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Bivariate copulas defined from matrices

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Abstract

We propose a semiparametric family of copulas based on a set of orthonormal functions and a matrix. This new copula permits to reach values of Spearman's Rho arbitrarily close to one without introducing a singular component. Moreover, it encompasses several extensions of FGM copulas as well as copulas based on partition of unity such as Bernstein or checkerboard copulas. It is also shown that projection of arbitrary densities of copulas onto tensor product bases can enter our framework. Finally, two estimators of copulas are introduced and their finite sample behaviours are compared on simulated data.

Keywords: Copulas, Semiparametric family, Coefficients of dependence, Estimation.

AMS classifications: 62H05, 62H20, 62H12.

1 Introduction

A bivariate copula defined on the unit square $[0,1]^2$ is a bivariate cumulative distribution function (cdf) with univariate uniform margins. Sklar's Theorem [35] states that any bivariate distribution with cdf H and marginal cdf F and G can be written $H(x,y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions. One of the most popular parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family [9, 13, 25] defined when $\theta \in [-1,1]$ by

$$C(u,v) = uv + \theta u(1-u)v(1-v). \tag{1}$$

A well-known limitation to this family is that it does not allow the modeling of large dependences since the associated Spearman's Rho is limited to $[-1/3, 1/3]$. A possible extension of the FGM family is to consider the semi-parametric family of symmetric copulas defined by

$$C(u,v) = uv + \theta\varphi(u)\varphi(v), \tag{2}$$

with $\theta \in [-1, 1]$. It was first introduced in [29], and extensively studied in [2, 3]. In particular, it can be shown that, for a properly chosen function φ, the range of Spearman's Rho is extended to $[-3/4, 3/4]$. In [4] an extension of (2) is proposed where θ is a univariate function. This modification allows the introduction of a singular component concentrated on the diagonal $v = u$ and extends the range of Spearman's Rho to $[-3/4, 1]$. We also refer to [6, 7] for another yet similar extensions.

Here, a new extension of (2) is proposed where, roughly speaking, the single parameter θ is replaced by a matrix and the function φ is replaced by a set of functions. This new copula permits to reach values of Spearman's Rho arbitrarily close to 1 without singular component. Moreover, it also encompasses copulas based on partition of unity such as Bernstein copula [32] or checkerboard copula [21, 22]. Finally, it is also shown that projection of arbitrary densities of copulas onto tensor product bases can enter our framework. We take profit of this property to propose new estimators of copula densities. This paper is organized as follows: The family of copula is introduced in Section 2 and some algebraic properties are established. Dependence properties are reviewed in Section 3 while approximation issues are highlighted in Section 4. Some links with existing copulas as well as new examples are presented in Section 5. Finally, two estimators of copula densities are proposed in Section 6 and their finite sample properties are illustrated in Section 7 on simulated data. Concluding remarks are drawn in Section 8. Proofs are postponed to the Appendix.

2 A new family of copulas

Throughout this paper, e_j denotes the jth vector of the canonical basis of $\mathbb{R}^p$ where $j = 1, \ldots, p$ and $p \geq 2$. Besides, $\langle f, g \rangle$ is the usual scalar product in L_2 while the associated norm is $\|f\| = \langle f, f \rangle^{1/2}$. We focus on the modeling of the copula density (denoted by c) rather than on the modeling of the copula (denoted by C) itself:

Definition 1 *Let $\phi : [0, 1] \to \mathbb{R}^p$ be a vector of p orthonormal functions such that $\phi_1(t) = 1$ for all $t \in [0, 1]$. Two sets are defined from ϕ:*

$$\begin{aligned} \mathcal{A}_\phi &= \left\{ A \in \mathbb{R}^{p \times p},\ Ae_1 = e_1, A^t\, e_1 = e_1, \forall (u, v) \in [0, 1]^2,\ \phi(u)^t\, A\phi(v) \geq 0 \right\}, \\ \mathcal{C}_\phi &= \left\{ c : [0, 1]^2 \to \mathbb{R},\ c(u, v) = \phi(u)^t\, A\phi(v),\ A \in \mathcal{A}_\phi \right\}, \end{aligned}$$

where x^t denotes the transposition of the vector x.

The next result establishes that all the functions of $\mathcal{C}_\phi$ are densities of copulas.

Proposition 1 *$\mathcal{C}_\phi$ is a non-empty set of copula densities.*

It is clear that $\mathcal{A}_\phi$ is not empty since $A_1 = e_1 e_1^t \in \mathcal{A}_\phi$. The associated function $c(u, v) = \phi(u)^t\, A_1 \phi(v) = 1 \in \mathcal{C}_\phi$ is the density of the independent copula. The remainder of the proof

is postponed to the Appendix. Let us also note that $\mathcal{A}_\phi$ is a subset of matrices with eigenvalue 1 associated with the eigenvector e_1 and with non-negative trace. Indeed, if $A \in \mathcal{A}_\phi$ then

$$\mathrm{tr}(A\phi(x)\phi(x)^t) = \mathrm{tr}(\phi(x)^t A\phi(x)) = \phi(x)^t A\phi(x) \geq 0$$

for all $x \in [0,1]$. Integrating with respect to x yields the result since, by assumption,

$$\int_0^1 \phi(x)\phi(x)^t \, dx = I_p, \tag{3}$$

where I_p is the $p \times p$ identity matrix.

Example 1 *If $p = 2$ then $A \in \mathcal{A}_\phi$ implies that $A = diag\{1, \theta\}$ with $\theta \geq -1$. The associated density of copulas can be written as $c(u,v) = 1 + \theta\phi(u)\phi(v)$ which corresponds to family (2).*

This family includes FGM copulas (1) which contains all copulas with both horizontal and vertical quadratic sections [30], the subfamily of symmetric copulas with cubic sections proposed in [26], equation (4.4), and some kernel extensions of FGM copulas introduced in [16, 20]. We refer to [10] for a method to construct admissible functions ϕ.
The following lemma will reveal useful to build densities of copulas in $\mathcal{C}_\phi$ without the orthogonality assumption on ϕ.

Lemma 1 *Let $\psi : [0,1] \to \mathbb{R}^p$ be a vector of p functions such that $\psi_1(t) = 1$ for all $t \in [0,1]$ and $\int_0^1 \psi(x)dx = e_1$. Let Γ be the Gram matrix defined as $\Gamma = \int_0^1 \psi(x)\psi(x)^t \, dx$ and $B \in \mathcal{A}_\psi$. Then, $A := \Gamma^{1/2} B \Gamma^{1/2} \in \mathcal{A}_\phi$ where $\phi := \Gamma^{-1/2}\psi$ fulfills the conditions of Definition 1 and $\phi(u)^t A\phi(v) = \psi(u)^t B\psi(v)$ for all $(u,v) \in [0,1]^2$.*

See Subsection 9.2 of the Appendix for a proof. A direct application of this lemma yields:

Example 2 *The family of copulas with cubic sections proposed in [26], Theorem 4.1 and given by*

$$C(u,v) = uv + uv(1-u)(1-v)[A_1 v(1-u) + A_2(1-v)(1-u) + B_1 uv + B_2 u(1-v)]$$

can be written in our formalism with Lemma 1. Here, $p = 3$, $\psi_1(t) = 1$, $\psi_2(t) = 1 - 4t + 3t^2$, $\psi_3(t) = 2t - 3t^2$, $\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/15 & 1/30 \\ 0 & 1/30 & 2/15 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2 & A_1 \\ 0 & B_2 & B_1 \end{pmatrix}$.

