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Bivariate copulas defined from matrices

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Abstract

We propose a semiparametric family of copulas based on a set of orthonormal functions and a matrix. This new copula permits to reach values of Spearman's Rho arbitrarily close to one without introducing a singular component. Moreover, it encompasses several extensions of FGM copulas as well as copulas based on partition of unity such as Bernstein or checkerboard copulas. Finally, it is also shown that projection of arbitrary densities of copulas onto tensor product bases can enter our framework.

Keywords: Copulas, Semiparametric family, Coefficients of dependence.

AMS classifications: 62H05, 62H20.

1 Introduction

A bivariate copula defined on the unit square $[0, 1]^2$ is a bivariate cumulative distribution function (cdf) with univariate uniform margins. Sklar's Theorem [32] states that any bivariate distribution with cdf H and marginal cdf F and G can be written $H(x, y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions. One of the most popular parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family [8, 12, 22] defined when $\theta \in [-1, 1]$ by

$$C(u, v) = uv + \theta u(1 - u)v(1 - v). \quad (1)$$

A well-known limitation to this family is that it does not allow the modeling of large dependences since the associated Spearman's Rho is limited to $[-1/3, 1/3]$. A possible extension of the FGM family is to consider the semi-parametric family of symmetric copulas defined by

$$C(u, v) = uv + \theta \varphi(u)\varphi(v), \quad (2)$$

with $\theta \in [-1, 1]$. It was first introduced in [26], and extensively studied in [1, 2]. In particular, it can be shown that, for a properly chosen function φ , the range of Spearman's Rho is extended to $[-3/4, 3/4]$. In [3] an extension of (2) is proposed where θ is a univariate function. This modification allows the introduction of a singular component concentrated on the diagonal $v = u$ and extends the range of Spearman's Rho to $[-3/4, 1]$. We also refer to [5, 6] for another yet similar extensions.

Here, a new extension of (2) is proposed where, roughly speaking, the single parameter θ is replaced by a matrix and the function φ is replaced by a set of functions. This new copula permits to reach values of Spearman's Rho arbitrarily close to 1 without singular component. Moreover, it also encompasses copulas based on partition of unity such as Bernstein copula [30] or checkerboard copula [19, 20]. Finally, it is also shown that projection of arbitrary densities of copulas onto tensor product bases can enter our framework. This paper is organized as follows: The family of copula is introduced in Section 2 and some first properties are given. Algebraic properties are established in Section 3, dependence properties are reviewed in Section 4 while important approximation issues are highlighted in Section 5. Some links with existing copulas as well as examples of new copulas are presented in Section 6. Concluding remarks are drawn Section 7. Proofs are postponed to the Appendix.

2 A new family of copulas

Throughout this paper, e_j denotes the j th vector of the canonical basis of \mathbb{R}^p where $j = 1, \dots, p$ and $p \geq 2$. Besides, $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ is the usual scalar product in $L_2([0, 1])$. The associated norm is denoted by $\|f\| = \langle f, f \rangle^{1/2}$. We focus on the modeling of the copula density (denoted by c) rather than on the modeling of the copula (denoted by C) itself:

Definition 1 Let $\phi : [0, 1] \rightarrow \mathbb{R}^p$ be a vector of p orthonormal functions such that $\phi_1(t) = 1$ for all $t \in [0, 1]$. Two sets are defined from ϕ :

$$\begin{aligned} \mathcal{A}_\phi &= \{A \in \mathbb{R}^{p \times p}, Ae_1 = e_1, {}^tAe_1 = e_1, \forall (u, v) \in [0, 1]^2, {}^t\phi(u)A\phi(v) \geq 0\}, \\ \mathcal{C}_\phi &= \{c : [0, 1]^2 \rightarrow \mathbb{R}, c(u, v) = {}^t\phi(u)A\phi(v), A \in \mathcal{A}_\phi\}. \end{aligned}$$

The next result establishes that all the functions of \mathcal{C}_ϕ are densities of copulas.

Proposition 1 \mathcal{C}_ϕ is a non-empty set of copula densities.

It is clear that \mathcal{A}_ϕ is not empty since $A_1 = {}^te_1e_1 \in \mathcal{A}_\phi$. The associated function $c(u, v) = {}^t\phi(u)A_1\phi(v) = 1 \in \mathcal{C}_\phi$ is the density of the independent copula. The remainder of the proof is postponed to the Appendix. Let us also note \mathcal{A}_ϕ is a subset of matrices with eigenvalue 1 associated

to the eigenvector e_1 and with non-negative trace. Indeed, if $A \in \mathcal{A}_\phi$ then

$$\text{tr}(A\phi(x) {}^t\phi(x)) = \text{tr}({}^t\phi(x)A\phi(x)) = {}^t\phi(x)A\phi(x) \geq 0$$

for all $x \in [0, 1]$. Integrating with respect to x yields the result since, by assumption,

$$\int_0^1 \phi(x) {}^t\phi(x) dx = I_p, \quad (3)$$

where I_p is the $p \times p$ identity matrix.

Example 1 If $p = 2$ then $A \in \mathcal{A}_\phi$ implies that $A = \text{diag}\{1, \theta\}$ with $\theta \geq -1$. The associated density of copulas can be written as $c(u, v) = 1 + \theta\phi(u)\phi(v)$ which corresponds to family (2).

This family includes FGM copulas (1) which contains all copulas with both horizontal and vertical quadratic sections [28], the subfamily of symmetric copulas with cubic sections proposed in [23], equation (4.4), and some kernel extensions of FGM copulas introduced in [15, 18]. We refer to [9] for a method to construct admissible functions ϕ .

The following lemma will reveal useful to build densities of copulas in \mathcal{C}_ϕ without the orthogonality assumption on ϕ .

