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Properties of the Ornstein-Uhlenbeck bridge

Sylvain Corlay*

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Abstract

This paper is a study of the properties of the Ornstein-Uhlenbeck bridge. We derive its canonical decomposition using filtration enlargement techniques, and its Karhunen-Loève expansion for any value of the initial variance and mean-reversion parameter (or mean-repulsion if negative). Finally, we present an application to the optimal functional quantization of the Ornstein-Uhlenbeck bridge.

Keywords: Ornstein-Uhlenbeck bridge, canonical decomposition, Karhunen-Loève, filtration enlargement, functional quantization

Introduction

Let θ and μ be two real numbers and $\sigma > 0$. The Ornstein-Uhlenbeck process of long-term mean μ, mean-reversion parameter θ and volatility σ is defined as the solution of the stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \tag{1}$$

where W is a standard Brownian motion and where $X_0 \overset{\mathcal{L}}{\sim} \mathcal{N}\left(x_0, \sigma_0^2\right)$ is assumed to be independent of W. The S.D.E. is solved by applying Itô's lemma to the process $U_t := X_t e^{\theta t}$. We obtain

$$X_t = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \int_0^t \sigma e^{\theta(u-t)} dW_u,$$

and

$$\mathbb{E}[X_t] = x_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right), \qquad \operatorname{cov}\left(X_t, X_s\right) = \begin{cases} \sigma_0^2 e^{-\theta(s+t)} + \sigma^2 e^{-\theta(s+t)} \frac{e^{2\theta s \wedge t} - 1}{2\theta} & \text{if} \quad \theta \neq 0, \\ \sigma_0^2 + \sigma^2 \; s \wedge t & \text{if} \quad \theta = 0. \end{cases}$$

Let us now consider a finite horizon $T > 0$. For $\theta \neq 0$, we have

$$\mathbb{E}\left[\left(X_t - \mathbb{E}[X_t|X_T]\right)\left(X_s - \mathbb{E}[X_s|X_T]\right)\right] = \operatorname{cov}(X_t, X_s) - \frac{\operatorname{cov}(X_s, X_T)\operatorname{cov}(X_t, X_T)}{\operatorname{Var}(X_T)}$$

$$= \frac{1}{2\theta} e^{-\theta(s+t)}\left(\left(2\theta\sigma_0^2 - \sigma^2\right) + \sigma^2 e^{2\theta s \wedge t}\right) - \frac{1}{2\theta} e^{-\theta T} \frac{\left(\sigma^2 e^{\theta t} + \left(2\theta\sigma_0^2 - \sigma^2\right) e^{-\theta t}\right)\left(\sigma^2 e^{\theta s} + \left(2\theta\sigma_0^2 - \sigma^2\right) e^{-\theta s}\right)}{\sigma^2 e^{\theta T} + \left(2\theta\sigma_0^2 - \sigma^2\right) e^{-\theta T}},$$

and if $\theta = 0$, $\mathbb{E}\left[\left(X_t - \mathbb{E}[X_t|X_T]\right)\left(X_s - \mathbb{E}[X_s|X_T]\right)\right] = \sigma_0^2 + \sigma^2 s \wedge t - \frac{\left(\sigma_0^2 + \sigma^2 t\right)\left(\sigma_0^2 + \sigma^2 s\right)}{\sigma_0^2 + \sigma^2 T}$.

1 Canonical decompositions in the enlarged filtration

1.1 Backgrounds on generalized bridges

Let $(X_t)_{t\in[0,T]}$ be a continuous centered Gaussian semimartingale starting from 0 on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F}^X$ its natural filtration. Fernique's theorem ensures that $\int_0^T \mathbb{E}\left[X_t^2\right] dt < +\infty$.

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As Alili in [1], we are interested in the conditioning with respect to a finite family $\overline{Z}_T := (Z_T^i)_{i\in I}$ of Gaussian random variables, which are the terminal values of processes of the form $Z_t^i = \int_0^t f_i(s)dX_s$, $i \in I$, for some finite set of bounded measurable functions $\overline{f} = (f_i)_{i\in I}$. A *generalized bridge* for $(X_t)_{t\in[0,T]}$ corresponding to $\overline{f}$ with end-point $\overline{z} = (z_i)_{i\in I}$ is a process $\left(X_t^{\overline{f},\overline{z}}\right)_{t\in[0,T]}$ with distribution $X^{\overline{f},\overline{z}} \overset{\mathcal{L}}{\sim} \mathcal{L}\left(X\middle|Z_T^i = z_i,\ i\in I\right)$.