More generally, iterated FGM families [15, 18, 23] where

$$C(u,v) = uv + \sum_{j=1}^{p} \theta_j (uv)^{\alpha_j} ((1-u)(1-v))^{\beta_j},$$

and $\{\alpha_j, \beta_j\} = \{[j/2] + 1, [(j+1)/2]\}$ can be shown to be particular cases of our family thanks to Lemma 1. More examples are given in Section 5.

Let T_ϕ the mapping defined for any copula density $c \in L_2([0,1]^2)$ by

$$T_\phi(c) = \int_0^1 \int_0^1 c(x,y)\phi(x)\phi(y)^t \, dxdy \in \mathbb{R}^{p\times p}. \tag{4}$$

The mapping T_ϕ permits to compute the matrix associated with any copula density $c \in \mathcal{C}_\phi$:

Proposition 2 *Each copula density $c \in \mathcal{C}_\phi$ is defined by an unique matrix A which is given by*

$$A = T_\phi(c) = \mathbb{E}_c(\phi(U)\phi(V)^t) = cov(\phi(U), \phi(V)) + e_1 e_1^t,$$

where (U,V) is a random pair with density c.

Let $\times$ denote the matrix product and let $\star$ denote the product of copulas introduced in [5] and defined in terms of densities as

$$c_A \star c_B(u,v) := \int_0^1 c_A(u,s)c_B(s,v)ds,$$

for all $(u,v) \in [0,1]^2$. The following stability properties can be established (see Appendix for a proof).

Proposition 3 *$\mathcal{A}_\phi$ is a convex set and $(\mathcal{A}_\phi, \times)$ is a semi-group. If, moreover, $I_p \in \mathcal{A}_\phi$ then $(\mathcal{A}_\phi, \times)$ is a monoid.*

Let us also consider the bivariate function $q(u,v) = \phi(u)^t \, \phi(v)$ defined for $(u,v) \in [0,1]^2$. The following result is the analogous of Proposition 3 for $\mathcal{C}_\phi$.

Proposition 4 *$\mathcal{C}_\phi$ is a closed convex set and $(\mathcal{C}_\phi, \times)$ is a semi-group. If, moreover, $q \in \mathcal{C}_\phi$ then $(\mathcal{C}_\phi, \times)$ is a monoid.*

In view of Propositions 2 – 4, it appears that $T_\phi(c_A \star c_B) = T_\phi(c_A)T_\phi(c_B)$ for all $(c_A, c_B) \in \mathcal{C}_\phi^2$ and thus:

Proposition 5 *T_ϕ is an isomorphism between $(\mathcal{A}_\phi, \times)$ and $(\mathcal{C}_\phi, \star)$.*

To summarize, it appears that $\mathcal{A}_\phi$ is stable with respect to matrix multiplication. Moreover, multiplying the matrices is equivalent to "multiplying" the copulas using the $\star$ product. Besides, from the results of [5], it is possible to build Markov processes by giving all the marginal distributions and a family of copulas satisfying a functional equation based on the $\star$ product and simpler than the Chapman-Kolmogorov (differential) equation. In our case, the isomorphism between $(\mathcal{A}_\phi, \times)$ and $(\mathcal{C}_\phi, \star)$ allows to further simplify the functional equation into a matrix equation. We refer to [5] for more details on this methodology.

Finally, the next lemma shows that it is possible to aggregate copulas of $\mathcal{C}_\phi$ with different number of orthogonal functions through the use of Cesàro summations.

Lemma 2 *Let $c_p \in \mathcal{C}_\phi$ with associated matrix A and consider the density of copula defined for all $(u,v) \in [0,1]^2$ and $q \geq 1$ by*

$$\bar{c}_q(u,v) := \frac{1}{q}\sum_{p=1}^{q} c_p(u,v).$$

Then, $\bar{c}_q \in \mathcal{C}_\phi$ with associated matrix B defined by $B_{ij} = (q+1-\max(i,j))A_{i,j}/q$ for all $(i,j) \in \{1,\ldots,q\}^2$.

An application of this lemma is provided in Paragraph 5.2.

3 Dependence properties

Several measures of association between the components of a random pair can be considered: the normalized volume [34], Kendall's Tau [28], paragraph 5.1.1, Gini's gamma [12], Blomqwist's medial correlation coefficient [28], paragraph 5.1.4, Spearman's footrule [12], and Spearman's Rho [28], paragraph 5.1.2. All these measures are invariant to strictly increasing functions. Kendall's Tau and Spearman's Rho can be interpreted as probabilities of concordance minus probabilities of discordance of two random pairs. Let us first focus on the Spearman's Rho. It can be written only in terms of the copula C:

$$\rho = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{5}$$

Note that ρ coincides with the correlation coefficient between the uniform marginal distributions. In the case of the copula introduced in Definition 1, it can be expressed thanks to the function ϕ and the matrix $A \in \mathcal{A}_\phi$ associated with its density.

Proposition 6 *Let (U,V) be a random pair with density of copula $c \in \mathcal{C}_\phi$ associated with the matrix $A \in \mathcal{A}_\phi$. The Spearman's Rho is given by $\rho = 12\mu^t A\mu - 3$ where $\mu = \int_0^1 x\phi(x)dx$.*

Similarly to the Spearman's Rho, the Kendall's Tau can be written only in terms of the copula C:

$$\tau = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1, \tag{6}$$

and in the framework of Definition 1, it can be expressed thanks to the function ϕ and the matrix $A \in \mathcal{A}_\phi$ associated with its density.

Proposition 7 *Let (U,V) be a random pair with density of copula $c \in \mathcal{C}_\phi$ associated with the matrix $A \in \mathcal{A}_\phi$. The Kendall's Tau is given by $\tau = 1 - 4tr(A^t\,\Theta A\Theta)$ where Θ is the $p \times p$ matrix defined by $\Theta = \int_0^1 \Psi(u)\phi(u)^t\,du$ and with $\Psi(u) = \int_0^u \phi(t)dt$.*

Let us note that Propositions 6 and 7 extend the results of [8], Theorem 25 established in the case of copulas based on partition of unity. Such copulas were introduced in [21, 22]. It is shown in Section 5 that they are particular cases of the family considered in Definition 1. Besides, Blomqwist's medial correlation coefficient, Gini's gamma and Spearman's footrule can also be rewritten in

terms of the copula C (see for instance [27]). All these coefficients thus benefit from closed form expressions similar to these of Propositions 6, 7 but we do not enter into details there.

Following [36], the total tail dependence along the diagonals between two random variables X and Y with respective cdf F and G can be quantified by

$$\Lambda = \begin{pmatrix} \lambda_{LU} & \lambda_{UU} \\ \lambda_{LL} & \lambda_{UL} \end{pmatrix} = \lim_{t \to 1^-} \begin{pmatrix} \mathbb{P}(G(Y) > t | F(X) < 1-t) & \mathbb{P}(G(Y) > t | F(X) > t) \\ \mathbb{P}(G(Y) < 1-t | F(X) < 1-t) & \mathbb{P}(G(Y) < 1-t | F(X) > t) \end{pmatrix}.$$

Again, all these coefficients can be written only in terms of the copula, for instance

$$\lambda_{UU} = \lim_{u \to 1^-} \frac{\bar{C}(u,u)}{1-u}, \tag{7}$$

where $\bar{C}$ is the survival copula, *i.e.* $\bar{C}(u,v) = 1 - u - v + C(u,v)$. In the family $\mathcal{C}_\phi$, the tail dependence along the diagonals is not possible:

Proposition 8 *Let (U,V) be a random pair with density of copula $c \in \mathcal{C}_\phi$. Then,* $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

4 Approximation properties

Let c be a density of copula in $L_2([0,1]^2)$. Recall that the mapping T_ϕ associates a $p \times p$ matrix to c via (4) and introduce:

$$P_\phi(c)(u,v) = \phi(u)^t \, T_\phi(c) \phi(v), \quad (u,v) \in [0,1]^2. \tag{8}$$

The next lemma gives a necessary and sufficient condition for $P_\phi(c) \in \mathcal{C}_\phi$:

Lemma 3

(i) Let $c \in L_2([0,1]^2)$ be an arbitrary density of copula. If $I_p \in \mathcal{A}_\phi$ then $P_\phi(c) \in \mathcal{C}_\phi$ and $P_\phi(c) = q \star c \star q$.

