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Stability properties for quasilinear parabolic equations with measure data and applications

Marie-Françoise BIDAUT-VERON*

Hung NGUYEN QUOC†

Abstract

Let Ω be a bounded domain of \mathbb{R}^N , and $Q = \Omega \times (0, T)$. We first study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $p > 1$, $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators $u \mapsto \mathcal{A}(u) = \operatorname{div}(A(x, t, \nabla u))$.

As an application, we consider perturbed problems *of type*

$$\begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{G}(u)$ may be an absorption or a source term. In the model case $\mathcal{G}(u) = \pm |u|^{q-1} u$ ($q > p - 1$), or \mathcal{G} has an exponential type. We give existence results when q is subcritical, or when the measure μ is good in time and satisfies suitable capacity conditions.

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1 Introduction

Let Ω be a bounded domain of \mathbb{R}^N , and $Q = \Omega \times (0, T)$, $T > 0$. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively. We are concerned with the problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and A is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for *a.e.* $(x, t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x, t, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, t, \xi)| \leq a(x, t) + c_2 |\xi|^{p-1}, \quad c_1, c_2 > 0, a \in L^{p'}(Q), \quad (1.2)$$

$$(A(x, t, \xi) - A(x, t, \zeta)) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta. \quad (1.3)$$

This includes the model problem

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.4)$$

where Δ_p is the p -Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p > 1$. As an application, we consider problems with a nonlinear term of order 0:

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.5)$$

where A is a Caratheodory function on $\Omega \times \mathbb{R}^N$, such that, for *a.e.* $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, \xi)| \leq c_2 |\xi|^{p-1}, \quad c_3, c_4 > 0, \quad (1.6)$$

$$(A(x, \xi) - A(x, \zeta)) \cdot (\xi - \zeta) > 0 \text{ if } \xi \neq \zeta, \quad (1.7)$$

and $\mathcal{G}(u)$ may be an absorption or a source term, and possibly depends on $(x, t) \in Q$. The model problem is the case where \mathcal{G} has a power-type $\mathcal{G}(u) = \pm |u|^{q-1} u$ ($q > p - 1$), or an exponential type.

First make a brief survey of the elliptic associated problem:

$$\begin{cases} -\operatorname{div}(A(x, \nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu \in \mathcal{M}_b(\Omega)$ and assumptions (1.6), (1.7). When $p = 2$, $A(x, \nabla u) = \nabla u$ existence and uniqueness are proved for general elliptic operators by duality methods in [58]. For $p > 2 - 1/N$, the existence of solutions in the sense of distributions is obtained in [23] and [24]. The condition

on p ensures that the gradient ∇u is well defined in $(L^1(\Omega))^N$. For general $p > 1$, new classes of solutions are introduced, first when $\mu \in L^1(\Omega)$, such as *entropy solutions*, and *renormalized solutions*, see [13], and also [57], and existence and uniqueness is obtained. For any $\mu \in \mathcal{M}_b(\Omega)$ the main work is done in [32, Theorems 3.1, 3.2], where not only existence is proved, but also a stability result, fundamental for applications. Uniqueness is still an open problem.

Next we make a brief survey about problem (1.1).

The first studies concern the case $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness is obtained by variational methods, see [45]. In the general case $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$, the pionner results come from [23], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \quad (1.8)$$

see also [55], [56], and [26]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces $u \in L^{p_c, \infty}(Q)$ and $|\nabla u| \in L^{m_c, \infty}(Q)$, where

$$p_c = p - 1 + \frac{p}{N}, \quad m_c = p - \frac{N}{N+1}. \quad (1.9)$$

This condition (1.8) ensures that u and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N+1)$. Uniqueness follows in the case $p = 2$, $A(x, t, \nabla u) = \nabla u$, by duality methods, see [48].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of solutions: *entropy solutions*, and *renormalized solutions*, see [19], [54], see also [3] for a semi-group approach.

Then a class of *regular* measures is studied in [33], where a notion of parabolic capacity c_p^Q is introduced, defined by

$$c_p^Q(E) = \inf_{E \subset U} \inf_{\text{open } \subset Q} \{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \},$$

for any Borel set $E \subset Q$, where

$$\begin{aligned} X &= L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega)), \\ W &= \{z : z \in X, z_t \in X'\}, \text{ embedded with the norm } \|u\|_W = \|u\|_X + \|u_t\|_{X'}. \end{aligned}$$

Let $\mathcal{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c_p^Q -capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \implies |\mu(E)| = 0.$$

Then existence and uniqueness of renormalized solutions holds for any measure $\mu \in \mathcal{M}_b(\Omega) \cap \mathcal{M}_0(Q)$, called *regular* (or *diffuse*) and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [34]; see also [20] for more general equations.

Next consider *any* measure $\mu \in \mathcal{M}_b(Q)$. Let $\mathcal{M}_s(Q)$ be the set of all bounded Radon measures on Q with support on a set of zero c_p^Q capacity, also called *singular*. Let $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$

be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$. From [33], μ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \quad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \quad (1.10)$$

and $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \quad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \quad (1.11)$$

and we write $\mu_0 = (f, g, h)$. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [19],[49]. In the range (1.8) the existence of a renormalized solution relative to the decomposition (1.11) is proved in [49], using suitable approximations of μ_0 and μ_s . Uniqueness is still open, as well as in the elliptic case.

Next consider the problem (1.5). First we consider the case of an *absorption term*: $\mathcal{G}(u)u \geq 0$.

