



HAL
open science

On the distribution functions of two oscillating sequences

Christoph Aistleitner, Markus Hofer, Manfred Madritsch

► **To cite this version:**

Christoph Aistleitner, Markus Hofer, Manfred Madritsch. On the distribution functions of two oscillating sequences. *Uniform Distribution Theory*, 2013, 8 (2), pp.157-169. hal-00875091

HAL Id: hal-00875091

<https://hal.science/hal-00875091>

Submitted on 21 Oct 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

On the distribution functions of two oscillating sequences

Christoph Aistleitner* Markus Hofer† Manfred Madritsch‡

Abstract

We investigate the set of all distribution functions of two special sequences on the unit interval, which involve logarithmic and trigonometric terms. We completely characterise the set of all distribution functions $G(x_n)$ for $(x_n)_{n \geq 1} = (\{\cos(\alpha n)^n\})_{n \geq 1}$ and arbitrary α , where $\{x\}$ denotes the fractional part of x . Furthermore we give a sufficient number-theoretic condition on α for which $(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}$ is uniformly distributed. Finally we calculate $G(x_n)$ in the case when $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

1 Introduction

In the present paper we consider the set of all distribution functions $G(x_n)$ of sequences $(x_n)_{n \geq 1}, x_n \in [0, 1)$. For an interval $I \subseteq [0, 1)$ we set $A(I, N, x_n)$ to be the number of hits of I among the first N elements of $(x_n)_{n \geq 1}$, i.e.,

$$A(I, N, x_n) = \#\{n \leq N : x_n \in I\} = \sum_{n=1}^N \mathbf{1}_I(x_n).$$

A non-decreasing function $g(x)$ satisfying $g(0) = 0, g(1) = 1$, is called a distribution function of a sequence $(x_n)_{n \geq 1}$ if there exists an increasing sequence $(N_k)_{k \geq 1}$ such that

$$g(x) = \lim_{k \rightarrow \infty} \frac{A([0, x), N_k, x_n)}{N_k}, \quad x \in [0, 1], \quad (1)$$

holds in every point of continuity of g . In the sequel $G(x_n)$ denotes the set of all functions for which (1) holds.

Furthermore a sequence $(x_n)_{n \geq 1}$ is said to have the asymptotic distribution function $g(x)$ if (1) holds for $N_k = k$. Then the set $G(x_n)$ reduces to a singleton. Moreover if $G(x_n) = \{x\}$ the sequence $(x_n)_{n \geq 1}$ is called uniformly distributed (u.d.).

Closely connected to distribution functions of sequences is the concept of the discrepancy of sequences. For a sequence $(x_n)_{n \geq 1}$ the discrepancy of the first N elements of $(x_n)_{n \geq 1}$ is given as

$$D_N((x_n)_{n \geq 1}) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta), N, x_n)}{N} - (\beta - \alpha) \right|.$$

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: aistleitner@math.tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23.

†Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: markus.hofer@tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23 and by the DOC [Doctoral Fellowship Programme of the Austrian Academy of Sciences].

‡Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: madritsch@tugraz.at.

Mathematics Subject Classification: 11K31 (11K06 11K38 11L07)

Keywords: Distribution modulo 1, uniform distribution, set of all distribution functions, oscillating sequences

Note that by a theorem of Weyl, see e.g. [?], it follows that $\lim_{N \rightarrow \infty} D_N((x_n)_{n \geq 1}) = 0$ holds if and only if that the sequence $(x_n)_{n \geq 1}$ is u.d. in $[0, 1)$. For general results on uniform distribution of sequences and discrepancy theory see [?], [?] or [?].

The following problem concerning the set of all distribution functions of a sequence is stated in the open problem section on the web site of *Uniform distribution theory*¹:

Find the set $G(x_n)$ for the following sequences:

$$(x_n)_{n \geq 1} = (\{\cos(n)^n\})_{n \geq 1}, \quad (2)$$

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}, \quad (3)$$

$$(x_n)_{n \geq 1} = (\{\cos(n + \log(n))\})_{n \geq 1}, \quad (4)$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.