(ii) Conversely, if $P_\phi(c) \in \mathcal{C}_\phi$ for all $c \in L_2([0,1]^2)$ then $I_p \in \mathcal{A}_\phi$.

Let $(c_1, c_2) \in L_2^2([0,1]^2)$ and let us consider the scalar product defined as

$$\prec c_1, c_2 \succ = \int_0^1 (c_1 \star c_2)(u,u) du = \int_0^1 \int_0^1 c_1(u,v) c_2(v,u) du dv.$$

Note that, for symmetric copulas, the above scalar product reduces to the L_2- scalar product

$$\langle c_1, c_2 \rangle = \int_0^1 \int_0^1 c_1(u,v) c_2(u,v) du dv.$$

In the case of densities of copulas in $\mathcal{C}_\phi$, the scalar product can be computed using the associated matrices:

Lemma 4 *Let $c_2 \in L_2([0,1]^2)$.*

(i) If $c_1 \in \mathcal{C}_\phi$ with associated matrix A, then $\prec c_1, c_2 \succ = tr(AT_\varphi(c_2))$.

(ii) If, moreover, $c_2 \in \mathcal{C}_\phi$ with associated matrix B, then $\prec c_1, c_2 \succ = tr(AB)$.

As a consequence of the above lemmas, we have:

Proposition 9 *P_ϕ is an orthogonal projection on $\mathcal{C}_\phi$ if and only if $I_p \in \mathcal{A}_\phi$.*

This result is now illustrated on the FGM family where explicit computations can be done:

Example 3 *It is well known that the FGM family is a particular case of Example 1 with $\phi(x) = \sqrt{3}(1-2x)$, $A = diag\{1, \theta\}$ and where $|\theta| \leq 1/3$. Here, $I_p \notin \mathcal{A}_\phi$ and thus the projection $P_\phi(c)$ of any density of copula c on the FGM family is not itself a density of copula in the general case. However, let us remark that $T_\phi(c) = diag\{1, \widetilde{\theta}\}$ and thus $P_\phi(c)(u,v) = 1 + 3\tilde{\theta}(1-2u)(1-2v)$ where*

$$\tilde{\theta} = \int_0^1 \int_0^1 c(x,y)\phi(x)\phi(y)dxdy = 3\int_0^1 \int_0^1 c(x,y)(1-2x)(1-2y)dxdy = \rho_c,$$

the Spearman's Rho associated with c. We thus have the following result:

- *If $|\rho_c| \leq 1/3$ then $P_\phi(c)$ is a FGM copula and $\rho_{P_\phi(c)} = \rho_c$,*
- *If $|\rho_c| > 1/3$ then $P_\phi(c)$ is not a copula.*

It appears from this example that it is possible to associate with any copula a FGM copula with the same Spearman's Rho ρ_c provided $|\rho_c| \leq 1/3$.

Suppose now that $\{\phi_i\}_{i\geq 1}$ is an orthonormal basis of $L_2([0,1])$. Then, $\{\phi_i \otimes \phi_j\}_{i,j\geq 1}$ is an orthonormal basis of $L_2([0,1]^2)$ where $\otimes$ denotes the tensor product, *i.e* $(f \otimes g)(u,v) := f(u)g(v)$ for all $(u,v) \in [0,1]^2$. Consequently, the L_2-projection of any $c \in L_2([0,1]^2)$ on $\{\phi_i \otimes \phi_j\}_{1\leq i,j\leq p}$ is given by

$$\tilde{c}_p(u,v) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_{i,j}\phi_i(u)\phi_j(v) = \phi(u)^t\, A\phi(v)$$

where $A = (a_{i,j})_{1\leq i,j\leq p}$ with $a_{i,j} = \int_0^1\int_0^1 c(x,y)\phi_i(x)\phi_j(y)dxdy = (T_\phi(c))_{i,j}$. This yields $A = T_\phi(c)$ and $\tilde{c}_p = P_\phi(c)$. In view of Lemma 3(i), it follows that $\tilde{c}_p \in \mathcal{C}_\phi$ if $I_p \in \mathcal{A}_\phi$. As a conclusion, the L_2-projection of any density of copula in $L_2([0,1]^2)$ on a tensor product basis can be written in our formalism and the following result holds:

Theorem 1 *Let $\{\phi_i\}_{1\leq i\leq p}$ be an orthonormal family of $L_2([0,1])$.*

(i) The projection P_ϕ on $\mathcal{C}_\phi$ introduced in (8) coincides with the L_2- projection on $\{\phi_i \otimes \phi_j\}_{1\leq i,j\leq p}$.

(ii) Moreover, these projections give rise to densities of copula in $\mathcal{C}_\phi$ if and only if $I_p \in \mathcal{A}_\phi$.

Classical approximation properties in $L_2([0,1]^2)$ yield:

Corollary 1 *Suppose the assumptions of Theorem 1 hold and let $c \in L_2([0,1]^2)$ be a density of copula. Then, $\|c - P_\phi(c)\| \to 0$ and $\rho(P_\phi(c)) \to \rho(c)$ as $p \to \infty$, where $\rho(P_\phi(c))$ and $\rho(c)$ denote respectively the Spearman's Rho associated with $P_\phi(c)$ and c.*

5 Examples

In paragraph 5.1, some examples of copulas found in the literature are shown to enter in our model. New families are exhibited in paragraph 5.2.

5.1 Copulas based on partition of unity

Recall that a collection of functions $\xi = (\xi_1, \ldots, \xi_p)^t$ is called a partition of unity [21, 22] if $\xi_i \geq 0$, $\int_0^1 \xi_i(x)dx = 1/p$ for all $i = 1, \ldots, p$ and $\sum_{i=1}^p \xi_i = 1$. It can be established that copulas based on partition of unity are particular cases of the proposed family:

Proposition 10 *Let $\xi = (\xi_1, \ldots, \xi_p)^t$ be a partition of unity, and let M be a $p \times p$ doubly stochastic matrix. Then, the function defined for all $(u,v) \in [0,1]^2$ by $c(u,v) = p\xi(u)^t M\xi(v)$ is a density of copula and $c \in \mathcal{C}_\phi$. Moreover, $\phi = (H\Gamma_\xi H^t)^{-1/2} H\xi$ where $s = e_1 + \cdots + e_p$, $H = I_p + e_1 s^t - s e_1^t$, and Γ_ξ is the Gram matrix associated with ξ.*

As an illustration, we have:

Example 4 *The Bernstein copula [32] is obtained by choosing $\xi_i(x) = C_{p-1}^{i-1} x^{i-1}(1-x)^{p-i}$ and*

$$M_{ij} = p\left\{C\left(\frac{i}{p}, \frac{j}{p}\right) - C\left(\frac{i-1}{p}, \frac{j}{p}\right) - C\left(\frac{i}{p}, \frac{j-1}{p}\right) + C\left(\frac{i-1}{p}, \frac{j-1}{p}\right)\right\}, \tag{9}$$

where C is an arbitrary copula.