Let us recall the case $p = 2$, $A(x, \nabla u) = \nabla u$ and $\mathcal{G}(u) = |u|^{q-1}u$ ($q > 1$). The first results concern the case $\mu = 0$ and u_0 is a Dirac mass in Ω , see [28]: existence holds if and only if $q < (N + 2)/N$. Then optimal results are given in [7], for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $C_{\alpha,k}$, ($\alpha, k > 1$) defined, for any Borel set $E \subset \mathbb{R}^N$, by

$$C_{\alpha,k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N), \quad G_\alpha * \varphi \geq \chi_E\},$$

where G_α is the Bessel kernel of order α ; and a capacity $c_{\mathbf{G},k}$ ($k > 1$) adapted to the operator of the heat equation of kernel $\mathbf{G}(x, t) = \chi_{(0,\infty)}(4\pi t)^{-N/2} e^{-|x|^2/4t}$: for any Borel set $E \subset \mathbb{R}^{N+1}$,

$$c_{\mathbf{G},k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}), \quad \mathbf{G} * \varphi \geq \chi_E\}.$$

From [7], there exists a solution if and only if μ does not charge the sets of $c_{\mathbf{G},q'}(E)$ capacity zero and u_0 does not charge the sets of $C_{2/q,q'}$ capacity zero. Observe that one can reduce to a zero initial data, by considering the measure $\mu + u_0 \otimes \delta_0^t$ in $\Omega \times (-T, T)$, where \otimes is the tensorial product and δ_0^t is the Dirac mass in time at 0.

For $p \neq 2$ such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case $\mu = 0$ with Ω bounded, or $\Omega = \mathbb{R}^N$. The case where u_0 is a Dirac mass in Ω is studied in [36], [40] when $p > 2$, and [29] when $p < 2$. Existence and uniqueness hold in the subcritical case $q < p_c$. If $q \geq p_c$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathcal{M}_b^+(\Omega)$, the existence is obtained in the sense of distributions in [60], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [16]. The case $\mu \in L^1(Q)$, $u_0 = 0$ is treated in [30], and $\mu \in L^1(Q)$, $u_0 = L^1(\Omega)$ in [4] where \mathcal{G} can be multivalued. The case $\mu \in \mathcal{M}_0(Q)$ is studied in [50], with a new formulation of the solutions, and existence and uniqueness is obtained for any function $\mathcal{G} \in C(\mathbb{R})$ such that $\mathcal{G}(u)u \geq 0$. Up to our knowledge, up to now no existence results have been obtained for a general measure $\mu \in \mathcal{M}_b(Q)$.

The case of a source term $\mathcal{G}(u) = -u^q$ with $u \geq 0$ has been treated in [6] for $p = 2$, where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general p .

2 Main results

In all the sequel we suppose that p satisfies (1.8). Then

$$X = L^p(0, T; W_0^{1,p}(\Omega)), \quad X' = L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

We first study problem (1.1). In Section 3 we give some approximations of $\mu \in \mathcal{M}_b(Q)$, useful for the applications. In Section 4 we recall the definition of renormalized solutions, that we call R-solutions of (1.1), relative to the decomposition (1.11) of μ_0 , and study some of their properties.

Our main result is a *stability theorem* for problem (1.1), proved in Section 5, extending to the parabolic case the stability result of [32, Theorem 3.4], and improving the result of [49]:

Theorem 2.1 *Let $A : Q \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.2), (1.3). Let $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

with $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$, $h \in X$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

with $f_n \in L^1(Q)$, $g_n \in (L^{p'}(Q))^N$, $h_n \in X$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q)$, $\rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n|(Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in X , $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology of measures; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$. Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} u_{n,t} - \operatorname{div}(A(x, t, \nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \quad (2.1)$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$. Then up to a subsequence, $\{u_n\}$ converges a.e. in Q to a R-solution u of (1.1), and $\{v_n\}$ converges a.e. in Q to $v = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla v_n\}$ converge respectively to $\nabla u, \nabla v$ a.e. in Q , and $\{T_k(u_n)\}, \{T_k(v_n)\}$ converge to $T_k(u), T_k(v)$ strongly in X for any $k > 0$.

In Section 6 we give applications to problems of type (1.5).

We first give an existence result of subcritical type, valid for any measure $\mu \in \mathcal{M}_b(Q)$:

Theorem 2.2 Let $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying (1.2), (1.3) with $a \equiv 0$. Let $(x, t, r) \mapsto \mathcal{G}(x, t, r)$ be a Caratheodory function on $Q \times \mathbb{R}$ and $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in \mathbb{R}^+ , such that

$$|\mathcal{G}(x, t, r)| \leq G(|r|) \quad \text{for a.e. } (x, t) \in Q \text{ and any } r \in \mathbb{R}, \quad (2.2)$$

$$\int_1^\infty G(s)s^{-1-p_c} ds < \infty. \quad (2.3)$$

(i) Suppose that $\mathcal{G}(x, t, r)r \geq 0$, for a.e. (x, t) in Q and any $r \in \mathbb{R}$. Then, for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a R -solution u of problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.4)$$

(ii) Suppose that $\mathcal{G}(x, t, r)r \leq 0$, for a.e. $(x, t) \in Q$ and any $r \in \mathbb{R}$, and $u_0 \geq 0, \mu \geq 0$. There exists $\varepsilon > 0$ such that for any $\lambda > 0$, any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$ with $\lambda + |\mu|(Q) + \|u_0\|_{L^1(\Omega)} \leq \varepsilon$, problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \lambda \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (2.5)$$

admits a nonnegative R -solution.

In particular for any $0 < q < p_c$, if $\mathcal{G}(u) = |u|^{q-1}u$, existence holds for any measure $\mu \in \mathcal{M}_b(Q)$; if $\mathcal{G}(u) = -|u|^{q-1}u$, existence holds for μ small enough. In the supercritical case $q \geq p_c$, the class of "admissible" measures, for which there exist solutions, is not known.