For the sequence in (4) this problem has already been solved by S. Steinerberger. One can find a short version of the proof in the open problem section of *Uniform distribution theory*¹. The fact that the sequence in (4) is not u.d. has previously been proved by Kuipers, see [?].

The outline of the rest of the paper is as follows: in the second section we characterise the set of all distribution functions of a general version of the sequence given in (2). We use the Koksma inequality, see e.g. [?], in the third section to find a sufficient condition on the parameter α for which $(x_n)_{n \geq 1}$ given in (3) is u.d. Furthermore we give a complete solution of the problem in the case $\frac{\alpha}{2\pi} \in \mathbb{Q}$.

2 The set of all distribution functions of $(\{\cos(\alpha n)^n\})_{n \geq 1}$

Former results on the sequence (2) are due to Bukor [?] and in more generality Luca [?]. Bukor showed that $(\cos(n)^n)_{n \geq 1}$ is everywhere dense in the interval $[-1, 1]$. The following theorem characterises the set of distribution functions $G(x_n)$ for

$$(x_n)_{n \geq 1} = (\{\cos(\alpha n)^n\})_{n \geq 1},$$

which is a generalised version of (2).

Theorem 2.1 For $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ we set $a = 3/4$, in the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ for p, q co-prime let

$$a = \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : (\cos(\alpha n))^n \geq 0\}}{N} = \begin{cases} \frac{q+1}{2q} + \frac{q-1}{4q}, & \text{if } 4 \mid (q-1), \\ \frac{q-1}{2q} + \frac{q+1}{4q}, & \text{if } 4 \nmid (q-1) \end{cases} \quad (5)$$

for q odd and let

$$\begin{aligned} a &= \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : (\cos(\alpha n))^n \geq 0\}}{N} \\ &= \frac{\#\{1 \leq n \leq q : (\cos(\alpha n))^n \geq 0\}}{q} = \begin{cases} \frac{1}{2} + \frac{q-2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \mid (q-2), \\ \frac{1}{2} + \frac{q+2}{4q}, & \text{if } 4 \nmid q \text{ and } 8 \nmid (q-2), \\ \frac{q+2}{2q} + \frac{1}{4}, & \text{if } 4 \mid q \text{ and } 8 \nmid q, \\ \frac{q+2}{2q} + \frac{q-4}{4q}, & \text{if } 8 \mid q \end{cases} \quad (6) \end{aligned}$$

¹Problem 1.10 in the open problem collection as of 11. December 2011 (<http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf>)

for q even.

Then the set of all distribution functions of $(x_n)_{n \geq 1}$ is given by $G(x_n) = \{g_a(x)\}$, with

$$g_a(x) = \begin{cases} 0, & \text{if } x = 0, \\ a, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Proof:

First we consider the case $\frac{\alpha}{2\pi} \notin \mathbb{Q}$. Let $\epsilon, \delta > 0$ be small and fixed. Then for sufficiently large $N_0 \in \mathbb{N}$ we have

$$\lambda(\{x \in [0, 1]: |\cos(2\pi x)^{N_0}| > \delta\}) < \epsilon,$$

where λ denotes the Lebesgue measure. By the fact that $(\{\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is u.d. in $[0, 1]$ if $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ (see e.g. [?]) it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : |\cos(\alpha n)^n| > \delta\}}{N} < \epsilon.$$

Furthermore since

$$\lambda(\{x \in [0, 1]: \cos(2\pi x) < 0\}) = \frac{1}{2}$$

and by the fact that also $(\{2\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is u.d. in $[0, 1]$ it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 0\}}{N} = \frac{1}{4}.$$

Noting that $|\cos(\alpha n)^n| \leq \delta$ and $\cos(\alpha n)^n < 0$ imply $\{\cos(\alpha n)^n\} \geq 1 - \delta$. Thus we have

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n > \delta\}}{N} \geq \frac{3}{4} - \epsilon$$

and

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \cos(\alpha n)^n < 1 - \delta\}}{N} \leq \frac{3}{4} + \epsilon$$

Since ϵ and δ are arbitrary, this proves the first part of the theorem.

