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# Properties of the Ornstein-Uhlenbeck bridge

Sylvain CORLAY\*

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## Abstract

This note is a study of the Ornstein-Uhlenbeck bridge and its main properties. We present the derivation of its canonical decomposition in the first section of the paper using filtration enlargement techniques. In the second section, we present its Karhunen-Loève expansion, for any value of the initial variance and mean-reversion parameter (or mean-repulsion if negative). In the third section, we present an application to the optimal functional quantization of the Ornstein-Uhlenbeck bridge.

**Keywords:** Ornstein-Uhlenbeck bridge, canonical decomposition, Karhunen-Loève, filtration enlargement, functional quantization

## Introduction

Let  $\theta$  and  $\mu$  be two real numbers and  $\sigma > 0$ . The Ornstein-Uhlenbeck process of long-term mean  $\mu$ , mean-reversion parameter  $\theta$  and volatility  $\sigma$  is defined as the solution of the stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad (1)$$

where  $W$  is a standard Brownian motion and where  $X_0 \stackrel{\mathcal{L}}{\sim} \mathcal{N}(x_0, \sigma_0^2)$  is assumed to be independent of  $W$ . The S.D.E. is solved by applying Itô's lemma to the process  $U_t := X_t e^{\theta t}$ . We obtain

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(u-t)} dW_u,$$

and

$$\mathbb{E}[X_t] = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}), \quad \text{cov}(X_t, X_s) = \begin{cases} \sigma_0^2 e^{-\theta(s+t)} + \sigma^2 e^{-\theta(s+t)} \frac{e^{2\theta s \wedge t} - 1}{2\theta} & \text{if } \theta \neq 0, \\ \sigma_0^2 + \sigma^2 s \wedge t & \text{if } \theta = 0. \end{cases}$$

Let us now consider a finite horizon  $T > 0$ . For  $\theta \neq 0$ , we have

$$\begin{aligned} \mathbb{E}[(X_t - \mathbb{E}[X_t|X_T])(X_s - \mathbb{E}[X_s|X_T])] &= \text{cov}(X_t, X_s) - \frac{\text{cov}(X_s, X_T) \text{cov}(X_t, X_T)}{\text{Var}(X_T)} \\ &= \frac{1}{2\theta} e^{-\theta(s+t)} ((2\theta\sigma_0^2 - \sigma^2) + \sigma^2 e^{2\theta s \wedge t}) - \frac{1}{2\theta} e^{-\theta T} \frac{(\sigma^2 e^{\theta t} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta t}) (\sigma^2 e^{\theta s} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta s})}{\sigma^2 e^{\theta T} + (2\theta\sigma_0^2 - \sigma^2) e^{-\theta T}}, \end{aligned}$$

and if  $\theta = 0$ ,

$$\mathbb{E}[(X_t - \mathbb{E}[X_t|X_T])(X_s - \mathbb{E}[X_s|X_T])] = \sigma_0^2 + \sigma^2 s \wedge t - \frac{(\sigma_0^2 + \sigma^2 t)(\sigma_0^2 + \sigma^2 s)}{\sigma_0^2 + \sigma^2 T}.$$

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# 1 Canonical decompositions in the enlarged filtration

## 1.1 Backgrounds on generalized bridges

Let  $(X_t)_{t \in [0, T]}$  be a continuous centered Gaussian semimartingale starting from 0 on  $(\Omega, \mathcal{A}, \mathbb{P})$  and  $\mathcal{F}^X$  its natural filtration. Fernique's theorem ensures that  $\int_0^T \mathbb{E} [X_t^2] dt < +\infty$  (see Janson [5]).