Lemma 1 Let $\psi : [0, 1] \rightarrow \mathbb{R}^p$ be a vector of p functions such that $\psi_1(t) = 1$ for all $t \in [0, 1]$ and $\int_0^1 \psi(x) dx = e_1$. Let Γ be the Gram matrix defined as $\Gamma = \int_0^1 \psi(x) {}^t\psi(x) dx$ and $B \in \mathcal{A}_\psi$. Then, $A := \Gamma^{1/2} B \Gamma^{1/2} \in \mathcal{A}_\phi$ where $\phi := \Gamma^{-1/2} \psi$ fulfills the conditions of Definition 1 and ${}^t\phi(u)A\phi(v) = {}^t\psi(u)B\psi(v)$ for all $(u, v) \in [0, 1]^2$.

See Subsection 8.2 of the Appendix for a proof. A direct application of this lemma yields:

Example 2 The parametric family of copulas with cubic sections proposed in [23], Theorem 4.1 and given by

$$C(u, v) = uv + uv(1-u)(1-v)[A_1v(1-u) + A_2(1-v)(1-u) + B_1uv + B_2u(1-v)]$$

can be written in our formalism with Lemma 1. Here, $p = 3$, $\psi_1(t) = 1$, $\psi_2(t) = 1 - 4t + 3t^3$,

$$\psi_3(t) = 2t - 3t^2, \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/15 & 1/5 \\ 0 & 1/5 & 4/5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2 & A_1 \\ 0 & B_2 & B_1 \end{pmatrix}.$$

More generally, iterated FGM families [17, 14, 21] where

$$C(u, v) = uv + \sum_{j=1}^p \theta_j (uv)^{\alpha_j} ((1-u)(1-v))^{\beta_j},$$

and $\{\alpha_j, \beta_j\} = \{[j/2] + 1, [(j+1)/2]\}$ can be shown to be particular cases of our family thanks to Lemma 1. More examples are given in Section 6.

Let T_ϕ the mapping defined for any copula density $c \in L_2([0, 1]^2)$ by

$$T_\phi(c) = \int_0^1 \int_0^1 c(x, y) \phi(x) {}^t\phi(y) dx dy \in \mathbb{R}^{p \times p}. \quad (4)$$

The mapping T_ϕ permits to compute the matrix associated to any copula density $c \in \mathcal{C}_\phi$:

Proposition 2 *Each copula density $c \in \mathcal{C}_\phi$ is defined by an unique matrix A which is given by*

$$A = T_\phi(c) = \mathbb{E}_c(\phi(U) {}^t\phi(V)) = \text{cov}(\phi(U), \phi(V)) + e_1 {}^t e_1,$$

where (U, V) is a random pair with density c .

3 Algebraic properties

Let \times denote the matrix product and let \star denote the product of copulas introduced in [4] and defined in terms of densities as

$$c_A \star c_B(u, v) := \int_0^1 c_A(u, s) c_B(s, v) ds,$$

for all $(u, v) \in [0, 1]^2$. The following stability properties can be established (see Appendix for a proof).

Proposition 3 *\mathcal{A}_ϕ is a convex set and $(\mathcal{A}_\phi, \times)$ is a semi-group. If, moreover, $I_p \in \mathcal{A}_\phi$ then $(\mathcal{A}_\phi, \times)$ is a monoid.*

Let us also consider the bivariate function

$$q(u, v) = {}^t\phi(u)\phi(v)$$

defined for $(u, v) \in [0, 1]^2$. The following result is the analogous of Proposition 3 for \mathcal{C}_ϕ .

Proposition 4 *\mathcal{C}_ϕ is a closed convex set and $(\mathcal{C}_\phi, \times)$ is a semi-group. If, moreover, $q \in \mathcal{C}_\phi$ then $(\mathcal{C}_\phi, \times)$ is a monoid.*

In view of Propositions 2 – 4, it appears that $T_\phi(c_A \star c_B) = T_\phi(c_A)T_\phi(c_B)$ for all $(c_A, c_B) \in \mathcal{C}_\phi^2$ and thus:

Proposition 5 *T_ϕ is an isomorphism between $(\mathcal{A}_\phi, \times)$ and $(\mathcal{C}_\phi, \star)$.*

To summarize, it appears that \mathcal{A}_ϕ is stable with respect to matrix multiplication. Moreover, multiplying the matrices is equivalent to “multiplying” the copulas using the \star product.

Finally, the next lemma shows that it is possible to aggregate copulas of \mathcal{C}_ϕ with different number of orthogonal functions through the use of Cesàro summations.

Lemma 2 Let $c_p \in \mathcal{C}_\phi$ with associated matrix A and consider the density of copula defined for all $(u, v) \in [0, 1]^2$ and $q \geq 1$ by

$$\bar{c}_q(u, v) := \frac{1}{q} \sum_{p=1}^q c_p(u, v).$$

Then, $\bar{c}_q \in \mathcal{C}_\phi$ with associated matrix B defined by $B_{ij} = (q + 1 - \max(i, j))A_{i,j}/q$ for all $(i, j) \in \{1, \dots, q\}^2$.

An application of this lemma is illustrated in Paragraph 6.2.1.

4 Dependence properties

Several measures of association between the components of a random pair can be considered: the normalized volume [31], Kendall's Tau [25], paragraph 5.1.1, Gini's gamma [11], Blomqwist's medial correlation coefficient [25], paragraph 5.1.4, Spearman's footrule [11], and Spearman's Rho [25], paragraph 5.1.2. All these measures are invariant to strictly increasing functions. Kendall's Tau and Spearman's Rho can be interpreted as probabilities of concordance minus probabilities of discordance of two random pairs. Let us first focus on the Spearman's Rho. It can be written only in terms of the copula C :

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3. \quad (5)$$

Note that ρ coincides with the correlation coefficient between the uniform marginal distributions. In the case of the copula introduced in Definition 1, it can be expressed thanks to the function ϕ and the matrix $A \in \mathcal{A}_\phi$ associated to its density.

Proposition 6 Let (U, V) be a random pair with density of copula $c \in \mathcal{C}_\phi$ associated to the matrix $A \in \mathcal{A}_\phi$. The Spearman's Rho is given by $\rho = 12 \int_0^1 x \phi(x) dx - 3$ where $\mu = \int_0^1 x \phi(x) dx$.