In the case of Bernstein copula, the basis $\phi = (H\Gamma_\xi H^t)^{-1/2} H\xi$ cannot be simplified. However, in the particular case where $\{\xi_1, \ldots, \xi_p\}$ is orthogonal and $\int_0^1 \xi_i^2(t)dt = \beta^2$ for all $i = 1, \ldots, p$, we have $\Gamma_\xi = \beta^2 I_p$ and therefore

$$H\Gamma_\xi H^t = \beta^2 \begin{pmatrix} p & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 2 & 1 & \ldots & \ldots & 1 \\ \vdots & 1 & 2 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 2 & 1 \\ 0 & 1 & \ldots & \ldots & 1 & 2 \end{pmatrix}.$$

The inverse square-root of this matrix benefits from a closed-form expression, and we thus have a simple linear relation between the two families of functions: $\phi = \Omega\xi$ with

$$\Omega = \frac{1}{\beta}\begin{pmatrix} p^{-1/2} & \dots & \dots & \dots & \dots & p^{-1/2} \\ -p^{-1/2} & \gamma & (\gamma-1) & \dots & \dots & (\gamma-1) \\ \vdots & (\gamma-1) & \gamma & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \gamma & (\gamma-1) \\ -p^{-1/2} & (\gamma-1) & \dots & \dots & (\gamma-1) & \gamma \end{pmatrix}, \tag{10}$$

where $\gamma = (p-2+p^{-1/2})/(p-1)$. Explicit computations can be achieved in case of the checkerboard copula:

Example 5 *The checkerboard copula [21, 22] is obtained by choosing $\xi_i(x) = \mathbb{I}\{x \in I_i\}$ where $\{I_i,\ i = 1, \dots, p\}$ is the equidistant partition of $[0,1]$ into p intervals and M_{ij} as in (9). Besides, $\Gamma_\xi = (1/p)I_p$ and therefore $\phi = \Omega\xi$ where Ω is given by (10) with $\beta = p^{-1/2}$.*

5.2 Orthogonal bases

Trigonometric basis. The trigonometric family is defined by $\phi_0(x) = 1$, $\phi_{2j-1}(x) = \sqrt{2}\sin(2\pi jx)$ and $\phi_{2j}(x) = \sqrt{2}\cos(2\pi jx)$ for all $j \geq 1$ and $x \in [0,1]$. It is orthonormal with respect to the usual scalar product on $L_2([0,1])$. Let $\theta \geq 0$ and consider the $(2p+1) \times (2p+1)$ matrix $A =$diag$\{1, \theta, \theta, \dots, \theta\}$. One has

$$c_{p,\theta}(u,v) := \phi(u)^t\, A\phi(v) = 1 - \theta + \theta D_p(u-v)$$

where D_p is the Dirichlet kernel given by

$$D_p(t) = \frac{\sin((2p+1)\pi t)}{\sin(\pi t)}.$$

It is then clear that $c_{p,\theta} \in \mathcal{C}_\phi$ if $\theta \leq 1/(1-D_p^-)$ where $D_p^- := \min_t D_p(t)$. Since $D_p^- < 0$, it follows that θ is upper bounded by $1/(1-D_p^-) < 1$ and thus $I_p \notin \mathcal{A}_\phi$ when ϕ is the trigonometric family. Besides, Proposition 6 yields that the associated Spearman's rho is

$$\rho_{p,\theta} = \frac{6\theta}{\pi^2}\sum_{j=1}^{p}\frac{1}{j^2}.$$

Numerical computations show that the maximum value is obtained for $p = 2$ for which $D_2^- = -1$ and $\rho_{2,1/2} = 15/(4\pi^2)$. This bound can be increased by introducing

$$\bar{c}_{q,\theta}(u,v) := \frac{1}{q}\sum_{p=0}^{q-1} c_{p,\theta}(u,v) = 1 - \theta + \theta F_q(u-v) \tag{11}$$

where F_q is the Fejér kernel [14] defined as

$$F_q(x) = \frac{1}{q}\sum_{p=0}^{q-1} D_p(x) = \frac{1}{q}\left(\frac{\sin(qx/2)}{\sin(x/2)}\right)^2.$$

Since this kernel is positive, it is readily seen that $\bar{c}_{q,\theta}$ is a density of copula for all $\theta \in [0,1]$. Besides, Lemma 2 entails that $\bar{c}_{q,\theta} \in \mathcal{C}_\phi$ for all $\theta \in [0,1]$. The associated Spearman's rho is

$$\bar{\rho}_{q,\theta} = \frac{1}{q}\sum_{p=0}^{q-1}\rho_{p,\theta} = \frac{6\theta}{\pi^2}\frac{1}{q}\sum_{p=1}^{q-1}\sum_{j=1}^{p}\frac{1}{j^2} = \frac{6\theta}{\pi^2}\left(\sum_{j=1}^{q-1}\frac{1}{j^2} - \frac{1}{q}\sum_{j=1}^{q-1}\frac{1}{j}\right).$$

Let us remark that $\bar{\rho}_{q,\theta} \to \theta$ as $q \to \infty$ and thus, arbitrary large dependences can be modeled.

The Haar basis. For each positive integer i, let us denote by J_i the interval $J_i = \left[\frac{p_i}{2^{q_i-1}}, \frac{p_i+1}{2^{q_i-1}}\right)$ where p_i and q_i are the integers uniquely determined by $i = 2^{q_i-1} + p_i$ and $0 \le p_i < 2^{q_i-1}$. The Haar basis [31] is defined by $\phi_0(t) = \mathbb{I}\{t \in [0,1]\}$ and $\phi_i(t) = 2^{\frac{q_i-1}{2}}\left(\mathbb{I}\{t \in J_{2i}\} - \mathbb{I}\{t \in J_{2i+1}\}\right)$, for $i = 1, \ldots, p$. In the following, it is assumed that p is a power of 2. Let $\theta \ge 0$ and consider the $p \times p$ matrix $A = \text{diag}\{1, \theta, \theta, \ldots, \theta\}$. One has

$$c_{p,\theta}(u,v) := \phi(u)^t\, A\phi(v) = 1 - \theta + \theta K_p(u,v) \tag{12}$$

where K_p is the Dirichlet kernel associated with the Haar basis

$$K_p(u,v) = p\sum_{i=1}^{p}\mathbb{I}\{(u,v) \in I_i^2\},$$

and recall that $\{I_i,\ i = 1, \ldots, p\}$ is the equidistant partition of $[0,1]$ into p intervals. It is thus clear that K_p can be rewritten as

$$K_p(u,v) = p\sum_{i=1}^{p}\sum_{j=1}^{p} M_{i,j}\xi_i(u)\xi_j(v)$$

where $M = I_p$ is the $p \times p$ identity matrix and $\xi_i(u) = \mathbb{I}\{u \in I_i\}$. It appears that K_p is the density of a copula based on partition of unity. In view of Proposition 10, it follows that $K_p \in \mathcal{C}_\phi$. Finally, (12) can be interpreted as a linear mixture of K_p and the independent copula, leading to $c_{p,\theta} \in \mathcal{C}_\phi$ for all $\theta \in [0,1]$. Thus, $I_p \in \mathcal{A}_\phi$ when ϕ is the Haar family. Straightforward calculations show that the associated Spearman's rho is given by $\rho_p(\theta) = \theta\left(1 - p^{-2}\right)$. Let us remark that $\rho_p(\theta) \to \theta$ as $p \to \infty$ and thus, arbitrary large dependences can be modeled.