As Alili in [1], we are interested in the conditioning with respect to a finite family  $\overline{Z}_T := (Z_T^i)_{i \in I}$  of Gaussian random variables, which are the terminal values of processes of the form  $Z_t^i = \int_0^t f_i(s) dX_s$ ,  $i \in I$ , for some finite set of bounded measurable functions  $\overline{f} = (f_i)_{i \in I}$ . A *generalized bridge* for  $(X_t)_{t \in [0, T]}$  corresponding to  $\overline{f}$  with end-point  $\overline{z} = (z_i)_{i \in I}$  is a process  $(X_t^{\overline{f}, \overline{z}})_{t \in [0, T]}$  with distribution  $X^{\overline{f}, \overline{z}} \stackrel{\mathcal{L}}{\sim} \mathcal{L}(X | Z_T^i = z_i, i \in I)$ .

### 1.1.1 Generalized bridges as semimartingales

Jirina's theorem ensures the existence of a transition kernel  $\nu_{\overline{Z}_T | ((X_t)_{t \in [0, s]})} : \mathcal{B}(\mathbb{R}^I) \times C^0([0, s], \mathbb{R}) \rightarrow \mathbb{R}_+$ , corresponding to the conditional distribution  $\mathcal{L}(\overline{Z}_T | ((X_t)_{t \in [0, s]}))$ . We make the assumption  $(\mathcal{H})$  that, for every  $s \in [0, T]$  and for every  $(t \mapsto x_t) \in C^0([0, s], \mathbb{R})$ , the probability measure  $\nu_{\overline{Z}_T | ((X_t)_{t \in [0, s]})}(d\overline{z}, (x_u)_{u \in [0, s]})$  is absolutely continuous with respect to the Lebesgue measure. We denote by  $\Pi_{(x_u)_{u \in [0, s]}, T}$  its density. The covariance matrix of this Gaussian distribution on  $\mathbb{R}^I$  writes

$$Q(s, T) := \mathbb{E} \left[ (\overline{Z}_T - \mathbb{E}[\overline{Z}_T | (X_u)_{u \in [0, s]}]) (\overline{Z}_T - \mathbb{E}[\overline{Z}_T | (X_u)_{u \in [0, s]}])^* \middle| (X_u)_{u \in [0, s]} \right].$$

Hence, this hypothesis is equivalent to assume that

$$Q(s, T) \text{ is invertible for every } s \in [0, T]. \quad (\mathcal{H})$$

**Theorem 1.1** (Radon-Nikodym derivative). *Under the  $(\mathcal{H})$  hypothesis, for any  $s \in [0, T]$ , and for  $\mathbb{P}_{\overline{Z}_T}$ -almost every  $\overline{z} \in \mathbb{R}^I$ ,  $\mathbb{P}[\cdot | \overline{Z}_T = \overline{z}]$  is equivalent to  $\mathbb{P}$  on  $\mathcal{F}_s^X$  and its Radon-Nikodym density is given by*

$$\frac{d\mathbb{P}[\cdot | \overline{Z}_T = \overline{z}]}{d\mathbb{P}} \Big|_{\mathcal{F}_s^X} = \frac{\Pi_{(X_u)_{u \in [0, s]}, T}(\overline{z})}{\Pi_{0, T}(\overline{z})}.$$

**Proposition 1.2** (Generalized bridges as semimartingales). *Let us define the filtration  $\mathcal{G}^{X, \overline{f}}$  by  $\mathcal{G}_t^{X, \overline{f}} := \sigma(\overline{Z}_T, \mathcal{F}_t^X)$ , the enlargement of the filtration  $\mathcal{F}^X$  corresponding to the above conditioning. We consider the stochastic process  $D_s^{\overline{z}} := \frac{d\mathbb{P}[\cdot | \overline{Z}_T = \overline{z}]}{d\mathbb{P}} \Big|_{\mathcal{F}_s^X} = \frac{\Pi_{(X_t)_{t \in [0, s]}, T}(\overline{z})}{\Pi_{0, T}(\overline{z})}$  for  $s \in [0, T]$ . Then, under the  $(\mathcal{H})$  hypothesis, and the assumption that  $D^{\overline{z}}$  is continuous,  $X$  is a continuous  $\mathcal{G}^{X, \overline{f}}$ -semimartingale on  $[0, T]$ .*

Theorem 1.1 and Proposition 1.2 were first established in the Brownian case in [1] and extended to Gaussian semimartingales in [2].