Similarly to the Spearman's Rho, the Kendall's Tau can be written only in terms of the copula C :

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1, \quad (6)$$

and in the framework of Definition 1, it can be expressed thanks to the function ϕ and the matrix $A \in \mathcal{A}_\phi$ associated to its density.

Proposition 7 Let (U, V) be a random pair with density of copula $c \in \mathcal{C}_\phi$ associated to the matrix $A \in \mathcal{A}_\phi$. The Kendall's Tau is given by $\tau = 1 - 4 \text{tr}(A \Theta A \Theta)$ where Θ is the $p \times p$ matrix defined by $\Theta = \int_0^1 \Psi(u) \phi(u) du$ and with $\Psi(u) = \int_0^u \phi(t) dt$.

Let us note that Propositions 6 and 7 extend the results of [7], Theorem 25 established in the case of copulas based on partition of unity. Such copulas were introduced in [19, 20]. It is shown in Section 6 that they are particular cases of the family considered in Definition 1. Besides, Blomqwist's medial correlation coefficient, Gini's gamma and Spearman's footrule can also be rewritten in

terms of the copula C (see for instance [24]). All these coefficients thus benefit from closed form expressions similar to these of Propositions 6, 7 and we do not enter into details there.

The upper tail dependence between two random variables X and Y with respective cdf F and G can be quantified by the upper tail dependence coefficient [16], paragraph 2.1.10, defined as:

$$\lambda = \lim_{t \rightarrow 1^-} \mathbb{P}(F(X) > t | G(Y) > t).$$

Again, this coefficient can be written only in terms of the copula (see [25], Theorem 5.4.2):

$$\lambda = \lim_{u \rightarrow 1^-} \frac{\bar{C}(u, u)}{1 - u}, \quad (7)$$

where \bar{C} is the survival copula, *i.e.* $\bar{C}(u, v) = 1 - u - v + C(u, v)$. In the family \mathcal{C}_ϕ , the upper tail dependence is not possible:

Proposition 8 *Let (U, V) be a random pair with density of copula $c \in \mathcal{C}_\phi$. Then, $\lambda = 0$.*

5 Approximation properties

Let c be a density of copula in $L_2([0, 1]^2)$. Recall that the mapping T_ϕ associates a $p \times p$ matrix to c via (4) and introduce:

$$P_\phi(c)(u, v) = {}^t\phi(u)T_\phi(c)\phi(v), \quad (u, v) \in [0, 1]^2. \quad (8)$$

The next lemma gives a necessary and sufficient condition for $P_\phi(c) \in \mathcal{C}_\phi$:

Lemma 3

(i) *Let $c \in L_2([0, 1]^2)$ be an arbitrary density of copula. If $I_p \in \mathcal{A}_\phi$ then $P_\phi(c) \in \mathcal{C}_\phi$ and $P_\phi(c) = q \star c \star q$.*

(ii) *Conversely, if $P_\phi(c) \in \mathcal{C}_\phi$ for all $c \in L_2([0, 1]^2)$ then $I_p \in \mathcal{A}_\phi$.*

Let $(c_1, c_2) \in L_2^2([0, 1]^2)$ and let us consider the scalar product defined as

$$\langle c_1, c_2 \rangle = \int_0^1 (c_1 \star c_2)(u, u) du = \int_0^1 \int_0^1 c_1(u, v) c_2(v, u) dudv.$$

Let us remark that, for symmetric copulas, the above scalar product reduces to the L_2 - scalar product

$$\langle c_1, c_2 \rangle = \int_0^1 \int_0^1 c_1(u, v) c_2(u, v) dudv.$$

In the case of densities of copulas in \mathcal{C}_ϕ , the scalar product can be computed using the associated matrices:

Lemma 4 Let $c_2 \in L_2([0, 1]^2)$.

- (i) If $c_1 \in \mathcal{C}_\phi$ with associated matrix A , then $\prec c_1, c_2 \succ = \text{tr}(AT_\phi(c_2))$.
- (ii) If, moreover, $c_2 \in \mathcal{C}_\phi$ with associated matrix B , then $\prec c_1, c_2 \succ = \text{tr}(AB)$.

As a consequence of the above lemmas, we have:

Proposition 9 P_ϕ is an orthogonal projection on \mathcal{C}_ϕ if and only if $I_p \in \mathcal{A}_\phi$.

This result is now illustrated on the FGM family where explicit computations can be done:

Example 3 It is well known that the FGM family is a particular case of Example 1 with $\phi(x) = \sqrt{3}(1 - 2x)$, $A = \text{diag}\{1, \theta\}$ and where $|\theta| \leq 1/3$. Here, $I_p \notin \mathcal{A}_\phi$ and thus the projection $P_\phi(c)$ of any density of copula c on the FGM family is not itself a density of copula in the general case. However, let us remark that $T_\phi(c) = \text{diag}\{1, \tilde{\theta}\}$ and thus $P_\phi(c)(u, v) = 1 + 3\tilde{\theta}(1 - 2u)(1 - 2v)$ where

$$\tilde{\theta} = \int_0^1 \int_0^1 c(x, y)\phi(x)\phi(y)dx dy = 3 \int_0^1 \int_0^1 c(x, y)(1 - 2x)(1 - 2y)dx dy = \rho_c,$$

the Spearman's Rho associated to c . We thus have the following result:

- If $|\rho_c| \leq 1/3$ then $P_\phi(c)$ is a FGM copula and $\rho_{P_\phi(c)} = \rho_c$,
- If $|\rho_c| > 1/3$ then $P_\phi(c)$ is not a copula.

It appears from this example that it is possible to associate to any copula a FGM copula with the same Spearman's Rho ρ_c provided $|\rho_c| \leq 1/3$.