1.1.1 Generalized bridges as semimartingales

Jirina's theorem ensures the existence of a transition kernel $\nu_{\overline{Z}_T|\left((X_t)_{t\in[0,s]}\right)} : \mathcal{B}(\mathbb{R}^I)\times C^0\left([0,s],\mathbb{R}\right) \to \mathbb{R}_+$, corresponding to the conditional distribution $\mathcal{L}\left(\overline{Z}_T\middle|\left((X_t)_{t\in[0,s]}\right)\right)$. We make the assumption $(\mathcal{H})$ that, for every $s \in [0,T)$ and for every $(t \mapsto x_t) \in C^0\left([0,s],\mathbb{R}\right)$, the probability measure $\nu_{\overline{Z}_T|\left((X_t)_{t\in[0,s]}\right)}\left(d\overline{z},(x_u)_{u\in[0,s]}\right)$ is absolutely continuous with respect to the Lebesgue measure. We denote by $\Pi_{(x_u)_{u\in[0,s]},T}$ its density. The covariance matrix of this Gaussian distribution on $\mathbb{R}^I$ is given by

$$Q(s,T) := \mathbb{E}\left[\left(\overline{Z}_T - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)\left(\overline{Z}_T - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)^*\middle|(X_u)_{u\in[0,s]}\right].$$

Hence, this hypothesis is equivalent to assume that

$$Q(s,T) \quad \text{is invertible for every } s \in [0,T). \tag{$\mathcal{H}$}$$

Theorem 1.1 (Radon-Nikodym derivative)**.** *Under the $(\mathcal{H})$ hypothesis, for any $s \in [0,T)$, and for $\mathbb{P}_{\overline{Z}_T}$-almost every $\overline{z} \in \mathbb{R}^I$, $\mathbb{P}\left[\cdot\middle|\overline{Z}_T = \overline{z}\right]$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_s^X$ and its Radon-Nikodym density is given by*

$$\left.\frac{d\mathbb{P}\left[\cdot\middle|\overline{Z}_T = \overline{z}\right]}{d\mathbb{P}}\right|_{\mathcal{F}_s^X} = \frac{\Pi_{(X_u)_{u\in[0,s]},T}(\overline{z})}{\Pi_{0,T}(\overline{z})}.$$

Proposition 1.2 (Generalized bridges as semimartingales)**.** *Let us define the filtration $\mathcal{G}^{X,\overline{f}}$ by $\mathcal{G}_t^{X,\overline{f}} := \sigma\left(\overline{Z}_T, \mathcal{F}_t^X\right)$, the enlargement of the filtration $\mathcal{F}^X$ corresponding to the above conditioning. We consider the stochastic process $D_s^{\overline{z}} := \frac{d\mathbb{P}\left[\cdot\middle|\overline{Z}_T=\overline{z}\right]}{d\mathbb{P}}_{|\mathcal{F}_s^X} = \frac{\Pi_{(X_t)_{t\in[0,s]},T}(\overline{z})}{\Pi_{0,T}(\overline{z})}$ for $s \in [0,T)$. Then, under the $(\mathcal{H})$ hypothesis, and the assumption that $D^{\overline{z}}$ is continuous, X is a continuous $\mathcal{G}^{X,\overline{f}}$-semimartingale on $[0,T)$.*

Theorem 1.1 and Proposition 1.2 were first established in the Brownian case in [1] and extended to Gaussian semimartingales in [2].

1.1.2 Canonical decomposition of generalized bridges

Now, as X is a continuous $\left(\mathcal{F}^X,\mathbb{P}\right)$-semimartingale, we write its canonical decomposition $X = V + M$. Following the lines of [2], we define $L_t^{\overline{z}} := \int_0^t \frac{d\Pi_{(X_u)_{u\in[0,s]},T}(\overline{z})}{\Pi_{(X_u)_{u\in[0,s]},T}(\overline{z})}$. Then

$$d\left\langle X, L^{\overline{z}}\right\rangle_s = d\left\langle X, \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,\cdot]}\right]\right\rangle_s Q(s,T)^{-1}\left(\overline{z} - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)^*.$$