Next we give new results relative to *measures that have a good behaviour in t* , based on recent results of [17] relative to the elliptic case. We recall the notions of (truncated) Wölf potential for any nonnegative measure $\omega \in \mathcal{M}^+(\mathbb{R}^N)$ any $R > 0, x_0 \in \mathbb{R}^N$,

$$\mathbf{W}_{1,p}^R[\omega](x_0) = \int_0^R (t^{p-N}\omega(B(x_0, t)))^{\frac{1}{p-1}} \frac{dt}{t}.$$

Any measure $\omega \in \mathcal{M}_b(\Omega)$ is identified with its extension by 0 to \mathbb{R}^N . In case of absorption, we obtain the following:

Theorem 2.3 Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N, q > p - 1, \mu \in \mathcal{M}_b(Q), f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that

$$|\mu| \leq \omega \otimes F, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega), F \in L^1((0, T)), F \geq 0,$$

and ω does not charge the sets of $C_{p, \frac{q}{q+1-p}}$ -capacity zero. Then there exists a R -solution u of problem

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + |u|^{q-1}u = f + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.6)$$

We show that some of these measures may not lie in $\mathcal{M}_0(Q)$, which improves the existence results of [50], see Proposition 3.3 and Remark 6.7. Otherwise our result can be extended to a more general function \mathcal{G} , see Remark 6.9. We also consider a source term:

Theorem 2.4 *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N$, $q > p-1$. Let $\mu \in \mathcal{M}_b^+(Q)$, and $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. Assume that*

$$\mu \leq \omega \otimes \chi_{(0,T)}, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega).$$

Then there exist $\lambda_0 = \lambda_0(N, p, q, c_3, c_4, \text{diam}\Omega)$ and $b_0 = b_0(N, p, q, c_3, c_4, \text{diam}\Omega)$ such that, if

$$\omega(E) \leq \lambda_0 C_{p, \frac{q}{q-p+1}}(E), \quad \forall E \text{ compact } \subset \mathbb{R}^N, \quad \|u_0\|_{\infty, \Omega} \leq b_0, \quad (2.7)$$

there exists a nonnegative R -solution u of problem

$$\begin{cases} u_t - \text{div}(A(x, \nabla u)) = u^q + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (2.8)$$

which satisfies, a.e. in Q ,

$$u(x, t) \leq C \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega](x) + 2\|u_0\|_{L^\infty}, \quad (2.9)$$

where $C = C(N, p, c_3, c_4)$.

Corresponding results in case where \mathcal{G} has exponential type are given at Theorems 6.10 and 6.15.

3 Approximations of measures

For any open set ϖ of \mathbb{R}^m and $F \in (L^k(\varpi))^\nu$, $k \in [1, \infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k, \varpi} = \|F\|_{(L^k(\varpi))^\nu}$.

First we give approximations of nonnegative measures in $\mathcal{M}_0(Q)$. We recall that any measure $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$ admits a decomposition under the form $\mu = (f, g, h)$ given by (1.11). Conversely, any measure of this form, such that $h \in L^\infty(Q)$, lies in $\mathcal{M}_0(Q)$, see [50, Proposition 3.1].

Lemma 3.1 *Let $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$.*

(i) Then, we can find a decomposition $\mu = (f, g, h)$ with $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$, $h \in X$ such that

$$\|f\|_{1, Q} + \|g\|_{p', Q} + \|h\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|g\|_{p', Q} + \|h\|_X \leq \varepsilon. \quad (3.1)$$

(ii) Furthermore, there exists a sequence of measures $\mu_n = (f_n, g_n, h_n)$, such that $f_n, g_n, h_n \in C_c^\infty(Q)$ and strongly converge to f, g, h in $L^1(Q)$, $(L^{p'}(Q))^N$ and X respectively, and μ_n converges to μ in the narrow topology, and satisfying

$$\|f_n\|_{1, Q} + \|g_n\|_{p', Q} + \|h_n\|_X \leq (1 + 2\varepsilon)\mu(Q), \quad \|g_n\|_{p', Q} + \|h_n\|_X \leq 2\varepsilon. \quad (3.2)$$

Proof. (i) Step 1. Case where μ has a compact support in Q . By [33], we can find a decomposition $\mu = (f, g, h)$ with f, g, h have a compact support in Q . Let $\{\varphi_n\}$ be sequence of mollifiers in \mathbb{R}^{N+1} . Then $\mu_n = \varphi_n * \mu \in C_c^\infty(Q)$ for n large enough. We see that $\mu_n(Q) = \mu(Q)$ and μ_n admits the decomposition $\mu_n = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$. Since $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, we have for n_0 large enough,

$$\|f - f_{n_0}\|_{1,Q} + \|g - g_{n_0}\|_{p',Q} + \|h - h_{n_0}\|_X \leq \varepsilon \min\{\mu(Q), 1\}.$$

Then we obtain a decomposition $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$, such that

$$\|\hat{f}\|_{1,Q} + \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq \varepsilon. \quad (3.3)$$

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q . Set $\tilde{\mu}_0 = \theta_0\mu$, and $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$, for any $n \geq 1$. Since $\tilde{\mu}_n \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ has compact support, by Step 1, we can find a decomposition $\tilde{\mu}_n = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ such that

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon)\mu_n(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2^{-n-1}\varepsilon.$$

Let $\bar{f}_n = \sum_{k=0}^n \tilde{f}_k$, $\bar{g}_n = \sum_{k=0}^n \tilde{g}_k$ and $\bar{h}_n = \sum_{k=0}^n \tilde{h}_k$. Clearly, $\theta_n\mu = (\bar{f}_n, \bar{g}_n, \bar{h}_n)$, and $\{\bar{f}_n\}, \{\bar{g}_n\}, \{\bar{h}_n\}$ converge strongly to some f, g, h , respectively in $L^1(Q), (L^{p'}(Q))^N, X$, with

$$\|\bar{f}_n\|_{1,Q} + \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X \leq \varepsilon.$$

Therefore, $\mu = (f, g, h)$ and (3.1) holds.