Suppose now that $\{\phi_i\}_{i \geq 1}$ is an orthonormal basis of $L_2([0, 1])$. Then, $\{\phi_i \otimes \phi_j\}_{i, j \geq 1}$ is an orthonormal basis of $L_2([0, 1]^2)$ where \otimes denotes the tensor product, i.e. $(f \otimes g)(u, v) := f(u)g(v)$ for all $(u, v) \in [0, 1]^2$. As a consequence, the L_2 -projection of any $c \in L_2([0, 1]^2)$ on $\{\phi_i \otimes \phi_j\}_{1 \leq i, j \leq p}$ is given by

$$\tilde{c}_p(u, v) = \sum_{i=1}^p \sum_{j=1}^p a_{i,j} \phi_i(u) \phi_j(v) = {}^t \phi(u) A \phi(v)$$

where $A = (a_{i,j})_{1 \leq i, j \leq p}$ with

$$a_{i,j} = \int_0^1 \int_0^1 c(x, y) \phi_i(x) \phi_j(y) dx dy = (T_\phi(c))_{i,j}.$$

This yields $A = T_\phi(c)$ and $\tilde{c}_p = P_\phi(c)$. In view of Lemma 3(i), it follows that $\tilde{c}_p \in \mathcal{C}_\phi$ if $I_p \in \mathcal{A}_\phi$. As a conclusion, the L_2 -projection of any density of copula in $L_2([0, 1]^2)$ on a tensor product basis can be written in our formalism and the following result holds:

Theorem 1 Let $\{\phi_i\}_{1 \leq i \leq p}$ be an orthonormal family of $L_2([0, 1])$.

- (i) The projection P_ϕ on \mathcal{C}_ϕ introduced in (8) coincides with the L_2 - projection on $\{\phi_i \otimes \phi_j\}_{1 \leq i, j \leq p}$.
- (ii) Moreover, these projections give rise to densities of copula in \mathcal{C}_ϕ if and only if $I_p \in \mathcal{A}_\phi$.

Classical approximation properties in $L_2([0, 1]^2)$ yield:

Corollary 1 Suppose the assumptions of Theorem 1 hold and let $c \in L_2([0, 1]^2)$.

Then, $\|c - P_\phi(c)\| \rightarrow 0$ and $\rho(P_\phi(c)) \rightarrow \rho(c)$ as $p \rightarrow \infty$, where $\rho(P_\phi(c))$ and $\rho(c)$ denote respectively the Spearman's Rho associated to $P_\phi(c)$ and c .

6 Examples

In paragraph 6.1, some examples of copulas found in the literature are shown to enter our model. New families are exhibited in paragraph 6.2.

6.1 Copulas based on partition of unity

Recall that a collection of functions $\xi = {}^t(\xi_1, \dots, \xi_p)$ is called a partition of unity [19, 20] if

- $\xi_i \geq 0$ for all $i = 1, \dots, p$,
- $\int_0^1 \xi_i(x) dx = 1/p$ for all $i = 1, \dots, p$ and
- $\sum_{i=1}^p \xi_i = 1$.

It can be established that copulas based on partition of unity are particular of the proposed family:

Proposition 10 Let $\xi = {}^t(\xi_1, \dots, \xi_p)$ be a partition of unity, and let M be a $p \times p$ doubly stochastic matrix. Then, the function defined for all $(u, v) \in [0, 1]^2$ by $c(u, v) = p {}^t\xi(u) M \xi(v)$ is a density of copula and $c \in \mathcal{C}_\phi$. Moreover, $\phi = (H \Gamma_\xi {}^t H)^{-1/2} H \xi$ where $s = e_1 + \dots + e_p$, $H = I_p + e_1 {}^t s - s {}^t e_1$, and Γ_ξ is the Gram matrix associated to ξ .

As an illustration, we have:

Example 4 The Bernstein copula [30] is obtained by choosing

$$\xi_i(x) = C_{p-1}^{i-1} x^{i-1} (1-x)^{p-i}$$

and

$$M_{ij} = p \left\{ C \left(\frac{i}{p}, \frac{j}{p} \right) - C \left(\frac{i-1}{p}, \frac{j}{p} \right) - C \left(\frac{i}{p}, \frac{j-1}{p} \right) + C \left(\frac{i-1}{p}, \frac{j-1}{p} \right) \right\}, \quad (9)$$

where C is an arbitrary copula.

In the case of Bernstein copula, the basis $\phi = (H\Gamma_\xi {}^tH)^{-1/2}H\xi$ cannot be simplified. However, in the particular case where $\{\xi_1, \dots, \xi_p\}$ is orthogonal and $\int_0^1 \xi_i^2(t)dt = \beta^2$ for all $i = 1, \dots, p$, we have $\Gamma_\xi = \beta^2 I_p$ and therefore

$$H\Gamma_\xi {}^tH = \beta^2 \begin{pmatrix} p & 0 & \dots & \dots & \dots & 0 \\ 0 & 2 & 1 & \dots & \dots & 1 \\ \vdots & 1 & 2 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 2 & 1 \\ 0 & 1 & \dots & \dots & 1 & 2 \end{pmatrix}.$$

The inverse square-root of this matrix benefits from a closed-form expression, and we thus have a simple linear relation between the two families of functions: $\phi = \Omega\xi$ with

$$\Omega = \frac{1}{\beta} \begin{pmatrix} p^{-1/2} & \dots & \dots & \dots & \dots & p^{-1/2} \\ -p^{-1/2} & \gamma & (\gamma-1) & \dots & \dots & (\gamma-1) \\ \vdots & (\gamma-1) & \gamma & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \gamma & (\gamma-1) \\ -p^{-1/2} & (\gamma-1) & \dots & \dots & (\gamma-1) & \gamma \end{pmatrix}, \quad (10)$$

where $\gamma = (p-2+p^{-1/2})/(p-1)$. Explicit computations can be achieved in case of the checkerboard copula:

Example 5 *The checkerboard copula [19, 20] is obtained by choosing $\xi_i(x) = \mathbb{I}\{x \in I_i\}$ where $\{I_i, i = 1, \dots, p\}$ is the equidistant partition of $[0, 1]$ into p intervals*

$$I_i = \left[\frac{i-1}{p}, \frac{i}{p} \right]$$

and M_{ij} as in (9). Besides, $\Gamma_\xi = (1/p)I_p$ and therefore $\phi = \Omega\xi$ where Ω is given by (10) with $\beta = p^{-1/2}$.