Hence $M - \left\langle X, L^{\overline{z}}\right\rangle$ is a $\left(\mathcal{G}^{X,\overline{f}}, \mathbb{P}\left[\cdot\middle|\overline{Z}_T = \overline{z}\right]\right)$-martingale, and the canonical decomposition satisfies $X = \left(V + \left\langle X, L^{\overline{z}}\right\rangle\right) + \left(M - \left\langle X, L^{\overline{z}}\right\rangle\right)$. The expression for $\left\langle X, L^{\overline{z}}\right\rangle$ can be simplified in the two following cases:

- *In the case where X is a martingale*, then $\forall j \in I$, $\mathbb{E}\left[Z_T^j\middle|(X_u)_{u\in[0,s]}\right] = \int_0^s f_j(u)dX_u$ so that

$$d\left\langle X, L^{\overline{z}}\right\rangle_s = \sum_{i\in I} f_i(s)\sum_{j\in I}\left(Q(s,T)^{-1}\right)_{ij}\left(z_j - \mathbb{E}\left[Z_T^j\middle|(X_u)_{u\in[0,s]}\right]\right)d\langle X\rangle_s. \tag{2}$$

- *In the case where X is a Markov process*, for every $j \in I$ there exists $g_j \in L^2([0,T])$ such that $\mathbb{E}\left[Z_T^j\middle|(X_u)_{u\in[0,s]}\right] = \int_0^s f_j(u)dX_u + g_j(s)X_s$. Hence, if one assumes that functions $(g_j)_{j\in I}$ have

finite-variations, which is the case if X is an Ornstein-Uhlenbeck process, then $d\left\langle X, \mathbb{E}\left[\overline{Z}_T \middle| (X_u)_{u\in[0,\cdot]}\right]\right\rangle_s = \left(\overline{f}(s)+\overline{g}(s)\right) d\langle X\rangle_s$, and thus

$$d\left\langle X, L^{\overline{z}}\right\rangle_s = \sum_{i\in I}\left(f_i(s)+g_i(s)\right)\sum_{j\in I}\left(Q(s,T)^{-1}\right)_{ij}\left(z_j - \mathbb{E}\left[Z_T^j \middle| (X_u)_{u\in[0,s]}\right]\right) d\langle X\rangle_s. \tag{3}$$

1.2 The Ornstein-Uhlenbeck bridge with a deterministic starting point

We perform the conditioning of a centered Ornstein-Uhlenbeck process X starting from 0 by $X_T = z$. With the same notation as the previous section, we have

$$d\left\langle X, L^z\right\rangle_s = d\left\langle X, \mathbb{E}\left[\overline{Z}_T \middle| (X_u)_{u\in[0,\cdot]}\right]\right\rangle_s Q(s,T)^{-1}\left(\overline{z} - E\left[\overline{Z}_T \middle| (X_u)_{u\in[0,s]}\right]\right).$$

We have $\mathbb{E}\left[X_T \middle| (X_u)_{u\in[0,s]}\right] = X_s e^{-\theta(T-s)}$ and thus $d\left\langle X, E\left[Z_T \middle| (X_u)_{u\in[0,\cdot]}\right]\right\rangle_s = e^{-\theta(T-s)} d\langle X\rangle_s = \sigma^2 e^{-\theta(T-s)} ds$. In this case, Equation (3) simplifies to

$$d\langle X, L^z\rangle_s = e^{-\theta(T-s)}\sigma^2 Q(s,T)^{-1}\left(\overline{z} - X_s e^{-\theta(T-s)}\right) ds. \tag{4}$$

Moreover

$$Q(s,T) = \mathbb{E}\left[\left(\int_s^T \sigma e^{\theta(s-T)} dW_u\right)^2\right] = \sigma^2 e^{-2\theta T}\int_s^T e^{2\theta s} ds,$$

and thus

$$Q(s,T)^{-1} = \begin{cases} \frac{2\theta}{\sigma^2} e^{-2\theta T}\frac{1}{e^{2\theta T} - e^{2\theta s}} & \text{if } \theta \neq 0, \\ \frac{1}{\sigma^2(T-s)} & \text{if } \theta = 0. \end{cases}$$

If $\theta \neq 0$, plugging this into (4) yields

$$d\langle X, L^z\rangle_s = e^{-\theta(T-s)} 2\theta e^{2\theta T}\frac{1}{e^{2\theta T} - e^{2\theta s}}\left(\overline{z} - X_s e^{-\theta(T-s)}\right) ds = 2\theta\frac{z e^{\theta(T+s)} + X_s e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}} ds. \tag{5}$$