(ii) We take a sequence $\{m_n\}$ in \mathbb{N} such that $f_n = \varphi_{m_n} * \bar{f}_n, g_n = \varphi_{m_n} * \bar{g}_n, h_n = \varphi_{m_n} * \bar{h}_n \in C_c^\infty(Q)$ and

$$\|f_n - \bar{f}_n\|_{1,Q} + \|g_n - \bar{g}_n\|_{p',Q} + \|h_n - \bar{h}_n\|_X \leq \frac{\varepsilon}{n+1} \min\{\mu(Q), 1\}.$$

Let $\mu_n = \varphi_{m_n} * (\theta_n\mu) = (f_n, g_n, h_n)$. Therefore, $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively. And (3.2) holds. Furthermore, $\{\mu_n\}$ converges weak-* to μ , and $\mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu(Q)$, thus $\{\mu_n\}$ converges in the narrow topology. \blacksquare

As a consequence, we get an approximation property for any measure $\mu \in \mathcal{M}_b^+(Q)$:

Proposition 3.2 *Let $\mu \in \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to μ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q), g_n, g \in (L^{p'}(Q))^N$ and $h_n, h \in X, \mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \quad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and $\{\mu_{n,s}\}$ converges to μ_s (strongly) in $\mathcal{M}_b(Q)$ and

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{n,s}(\Omega) \leq (1 + \varepsilon)\mu(Q), \quad \text{and } \|g_n\|_{p',Q} + \|h_n\|_X \leq \varepsilon. \quad (3.4)$$

Proof. Since $\{\mu_n\}$ is nondecreasing, then $\{\mu_{n,0}\}, \{\mu_{n,s}\}$ are too. Clearly, $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$. Hence, $\{\mu_{n,s}\}$ converge to μ_s and $\{\mu_{n,0}\}$ converge to μ_0 (strongly) in $\mathcal{M}_b(Q)$. Set $\tilde{\mu}_{0,0} = \mu_{0,0}$, and $\tilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for any $n \geq 1$. By Lemma 3.1, (i), we can find $\tilde{f}_n \in L^1(Q)$, $\tilde{g}_n \in (L^{p'}(Q))^N$ and $\tilde{h}_n \in X$ such that $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ and

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon)\tilde{\mu}_{n,0}(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2^{-n-1}\varepsilon.$$

Let $f_n = \sum_{k=0}^n \tilde{f}_k$, $G_n = \sum_{k=0}^n \tilde{g}_k$ and $h_n = \sum_{k=0}^n \tilde{h}_k$. Clearly, $\mu_{n,0} = (f_n, g_n, h_n)$ and the convergence properties hold with (3.4), since

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X \leq (1 + \varepsilon)\mu_0(Q).$$

In Section 6 we consider some measures $\mu \in \mathcal{M}_b(Q)$ which satisfy $|\mu| \leq \omega \otimes F$, with $\omega \in \mathcal{M}_b(\Omega)$ and $F \in L^1((0, T))$, $F \geq 0$. It is interesting to compare the properties of $\omega \otimes F$ and ω : ■

Let c_p^Ω be the elliptic capacity in Ω defined by

$$c_p^\Omega(K) = \inf \left\{ \int_\Omega |\nabla \varphi|^p : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\},$$

for any compact set $K \subset \Omega$.

Let $\mathcal{M}_{0,e}(\Omega)$ be the set of Radon measures ω on that do not charge the sets of zero c_p^Ω -capacity. Then $\mathcal{M}_b(\Omega) \cap \mathcal{M}_{0,e}(\Omega)$ is characterised as the set of measures $\omega \in \mathcal{M}_b(\Omega)$ which can be written under the form $\tilde{f} - \operatorname{div} \tilde{g}$ with $\tilde{f} \in L^1(\Omega)$ and $\tilde{g} \in (L^{p'}(\Omega))^N$, see [25].

Proposition 3.3 *For any $F \in L^1((0, T))$ with $\int_0^T F(t)dt \neq 0$, and $\omega \in \mathcal{M}_b(\Omega)$,*

$$\omega \in \mathcal{M}_{0,e}(\Omega) \iff \omega \otimes F \in \mathcal{M}_0(Q).$$

Proof. Assume that $\omega \otimes F \in \mathcal{M}_0(Q)$. Then, there exist $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$ and $h \in X$, such that

$$\int_Q \varphi(x, t)F(t)d\omega(x)dt = \int_Q \varphi(x, t)f(x, t)dxdt + \int_Q g(x, t) \cdot \nabla \varphi(x, t)dxdt - \int_Q h(x, t)\varphi_t(t, x)dxdt, \quad (3.5)$$

for all $\varphi \in C_c^\infty(\Omega \times [0, T])$, see [50, Lemma 2.24 and Theorem 2.27]. By choosing $\varphi(x, t) = \varphi(x) \in C_c^\infty(\Omega)$ and using Fubini's Theorem, (3.5) is rewritten as

$$\int_\Omega \varphi(x)d\omega(x) = \int_\Omega \varphi(x)\tilde{f}(x)dx + \int_\Omega \tilde{g}(x) \cdot \nabla \varphi(x)dx,$$

where $\tilde{f}(x) = \left(\int_0^T F(t)dt \right)^{-1} \int_0^T f(x, t)dt \in L^1(\Omega)$ and $\tilde{g}(x) = \left(\int_0^T F(t)dt \right)^{-1} \int_0^T g(x, t)dt \in (L^{p'}(\Omega))^N$; hence $\omega \in \mathcal{M}_{0,e}(\Omega)$.

Conversely, assume that $\omega = \tilde{f} - \operatorname{div} \tilde{g} \in \mathcal{M}_{0,\varepsilon}(\Omega)$, with $\tilde{f} \in L^1(\Omega)$ and $\tilde{g} \in \left(L^{p'}(\Omega)\right)^N$. So $\omega \otimes T_n(F) = f_n - \operatorname{div} g_n$, with $f_n = \tilde{f}T_n(F) \in L^1(Q)$ and $g_n = \tilde{g}T_n(F) \in \left(L^{p'}(Q)\right)^N$. Then $\omega \otimes T_n(F)$ admits the decomposition (f_n, g_n, h) , with $h = 0 \in L^\infty(Q)$, thus $\omega \otimes T_n(F) \in \mathcal{M}_0(Q)$. And $\{\omega \otimes T_n(F)\}$ converges to $\omega \otimes F$ strongly in $\mathcal{M}_b(Q)$, since $\|\omega \otimes (F - T_n(F))\|_{\mathcal{M}_b(Q)} \leq \|\omega\|_{\mathcal{M}_b(\Omega)} \|F - T_n(F)\|_{L^1((0,T))}$. Then $\omega \otimes F \in \mathcal{M}_0(Q)$, since $\mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$ is strongly closed in $\mathcal{M}_b(Q)$. \blacksquare

4 Renormalized solutions of problem (1.1)

4.1 Notations and Definition

For any function $f \in L^1(Q)$, we write $\int_Q f$ instead of $\int_Q f dxdt$, and for any measurable set $E \subset Q$, $\int_E f$ instead of $\int_E f dxdt$.