6.2 Orthogonal bases

6.2.1 Trigonometric basis

The trigonometric family is defined by $\phi_0(x) = 1$, $\phi_{2j-1}(x) = \sqrt{2} \sin(2\pi jx)$ and $\phi_{2j}(x) = \sqrt{2} \cos(2\pi jx)$ for all $j \geq 1$ and $x \in [0, 1]$. It is orthonormal with respect to the usual scalar product on $L_2([0, 1])$.

Let $\theta \geq 0$ and consider the $(2p+1) \times (2p+1)$ matrix $A = \text{diag}\{1, \theta, \theta, \dots, \theta\}$. One has

$$c_{p,\theta}(u, v) := {}^t\phi(u)A\phi(v) = 1 - \theta + \theta D_p(u - v)$$

where D_p is the Dirichlet kernel given by

$$D_p(t) = \frac{\sin((2p+1)\pi t)}{\sin(\pi t)}.$$

It is then clear that $c_{p,\theta} \in \mathcal{C}_\phi$ if $\theta \leq 1/(1 - D_p^-)$ where $D_p^- := \min_t D_p(t)$. Since $D_p^- < 0$, it follows that θ is upper bounded by $1/(1 - D_p^-) < 1$ and thus $I_p \notin \mathcal{A}_\phi$ when ϕ is the trigonometric family. Besides, Proposition 6 shows that the associated Spearman's rho is

$$\rho_{p,\theta} = \frac{6\theta}{\pi^2} \sum_{j=1}^p \frac{1}{j^2}.$$

Numerical computations show that the maximum value is obtained for $p = 2$ for which $D_2^- = -1$ and $\rho_{2,1/2} = 15/(4\pi^2)$. This bound can be increased by introducing

$$\bar{c}_{q,\theta}(u, v) := \frac{1}{q} \sum_{p=0}^{q-1} c_{p,\theta}(u, v) = 1 - \theta + \theta F_q(u - v)$$

where F_q is the Fejér kernel [13] defined as

$$F_q(x) = \frac{1}{q} \sum_{p=0}^{q-1} D_p(x) = \frac{1}{q} \left(\frac{\sin(qx/2)}{\sin(x/2)} \right)^2.$$

Since this kernel is positive, it is readily seen that $\bar{c}_{q,\theta}$ is a density of copula for all $\theta \in [0, 1]$. Besides, Lemma 2 entails that $\bar{c}_{p,\theta} \in \mathcal{C}_\phi$ for all $\theta \in [0, 1]$. The associated Spearman's rho is

$$\bar{\rho}_{q,\theta} = \frac{1}{q} \sum_{p=0}^{q-1} \rho_{p,\theta} = \frac{6\theta}{\pi^2} \frac{1}{q} \sum_{p=0}^{q-1} \sum_{j=1}^p \frac{1}{j^2} = \frac{6\theta}{\pi^2} \left(\sum_{j=1}^{q-1} \frac{1}{j^2} - \frac{1}{q} \sum_{j=1}^{q-1} \frac{1}{j} \right).$$

Let us remark that $\bar{\rho}_{q,\theta} \rightarrow \theta$ as $q \rightarrow \infty$ and thus, arbitrary large dependences can be modeled.

6.2.2 The Haar basis

For each positive integer i , let us denote by J_i the interval $J_i = [\frac{p_i}{2^{q_i-1}}, \frac{p_i+1}{2^{q_i-1}})$ where p_i and q_i are the integers uniquely determined by $i = 2^{q_i-1} + p_i$ and $0 \leq p_i < 2^{q_i-1}$. The Haar basis [29] is defined by

$$\phi_0(t) = \mathbb{I}\{t \in [0, 1]\}, \quad \phi_i(t) = 2^{\frac{q_i-1}{2}} (\mathbb{I}\{t \in J_{2i}\} - \mathbb{I}\{t \in J_{2i+1}\}),$$

for $i = 1, \dots, p$. In the following, it is assumed that p is a power of 2. Let $\theta \geq 0$ and consider the $p \times p$ matrix $A = \text{diag}\{1, \theta, \theta, \dots, \theta\}$. One has

$$c_{p,\theta}(u, v) := {}^t\phi(u)A\phi(v) = 1 - \theta + \theta K_p(u, v) \tag{11}$$

where K_p is the Dirichlet kernel associated to the Haar basis

$$K_p(u, v) = p \sum_{i=1}^p \mathbb{I}\{(u, v) \in I_i^2\},$$

and recall that $\{I_i, i = 1, \dots, p\}$ is the equidistant partition of $[0, 1]$ into p intervals. It is thus clear that K_p can be rewritten as

$$K_p(u, v) = p \sum_{i=1}^p \sum_{j=1}^p M_{i,j} \xi_i(u) \xi_j(v)$$

where $M = I_p$ the $p \times p$ identity matrix and $\xi_i(u) = \mathbb{I}\{u \in I_i\}$. It appears that K_p is the density of a copula based on partition of unity. In view of Proposition 10, it follows that $K_p \in \mathcal{C}_\phi$. Finally, the Haar copula (11) can be interpreted as a linear mixture of K_p and the independent copula, leading to $c_{p,\theta} \in \mathcal{C}_\phi$ for all $\theta \in [0, 1]$. Thus, $I_p \in \mathcal{A}_\phi$ when ϕ is the Haar family. Straightforward calculations show that the associated Spearman's rho is given by

$$\rho_p(\theta) = \theta \left(1 - \frac{1}{p^2} \right).$$

Let us remark that $\rho_p(\theta) \rightarrow \theta$ as $p \rightarrow \infty$ and thus, arbitrary large dependences can be modeled.