Finally, we obtain the following canonical decomposition

$$dX_t = -\theta X_t dt + 2\theta\frac{z e^{\theta(T+t)} + X_t e^{2\theta t}}{e^{2\theta T} - e^{2\theta t}} dt + \underbrace{\left(\theta X_t dt + dX_t + d\langle X, L^z\rangle_t\right)}_{(\mathcal{G}^W, \mathbb{P}[\cdot|Z_T = z])\text{-martingale}}.$$

By Lévy's characterization of Brownian motion, the martingale part is of the form $\sigma d\widetilde{W}_t$ where $\widetilde{W}$ is a $\left(\mathcal{G}^W, \mathbb{P}[\cdot|Z_T = z]\right)$-Brownian motion. When $\theta = 0$, we obtain the classical Brownian bridge $dX_t = \frac{z - X_t}{T-t} dt + \sigma d\widetilde{W}_t$.

1.3 The corresponding generalized Brownian bridge

Still in the case where $\sigma_0^2 = 0$, the Ornstein-Uhlenbeck process X_t is in one-to-one correspondence with $Z_t := \int_0^T e^{\theta u} dW_u$. Hence, the Brownian motion W conditioned by $Z_T = z$ is a generalized Brownian bridge in the sense of Alili [1] corresponding to $(f_i)_{i\in I} = (f : s \mapsto e^{\theta s})$, and thus we know its canonical decomposition. We have $\mathbb{E}\left[Z_T \middle| (W_u)_{u\in[0,t]}\right] = \int_0^t e^{\theta u} dW_u$, and $Z_T - \mathbb{E}\left[Z_T \middle| (W_u)_{u\in[0,t]}\right] = \int_t^T e^{\theta u} dW_u$, which yields

$$Q(s,T) = \mathbb{E}\left[\left(\int_s^T e^{\theta u} dW_u\right)^2\right] = \int_s^T e^{2\theta u} du.$$

Hence, if $\theta \neq 0$, $Q(s,T)^{-1} = \frac{2\theta}{e^{2\theta T} - e^{2\theta s}}$. Otherwise, $Q(s,T)^{-1} = \frac{1}{T-s}$.

$$\begin{aligned} d\langle W, L^z\rangle_s &= f(s) Q(s,T)^{-1}\left(z - \textstyle\int_0^s f(u) dW_u\right) ds \\ &= \begin{cases} \frac{2\theta e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}}\left(z e^{-\theta s} - \frac{X_s}{\sigma}\right) ds = \frac{2\theta e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}}\frac{X_T e^{\theta(T-s)} - X_s}{\sigma} ds & \text{if } \theta \neq 0, \\ \frac{X_T - X_s}{\sigma(T-s)} ds & \text{if } \theta = 0. \end{cases} \end{aligned}$$

Plugging this into Equation (6) yields the canonical decomposition of the corresponding generalized bridge.

$$W = \langle W, L^z\rangle \quad + \quad \underbrace{\left(W - \langle W, L^z\rangle\right)}_{=\widetilde{W},\ (\mathcal{G}^W, \mathbb{P}[\cdot|Z_T = z])\text{-Brownian motion}}. \tag{6}$$

1.4 The case of a non-deterministic starting point ($\sigma_0^2 \neq 0$)

The conditional distribution of X_0 knowing X_T, $\mathcal{N}\left(\mathbb{E}[X_0|X_T], \mathrm{Var}(X_0|X_T)\right)$ is explicit from the covariance and expectation of the Ornstein-Uhlenbeck bridge given in the introduction.

From Equation (1), we can write $X_t = \left(X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right)\right) \overset{\perp}{+} \underbrace{\int_0^t \sigma e^{\theta(u-t)} dW_u}_{=:\widetilde{X}_t}$. $\widetilde{X}_t$ depends on X_T only through its dependence on $\widetilde{X}_T = X_T - X_0 e^{-\theta T} - \mu\left(1 - e^{-\theta t}\right)$. Thus, we just need to plug $z = X_T - X_0 e^{-\theta T} - \mu\left(1 - e^{-\theta t}\right)$ in (5) to obtain the canonical decomposition of $\widetilde{X}$ with respect to $\left(\mathcal{G}^W, \mathbb{P}[\cdot|X_T]\right)$.

2 Karhunen-Loève expansions

In this section, we derive the Karhunen-Loève expansion of the Ornstein-Uhlenbeck bridge of any initial variance or mean-reversion parameter. The method of derivation is the same as the one used for the Ornstein-Uhlenbeck process in [3].