In the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ we easily see that $(\{\cos(\alpha n)\})_{n \geq 1}$ takes only finitely many different values of the form $\cos\left(2\pi\frac{j}{q}\right)$, $j = 1, \dots, q$, which appear periodically with period q .

Consider the case that q is odd. We want to calculate k given as

$$k = \#\left\{j \in \{1, \dots, q\} : \cos\left(2\pi\frac{j}{q}\right) \geq 0\right\}.$$

By $\cos\left(2\pi\frac{q}{q}\right) = 1$ and

$$\cos\left(2\pi\frac{j}{q}\right) = \cos\left(2\pi\frac{q-j}{q}\right), \quad \text{for } j = 1, \dots, q,$$

it follows that $k = 2l - 1$, where l is the maximal value in $\{1, \dots, q\}$ for which

$$2\pi\frac{l}{q} < \frac{\pi}{2} \Leftrightarrow l < \frac{q}{4}.$$

Thus if $4 \mid (q - 1)$ we have $l = \frac{q-1}{4}$ and it follows $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q+1}{2}$ values of $j = 1, \dots, q$.

Similarly if $4 \nmid (q - 1)$ we have $l = \frac{q-3}{4}$ and $\cos\left(2\pi\frac{j}{q}\right) \geq 0$ for $\frac{q-1}{2}$ values of $j = 1, \dots, q$.

Since q is odd it follows for every $j = 1, \dots, q$ that the term $\cos\left(2\pi\frac{j}{q}\right)$ appears in $(\cos(\alpha n)^n)_{n \geq 1}$ alternating with odd and even exponent, thus for i such that $\cos(\alpha i) < 0$ it follows that

$$\lim_{N \rightarrow \infty} \frac{\#\{n \leq N : \cos(\alpha i)^{qn+i} > 0\}}{N} = \frac{1}{2}.$$

This, together with the above considerations, proves (5).

Assume now that q is even and $4 \nmid q$. Then $\cos(\alpha i) \neq 0$ for $i = 1, \dots, q$ and by similar considerations as above we have $\cos(\alpha i) > 0$ for $\frac{q}{2}$ values of i and $\frac{q}{2}$ is an odd number. Furthermore for every $j = 1, \dots, q$, the exponent of the term $\cos\left(2\pi\frac{j}{q}\right)$ in $(\cos(\alpha n)^n)_{n \geq 1}$ is either always odd or always even. Moreover it is easy to see that the exponents of $\cos\left(2\pi\frac{j}{q}\right)$ and $\cos\left(2\pi\frac{j+1}{q}\right)$ in $(\cos(\alpha n)^n)_{n \geq 1}$ cannot be both even or both odd. Thus the number of j 's such that $\cos\left(2\pi\frac{j}{q}\right) < 0$ and which appear with even exponent is $\frac{q-2}{4}$ when $8 \mid (q-2)$ and $\frac{q+2}{4}$ when $8 \nmid (q-2)$.

In the case $4 \mid q$, it follows that $\cos(\alpha i) = 0$ for two values of i . Thus $\cos(\alpha i) \geq 0$ for $\frac{q+2}{2}$ values of $i = 1, \dots, q$. Formula (6) follows now by the similar arguments as above. This completes the proof. \square

3 The set of all distribution functions of $(\{\log(n) \cos(\alpha n)\})_{n \geq 1}$

Properties of the sequence (3) have been studied by Berend, Boshernitzan and Kolesnik [?]. They proved the denseness of (3) in $[0, 1)$ for arbitrary α . Furthermore they also concluded in [?] that sequences of the form

$$(\{n^a \log^b(n) \cos(2\pi n\alpha)\})_{n \geq 1} \tag{7}$$

are dense in $[0, 1)$, provided α is irrational and either $a > 0$ or $a = 0, b > 0$. Moreover they showed that the sequence in (7) is u.d. for α irrational and either $a > 0$ or $a = 0, b > 1$. Additionally they remarked without proof that the sequence (3) is not u.d. for uncountably many α , see [?].

The next theorem gives a condition on the parameter α which implies the u.d. property for the sequence in (3).