6 Estimation

Let $(U_1, V_1), \ldots, (U_n, V_n)$ be independent copies of a random pair (U, V) from a density $c \in \mathcal{C}_\phi$ associated with a matrix A. Assume that the function ϕ is known. Then, estimating c reduces to

estimating A. Proposition 2 provides two interpretations of A in terms of a covariance matrices and thus two possible estimators:

$$\begin{aligned}\hat{A}_{1,n} &= \frac{1}{n}\sum_{i=1}^{n}\phi(U_i)\phi(V_i)^t,\\ \hat{A}_{2,n} &= \frac{1}{n}\sum_{i=1}^{n}(\phi(U_i)-e_1)(\phi(V_i)-e_1)^t + e_1e_1^t.\end{aligned}$$

In both cases, the corresponding estimated density is given by $\hat{c}_{j,n}(u,v) = \phi(u)^t\,\hat{A}_{j,n}\phi(v)$, $j \in \{1,2\}$ and can be simplified as

$$\begin{aligned}\hat{c}_{1,n}(u,v) &= \frac{1}{n}\sum_{i=1}^{n}q(u,U_i)q(v,V_i),\\ \hat{c}_{2,n}(u,v) &= 1+\frac{1}{n}\sum_{i=1}^{n}(q(u,U_i)-1)(q(v,V_i)-1).\end{aligned}$$

For a fixed value of n, one cannot guaranty that these estimators belong to $\mathcal{C}_\phi$. Nevertheless, one can prove that the margins of $\hat{c}_{2,n}$ are uniform: $\int_0^1 \hat{c}_{2,n}(u,t)dt = \int_0^1 \hat{c}_{2,n}(t,v)dt = 1$ for all $(u,v) \in [0,1]^2$. Besides, if $I_p \in \mathcal{A}_\phi$ then $\hat{c}_{1,n}(u,v) \geq 0$ for all $(u,v) \in [0,1]^2$. The next result can also be readily established:

Proposition 11 *Let $c \in \mathcal{C}_\phi$. For all $j \in \{1,2\}$,*

(i) $\mathbb{E}(\hat{c}_{j,n}(u,v)) = c(u,v)$ *for all* $n \geq 1$.

(ii) $\sqrt{n}(\hat{c}_{j,n}(u,v)-c(u,v))$ *converges in distribution to a centered Gaussian distribution* $\mathcal{N}(0,\sigma_j^2(u,v))$ *as* $n \to \infty$, *where*

$$\begin{aligned}\sigma_1^2(u,v) &= \zeta_1(u)^t\,A\zeta_1(v) - c(u,v)^2 \quad \textit{with} \quad \zeta_1(t) = \int_0^1 q^2(x,t)\phi(x)dx,\\ \sigma_2^2(u,v) &= \zeta_2(u)^t\,A\zeta_2(v) - (c(u,v)-1)^2 \quad \textit{with} \quad \zeta_2(t) = \int_0^1 (q(x,t)-1)^2\phi(x)dx.\end{aligned}$$

More generally, both estimators $\hat{c}_{1,n}$ and $\hat{c}_{2,n}$ can still be used even for densities of copulas $c \notin \mathcal{C}_\phi$. Letting $\tilde{c}_p = P_\phi(c)$ and $\hat{c}_p \in \{\hat{c}_{1,n}, \hat{c}_{2,n}\}$, the straightfoward decomposition holds: $(\hat{c}_p - c) = (\hat{c}_p - \tilde{c}_p) + (\tilde{c}_p - c)$. The first term can be interpreted as an estimation error, it is controlled (when p if fixed and $n \to \infty$) by Proposition 11. The second term is an approximation error, independent of n, it is controlled by Corollary 1 (when $p \to \infty$). The asymptotic properties when both $p \to \infty$ and $n \to \infty$ depend on the chosen basis of functions. They can be derived thanks to the following classical result:

Proposition 12 *Let $\{\phi_i\}_{i\geq 1}$ be an orthonormal basis of $L_2([0,1])$ and consider $c \in L_2([0,1]^2)$. Set $A_p = T_\phi(c)$ and $\tilde{c}_p = P_\phi(c)$ for the sake of simplicity. Then, for all $j \in \{1,2\}$,*

$$\|\hat{c}_{j,n} - c\|^2 = \|\hat{A}_{j,n} - A_p\|_F^2 + \|\tilde{c}_p - c\|^2,$$

where $\|M\|_F = \sqrt{tr(MM^t)}$ is the Frobenius norm of the matrix M.

7 Numerical experiments

The proposed estimators $\hat{c}_{1,n}$ and $\hat{c}_{2,n}$ are compared on simulated data. Both estimators are built on the trigonometric basis, see Subsection 5.2. The value of p is selected by minimizing the distance between the empirical Spearman's Rho and the estimated one under the model: $\hat{\rho}_{j,n} = 12\mu^t \hat{A}_{j,n}\, \mu - 3$, $j \in \{1, 2\}$, see Proposition 6. The comparison is achieved on three models: the Fejér copula (11) with $q = 5$, the Gaussian copula and Ali-Mikhail-Haq (AMH) copula [1] with a sample size fixed to $n = 500$. For each of the above copula models, five values of the dependence parameter were tested in order to explore a wide range of possible Spearman's Rho values: $[0.06, 0.55]$ for the Fejér copula, $[-0.9, 0.9]$ for the Gaussian copula and $[-0.25, 0.41]$ for the AMH one. Let us highlight that Fejér copula belongs to our family while Gaussian and AMH copulas do not. The comparison of the estimation results in both situations should illustrate the approximation ability of the considered family of copulas $\mathcal{C}_\phi$. The experiments where conducted on $N = 100$ replications of the 3×5 datasets. The empirical mean $\bar{\rho}_j$ as well as the sum-of-squared-errors ε_j corresponding to each estimator $\hat{\rho}_{j,n}$, $j \in \{1, 2\}$ are computed on the N replications and reported in Table 1. It appears that both estimators are able to estimate the Spearman's Rho in a reliable way, even though large values of $|\rho|$ are slightly under-estimated. Clearly, $\hat{c}_{2,n}$ performs better than $\hat{c}_{1,n}$. The second estimator yields much smaller errors than the first one. Let us also stress that the sum-of-squared-errors associated with $\hat{c}_{2,n}$ is nearly constant for all considered copula models and Spearman's Rho values. This phenomenon shows that the considered family may be suitable for estimating various dependence structures. The above results are confirmed by Figure 1. It displays the boxplots of $(\hat{\rho}_{j,n} - \rho)$ computed on the replications of the 3×5 datasets for each estimator ($j \in \{1, 2\}$). The estimation errors associated with $\hat{c}_{2,n}$ are nearly centered and benefit from a smaller variance than those associated to $\hat{c}_{1,n}$.