We set $T_k(r) = \max\{\min\{r, k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. We recall that if u is a measurable function defined and finite a.e. in Q , such that $T_k(u) \in X$ for any $k > 0$, there exists a measurable function w from Q into \mathbb{R}^N such that $\nabla T_k(u) = \chi_{|u| \leq k} w$, a.e. in Q , and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, and (f, g, h) be a decomposition of μ_0 given by (1.11), and $\widehat{\mu}_0 = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu}_0 \notin \mathcal{M}(Q)$, but we write, for convenience,

$$\int_Q w d\widehat{\mu}_0 := \int_Q (fw + g \cdot \nabla w), \quad \forall w \in X \cap L^\infty(Q).$$

Definition 4.1 Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function u is a **renormalized solution**, called **R-solution** of (1.1) if there exists a decomposition (f, g, h) of μ_0 such that

$$v = u - h \in L^\sigma(0, T; W_0^{1,\sigma}(\Omega) \cap L^\infty(0, T; L^1(\Omega))), \quad \forall \sigma \in [1, m_c); \quad T_k(v) \in X, \quad \forall k > 0, \quad (4.1)$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} , and $S(0) = 0$,

$$-\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(v)\varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q S'(v)\varphi d\widehat{\mu}_0, \quad (4.2)$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(T, \cdot) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla v = \int_Q \phi d\mu_s^+ \quad (4.3)$$

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{-m \geq v > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla v = \int_Q \phi d\mu_s^-. \quad (4.4)$$

Remark 4.2 As a consequence, $S(v) \in C([0, T]; L^1(\Omega))$ and $S(v)(0, \cdot) = S(u_0)$ in Ω ; and u satisfies the equation

$$(S(v))_t - \operatorname{div}(S'(v)A(x, t, \nabla u)) + S''(v)A(x, t, \nabla u) \cdot \nabla v = fS'(v) - \operatorname{div}(gS'(v)) + S''(v)g \cdot \nabla v, \quad (4.5)$$

in the sense of distributions in Q , see [49, Remark 3]. Moreover

$$\begin{aligned} \|S(v)_t\|_{X'+L^1(Q)} &\leq \left\| \operatorname{div}(S'(v)A(x, t, \nabla u)) \right\|_{X'} + \left\| S''(v)A(x, t, \nabla u) \cdot \nabla v \right\|_{1,Q} \\ &\quad + \left\| S'(v)f \right\|_{1,Q} + \left\| g \cdot S''(v)\nabla v \right\|_{1,Q} + \left\| \operatorname{div}(S'(v)g) \right\|_{X'}. \end{aligned}$$

Thus, if $[-M, M] \supset \operatorname{supp} S'$,

$$\begin{aligned} \left\| S''(v)A(x, t, \nabla u) \cdot \nabla v \right\|_{1,Q} &\leq \|S\|_{W^{2,\infty}(\mathbb{R})} \left(\|A(x, t, \nabla u)\chi_{|v| \leq M}\|_{p',Q}^{p'} + \|\nabla T_M(v)\|_{p,Q}^p \right) \\ &\leq C \|S\|_{W^{2,\infty}(\mathbb{R})} \left(\|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q} + \|a\|_{p',Q}^{p'} + \|\nabla T_M(v)\|_{p,Q}^p \right) \end{aligned}$$

thus

$$\begin{aligned} \|S(v)_t\|_{X'+L^1(Q)} &\leq C \|S\|_{W^{2,\infty}(\mathbb{R})} \left(\|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q}^{1/p'} + \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q} + \|\nabla T_M(v)\|_{p,Q}^p \right. \\ &\quad \left. + \|a\|_{p',Q} + \|a\|_{p',Q}^{p'} + \|f\|_{1,Q} + \|g\|_{p',Q} \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q}^{1/p} + \|g\|_{p',Q} \right) \quad (4.6) \end{aligned}$$

We also deduce that, for any $\varphi \in X \cap L^\infty(Q)$, such that $\varphi_t \in X' + L^1(Q)$,

$$\begin{aligned} \int_\Omega S(v(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi \\ + \int_Q S''(v)A(x, t, \nabla u) \cdot \nabla v \varphi = \int_Q S'(v)\varphi d\widehat{\mu}_0. \quad (4.7) \end{aligned}$$

Remark 4.3 Let u, v satisfying (4.1). It is easy to see that the condition (4.3) (resp. (4.4)) is equivalent to

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^+ \quad (4.8)$$

resp.

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \geq v > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^-. \quad (4.9)$$

In particular, for any $\varphi \in L^{p'}(Q)$ there holds

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla u| \varphi = 0, \quad \lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla v| \varphi = 0. \quad (4.10)$$

Remark 4.4 (i) Any function $U \in X$ such that $U_t \in X' + L^1(Q)$ admits a unique c_p^Q -quasi continuous representative, defined c_p^Q -quasi a.e. in Q , still denoted U . Furthermore, if $U \in L^\infty(Q)$, then for any $\mu_0 \in \mathcal{M}_0(Q)$, there holds $U \in L^\infty(Q, d\mu_0)$, see [49, Theorem 3 and Corollary 1].

(ii) Let u be any R -solution of problem (1.1). Then, $v = u - h$ admits a c_p^Q -quasi continuous functions representative which is finite c_p^Q -quasi a.e. in Q , and u satisfies definition 4.1 for every decomposition $(\tilde{f}, \tilde{g}, \tilde{h})$ such that $h - \tilde{h} \in L^\infty(Q)$, see [49, Proposition 3 and Theorem 4].