Theorem 3.1 *Let α be such that the discrepancy of the sequence $(z_n)_{n \geq 1} = (\{\frac{\alpha}{2\pi}n\})_{n \geq 1}$ is of asymptotic order $o\left(\frac{1}{\log(N)}\right)$. Then the sequence*

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}$$

is u.d.

Remark 3.1 *It is well-known that there is a close connection between the coefficients of the continued fraction expansion of α and the asymptotic order of the discrepancy of $(\{\alpha n\})_{n \geq 1}$. By a classical result of Khintchine [?] from the metric theory of continued fractions, the discrepancy of $(\{\alpha n\})_{n \geq 1}$ satisfies*

$$D_N(\{\alpha\}, \dots, \{\alpha N\}) = \mathcal{O}(N^{-1}(\log \log N)^{1+\epsilon})$$

as $N \rightarrow \infty$ for almost all α . Thus the conclusion of Theorem 3.1 holds for almost all α .

Proof:

We have to show that for every interval $[a, b] \subseteq [0, 1]$

$$\lim_{N \rightarrow \infty} \frac{A([a, b], N, x_n)}{N} = b - a$$

holds. Fix N sufficiently large and $\epsilon, \delta > 0$ small and let the sequence $(y_n)_{n \geq \epsilon N + 1}$ be given by

$$(y_n)_{n \geq \epsilon N + 1} = \{\log(\epsilon N(1 + \delta)^{k(n)}) \cos(\alpha n)\}_{n \geq \epsilon N + 1}$$

where $k(n): \{1, \dots, N\} \rightarrow \mathbb{N}$ is such that

$$\epsilon N(1 + \delta)^{k(n)-1} < n \leq \epsilon N(1 + \delta)^{k(n)}.$$

For the sake of simplicity we assume that ϵ and δ are chosen such that $\epsilon N \in \mathbb{N}$ and $\epsilon N(1 + \delta)^{k(N)} = N$. Note that $k(n)$ is uniformly bounded from above for all $1 < n \leq N$ by

$$k(n) \leq \frac{-\log(\epsilon)}{\log(1 + \delta)}.$$

We have

$$\begin{aligned} & \left| \frac{1}{N} \left(\sum_{k=1}^N \mathbf{1}_{\{x_k \in [a, b]\}} \right) - (b - a) \right| \\ &= \left| \frac{1}{N} \left(\sum_{k=1}^{\epsilon N} \mathbf{1}_{\{x_k \in [a, b]\}} \right) + \frac{1}{N} \left(\sum_{k=\epsilon N + 1}^N (\mathbf{1}_{\{x_k \in [a, b]\}} - \mathbf{1}_{\{y_k \in [a, b]\}} + \mathbf{1}_{\{y_k \in [a, b]\}}) \right) - (b - a) \right| \\ &\leq \left| \frac{1}{N} \left(\sum_{k=1}^{\epsilon N} \mathbf{1}_{\{x_k \in [a, b]\}} \right) \right| + \left| \frac{1}{N} \left(\sum_{k=\epsilon N + 1}^N (\mathbf{1}_{\{x_k \in [a, b]\}} - \mathbf{1}_{\{y_k \in [a, b]\}}) \right) \right| \\ &\quad + \left| \frac{1}{N} \left(\sum_{k=\epsilon N + 1}^N \mathbf{1}_{\{y_k \in [a, b]\}} \right) - (b - a) \right|. \end{aligned} \tag{8}$$

Furthermore note that we can write

$$\cos(\alpha n) = \cos \left(2\pi \left(\left\lfloor \frac{\alpha n}{2\pi} \right\rfloor + \left\{ \frac{\alpha n}{2\pi} \right\} \right) \right) = \cos \left(2\pi \left\{ \frac{\alpha n}{2\pi} \right\} \right),$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x .