ρ	$\bar{\rho}_1$	ε_1	$\bar{\rho}_2$	ε_2	ρ	$\bar{\rho}_1$	ε_1	$\bar{\rho}_2$	ε_2	ρ	$\bar{\rho}_1$	ε_1	$\bar{\rho}_2$	ε_2
0.06	0.05	0.49	0.06	0.20	-0.9	-0.87	0.21	-0.87	0.16	-0.25	-0.22	0.48	-0.25	0.15
0.18	0.16	0.75	0.18	0.23	-0.5	-0.46	0.51	-0.48	0.25	-0.15	-0.12	0.63	-0.14	0.19
0.30	0.27	0.57	0.29	0.19	0.0	0.00	0.53	0.00	0.13	0.00	-0.00	0.45	0.00	0.20
0.42	0.39	0.90	0.42	0.22	0.5	0.43	1.25	0.47	0.26	0.19	0.16	0.70	0.19	0.21
0.55	0.53	0.55	0.54	0.20	0.9	0.83	1.52	0.87	0.13	0.41	0.35	1.02	0.40	0.21

Table 1: Estimation results on three simulated copulas (Fejér (left), Gaussian (center) and AMH (right)) for different values of Spearman's Rho (ρ). For each estimator $\hat{c}_{j,n}$, $j \in \{1, 2\}$, the empirical mean of the estimated Spearman's Rho ($\bar{\rho}_j$) is computed as well as the associated sum-of-squared-errors (ε_j).

8 Conclusion and further work

We proposed a new family of bivariate copulas defined from a matrix and a family of orthogonal functions. High dependences can be modeled without introducing singular components. It has also been shown that this family can be used for approximating any density of copula. As a consequence, it appears as a good tool for modeling bivariate data. The extension to higher dimensional copula could be achieved by using high dimensional arrays. The computational issues involved in the estimation procedure could be overcomed using the approach of [33] to avoid multidimensional summations for the Bernstein copula. The extension to high dimension could also be done using one-factor copula models [19] similarly to [24] which permit to avoid the curse of dimensionality effects.

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9 Appendix

9.1 Proofs of main results

Proof of Proposition 1. Let $c \in \mathcal{C}_\phi$. Clearly, $c(u,v) \geq 0$ for all $(u,v) \in [0,1]^2$ from the definition of $\mathcal{A}_\phi$. It only remains to prove that the margins of c are standard uniform distributions. To this end, let us remark that

$$\int_0^1 \phi(v)dv = e_1, \tag{13}$$

since for all $j = 1, \ldots, p$, we have $\int_0^1 \phi_j(v)dv = \langle e_1, e_j \rangle = \delta_{1j}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. As a consequence,

$$\int_0^1 c(u,v)dv = \phi(u)^t A \int_0^1 \phi(v)dv = \phi(u)^t Ae_1 = \phi(u)^t e_1 = \phi_1(u) = 1.$$

The proof of $\int_0^1 c(u,v)du = 1$ is similar. □

Proof of Proposition 2. Let $c \in \mathcal{C}_\phi$ such that $c(u,v) = \phi(u)^t A\phi(v) = \phi(u)^t B\phi(v)$ for all $(u,v) \in [0,1]^2$ and for some $(A,B) \in \mathcal{A}_\phi^2$. It follows that

$$c(u,v)\phi(u)\phi(v)^t = \phi(u)\phi(u)^t A\phi(v)\phi(v)^t = \phi(u)\phi(u)^t B\phi(v)\phi(v)^t,$$

and integrating with respect to u and v yields $T_\phi(c) = A = B$ in view of (3). Remark that, if moreover, $c \in \mathcal{C}_\phi$ with associated matrix $A \in \mathcal{A}_\phi$ then

$$\mathbb{E}_c(\phi(U)) = \int_0^1 \int_0^1 \phi(u)\phi(u)^t A\phi(v)dudv = A \int_0^1 \phi(v)dv = Ae_1 = e_1$$

in view of (3) and (13). Thus, the matrix A can also be interpreted as $A = \text{cov}(\phi(U), \phi(V)) + e_1 e_1^t$. □

Proof of Proposition 3. It is clear that $\mathcal{A}_\phi$ is convex. Let us prove that $(\mathcal{A}_\phi, \times)$ is a semi-group. Since the product $\times$ is associative, it only remains to establish that $(A,B) \in \mathcal{A}_\phi^2$ entails $A \times B \in \mathcal{A}_\phi$. First, (3) entails

$$\begin{aligned} \phi(u)^t AB\phi(v) &= \phi(u)^t A \left\{ \int_0^1 \phi(y)\phi(y)^t dy \right\} B\phi(v) \\ &= \int_0^1 \left\{ \phi(u)^t A\phi(y) \right\} \left\{ \phi(y)^t B\phi(v) \right\} dy \geq 0, \end{aligned}$$

and second it is easily seen that $ABe_1 = e_1$ and $(AB)^t e_1 = e_1$. Finally, $I_p \in \mathcal{A}_\phi$ and $(\mathcal{A}_\phi, \times)$ is a semi-group both imply that $(\mathcal{A}_\phi, \times)$ is a monoid. □

Proof of Proposition 4. It is clear that $\mathcal{C}_\phi$ is convex. Let us prove that $(\mathcal{C}_\phi, \star)$ is a semi-group. Since the product $\star$ is associative (see [5], Theorem 2.4), it only remains to establish that $(c_A, c_B) \in \mathcal{C}_\phi^2$ entails $c_A \star c_B \in \mathcal{C}_\phi$:

$$\begin{aligned} c_A \star c_B(u,v) &= \int_0^1 \phi(u)^t\, A\phi(s)\phi(s)^t\, B\phi(v)ds, \\ &= \phi(u)^t\, A\left\{\int_0^1 \phi(s)\phi(s)^t\, ds\right\} B\phi(v) = \phi(u)^t\, AB\phi(v). \end{aligned}$$

This proves that $(c_A, c_B) \in \mathcal{C}_\phi^2$ and is associated with the matrix $AB \in \mathcal{C}_\phi$, see Proposition 3. Finally, $q \in \mathcal{C}_\phi$ and $(\mathcal{C}_\phi, \star)$ is a semi-group both imply that $(\mathcal{C}_\phi, \star)$ is a monoid. □

Proof of Proposition 6. Let C denote the copula associated with the density $c \in \mathcal{C}_\phi$. Introducing $\Psi(u) = \int_0^u \phi(t)dt$ and $\gamma = \int_0^1 \Psi(u)du$, we have $C(u,v) = \Psi(u)^t\, A\Psi(v)$ and (5) leads to $\rho = 12\gamma^t\, A\gamma - 3$. A partial integration yields $\gamma = e_1 - \mu$ and

$$(e_1 - \mu)^t\, A(e_1 - \mu) = e_1^t\, Ae_1 - \mu^t\, (A + A^t)e_1 + \mu^t\, A\mu = 1 - 2\mu_1 + \mu^t\, A\mu$$

with $\mu_1 = \int_0^1 tdt = 1/2$ and the result follows. □

Proof of Proposition 7. Let C denote the copula associated with the density $c \in \mathcal{C}_\phi$. Standard algebra gives

$$\begin{aligned} \int_0^1\!\!\int_0^1 C(u,v)dC(u,v) &= \int_0^1\!\!\int_0^1 \Psi(u)^t\, A\Psi(v)\phi(u)^t\, A\phi(v)dudv \\ &= \int_0^1\!\!\int_0^1 \Psi(v)^t\, A^t\, \Psi(u)\phi(u)^t\, A\phi(v)dudv \\ &= \int_0^1 \Psi(v)^t\, A^t\, \Theta A\phi(v)dv \\ &= \int_0^1 \operatorname{tr}\left(\Psi(v)^t\, A^t\, \Theta A\phi(v)\right) dv \\ &= \int_0^1 \operatorname{tr}\left(A^t\, \Theta A\phi(v)\Psi(v)^t\right) dv \\ &= \operatorname{tr}\left(A^t\, \Theta A\Theta^t\right). \end{aligned}$$