4.2 Steklov and Landes approximations

A main difficulty for proving Theorem 2.1 is the choice of admissible test functions (S, φ) in (4.2), valid for any R -solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

Definition 4.5 Let $\varepsilon \in (0, T)$ and $z \in L_{loc}^1(Q)$. For any $l \in (0, \varepsilon)$ we define the **Steklov time-averages** $[z]_l, [z]_{-l}$ of z by

$$[z]_l(x, t) = \frac{1}{l} \int_t^{t+l} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (0, T - \varepsilon),$$

$$[z]_{-l}(x, t) = \frac{1}{l} \int_{t-l}^t z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for R -solutions can be found in [22]. Recall some properties, given in [50]. Let $\varepsilon \in (0, T)$, and $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0, T])$, $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0, T])$ with $\text{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T - \varepsilon]$, $\text{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$. There holds

- (i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.
- (ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \rightarrow 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in X , and a.e. in Q ; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.
- (iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \rightarrow 0$, there exists a subsequence $\{l_\nu\}$ such that $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$ converge to z , c_p^Q -quasi everywhere in Q .

Next we recall the approximation introduced in [42], used in [30], [26], [21]:

Definition 4.6 Let $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Let u be a R -solution of (1.1), and $v = u - h$ given at (4.1), and $k > 0$. For any $\nu \in \mathbb{N}$, the **Landes-time approximation** $\langle T_k(v) \rangle_\nu$ of the truncate function $T_k(v)$ is defined in the following way:

Let $\{z_\nu\}$ be a sequence of functions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that $\|z_\nu\|_{\infty, \Omega} \leq k$, $\{z_\nu\}$ converges to $T_k(u_0)$ a.e. in Ω , and $\nu^{-1}\|z_\nu\|_{W_0^{1,p}(\Omega)}^p$ converges to 0. Then, $\langle T_k(v) \rangle_\nu$ is the unique solution of the problem

$$(\langle T_k(v) \rangle_\nu)_t = \nu(T_k(v) - \langle T_k(v) \rangle_\nu) \quad \text{in the sense of distributions,} \quad \langle T_k(v) \rangle_\nu(0) = z_\nu, \quad \text{in } \Omega.$$

Therefore, $\langle T_k(v) \rangle_\nu \in X \cap L^\infty(Q)$ and $(T_k(v))_\nu \in X$, see [42]. Furthermore, up to subsequences, $\{\langle T_k(v) \rangle_\nu\}$ converges to $T_k(v)$ strongly in X and a.e. in Q , and $\|(\langle T_k(v) \rangle_\nu)\|_{L^\infty(Q)} \leq k$.

4.3 First properties

In the sequel we use the following notations: for any function $J \in W^{1,\infty}(\mathbb{R})$, nondecreasing with $J(0) = 0$, we set

$$\bar{J}(r) = \int_0^r J(\tau) d\tau, \quad \mathcal{J}(r) = \int_0^r J'(\tau) \tau d\tau. \quad (4.11)$$

It is easy to verify that $\mathcal{J}(r) \geq 0$,

$$\mathcal{J}(r) + \bar{J}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geq s(J(r) - J(s)) \quad \forall r, s \in \mathbb{R}. \quad (4.12)$$

In particular we define, for any $k > 0$, and any $r \in \mathbb{R}$,

$$\bar{T}_k(r) = \int_0^r T_k(\tau) d\tau, \quad \mathcal{T}_k(r) = \int_0^r T_k'(\tau) \tau d\tau, \quad (4.13)$$

and we use several times a truncature used in [32]:

$$H_m(r) = \chi_{[-m,m]}(r) + \frac{2m - |s|}{m} \chi_{m < |s| \leq 2m}(r), \quad \bar{H}_m(r) = \int_0^r H_m(\tau) d\tau. \quad (4.14)$$

The next Lemma allows to extend the range of the test functions in (4.2). Its proof, given in the Appendix, is obtained by Steklov approximation of the solutions.

Lemma 4.7 *Let u be a R -solution of problem (1.1). Let $J \in W^{1,\infty}(\mathbb{R})$ be nondecreasing with $J(0) = 0$, and \bar{J} defined by (4.11). Then,*

$$\begin{aligned} & \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla (\xi J(S(v))) + \int_Q S''(v) A(x, t, \nabla u) \cdot \nabla v \xi J(S(v)) \\ & - \int_\Omega \xi(0) J(S(u_0)) S(u_0) - \int_Q \xi_t \bar{J}(S(v)) \\ & \leq \int_Q S'(v) \xi J(S(v)) d\widehat{\mu}_0, \end{aligned} \quad (4.15)$$

for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} and $S(0) = 0$, and for any $\xi \in C^1(Q) \cap W^{1,\infty}(Q)$, $\xi \geq 0$.

Next we give estimates of the gradient, following the first estimates of [26], see also [33], [49, Proposition 2], [43].

Proposition 4.8 *If u is a R -solution of problem (1.1), then there exists $c = c(p)$ such that, for any $k \geq 1$ and $\ell \geq 0$,*

$$\int_{\ell \leq |v| \leq \ell+k} |\nabla u|^p + \int_{\ell \leq |v| \leq \ell+k} |\nabla v|^p \leq ckM \quad (4.16)$$

and

$$\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq c(M + |\Omega|), \quad (4.17)$$

where

$$M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + \|a\|_{p',Q}^{p'}.$$

As a consequence, for any $k \geq 1$,

$$\text{meas}\{|v| > k\} \leq C_1 M_1 k^{-pc}, \quad \text{meas}\{|\nabla v| > k\} \leq C_2 M_2 k^{-mc}, \quad (4.18)$$

$$\text{meas}\{|u| > k\} \leq C_3 M_2 k^{-pc}, \quad \text{meas}\{|\nabla u| > k\} \leq C_4 M_2 k^{-mc}, \quad (4.19)$$

where $C_i = C_i(N, p, c_1, c_2)$, $i = 1-4$, and $M_1 = (M + |\Omega|)^{\frac{p}{N}} M$ and $M_2 = M_1 + M$.