Thus we get for the third term on the right hand-side of (8) by using the Koksma inequality for the functions $f_i(x) = \mathbf{1}_{\{\log(\epsilon N(1 + \delta)^i) \cos(2\pi x) \in [a, b]\}}$ and the sequence $(z_n)_{n \geq 1}$ that

$$\begin{aligned} & \left| \frac{1}{N} \left(\sum_{k=\epsilon N + 1}^N \mathbf{1}_{\{y_k \in [a, b]\}} \right) - (b - a) \right| \\ &\leq \sum_{i=1}^{k(N)} \frac{c_i N}{N} \left| \frac{1}{c_i N} \left(\sum_{\epsilon N(1 + \delta)^{(i-1)} < k \leq \epsilon N(1 + \delta)^i} \mathbf{1}_{\{y_k \in [a, b]\}} \right) - (b - a) \right| + \left| \frac{\epsilon N}{N} (b - a) \right| \\ &\leq \sum_{i=1}^{k(N)} \frac{c_i N}{N} V(f_i(x)) D_{N_i}((z_n)_{\epsilon N(1 + \delta)^{(i-1)} < n \leq \epsilon N(1 + \delta)^i}) + \epsilon(b - a) \end{aligned}$$

$$\begin{aligned}
&\leq k(N)c_{k(N)}V(f_{k(N)}(x))D_{N_i}((z_n)_{\epsilon N(1+\delta)^{(i-1)} < n \leq \epsilon N(1+\delta)^i}) + \epsilon(b-a) \\
&= \epsilon(b-a) + o\left(\frac{\log(\epsilon N(1+\delta)^{k(N)})}{\log(N)}\right) = \epsilon(b-a) + o\left(\frac{\log(N)}{\log(N)}\right) = \epsilon(b-a) + o(1), \quad (9)
\end{aligned}$$

where $c_i = \epsilon((1+\delta)^i - (1+\delta)^{(i-1)})$, $i = 1, \dots, k(N)$, $V(f(x))$ denotes the variation of the function f and $D_N((z_n)_{a \leq n \leq b})$ denotes the discrepancy of N consecutive elements of the sequence $(z_n)_{a \leq n \leq b}$. Note that for any $M \geq \epsilon N$ for some fixed $\epsilon > 0$ we have $1/\log M = O(1/\log N)$. This implies $D_M = O(1/\log N)$, uniformly for all $M \geq \epsilon N$. For $n \geq \epsilon N$ we have

$$\begin{aligned}
|x_n - y_n| &= \left| \log(n) \cos(\alpha n) - \log(\epsilon N(1+\delta)^{k(n)}) \cos(\alpha n) \right| \\
&= (\log(\epsilon N(1+\delta)^{k(n)}) - \log(n)) |\cos(\alpha n)| \\
&\leq (\log(\epsilon N(1+\delta)^{k(n)}) - \log(\epsilon N(1+\delta)^{k(n)-1})) \\
&= \log(1+\delta).
\end{aligned}$$

By (9) it follows for sufficiently large N that the number of $\epsilon N < n \leq N$ for which $|\mathbf{1}_{\{x_n \in [a,b]\}} - \mathbf{1}_{\{y_n \in [a,b]\}}| = 1$ is at most $5 \log(1+\delta)N$. Thus for the second term on the right hand-side of (8) we get

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \left(\sum_{k=\epsilon N+1}^N (\mathbf{1}_{\{x_k \in [a,b]\}} - \mathbf{1}_{\{y_k \in [a,b]\}}) \right) \right| \leq 5 \log(1+\delta).$$

Finally it follows for (8) that

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \left| \frac{1}{N} \left(\sum_{k=1}^{\epsilon N} \mathbf{1}_{\{x_k \in [a,b]\}} \right) \right| + \lim_{N \rightarrow \infty} \left| \frac{1}{N} \left(\sum_{k=\epsilon N+1}^N \mathbf{1}_{\{x_k \in [a,b]\}} - \mathbf{1}_{\{y_k \in [a,b]\}} \right) \right| \\
&+ \lim_{N \rightarrow \infty} \left| \frac{1}{N} \left(\sum_{k=\epsilon N+1}^N \mathbf{1}_{\{y_k \in [a,b]\}} \right) - (b-a) \right| \\
&\leq \epsilon + 5 \log(1+\delta) + \epsilon(b-a) + o(1) = 2\epsilon + 5 \log(1+\delta)
\end{aligned}$$

which proves the theorem. \square

The following lemma by Pólya and Szegő [?] characterises the set of distribution functions $G(\{c \log(n)\})$ and is the main tool for the proof of Theorem 3.2.