### 1.1.2 Canonical decomposition of generalized bridges

Now, as  $X$  is a continuous  $(\mathcal{F}^X, \mathbb{P})$ -semimartingale, we write its canonical decomposition  $X = V + M$ . Following the lines of [2], we define  $L_t^{\overline{z}} := \int_0^t \frac{d\Pi_{(X_u)_{u \in [0, s]}, T}(\overline{z})}{\Pi_{(X_u)_{u \in [0, s]}, T}(\overline{z})}$ . Then

$$d\langle X, L^{\overline{z}} \rangle_s = d\langle X, \mathbb{E}[\overline{Z}_T | (X_u)_{u \in [0, \cdot]}] \rangle_s Q(s, T)^{-1} (\overline{z} - \mathbb{E}[\overline{Z}_T | (X_u)_{u \in [0, s]}])^*.$$

Hence  $M - \langle X, L^{\overline{z}} \rangle$  is a  $(\mathcal{G}^{X, \overline{f}}, \mathbb{P}[\cdot | \overline{Z}_T = \overline{z}])$ -martingale, and the canonical decomposition writes  $X = (V + \langle X, L^{\overline{z}} \rangle) + (M - \langle X, L^{\overline{z}} \rangle)$ . The expression for  $\langle X, L^{\overline{z}} \rangle$  can be simplified in the two following cases:

- In the case where  $X$  is a martingale, then  $\forall j \in I$ ,  $\mathbb{E} \left[ Z_T^j \middle| (X_u)_{u \in [0, s]} \right] = \int_0^s f_j(u) dX_u$  so that

$$d \langle X, L^{\bar{z}} \rangle_s = \sum_{i \in I} f_i(s) \sum_{j \in I} (Q(s, T)^{-1})_{ij} \left( z_j - \mathbb{E} \left[ Z_T^j \middle| (X_u)_{u \in [0, s]} \right] \right) d \langle X \rangle_s. \quad (2)$$

- In the case where  $X$  is a Markov process, for every  $j \in I$  there exists  $g_j \in L^2([0, T])$  such that  $\mathbb{E} \left[ Z_T^j \middle| (X_u)_{u \in [0, s]} \right] = \int_0^s f_j(u) dX_u + g_j(s) X_s$ . Hence, if one assumes that functions  $(g_j)_{j \in I}$  have finite-  
variations, which is the case if  $X$  is an Ornstein-Uhlenbeck process, then  $d \langle X, \mathbb{E} \left[ \bar{Z}_T \middle| (X_u)_{u \in [0, \cdot]} \right] \rangle_s = (\bar{f}(s) + \bar{g}(s)) d \langle X \rangle_s$ , and thus

$$d \langle X, L^{\bar{z}} \rangle_s = \sum_{i \in I} (f_i(s) + g_i(s)) \sum_{j \in I} (Q(s, T)^{-1})_{ij} \left( z_j - \mathbb{E} \left[ Z_T^j \middle| (X_u)_{u \in [0, s]} \right] \right) d \langle X \rangle_s. \quad (3)$$

## 1.2 The Ornstein-Uhlenbeck bridge with a deterministic starting point ( $\sigma_0^2 = 0$ )

We perform the conditioning of a centered Ornstein-Uhlenbeck process  $X$  starting from 0 by  $X_T = z$ . With the same notation as the previous section, we have

$$d \langle X, L^z \rangle_s = d \langle X, \mathbb{E} \left[ \bar{Z}_T \middle| (X_u)_{u \in [0, \cdot]} \right] \rangle_s Q(s, T)^{-1} (\bar{z} - \mathbb{E} \left[ \bar{Z}_T \middle| (X_u)_{u \in [0, s]} \right]).$$