Proof. Set for any $r \in \mathbb{R}$, and $m, k, \ell > 0$,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For $m > k + \ell$, we can choose $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$ as test functions in (4.15), where $\overline{H_m}$ is defined at (4.14) and $\xi \in C^1([0, T])$ with values in $[0, 1]$, independent on x . Since $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$ for all $r \in \mathbb{R}$, we obtain

$$\begin{aligned} & - \int_{\Omega} \xi(0) T_{k,\ell}(u_0) \overline{H_m}(u_0) - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(v)) \\ & + \int_{\{\ell \leq |v| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla v - \frac{k}{m} \int_{\{m \leq |v| < 2m\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_Q H_m(v) \xi T_{k,\ell}(v) d\widehat{\mu}_0. \end{aligned}$$

And

$$\int_Q H_m(v) \xi T_{k,\ell}(v) d\widehat{\mu}_0 = \int_Q H_m(v) \xi T_{k,\ell}(v) f + \int_{\{\ell \leq |v| < \ell+k\}} \xi \nabla v \cdot g - \frac{k}{m} \int_{\{m \leq |v| < 2m\}} \xi \nabla v \cdot g.$$

Let $m \rightarrow \infty$; then, for any $k \geq 1$, since $v \in L^1(Q)$ and from (4.3), (4.4), and (4.10), we find

$$- \int_Q \xi_t \overline{T_{k,\ell}}(v) + \int_{\{\ell \leq |v| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_{\{\ell \leq |v| < \ell+k\}} \xi \nabla v \cdot g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \quad (4.20)$$

Next, we take $\xi \equiv 1$. We verify that there exists $c = c(p)$ such that

$$A(x, t, \nabla u) \cdot \nabla v - \nabla v \cdot g \geq \frac{c_1}{4} (|\nabla u|^p + |\nabla v|^p) - c(|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

where c_1 is the constant in (1.2). Hence (4.16) follows. Thus, from (4.20) and the Hölder inequality, we get, with another constant c , for any $\xi \in C^1([0, T])$ with values in $[0, 1]$,

$$- \int_Q \xi_t \overline{T_{k,\ell}}(v) \leq ckM$$

Thus $\int_\Omega \overline{T_{k,\ell}}(v)(t) \leq ckM$, for a.e. $t \in (0, T)$. We deduce (4.17) by taking $k = 1, \ell = 0$, since $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geq |r| - 1$, for any $r \in \mathbb{R}$.

Next, from the Gagliardo-Nirenberg embedding Theorem, we have

$$\int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leq C_1 \|v\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} \int_Q |\nabla T_k(v)|^p,$$

where $C_1 = C_1(N, p)$. Then, from (4.16) and (4.17), we get, for any $k \geq 1$,

$$\text{meas} \{|v| > k\} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leq C \|v\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(v)|^p \leq C_2 M_1 k^{-pc},$$

with $C_2 = C_2(N, p, c_1, c_2)$. We obtain

$$\begin{aligned} \text{meas} \{|\nabla v| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas} (\{|\nabla v|^p > s\}) ds \\ &\leq \text{meas} \left\{ |v| > k^{\frac{N}{N+1}} \right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas} \left(\left\{ |\nabla v|^p > s, |v| \leq k^{\frac{N}{N+1}} \right\} \right) ds \\ &\leq C_2 M_1 k^{-mc} + \frac{1}{k^p} \int_{|v| \leq k^{\frac{N}{N+1}}} |\nabla v|^p \leq C_2 M_2 k^{-mc}, \end{aligned}$$

with $C_3 = C_3(N, p, c_1, c_2)$. Furthermore, for any $k \geq 1$,

$$\text{meas} \{|h| > k\} + \text{meas} \{|\nabla h| > k\} \leq C_4 k^{-p} \|h\|_X^p,$$

where $C_4 = C_4(N, p, c_1, c_2)$. Therefore, we easily get (4.19). \blacksquare

Remark 4.9 If $\mu \in L^1(Q)$ and $a \equiv 0$ in (1.2), then (4.16) holds for all $k > 0$ and the term $|\Omega|$ in inequality (4.17) can be removed where $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$. Furthermore, (4.19) is stated as follows:

$$\text{meas} \{|u| > k\} \leq C_3 M^{\frac{p+N}{N}} k^{-pc}, \quad \text{meas} \{|\nabla u| > k\} \leq C_4 M^{\frac{N+2}{N+1}} k^{-mc}, \forall k > 0. \quad (4.21)$$

To see last inequality, we do in the following way:

$$\begin{aligned}
\text{meas } \{|\nabla v| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas } (\{|\nabla v|^p > s\}) ds \\
&\leq \text{meas } \left\{ |v| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}} \right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas } \left\{ |\nabla v|^p > s, |v| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}} \right\} ds \\
&\leq C_4 M^{\frac{N+2}{N+1}} k^{-m_c}.
\end{aligned}$$

Proposition 4.10 Let $\{\mu_n\} \subset \mathcal{M}_b(Q)$, and $\{u_{0,n}\} \subset L^1(\Omega)$, with

$$\sup_n |\mu_n|(Q) < \infty, \text{ and } \sup_n \|u_{0,n}\|_{1,\Omega} < \infty.$$

Let u_n be a R -solution of (1.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ and $u_{0,n}$, relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$, and $v_n = u_n - h_n$. Assume that $\{f_n\}$ is bounded in $L^1(Q)$, $\{g_n\}$ bounded in $(L^{p'}(Q))^N$ and $\{h_n\}$ bounded in X .

Then, up to a subsequence, $\{v_n\}$ converges a.e. to a function v , such that $T_k(v) \in X$ and $v \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$ for any $\sigma \in [1, m_c)$. And

- (i) $\{v_n\}$ converges to v strongly in $L^\sigma(Q)$ for any $\sigma \in [1, m_c)$, and $\sup \|v_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$,
- (ii) $\sup_{k>0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(v_n)|^p < \infty$,
- (iii) $\{T_k(v_n)\}$ converges to $T_k(v)$ weakly in X , for any $k > 0$,
- (iv) $\{A(x, t, \nabla(T_k(v_n) + h_n))\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$.