Besides, a partial integration shows that $\Theta^t = e_1 e_1^t - \Theta$. It follows that

$$\begin{aligned}
\operatorname{tr}\left(A^t\,\Theta A \Theta^t\right) &= \operatorname{tr}\left(A^t\,\Theta A e_1 e_1^t\right) - \operatorname{tr}\left(A^t\,\Theta A \Theta\right) \\
&= \operatorname{tr}\left(A^t\,\Theta e_1 e_1^t\right) - \operatorname{tr}\left(A^t\,\Theta A \Theta\right) \\
&= \operatorname{tr}\left(\Theta e_1 e_1^t\,A^t\right) - \operatorname{tr}\left(A^t\,\Theta A \Theta\right) \\
&= \operatorname{tr}\left(\Theta e_1 e_1^t\right) - \operatorname{tr}\left(A^t\,\Theta A \Theta\right) \\
&= \Theta_{1,1} - \operatorname{tr}\left(A^t\,\Theta A \Theta\right) \\
&= 1/2 - \operatorname{tr}\left(A^t\,\Theta A \Theta\right).
\end{aligned}$$

The conclusion follows from (6). □

Proof of Proposition 8. Let C denote the copula associated with the density $c \in \mathcal{C}_\phi$. Recalling that $\Psi(u) = \int_0^u \phi(t)dt$, we have $C(u,v) = \Psi(u)^t\,A\Psi(v)$. In view of (13), $\Psi(1) = e_1$ leading to $C(1,1) = 1$ and thus (7) can be rewritten as

$$\lambda_{UU} = 2 - \lim_{u \to 1^-} \frac{C(u,u) - C(1,1)}{u-1} = 2 - \left.\frac{\partial C(u,u)}{\partial u}\right|_{u=1}.$$

Straightforward calculations show that

$$\left.\frac{\partial C(u,u)}{\partial u}\right|_{u=1} = \phi(1)^t\,(A + A^t)\Psi(1) = \phi(1)^t\,(A + A^t)e_1 = 2\phi(1)^t\,e_1 = 2,$$

and consequently $\lambda_{UU} = 0$. The proof for the other terms of the matrix Λ is similar. □

Proof of Proposition 9. Let us suppose first that $I_p \in \mathcal{A}_\phi$. Let us first remark that, from Proposition 4, $\mathcal{C}_\phi$ is a convex and closed subset of $L_2([0,1]^2)$. Second, it is clear from Lemma 3(i) that P_ϕ is idempotent:

$$P_\phi(P_\phi(c)) = q \star P(c) \star q = q \star q \star c \star q \star q = q \star c \star q = P(c),$$

for all $c \in L_2([0,1]^2)$ since $q \star q = q$ from (3). Third, let $c \in L_2([0,1]^2)$ and $s \in \mathcal{C}_\phi$ with associated matrix A. Our aim is to prove that $\prec c - P_\phi(c), s \succ = 0$. In view of Lemma 4,

$$\prec c - P_\phi(c), s \succ = \operatorname{tr}(T_\phi(c)A) - \operatorname{tr}(T_\phi(c)A) = 0$$

and the direct part of the result is proved. Conversely, if P_ϕ is a projection on $\mathcal{C}_\phi$ then, necessarily, $P_\phi(q) \in \mathcal{C}_\phi$. Besides, $P_\phi(q) = q$ and thus $q \in \mathcal{C}_\phi$ entailing $I_p \in \mathcal{A}_\phi$. The converse part of the result is proved. □

Proof of Corollary 1. First, it is clear that $\|c - P_\phi(c)\| \to 0$ as $p \to \infty$, since, in view of Theorem 1(i), $\tilde{c}_p := P_\phi(c)$ can be interpreted as a L_2- projection of c. From (5), it follows that

$$\rho(\tilde{c}_p) - \rho(c) = \int_{[0,1]^4} (\tilde{c}_p(x,y) - c(x,y))\mathbb{I}\{x \le u\}\mathbb{I}\{y \le v\} dxdydudv$$

and Cauchy-Schwarz inequality yields

$$|\rho(\tilde{c}_p) - \rho(c)| \le \|\tilde{c}_p - c\| \left(\int_{[0,1]^4} \mathbb{I}\{x \le u\}\mathbb{I}\{y \le v\} dxdydudv \right)^{1/2} = \frac{1}{2}\|\tilde{c}_p - c\|$$

and the conclusion follows. □

Proof of Proposition 10. Let $\psi = H\xi$ and $B = pH^{-t}MH^{-1}$ where H^{-t} denotes the transposition of the inverse of H. First, it is easily seen that

$$c(u,v) = p\xi(u)^t\, M\xi(v) = \psi(u)^t\, B\psi(v),$$

with $\psi_1(t) = \sum_{i=1}^p \xi_i(t) = 1$ and $\int_0^1 \psi(t)dt = H\int_0^1 \xi(t)dt = \frac{1}{p}Hs = e_1$. Besides, since $\xi_i \ge 0$ for all $i = 1, \ldots, p$ and M is a doubly stochastic matrix, it is clear that $c(u,v) \ge 0$ for all $(u,v) \in [0,1]^2$. Second, standard algebra shows that $H^{-1}e_1 = s/p$ and $H^{-t}s = e_1$. As a consequence,

$$Be_1 = pH^{-t}MH^{-1}e_1 = H^{-t}Ms = H^{-t}s = e_1$$

and similarly $B^t\, e_1 = e_1$ leading to $B \in \mathcal{A}_\psi$. Lemma 1 entails that $A := \Gamma^{1/2}B\Gamma^{1/2} \in \mathcal{A}_\phi$ where $\phi := \Gamma^{-1/2}\psi$ fullfills the conditions of Definition 1 and the density of copula

$$c(u,v) = p\xi(u)^t\, M\xi(v) = \psi(u)^t\, B\psi(v) = \phi(u)^t\, A\phi(v)$$

belongs to $\mathcal{C}_\phi$. □

Proof of Proposition 11. The proof is a consequence of $\mathbb{E}(q(u,U)) = 1$, $\mathbb{E}(q(u,U)q(v,V)) = c(u,v)$ and of the Central-Limit Theorem. □

Proof of Proposition 12. As a consequence of the properties of the L_2- projection, $\|\hat{c}_{j,n} - c\|^2 = \|\hat{c}_{j,n} - \tilde{c}_p\|^2 + \|\tilde{c}_p - c\|^2$, and the result follows from $\|\hat{c}_{j,n} - \tilde{c}_p\|^2 = \|\hat{A}_{j,n} - A_p\|_F^2$ (the proof being similar to the one of Lemma 4). □

9.2 Proofs of auxiliary results

Proof of Lemma 1. Let us first remark that $\Gamma e_1 = e_1$ and $\Gamma^t\, e_1 = e_1$. Consequently, there exists a square root $\Gamma^{1/2}$ of Γ such that $\Gamma^{1/2}e_1 = e_1$ and $(\Gamma^{1/2})^t e_1 = e_1$. It follows that $Ae_1 = e_1$, $A^t\, e_1 = e_1$ and that, for all $(u,v) \in [0,1]^2$, $\phi(u)^t\, A\phi(v) = \psi(u)^t\, B\psi(v)$, it is thus clear that $A \in \mathcal{A}_\phi$ with $\phi_1(x) = 1$ for all $x \in [0,1]$ and ϕ is orthonormal. □