We have  $\mathbb{E} \left[ X_T \middle| (X_u)_{u \in [0, s]} \right] = X_s e^{-\theta(T-s)}$  and thus  $d \langle X, \mathbb{E} \left[ \bar{Z}_T \middle| (X_u)_{u \in [0, \cdot]} \right] \rangle_s = e^{-\theta(T-s)} d \langle X \rangle_s = \sigma^2 e^{-\theta(T-s)} ds$ . In this case, Equation (3) simplifies to

$$d \langle X, L^z \rangle_s = e^{-\theta(T-s)} \sigma^2 Q(s, T)^{-1} \left( \bar{z} - X_s e^{-\theta(T-s)} \right) ds. \quad (4)$$

Moreover

$$Q(s, T) = \mathbb{E} \left[ \left( \int_s^T \sigma e^{\theta(s-T)} dW_u \right)^2 \right] = \sigma^2 e^{-2\theta T} \int_s^T e^{2\theta s} ds,$$

and thus

$$Q(s, T)^{-1} = \begin{cases} \frac{2\theta}{\sigma^2} e^{-2\theta T} \frac{1}{e^{2\theta T} - e^{2\theta s}} & \text{if } \theta \neq 0, \\ \frac{1}{\sigma^2(T-s)} & \text{if } \theta = 0. \end{cases}$$

If  $\theta \neq 0$ , plugging this into (4) yields

$$d \langle X, L^z \rangle_s = e^{-\theta(T-s)} 2\theta e^{2\theta T} \frac{1}{e^{2\theta T} - e^{2\theta s}} \left( \bar{z} - X_s e^{-\theta(T-s)} \right) ds = 2\theta \frac{z e^{\theta(T+s)} + X_s e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}} ds. \quad (5)$$

Finally, we obtain the following canonical decomposition

$$dX_t = -\theta X_t dt + 2\theta \frac{z e^{\theta(T+t)} + X_t e^{2\theta t}}{e^{2\theta T} - e^{2\theta t}} dt + \underbrace{(\theta X_t dt + dX_t + d \langle X, L^z \rangle_t)}_{(\mathcal{G}^W, \mathbb{P}[\cdot | Z_T = z])\text{-martingale}}.$$

By Lévy's characterization of Brownian motion, the martingale part is of the form  $\sigma d\widetilde{W}_t$  where  $\widetilde{W}$  is a  $(\mathcal{G}^W, \mathbb{P}[\cdot | Z_T = z])$ -Brownian motion. When  $\theta = 0$ , we obtain the classical Brownian bridge

$$dX_t = \frac{z - X_t}{T - t} dt + \sigma d\widetilde{W}_t.$$

### 1.3 The corresponding generalized Brownian bridge

Still in the case where  $\sigma_0^2 = 0$ , the Ornstein-Uhlenbeck process  $X_t$  is in one-to-one correspondence with  $Z_t := \int_0^T e^{\theta u} dW_u$ . Hence, the Brownian motion  $W$  conditioned by  $Z_T = z$  is a generalized Brownian bridge in the sense of Alili [1] corresponding to  $(f_i)_{i \in I} = (f : s \mapsto e^{\theta s})$ , and thus we know its canonical decomposition. We have  $\mathbb{E}[Z_T | (W_u)_{u \in [0, t]}] = \int_0^t e^{\theta u} dW_u$ , and  $Z_T - \mathbb{E}[Z_T | (W_u)_{u \in [0, t]}] = \int_t^T e^{\theta u} dW_u$ , which yields

$$Q(s, T) = \mathbb{E} \left[ \left( \int_s^T e^{\theta u} dW_u \right)^2 \right] = \int_s^T e^{2\theta u} du.$$

Hence, if  $\theta \neq 0$ ,  $Q(s, T)^{-1} = \frac{2\theta}{e^{2\theta T} - e^{2\theta s}}$ . Otherwise,  $Q(s, T)^{-1} = \frac{1}{T-s}$ .