Lemma 3.1 (Pólya and Szegő) *The sequence $(x_n)_{n \geq 1} = (\{c \log n\})_{n \geq 1}$, $c > 0$, has distribution functions of the form*

$$g_{\beta,c}(x) = \frac{e^{\frac{\min(x,\beta)}{c}} - 1}{e^{\frac{\beta}{c}} - 1} + \frac{1}{e^{\frac{\beta}{c}}} \frac{e^{\frac{x}{c}} - 1}{e^{\frac{1}{c}} - 1}, \quad (10)$$

where $\lim_{k \rightarrow \infty} \{c \log N_k\} = \beta$ implies $F_{N_k}(x) \rightarrow g_{\beta}(x)$ and $F_N(x) = \frac{\#\{n \leq N; x_n \in [0,x]\}}{N}$. Moreover $G(\{c \log(n)\})$ is the set of all distribution functions of the form (10).

Remark 3.2 *Note that it follows as a corollary of Lemma 3.1 that the sequence $(x_n)_{n \geq 1} = (\{c \log(n)\})_{n \geq 1}$, $c < 0$, has distribution functions of the form $1 - g_{\beta,|c|}(1-x)$, where $\lim_{k \rightarrow \infty} \{|c| \log N_k\} = \beta$ and $g_{\beta,|c|}(x)$ is given in (10). Thus we define the function $f_{\beta,c}(x)$ as*

$$f_{\beta,c}(x) = \begin{cases} g_{\beta,c}(x), & \text{if } c > 0, \\ 1 - g_{\beta,|c|}(1-x) & \text{if } c < 0, \\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c = 0, \end{cases} \quad (11)$$

where $\lim_{k \rightarrow \infty} \{ |c| \log N_k \} = \beta$.

Theorem 3.2 Let $\frac{p}{q} := \frac{\alpha}{2\pi} \in \mathbb{Q}$ where p, q are co-prime and

$$(x_n)_{n \geq 1} = (\{\log(n) \cos(\alpha n)\})_{n \geq 1}.$$

Then for a fixed subsequence $(N_k)_{k \geq 1}$ of \mathbb{N} with

$$\lim_{k \rightarrow \infty} \{ \cos(\alpha i) \log N_k \} = \beta_i, \quad \text{for } i = 1, \dots, q, \quad (12)$$

the asymptotic distribution of $(x_n)_{n \geq 1}$ along the subsequence $(N_k)_{k \geq 1}$ is given by

$$f(x) = \frac{1}{q} \sum_{i=1}^q h_{q, \beta_i, c_i}(x), \quad (13)$$

where $h_{q, \beta_i, c_i}(x)$ is given in (11),

$$h_{q, \beta_i, c_i}(x) = \begin{cases} f_{\beta_i, c_i}(x + 1 - \nu_i) - f_{\beta_i, c_i}(1 - \nu_i), & \text{if } 0 \leq x \leq \nu_i \text{ and } c_i > 0, \\ f_{\beta_i, c_i}(x - \nu_i) + 1 - f_{\beta_i, c_i}(1 - \nu_i), & \text{if } \nu_i \leq x \leq 1 \text{ and } c_i > 0, \\ f_{\beta_i, c_i}(x + \nu_i) - f_{\beta_i, c_i}(\nu_i), & \text{if } 0 \leq x \leq 1 - \nu_i \text{ and } c_i < 0, \\ f_{\beta_i, c_i}(x - (1 - \nu_i)) + 1 - f_{\beta_i, c_i}(\nu_i), & \text{if } 1 - \nu_i \leq x \leq 1 \text{ and } c_i < 0, \\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c_i = 0, \end{cases} \quad (14)$$

where $\nu_i = \{ |c_i| \log(q) \}$, $c_i = \cos(\alpha i)$, $\lim_{n \rightarrow \infty} \{ \cos(\alpha i) \log N_k \} = \beta_i$ and $f_{\beta, c}(x)$ is given in (11). Moreover, the set $G(x_n)$ is the set of all distribution functions of the form (13) for those $(\beta_1, \dots, \beta_q)$ for which a subsequence $(N_k)_{k \geq 1}$ satisfying (12) exists.