Proof of Lemma 2. Let $c_p \in \mathcal{C}_\phi$ with associated matrix A and consider the density of copula defined for all $(u,v) \in [0,1]^2$ and $q \geq 1$ by

$$\begin{aligned}
\bar{c}_q(u,v) &:= \frac{1}{q}\sum_{p=1}^{q} c_p(u,v) \\
&= \frac{1}{q}\sum_{p=1}^{q}\sum_{i=1}^{q}\sum_{j=1}^{q} A_{i,j}\phi_i(u)\phi_j(v)\mathbb{I}\{i \leq p\}\mathbb{I}\{j \leq p\} \\
&= \sum_{i=1}^{q}\sum_{j=1}^{q} A_{i,j}\left(\frac{1}{q}\sum_{p=1}^{q}\mathbb{I}\{\max(i,j) \leq p\}\right)\phi_i(u)\phi_j(v) \\
&= \sum_{i=1}^{q}\sum_{j=1}^{q} A_{i,j}\left(\frac{q+1-\max(i,j)}{q}\right)\phi_i(u)\phi_j(v) \\
&=: \sum_{i=1}^{q}\sum_{j=1}^{q} B_{i,j}\phi_i(u)\phi_j(v).
\end{aligned}$$

It is clear from its definition that $\bar{c}_q$ is a density of copula. Therefore, $\bar{c}_q \in \mathcal{C}_\phi$ and the result is proved. □

Proof of Lemma 3. (i) Let $c \in L_2([0,1]^2)$ and suppose $I_p \in \mathcal{A}_\phi$. Then $q \in \mathcal{C}_\phi$ and

$$\begin{aligned}
\phi(u)^t\, T_\phi(c)\phi(v) &= \phi(u)^t\left\{\int_0^1\int_0^1 \phi(x)c(x,y)\phi(y)^t\, dxdy\right\}\phi(v) \\
&= \int_0^1\int_0^1 q(u,x)c(x,y)q(y,v)dxdy \\
&= (q \star c \star q)(u,v),
\end{aligned}$$

which proves that $\phi(u)^t\, T_\phi(c)\phi(v) \geq 0$ for all $(u,v) \in [0,1]^2$. Moreover,

$$\begin{aligned}
T_\phi(c)e_1 &= \int_0^1\int_0^1 \phi(x)c(x,y)\left\{\phi(y)e_1\right\}^t dydx = \int_0^1\int_0^1 \phi(x)c(x,y)dydx \\
&= \int_0^1 \phi(x)dx = e_1,
\end{aligned}$$

in view of (13) and similarly $T_\phi(c)^t\, e_1 = e_1$. As a conclusion, $T_\phi(c) \in \mathcal{A}_\phi$ and thus $P_\phi(c) \in \mathcal{C}_\phi$.
(ii) Conversely, if $P_\phi(c) \in \mathcal{C}_\phi$ for all $c \in L_2([0,1]^2)$, we have in particular $P_\phi(q) = q \in \mathcal{C}_\phi$ and thus $I_p \in \mathcal{A}_\phi$. □

Proof of Lemma 4. (i) Let $c_2 \in L_2([0,1]^2)$ and let $c_1 \in \mathcal{C}_\phi$ with associated matrix A. By definition,

$$\begin{aligned}
\prec c_1, c_2 \succ &= \int_0^1\int_0^1 c_1(u,v)c_2(v,u)dudv \\
&= \int_0^1\int_0^1 \phi(u)^t\, A\phi(v)c_2(v,u)dudv \\
&= \int_0^1\int_0^1 \operatorname{tr}\left\{\phi(u)^t\, A\phi(v)c_2(v,u)\right\} dudv \\
&= \int_0^1\int_0^1 \operatorname{tr}\left\{A\phi(v)c_2(v,u)\phi(u)^t\right\} dudv = \operatorname{tr}\left\{AT_\varphi(c_2)\right\}.
\end{aligned}$$

(ii) If, moreover, $c_2 \in \mathcal{C}_\phi$ with associated matrix B then $T_\varphi(c_2) = B$ and the conclusion follows. $\square$

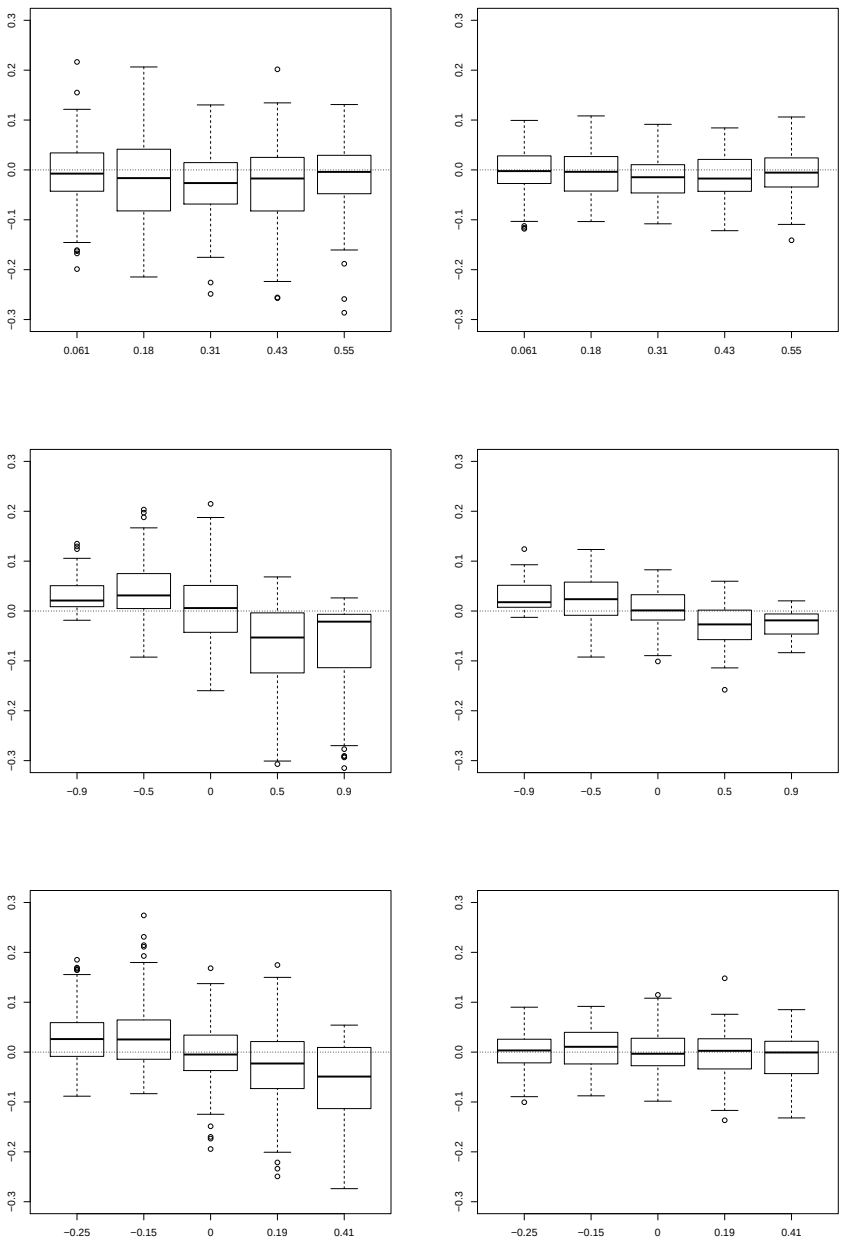

Figure 1: Boxplots of the estimation errors $(\hat{\rho}_{j,n} - \rho)$. Left: first estimator $(j = 1)$, right: second estimator $(j = 2)$. Top: Féjer copula, center: Gaussian copula, bottom: AMH copula. The theoretical values of ρ are displayed on the horizontal axes.