$$\begin{aligned} d\langle W, L^z \rangle_s &= f(s)Q(s, T)^{-1} \left( z - \int_0^s f(u) dW_u \right) ds \\ &= \begin{cases} \frac{2\theta e^{2\theta s}}{e^{2\theta T} - e^{2\theta s}} \left( z e^{-\theta s} - \frac{X_s}{\sigma} \right) ds & \text{if } \theta \neq 0, \\ \frac{X_T - X_s}{\sigma(T-s)} ds & \text{if } \theta = 0. \end{cases} \end{aligned}$$

Plugging this into Equation (6) yields the canonical decomposition of the corresponding generalized bridge.

$$\begin{aligned} W &= \langle W, L^z \rangle + \underbrace{(W - \langle W, L^z \rangle)}_{= \tilde{W}, (G^W, \mathbb{P}[\cdot | Z_T = z])\text{-Brownian motion}}. \end{aligned} \quad (6)$$

If we plug this into the S.D.E. (1), we retrieve the canonical decomposition derived in the previous section.

### 1.4 The case of a non-deterministic starting point ( $\sigma_0^2 \neq 0$ )

The conditional distribution of  $X_0$  knowing  $X_T$ ,  $\mathcal{N}(\mathbb{E}[X_0 | X_T], \text{Var}(X_0 | X_T))$  is explicit from the covariance and expectation of the Ornstein-Uhlenbeck bridge given in the introduction.

From Equation (1), we can write  $X_t = (X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})) + \underbrace{\int_0^t \sigma e^{\theta(u-t)} dW_u}_{=: \tilde{X}_t}$ .  $\tilde{X}_t$  depends on

$X_T$  only through its dependence on  $\tilde{X}_T = X_T - X_0 e^{-\theta T} - \mu(1 - e^{-\theta T})$ . Thus, we just need to plug  $z = X_T - X_0 e^{-\theta T} - \mu(1 - e^{-\theta T})$  in (5) to obtain the canonical decomposition of  $\tilde{X}$  with respect to  $(G^W, \mathbb{P}[\cdot | X_T])$ .

## 2 Karhunen-Loève expansions

In this section, we derive the Karhunen-Loève expansion of the Ornstein-Uhlenbeck bridge of any initial variance or mean-reversion parameter. The method of derivation is the same as the one used for the Ornstein-Uhlenbeck process in [3].

The covariance operator of the Ornstein-Uhlenbeck bridge is defined by  $T^{OB} f(s) := \int_0^T c(s, t) f(t) dt$ , where  $c(s, t)$  is the covariance function  $\mathbb{E}[(X_t - \mathbb{E}[X_t | X_T])(X_s - \mathbb{E}[X_s | X_T])]$  given in the introduction.

**Proposition 2.1.** *If  $f \in C([0, T])$  and  $g := T^{OB} f$ , then*

$$\begin{aligned} g'' - \theta^2 g &= -\sigma^2 f \\ g(T) &= 0, \quad \sigma_0^2 g'(0) = (\sigma^2 - \theta \sigma^2) g(0) \end{aligned}$$

*Conversely, if  $f \in C([0, T])$  and  $g \in C^2([0, T])$  satisfy these three properties, then  $g = T^{OB} f$ .*

This is proved by differentiating twice under the integral signs and evaluating at  $t = 0$  and  $t = T$ .

As a consequence,  $T^{OB}f = \lambda f$  is equivalent to  $g$  being the solution of the O.D.E.

$$\begin{aligned} \lambda g'' + (\sigma^2 - \lambda\theta^2)g &= 0 \\ g(T) = 0, \quad \sigma_0^2 g'(0) &= (\sigma^2 - \theta\sigma_0^2)g(0). \end{aligned}$$

