \lambda \theta^2) g = 0$ with the same boundary conditions. Hence the Karhunen-Loève eigenvalues and unit eigenfunctions of the Ornstein-Uhlenbeck bridge are

$$\lambda_n^{OB} = \frac{\sigma^2}{w_n^2 + \theta^2} \quad \text{and} \quad e_n^{OB}(t) = \left(\frac{T}{2} - \frac{1}{4w_n} \sin(2w_n T) \right)^{-1/2} \sin(w_n(t - T)), \quad (6)$$

where $(w_n)_{n \geq 1}$ are the strictly positive and increasingly sorted solutions to

$$(\sigma^2 - \theta \sigma_0^2) \sin(wT) = -w \sigma_0^2 \cos(wT). \quad (7)$$

1. **Deterministic starting point** ($\sigma_0 = 0$)

Equation (7) then amounts to $\sin(wT) = 0$, and thus $w_n = \frac{n\pi}{T}$ for $n \geq 1$. (This case has already been derived in [6].)

2. **Non-deterministic starting point** ($\sigma_0 \neq 0$)

(a) If $\sigma^2 = \theta\sigma_0^2$, Equation (7) amounts to $\cos(wT) = 0$, and thus the increasingly sorted positive solutions are $w_n = \frac{n\pi}{T} - \frac{\pi}{2T}$ for $n \geq 1$.

(b) If $\theta\sigma_0^2 < \sigma^2$, Equation (7) amounts to $\underbrace{(\sigma^2 - \sigma_0^2\theta)}_{>0} \tan(wT) = -\sigma_0^2 w$,

and thus the increasingly sorted positive solutions satisfy $w_n \in]\frac{n\pi}{T} - \frac{\pi}{2T}, \frac{n\pi}{T}[$ for $n \geq 1$.

(c) If $\theta\sigma_0^2 > \sigma^2$, Equation (7) amounts to $\underbrace{(\sigma_0^2\theta - \sigma^2)}_{>0} \tan(wT) = \sigma_0^2 w$.

There is a unique solution in each interval of the form $]\frac{k\pi}{T}, \frac{k\pi}{T} + \frac{\pi}{2T}[$ for $k \geq 1$. There is another solution on $]0, \frac{\pi}{2T}[$ if and only if $\sigma_0^2 > \sigma_0^2\theta - \sigma^2$.

In cases (b) and (c), the numerical value can then be computed using a root-finding method on the corresponding intervals. This is illustrated in Figure 1. Using a certain rational approximation of \tan proposed in [5], a closed-form approximation of the solution is obtained as the root of a third-order polynomial and can be used as a starting point for the root-finding method.

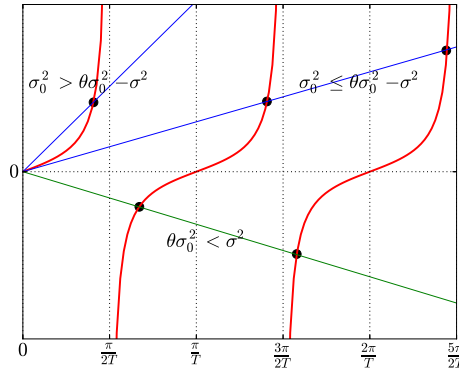


Figure 1: Solutions to Equation (7) in cases (b) and (c).

3 Functional quantization

The quantization of a random variable X valued in a reflexive separable Banach space $(E, |\cdot|)$ consists in its approximation by Y that takes finitely many values in E . We measure the resulting discretization error with the L^2 norm of the difference $|X - Y|$. If we settle on a fixed maximum cardinal N for $Y(\Omega)$, the minimization of the error reduces to the optimization problem.

$$\mathcal{E}_N(X, |\cdot|) = \min \left\{ \| |X - \text{Proj}_\Gamma(X)| \|_2, \Gamma \subset E \text{ such that } |\Gamma| \leq N \right\}, \quad (8)$$

A solution of (8) is an L^2 -optimal quantizer of X .