The covariance operator of the Ornstein-Uhlenbeck bridge is defined by $T^{OB} f\,(s) := \int_0^T c(s,t) f(t) dt$, where $c(s,t)$ is the covariance function $\mathbb{E}[(X_t - \mathbb{E}[X_t|X_T])(X_s - \mathbb{E}[X_s|X_T])]$ given in the introduction.

Proposition 2.1. *If $f \in C([0,T])$ and $g := T^{OB} f$, then*

$$\begin{array}{c} g'' - \theta^2 g = -\sigma^2 f \\ g(T) = 0, \qquad \sigma_0^2 g'(0) = \left(\sigma^2 - \theta\sigma^2\right) g(0) \end{array}$$

Conversely, if $f \in C([0,T])$ and $g \in C^2([0,T])$ satisfy these three properties, then $g = T^{OB} f$.

This is proved by differentiating twice under the integral signs and evaluating at $t = 0$ and $t = T$.

As a consequence, $T^{OB} f = \lambda f$ is equivalent to g being the solution of the O.D.E.

$$\begin{array}{c} \lambda g'' + (\sigma^2 - \lambda\theta^2) g = 0 \\ g(T) = 0, \qquad \sigma_0^2 g'(0) = (\sigma^2 - \theta\sigma_0^2) g(0). \end{array}$$

Hence, the Karhunen-Loève eigenvalues and unit eigenfunctions of the Ornstein-Uhlenbeck bridge are

$$\lambda_n^{OB} = \frac{\sigma^2}{w_n^2 + \theta^2} \qquad \text{and} \qquad e_n^{OB}(t) = \left(\frac{T}{2} - \frac{1}{4w_n}\sin\left(2w_n T\right)\right)^{-1/2} \sin(w_n(t-T)), \tag{7}$$

where $(w_n)_{n\geq 1}$ are the strictly positive and increasingly sorted solutions to

$$(\sigma^2 - \theta\sigma_0^2)\sin(wT) = -w\sigma_0^2 \cos(wT). \tag{8}$$

1. **Deterministic starting point** ($\sigma_0 = 0$)

   Equation (8) then amounts to $\sin(wT) = 0$, and thus $w_n = \frac{n\pi}{T}$ for $n \geq 1$. (This case has already been derived in [4].)

2. **Non-deterministic starting point** ($\sigma_0 \neq 0$)

   (a) If $\sigma^2 = \theta\sigma_0^2$, Equation (8) amounts to

   $$\cos(wT) = 0,$$

   and thus the increasingly sorted positive solutions are $w_n = \frac{n\pi}{T} - \frac{\pi}{2T}$ for $n \geq 1$.

   (b) If $\theta\sigma_0^2 < \sigma^2$, Equation (8) amounts to

   $$\underbrace{\left(\sigma^2 - \sigma_0^2\theta\right)}_{>0} \tan(wT) = -\sigma_0^2 w,$$

   and thus the increasingly sorted positive solutions are $w_n \in \left]\frac{n\pi}{T} - \frac{\pi}{2T}, \frac{n\pi}{T}\right[$ for $n \geq 1$.

(c) If $\theta\sigma_0^2 > \sigma^2$, Equation (8) amounts to

$$\underbrace{\left(\sigma_0^2\theta - \sigma^2\right)}_{>0}\tan(wT) = \sigma_0^2 w.$$

There is a unique solution in each interval of the form $\left]\frac{k\pi}{T}, \frac{k\pi}{T} + \frac{\pi}{2T}\right[$ for $k \geq 1$. There is another solution on $\left]0, \frac{\pi}{2T}\right[$ if and only if $\sigma_0^2 > \sigma_0^2\theta - \sigma^2$.

In cases (b) and (c), the numerical value can then be computed using a root-finding method on the corresponding interval. This is illustrated in Figure 1.

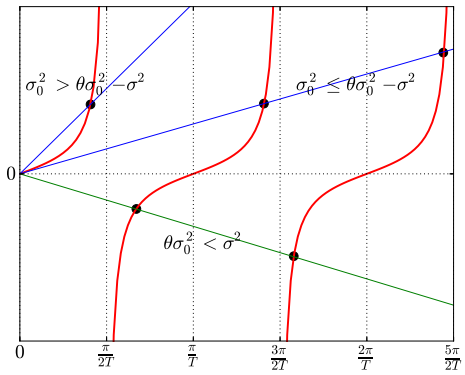

Figure 1: Solutions to Equation (8) in cases (b) and (c).