7 Conclusion and further work

We proposed a new family of copulas defined from a matrix and a family of orthogonal functions. High dependences can be modeled without introducing singular components. It has also been shown that this family can be used for approximating any density of copula. As a consequence, it appears as a good tool for modeling bivariate data. To this aim, consider $(U_1, V_1), \dots, (U_n, V_n)$ independent copies of a random pair (U, V) from a density $c \in \mathcal{C}_\phi$ associated to a matrix A . Assume that the function ϕ is known. Then, estimating c reduces to estimating A . Proposition 2 provides two interpretations of A in terms of a covariance matrices and thus two possible estimators:

$$\begin{aligned} \hat{A}_{1,n} &= \frac{1}{n} \sum_{i=1}^n \phi(U_i) {}^t\phi(V_i), \\ \hat{A}_{2,n} &= \frac{1}{n} \sum_{i=1}^n (\phi(U_i) - e_1) {}^t(\phi(V_i) - e_1) + e_1 {}^t e_1. \end{aligned}$$

In both cases, the corresponding estimated density is given by $\hat{c}_{j,n}(u, v) = {}^t\phi(u) \hat{A}_{j,n} \phi(v)$, $j \in \{1, 2\}$ and can be simplified as

$$\begin{aligned} \hat{c}_{1,n}(u, v) &= \frac{1}{n} \sum_{i=1}^n q(u, U_i) q(v, V_i), \\ \hat{c}_{2,n}(u, v) &= 1 + \frac{1}{n} \sum_{i=1}^n (q(u, U_i) - 1)(q(v, V_i) - 1). \end{aligned}$$

From the theoretical point of view, both estimators are unbiased and asymptotically Gaussian. Our future work will consist in investigating their finite sample behavior compared to rank-based methods [10].

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8 Appendix

8.1 Proofs of main results

Proof of Proposition 1. Let $c \in \mathcal{C}_\phi$. Clearly, $c(u, v) \geq 0$ for all $(u, v) \in [0, 1]^2$ from the definition of \mathcal{A}_ϕ . It only remains to prove that the margins of c are standard uniform distributions. To this end, let us remark that

$$\int_0^1 \phi(v)dv = e_1, \quad (12)$$

since for all $j = 1, \dots, p$, we have

$$\int_0^1 \phi_j(v)dv = \langle e_1, e_j \rangle = \delta_{1j},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. As a consequence,

$$\int_0^1 c(u, v)dv = {}^t\phi(u)A \int_0^1 \phi(v)dv = {}^t\phi(u)Ae_1 = {}^t\phi(u)e_1 = \phi_1(u) = 1.$$

The proof of $\int_0^1 c(u, v)du = 1$ is similar. □

Proof of Proposition 2. Let $c \in \mathcal{C}_\phi$ such that $c(u, v) = {}^t\phi(u)A\phi(v) = {}^t\phi(u)B\phi(v)$ for all $(u, v) \in [0, 1]^2$ and for some $(A, B) \in \mathcal{A}_\phi^2$. It follows that

$$c(u, v)\phi(u) {}^t\phi(v) = \phi(u) {}^t\phi(u)A\phi(v) {}^t\phi(v) = \phi(u) {}^t\phi(u)B\phi(v) {}^t\phi(v),$$

and integrating with respect to u and v yields $T_\phi(c) = A = B$ in view of (3). Remark also that, if moreover, $c \in \mathcal{C}_\phi$ with associated matrix $A \in \mathcal{A}_\phi$ then

$$\mathbb{E}_c(\phi(U)) = \int_0^1 \int_0^1 \phi(u) {}^t\phi(u)A\phi(v)dudv = A \int_0^1 \phi(v)dv = Ae_1 = e_1$$

in view of (3) and (12). As a consequence, the matrix A can also be interpreted as $A = \text{cov}(\phi(U), \phi(V)) + e_1 {}^t e_1$. □

Proof of Proposition 3. It is clear that \mathcal{A}_ϕ is convex. Let us prove that $(\mathcal{A}_\phi, \times)$ is a semi-group. Since the product \times is associative, it only remains to establish that $(A, B) \in \mathcal{A}_\phi^2$ entails $A \times B \in \mathcal{A}_\phi$. First, (3) entails

$$\begin{aligned} {}^t\phi(u)AB\phi(v) &= {}^t\phi(u)A \left\{ \int_0^1 \phi(y) {}^t\phi(y) dy \right\} B\phi(v) \\ &= \int_0^1 \{ {}^t\phi(u)A\phi(y) \} \{ {}^t\phi(y)B\phi(v) \} dy \geq 0, \end{aligned}$$

and second it is easily seen that $ABe_1 = e_1$ and ${}^t(AB)e_1 = e_1$. Finally, $I_p \in \mathcal{A}_\phi$ and $(\mathcal{A}_\phi, \times)$ is a semi-group both imply that $(\mathcal{A}_\phi, \times)$ is a monoid. \square

Proof of Proposition 4. It is clear that \mathcal{C}_ϕ is convex. Let us prove that $(\mathcal{C}_\phi, \star)$ is a semi-group. Since the product \star is associative (see [4], Theorem 2.4), it only remains to establish that $(c_A, c_B) \in \mathcal{C}_\phi^2$ entails $c_A \star c_B \in \mathcal{C}_\phi$:

$$\begin{aligned} c_A \star c_B(u, v) &= \int_0^1 {}^t\phi(u)A\phi(s) {}^t\phi(s)B\phi(v) ds, \\ &= {}^t\phi(u)A \left\{ \int_0^1 \phi(s) {}^t\phi(s) ds \right\} B\phi(v) = {}^t\phi(u)AB\phi(v). \end{aligned}$$

This proves that $(c_A, c_B) \in \mathcal{C}_\phi^2$ and is associated to the matrix $AB \in \mathcal{C}_\phi$, see Proposition 3. Finally, $q \in \mathcal{C}_\phi$ and $(\mathcal{C}_\phi, \star)$ is a semi-group both imply that $(\mathcal{C}_\phi, \star)$ is a monoid. \square