Hence, the Karhunen-Loève eigenvalues and unit eigenfunctions of the Ornstein-Uhlenbeck bridge are

$$\lambda_n^{OB} = \frac{\sigma^2}{w_n^2 + \theta^2} \quad \text{and} \quad e_n^{OB}(t) = \left( \frac{T}{2} - \frac{1}{4w_n} \sin(2w_n T) \right)^{-1/2} \sin(w_n(t - T)), \quad (7)$$

where  $(w_n)_{n \geq 1}$  are the strictly positive and increasingly sorted solutions to

$$(\sigma^2 - \theta\sigma_0^2) \sin(wT) = -w\sigma_0^2 \cos(wT). \quad (8)$$

1. **Deterministic starting point** ( $\sigma_0 = 0$ )

Equation (8) then amounts to  $\sin(wT) = 0$ , and thus  $w_n = \frac{n\pi}{T}$  for  $n \geq 1$ . (This case is derived in [4].)

2. **Non-deterministic starting point** ( $\sigma_0 \neq 0$ )

(a) If  $\sigma^2 = \theta\sigma_0^2$ , Equation (8) amounts to

$$\cos(wT) = 0,$$

and thus the increasingly sorted positive solutions are  $w_n = \frac{n\pi}{T} - \frac{\pi}{2T}$  for  $n \geq 1$ .

(b) If  $\theta\sigma_0^2 < \sigma^2$ , Equation (8) amounts to

$$\underbrace{(\sigma^2 - \sigma_0^2\theta)}_{>0} \tan(wT) = -\sigma_0^2 w,$$

and thus the increasingly sorted positive solutions are  $w_n \in ]\frac{n\pi}{T} - \frac{\pi}{2T}, \frac{n\pi}{T}[$  for  $n \geq 1$ .

(c) If  $\theta\sigma_0^2 > \sigma^2$ , Equation (8) amounts to

$$\underbrace{(\sigma_0^2\theta - \sigma^2)}_{>0} \tan(wT) = \sigma_0^2 w.$$

There is a unique solution in each interval of the form  $]\frac{k\pi}{T}, \frac{k\pi}{T} + \frac{\pi}{2T}[$  for  $k \geq 1$ . There is another solution on  $]0, \frac{\pi}{2T}[$  if and only if  $\sigma_0^2 > \sigma^2\theta - \sigma^2$ .

In cases (b) and (c), the numerical value can then be computed using a root-finding method on the corresponding interval. This is illustrated in Figure 1.

### 3 Functional quantization

The quantization of a random variable  $X$  valued in a reflexive separable Banach space  $(E, |\cdot|)$  consists in its approximation by  $Y$  that takes finitely many values in  $E$ . We measure the resulting discretization error with the  $L^2$  norm of the difference  $|X - Y|$ . If we settle on a fixed maximum cardinal  $N$  for  $Y(\Omega)$ , the minimization of the error reduces to the optimization problem.

$$\mathcal{E}_N(X, |\cdot|) = \min \{ \| |X - \text{Proj}_\Gamma(X)| \|_2, \Gamma \subset E \text{ such that } |\Gamma| \leq N \}, \quad (9)$$

A solution of (9) is an  $L^2$ -optimal quantizer of  $X$ .

Now let  $X$  be a bi-measurable stochastic process on  $[0, T]$  verifying  $\int_0^T \mathbb{E} [|X_t|^2] dt < \infty$ , which we see as a random variable valued in the Hilbert space  $H = L^2([0, T])$ . We assume that its covariance function  $\Gamma^X$  is continuous. In the seminal paper [7], it is shown that, in the centered Gaussian case, linear subspaces  $U$  of  $H$

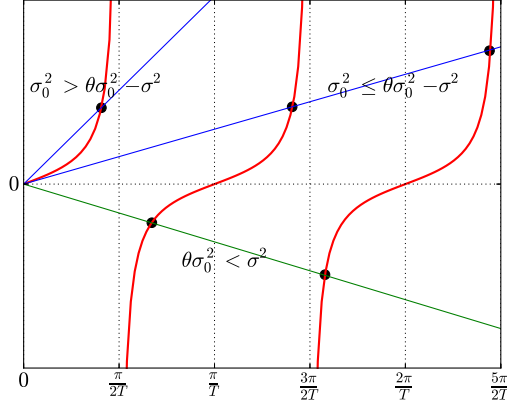


Figure 1: Solutions to Equation (8) in cases (b) and (c).

spanned by  $N$ -stationary quantizers correspond to principal components of  $X$ , in other words, are spanned by eigenvectors of the covariance operator of  $X$ , that is, its Karhunen-Loève eigenfunctions  $(e_n^X)_{n \geq 1}$ .