Remark 3.3 For arbitrary q , it is a difficult problem to determine all possible vectors $(\beta_1, \dots, \beta_q)$ for which there exists a subsequence $(N_k)_{k \geq 1}$ for which (12) holds, because there can exist non-trivial linear relations between the values $\cos(\alpha i)$, $i = 1, \dots, q$ (depending on number-theoretic properties of q). We will not further investigate this issue, the interested reader is referred to a Galois theoretic approach to this problem by Girstmair [?].

Proof:

As mentioned in the proof of Theorem 2.1 the function $\cos(\alpha n)$, $n \in \mathbb{N}$ takes only finitely many different values which appear in a period of length q . Let $(N_k)_{k \geq 1}$ be a sequence of \mathbb{N} for which (12) holds for some numbers β_1, \dots, β_q . We are interested in the asymptotic behavior of $(x_n)_{n \geq 1}$ and by

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{A([a, b], N_k, x_n)}{N_k} &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^q A([a, b], \lfloor N_k/q \rfloor, \cos(\alpha i) \log(qn + i))}{N_k} \\ &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^q A([a, b], \lfloor N_k/q \rfloor, \cos(\alpha i) \log(qn))}{N_k}, \end{aligned}$$

we get that the limit distribution of $(x_n)_{n \geq 1}$ is a mixture of limit distributions of q sequences of the form

$$(z_n^i)_{n \geq 1} = (\{\log(qn) \cos(\alpha i)\})_{n \geq 1}, \quad \text{for } i \in \{1, \dots, q\}.$$

Since $\log(qn) \cos(\alpha i) = \log(q) \cos(\alpha i) + \log(n) \cos(\alpha i)$ we get that the distribution function of $(z_n^i)_{n \geq 1}$ is given as the distribution function of $\log(n) \cos(\alpha i)$ shifted by a constant and thus it is easy to see that the right hand-side of (14) is the distribution function of $(z_n^i)_{n \geq 1}$. This proves (13).

In order to prove that all functions in $G(x_n)$ can be characterised by (13) we use the Bolzano-Weierstrass

theorem. Assume that $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ does not converge for at least one i , then it follows that there are at least two convergent subsequences of $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ which have different limits. If the limit distributions along all these subsequences are equal then the limit distribution of $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ along the whole sequence $(N_k)_{k \geq 1}$ is (if it exists) can be written as $f_{\beta,c}(h(x))$ as the limit along a sequence for which (12) holds. If at least two limit distributions along subsequences are not equal the limit distribution along $(\{\cos(\alpha i) \log N_k\})_{k \geq 1}$ does not exist. \square

Remark 3.4 We illustrate the set $G(x_n)$ in the simple case when $\alpha = \pi$, which means that $p = 1$ and $q = 2$ in the notation of Theorem 3.2. Note that $\cos(\alpha) = -1$ and $\cos(2\alpha) = 1$, thus as a sequence $(N_k)_{k \geq 1}$ which satisfies (12) one can choose for example $N_k = \lfloor \exp(k + \beta_1) \rfloor$. In this case we have

$$\lim_{k \rightarrow \infty} \{\cos(\alpha) \log(N_k)\} = \beta_1$$

and

$$\lim_{k \rightarrow \infty} \{\cos(2\alpha) \log(N_k)\} = \beta_2 = 1 - \beta_1.$$

We can calculate the corresponding distribution function by using the previous theorem. Figure 1 illustrates the range of distribution functions which one can achieve by varying β .



Figure 1: Distribution functions of $(x_n)_{n \geq 1}$ for $\alpha = \pi$ and $\beta = 0, \frac{1}{10}, \dots, \frac{9}{10}$.

ACKNOWLEDGEMENTS

The authors want to thank Kurt Girstmair from University of Innsbruck for helpful remarks on linear relations between the values of trigonometric functions. They also appreciate the interesting and fruitful discussions with Volker Ziegler from TU Graz on this topic.