Proof of Proposition 6. Let C denote the copula associated to the density $c \in \mathcal{C}_\phi$. Introducing $\Psi(u) = \int_0^u \phi(t) dt$ and $\gamma = \int_0^1 \Psi(u) du$, we have $C(u, v) = {}^t\Psi(u)A\Psi(v)$ and (5) leads to $\rho = 12 {}^t\gamma A\gamma - 3$. A partial integration yields $\gamma = e_1 - \mu$ and

$${}^t(e_1 - \mu)A(e_1 - \mu) = {}^te_1 A e_1 - {}^t\mu(A + {}^tA)e_1 + {}^t\mu A \mu = 1 - 2 {}^t\mu_1 + {}^t\mu A \mu$$

with $\mu_1 = \int_0^1 t dt = 1/2$ and the result follows. \square

Proof of Proposition 7. Let C denote the copula associated to the density $c \in \mathcal{C}_\phi$. Standard algebra gives

$$\begin{aligned}
\int_0^1 \int_0^1 C(u, v) dC(u, v) &= \int_0^1 \int_0^1 {}^t\Psi(u) A \Psi(v) {}^t\phi(u) A \phi(v) dudv \\
&= \int_0^1 \int_0^1 {}^t\Psi(v) {}^tA \Psi(u) {}^t\phi(u) A \phi(v) dudv \\
&= \int_0^1 {}^t\Psi(v) {}^tA \Theta A \phi(v) dv \\
&= \int_0^1 \text{tr} ({}^t\Psi(v) {}^tA \Theta A \phi(v)) dv \\
&= \int_0^1 \text{tr} ({}^tA \Theta A \phi(v) {}^t\Psi(v)) dv \\
&= \text{tr} ({}^tA \Theta A {}^t\Theta).
\end{aligned}$$

Besides, a partial integration shows that ${}^t\Theta = e_1 {}^t e_1 - \Theta$. It follows that

$$\begin{aligned}
\text{tr} ({}^tA \Theta A {}^t\Theta) &= \text{tr} ({}^tA \Theta A e_1 {}^t e_1) - \text{tr} ({}^tA \Theta A \Theta) \\
&= \text{tr} ({}^tA \Theta e_1 {}^t e_1) - \text{tr} ({}^tA \Theta A \Theta) \\
&= \text{tr} (\Theta e_1 {}^t e_1 {}^t A) - \text{tr} ({}^tA \Theta A \Theta) \\
&= \text{tr} (\Theta e_1 {}^t e_1) - \text{tr} ({}^tA \Theta A \Theta) \\
&= \Theta_{1,1} - \text{tr} ({}^tA \Theta A \Theta) \\
&= 1/2 - \text{tr} ({}^tA \Theta A \Theta).
\end{aligned}$$

The conclusion follows from (6). □

Proof of Proposition 8. Let C denote the copula associated to the density $c \in \mathcal{C}_\phi$. Recalling that $\Psi(u) = \int_0^u \phi(t) dt$, we have $C(u, v) = {}^t\Psi(u) A \Psi(v)$. In view of (12), $\Psi(1) = e_1$ leading to $C(1, 1) = 1$ and thus (7) can be rewritten as

$$\lambda = 2 - \lim_{u \rightarrow 1^-} \frac{C(u, u) - C(1, 1)}{u - 1} = 2 - \left. \frac{\partial C(u, u)}{\partial u} \right|_{u=1}.$$

Straightforward calculations show that

$$\left. \frac{\partial C(u, u)}{\partial u} \right|_{u=1} = {}^t\phi(1)(A + {}^tA)\Psi(1) = {}^t\phi(1)(A + {}^tA)e_1 = 2 {}^t\phi(1)e_1 = 2,$$

and consequently $\lambda = 0$. □

Proof of Proposition 9. Let us suppose first that $I_p \in \mathcal{A}_\phi$. Let us first remark that, from Proposition 4, \mathcal{C}_ϕ is a convex and closed subset of $L_2([0, 1]^2)$. Second, it is clear from Lemma 3(i)

that P_ϕ is idempotent:

$$P_\phi(P_\phi(c)) = q \star P(c) \star q = q \star q \star c \star q \star q = q \star c \star q = P(c),$$

for all $c \in L_2([0, 1]^2)$ since $q \star q = q$ from (3). Third, let $c \in L_2([0, 1]^2)$ and $s \in \mathcal{C}_\phi$ with associated matrix A . Our aim is to prove that $\langle c - P_\phi(c), s \rangle = 0$. In view of Lemma 4,

$$\langle c - P_\phi(c), s \rangle = \text{tr}(T_\phi(c)A) - \text{tr}(T_\phi(c)A) = 0$$

and the direct part of the result is proved. Conversely, if P_ϕ is a projection on \mathcal{C}_ϕ then, necessarily, $P_\phi(q) \in \mathcal{C}_\phi$. Besides, $P_\phi(q) = q$ and thus $q \in \mathcal{C}_\phi$ entailing $I_p \in \mathcal{A}_\phi$. The converse part of the result is proved. \square

Proof of Corollary 1. First, it is clear that $\|c - P_\phi(c)\| \rightarrow 0$ as $p \rightarrow \infty$, since, in view of Theorem 1(i), $\tilde{c}_p := P_\phi(c)$ can be interpreted as a L_2 - projection of c . From (5), it follows that

$$\rho(\tilde{c}_p) - \rho(c) = \int_{[0,1]^4} (\tilde{c}_p(x, y) - c(x, y)) \mathbb{I}\{x \leq u\} \mathbb{I}\{y \leq v\} dx dy dudv$$

and Cauchy-Schwarz inequality yields

$$|\rho(\tilde{c}_p) - \rho(c)| \leq \|\tilde{c}_p - c\| \left(\int_{[0,1]^4} \mathbb{I}\{x \leq u\} \mathbb{I}\{y \leq v\} dx dy dudv \right)^{1/2} = \frac{1}{2} \|\tilde{c}_p - c\|$$

and the conclusion follows. \square

Proof of Proposition 10. Let $\psi = H\xi$ and $B = p {}^t H^{-1} M H^{-1}$. First, it is easily seen that