3 Functional quantization

The quantization of a random variable X valued in a reflexive separable Banach space $(E, |\cdot|)$ consists in its approximation by Y that takes finitely many values in E. We measure the resulting discretization error with the L^2 norm of the difference $|X - Y|$. If we settle on a fixed maximum cardinal N for $Y(\Omega)$, the minimization of the error reduces to the optimization problem.

$$\mathcal{E}_N(X, |\cdot|) = \min\left\{\left\| |X - \mathrm{Proj}_\Gamma(X)| \right\|_2,\ \Gamma \subset E \text{ such that } |\Gamma| \leq N\right\}, \tag{9}$$

A solution of (9) is an L^2-*optimal quantizer* of X.

Now let X be a bi-measurable stochastic process on $[0,T]$ verifying $\int_0^T \mathbb{E}\left[|X_t|^2\right] dt < \infty$, which we see as a random variable valued in the Hilbert space $H = L^2([0,T])$. We assume that its covariance function Γ^X is continuous. In the seminal paper [6], it is shown that, in the centered Gaussian case, linear subspaces U of H spanned by N-stationary quantizers correspond to principal components of X, in other words, are spanned by eigenvectors of the covariance operator of X, that is, its Karhunen-Loève eigenfunctions $\left(e_n^X\right)_{n\geq 1}$.

To perform optimal quantization, the Karhunen-Loève expansion is first truncated at a fixed order m and then the $\mathbb{R}^m$-valued Gaussian vector constituted of the m first coordinates of the process on its Karhunen-Loève decomposition is quantized. We have to determine the optimal rank of truncation $d^X(N)$ (the quantization dimension) and the optimal $d^X(N)$-dimensional quantizer of the first coordinates, $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}\left(0, \lambda_j^X\right)$. The minimal quadratic distortion $\mathcal{E}_N(X)$ is given by

$$\mathcal{E}_N(X)^2 = \sum_{j\geq m+1} \lambda_j^X + \mathcal{E}_N\left(\bigotimes_{j=1}^m \mathcal{N}\left(0, \lambda_j^X\right)\right)^2.$$

If the eigensystem $\left(e_n^X, \lambda_n^X\right)_{1\leq n\leq d^X(N)}$ is known, we just need to perform the finite-dimensional quantization of $\bigotimes_{j=1}^m \mathcal{N}\left(0, \lambda_j^X\right)$. Various algorithms have been devised to deal with this problem, among others,

Lloyd's algorithm [5] and the Competitive Learning Vector Quantization (CLVQ) [7]. In Figures 2 and 3, we show optimal quantizers of the Orntein-Uhlenbeck bridge for different initial variances, mean-reversion parameters, volatilities and maturities.

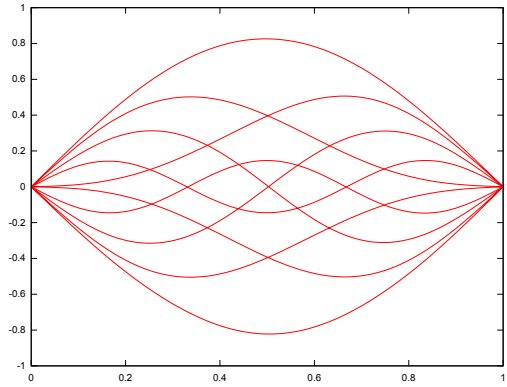

Figure 2: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters $T = 1$, $\theta = 1$, $\sigma^2 = 1$, $\sigma_0^2 = 0$, $x_0 = \mu = X_T = 0$ and $N = 10$.

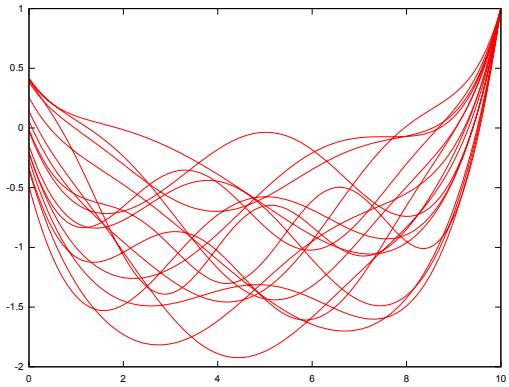

Figure 3: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters $T = 10$, $\theta = 1$, $\sigma^2 = 1$, $\sigma_0^2 = 1/2$, $x_0 = 0$, $\mu = -1$, $X_T = 1$ and $N = 16$.

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