Now let X be a bi-measurable stochastic process on $[0, T]$ verifying $\int_0^T \mathbb{E} [|X_t|^2] dt < \infty$, which we see as a random variable valued in the Hilbert space $H = L^2([0, T])$. We assume that its covariance function Γ^X is continuous. In the seminal paper [10], it is shown that, in the centered Gaussian case, linear subspaces U of H spanned by N -stationary quantizers correspond to principal components of X , in other words, are spanned by eigenvectors of the covariance operator of X , that is, its Karhunen-Loève eigenfunctions $(e_n^X)_{n \geq 1}$.

To perform optimal quantization, the Karhunen-Loève expansion is first truncated at a fixed order m and then the \mathbb{R}^m -valued Gaussian vector constituted of the m first coordinates of the process on its

Karhunen-Loève decomposition is quantized. We have to determine the optimal rank of truncation $d^X(N)$ (the quantization dimension) and the optimal $d^X(N)$ -dimensional quantizer of the first coordinates, $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}(0, \lambda_j^X)$. The minimal quadratic distortion $\mathcal{E}_N(X)$ is given by

$$\mathcal{E}_N(X)^2 = \sum_{j \geq m+1} \lambda_j^X + \mathcal{E}_N \left(\bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X) \right)^2.$$

If the eigensystem $(e_n^X, \lambda_n^X)_{1 \leq n \leq d^X(N)}$ is known, we just need to perform the finite-dimensional quantization of $\bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X)$. Various algorithms have been devised to deal with this problem, among others, Lloyd's algorithm [9] and the Competitive Learning Vector Quantization (CLVQ) [12]. In Figures 2 and 3, we show optimal quantizers of the Ornstein-Uhlenbeck bridge for different initial variances, mean-reversion parameters, volatilities and maturities.

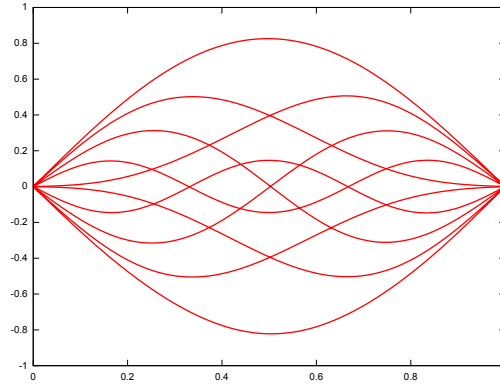


Figure 2: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters $T = 1$, $\theta = 1$, $\sigma^2 = 1$, $\sigma_0^2 = 0$, $x_0 = \mu = X_T = 0$ and $N = 10$.

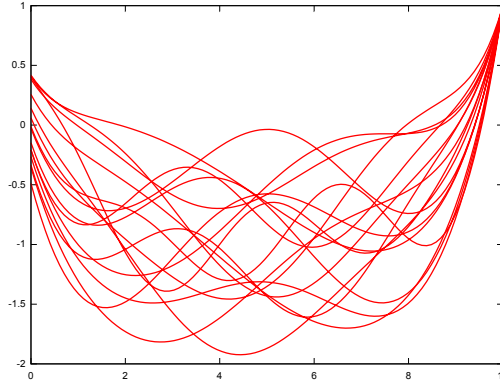


Figure 3: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters $T = 10$, $\theta = 1$, $\sigma^2 = 1$, $\sigma_0^2 = 1/2$, $x_0 = 0$, $\mu = -1$, $X_T = 1$ and $N = 16$.

Moreover, using the rate of decay of the Karhunen-Loève eigenvalues of the Ornstein-Uhlenbeck bridge, and [11, Theorem 2.2], we see that the optimal quadratic quantization error of level N of the Ornstein-Uhlenbeck bridge $\mathcal{E}_N(X)$ satisfies $\mathcal{E}_N(X) \sim K \log(N)^{-1/2}$ as $N \rightarrow \infty$, for some $K > 0$.

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