$$c(u, v) = p {}^t \xi(u) M \xi(v) = {}^t \psi(u) B \psi(v),$$

with

$$\psi_1(t) = \sum_{i=1}^p \xi_i(t) = 1$$

and

$$\int_0^1 \psi(t) dt = H \int_0^1 \xi(t) dt = \frac{1}{p} H s = e_1.$$

Besides, since $\xi_i \geq 0$ for all $i = 1, \dots, p$ and M is a doubly stochastic matrix, it is clear that $c(u, v) \geq 0$ for all $(u, v) \in [0, 1]^2$. Second, standard algebra shows that $H^{-1} e_1 = s/p$ and ${}^t H^{-1} s = e_1$. As a consequence,

$$B e_1 = p {}^t H^{-1} M H^{-1} e_1 = {}^t H^{-1} M s = {}^t H^{-1} s = e_1$$

and similarly ${}^t B e_1 = e_1$ leading to $B \in \mathcal{A}_\psi$. Lemma 1 entails that $A := \Gamma^{1/2} B \Gamma^{1/2} \in \mathcal{A}_\phi$ where $\phi := \Gamma^{-1/2} \psi$ fullfills the conditions of Definition 1 and the density of copula

$$c(u, v) = p {}^t \xi(u) M \xi(v) = {}^t \psi(u) B \psi(v) = {}^t \phi(u) A \phi(v)$$

belongs to \mathcal{C}_ϕ . \square

8.2 Proofs of auxiliary results

Proof of Lemma 1. Let us first remark that $\Gamma e_1 = e_1$ and ${}^t\Gamma e_1 = e_1$. Consequently, there exists a square root $\Gamma^{1/2}$ of Γ such that $\Gamma^{1/2}e_1 = e_1$ and ${}^t\Gamma^{1/2}e_1 = e_1$. It follows that $Ae_1 = e_1$, ${}^tAe_1 = e_1$ and that, for all $(u, v) \in [0, 1]^2$, ${}^t\phi(u)A\phi(v) = {}^t\psi(u)B\psi(v)$, it is thus clear that $A \in \mathcal{A}_\phi$ with $\phi_1(x) = 1$ for all $x \in [0, 1]$ and ϕ is orthonormal. \square

Proof of Lemma 2. Let $c_p \in \mathcal{C}_\phi$ with associated matrix A and consider the density of copula defined for all $(u, v) \in [0, 1]^2$ and $q \geq 1$ by

$$\begin{aligned} \bar{c}_q(u, v) &:= \frac{1}{q} \sum_{p=1}^q c_p(u, v) \\ &= \frac{1}{q} \sum_{p=1}^q \sum_{i=1}^q \sum_{j=1}^q A_{i,j} \phi_i(u) \phi_j(v) \mathbb{I}\{i \leq p\} \mathbb{I}\{j \leq p\} \\ &= \sum_{i=1}^q \sum_{j=1}^q A_{i,j} \left(\frac{1}{q} \sum_{p=1}^q \mathbb{I}\{\max(i, j) \leq p\} \right) \phi_i(u) \phi_j(v) \\ &= \sum_{i=1}^q \sum_{j=1}^q A_{i,j} \left(\frac{q+1 - \max(i, j)}{q} \right) \phi_i(u) \phi_j(v) \\ &=: \sum_{i=1}^q \sum_{j=1}^q B_{i,j} \phi_i(u) \phi_j(v). \end{aligned}$$

It is clear from its definition that \bar{c}_q is a density of copula. Therefore, $\bar{c}_q \in \mathcal{C}_\phi$ and the result is proved. \square

Proof of Lemma 3. (i) Let $c \in L_2([0, 1]^2)$ and suppose $I_p \in \mathcal{A}_\phi$. Then $q \in \mathcal{C}_\phi$ and

$$\begin{aligned} {}^t\phi(u)T_\phi(c)\phi(v) &= {}^t\phi(u) \left\{ \int_0^1 \int_0^1 \phi(x)c(x, y) {}^t\phi(y) dx dy \right\} \phi(v) \\ &= \int_0^1 \int_0^1 q(u, x)c(x, y)q(y, v) dx dy \\ &= (q \star c \star q)(u, v), \end{aligned}$$

which proves that ${}^t\phi(u)T_\phi(c)\phi(v) \geq 0$ for all $(u, v) \in [0, 1]^2$. Moreover,

$$\begin{aligned} T_\phi(c)e_1 &= \int_0^1 \int_0^1 \phi(x)c(x, y) {}^t\{\phi(y)e_1\} dy dx = \int_0^1 \int_0^1 \phi(x)c(x, y) dy dx \\ &= \int_0^1 \phi(x) dx = e_1, \end{aligned}$$

in view of (12) and similarly ${}^tT_\phi(c)e_1 = e_1$. As a conclusion, $T_\phi(c) \in \mathcal{A}_\phi$ and thus $P_\phi(c) \in \mathcal{C}_\phi$.

(ii) Conversely, if $P_\phi(c) \in \mathcal{C}_\phi$ for all $c \in L_2([0, 1]^2)$, we have in particular $P_\phi(q) = q \in \mathcal{C}_\phi$ and thus $I_p \in \mathcal{A}_\phi$. \square

Proof of Lemma 4. (i) Let $c_2 \in L_2([0, 1]^2)$ and let $c_1 \in \mathcal{C}_\phi$ with associated matrix A . By definition,

$$\begin{aligned}
\langle c_1, c_2 \rangle &= \int_0^1 \int_0^1 c_1(u, v) c_2(v, u) dudv \\
&= \int_0^1 \int_0^1 {}^t\phi(u) A \phi(v) c_2(v, u) dudv \\
&= \int_0^1 \int_0^1 \text{tr} \{ {}^t\phi(u) A \phi(v) c_2(v, u) \} dudv \\
&= \int_0^1 \int_0^1 \text{tr} \{ A \phi(v) c_2(v, u) {}^t\phi(u) \} dudv = \text{tr} \{ AT_\phi(c_2) \}.
\end{aligned}$$

(ii) If, moreover, $c_2 \in \mathcal{C}_\phi$ with associated matrix B then $T_\phi(c_2) = B$ and the conclusion follows. \square