To perform optimal quantization, the Karhunen-Loève expansion is first truncated at a fixed order  $m$  and then the  $\mathbb{R}^m$ -valued Gaussian vector constituted of the  $m$  first coordinates of the process on its Karhunen-Loève decomposition is quantized. We have to determine the optimal rank of truncation  $d^X(N)$  (the quantization dimension) and the optimal  $d^X(N)$ -dimensional quantizer of the first coordinates,  $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}(0, \lambda_j^X)$ . The minimal quadratic distortion  $\mathcal{E}_N(X)$  writes

$$\mathcal{E}_N(X)^2 = \sum_{j \geq m+1} \lambda_j^X + \mathcal{E}_N \left( \bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X) \right)^2.$$

If the eigensystem  $(e_n^X, \lambda_n^X)_{1 \leq n \leq d^X(N)}$  is known, we just need to perform the finite-dimensional quantization of  $\bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X)$ . Various algorithms have been devised to deal with this problem, among others, Lloyd's algorithm [6] and the Competitive Learning Vector Quantization (CLVQ). A review of these methods is available in [8]. In Figures 2 and 3, we show optimal quantizers of the Ornstein-Uhlenbeck bridge for different initial variances, mean-reversion parameters, volatilities and maturities.

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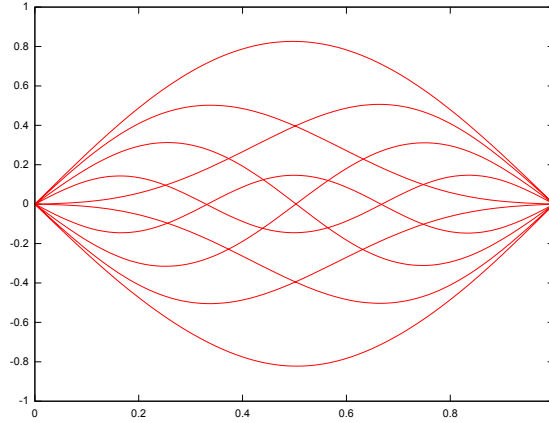


Figure 2: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters  $T = 1$ ,  $\theta = 1$ ,  $\sigma^2 = 1$ ,  $\sigma_0^2 = 0$ ,  $x_0 = \mu = X_T = 0$  and  $N = 10$ .

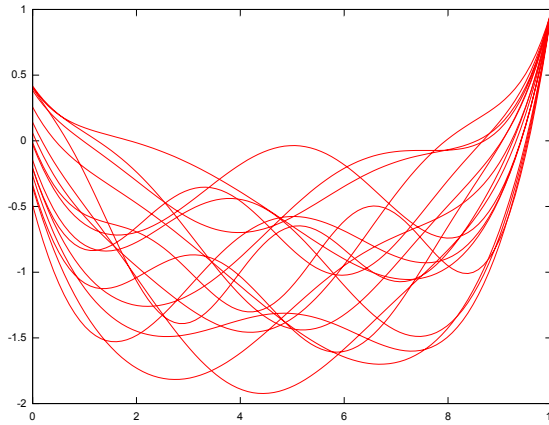


Figure 3: Optimal quantization of the Ornstein-Uhlenbeck bridge with parameters  $T = 10$ ,  $\theta = 1$ ,  $\sigma^2 = 1$ ,  $\sigma_0^2 = 1/2$ ,  $x_0 = 0$ ,  $\mu = -1$ ,  $X_T = 1$  and  $N = 16$ .

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