Proof. Take $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} and $S(0) = 0$. We combine (4.6) with (4.16), and deduce that $\{S(v_n)_t\}$ is bounded in $X' + L^1(Q)$ and $\{S(v_n)\}$ bounded in X . Hence, $\{S(v_n)\}$ is relatively compact in $L^1(Q)$. On the other hand, we choose $S = S_k$ such that $S_k(z) = z$, if $|z| < k$ and $S_k(z) = 2k \text{ sign} z$, if $|z| > 2k$. Thanks to (4.17), we obtain

$$\begin{aligned}
\text{meas } \{|v_n - v_m| > \sigma\} &\leq \text{meas } \{|v_n| > k\} + \text{meas } \{|v_m| > k\} + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\} \\
&\leq \frac{1}{k} (\|v_n\|_{1,Q} + \|v_m\|_{1,Q}) + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\} \\
&\leq \frac{C}{k} + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\}.
\end{aligned} \tag{4.22}$$

Thus, up to a subsequence $\{u_n\}$ is a Cauchy sequence in measure, and converges a.e. in Q to a function u . Thus, $\{T_k(v_n)\}$ converges to $T_k(v)$ weakly in X , since $\sup_n \|T_k(v_n)\|_X < \infty$ for any $k > 0$. And $\{|\nabla(T_k(v_n) + h_n)|^{p-2} \nabla(T_k(v_n) + h_n)\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$. Furthermore, from (4.18), $\{v_n\}$ converges to v strongly in $L^\sigma(Q)$, for any $\sigma < p_c$. \blacksquare

5 The convergence theorem

We first recall some properties of the measures, see [49, Lemma 5], [32].

Proposition 5.1 *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$, where μ_s^+ and μ_s^- are concentrated, respectively, on two disjoint sets E^+ and E^- of zero c_p^Q -capacity. Then, for any $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_s^-(E^- \setminus K_\delta^-) \leq \delta,$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$ with values in $[0, 1]$, such that $\psi_\delta^+, \psi_\delta^- = 1$ respectively on K_δ^+, K_δ^- , and $\text{supp}(\psi_\delta^+) \cap \text{supp}(\psi_\delta^-) = \emptyset$, and

$$\|\psi_\delta^+\|_X + \|(\psi_\delta^+)_t\|_{X'+L^1(Q)} \leq \delta, \quad \|\psi_\delta^-\|_X + \|(\psi_\delta^-)_t\|_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions $(\psi_\delta^+)_t = (\psi_\delta^+)_t^1 + (\psi_\delta^+)_t^2$ and $(\psi_\delta^-)_t = (\psi_\delta^-)_t^1 + (\psi_\delta^-)_t^2$ in $X' + L^1(Q)$, such that

$$\|(\psi_\delta^+)_t^1\|_{X'} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{1,Q} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^1\|_{X'} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{1,Q} \leq \frac{\delta}{3}. \quad (5.1)$$

Both $\{\psi_\delta^+\}$ and $\{\psi_\delta^-\}$ converge to 0, *-weakly in $L^\infty(Q)$, and strongly in $L^1(Q)$ and up to subsequences, a.e. in Q , as δ tends to 0.

Moreover if ρ_n and η_n are as in Theorem 2.1, we have, for any $\delta, \delta_1, \delta_2 > 0$,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n, \delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \quad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \quad (5.2)$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \quad (5.3)$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \quad (5.4)$$

Hereafter, if $n, \varepsilon, \dots, \nu$ are real numbers, and a function ϕ depends on $n, \varepsilon, \dots, \nu$ and eventual other parameters $\alpha, \beta, \dots, \gamma$, and $n \rightarrow n_0, \varepsilon \rightarrow \varepsilon_0, \dots, \nu \rightarrow \nu_0$, we write $\phi = \omega(n, \varepsilon, \dots, \nu)$, then this means $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} |\phi| = 0$, when the parameters $\alpha, \beta, \dots, \gamma$ are fixed. In the same way, $\phi \leq \omega(n, \varepsilon, \delta, \dots, \nu)$ means $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} \phi \leq 0$, and $\phi \geq \omega(n, \varepsilon, \dots, \nu)$ means $-\phi \leq \omega(n, \varepsilon, \dots, \nu)$.

Remark 5.2 *In the sequel we use a convergence property, consequence of the Dunford-Pettis theorem, still used in [32]: If $\{a_n\}$ is a sequence in $L^1(Q)$ converging to a weakly in $L^1(Q)$ and $\{b_n\}$ a bounded sequence in $L^\infty(Q)$ converging to b , a.e. in Q , then $\lim_{n \rightarrow \infty} \int_Q a_n b_n = \int_Q ab$.*

Next we prove Theorem 2.1.

Scheme of the proof. Let $\{\mu_n\}, \{u_{0,n}\}$ and $\{u_n\}$ satisfying the assumptions of Theorem 2.1. Then we can apply Proposition 4.10. Setting $v_n = u_n - h_n$, up to subsequences, $\{u_n\}$ converges *a.e.* in Q to some function u , and $\{v_n\}$ converges *a.e.* to $v = u - h$, such that $T_k(v) \in X$ and $v \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$ for every $\sigma \in [1, m_c)$. And $\{v_n\}$ satisfies the conclusions (i) to (iv) of Proposition 4.10. We have

$$\begin{aligned} \mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-, \end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \quad (5.5)$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathcal{M}_b^+(Q) \cap \mathcal{M}_0(Q), \quad \text{and} \quad \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}. \quad (5.6)$$

Let E^+, E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. For any $\delta_1, \delta_2 > 0$, let $\psi_{\delta_1}^+, \psi_{\delta_2}^+$ and $\psi_{\delta_1}^-, \psi_{\delta_2}^-$ as in Proposition 5.1 and set

$$\Phi_{\delta_1, \delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near E

$$I_1 := \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2), \quad (5.7)$$

and far from E ,

$$I_2 := \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2). \quad (5.8)$$

Then it follows that

$$\overline{\lim}_{n, \nu} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq 0, \quad (5.9)$$

which implies

$$\overline{\lim}_{n \rightarrow \infty} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - T_k(v)) \leq 0, \quad (5.10)$$

since $\{\langle T_k(v) \rangle_\nu\}$ converges to $T_k(v)$ in X . On the other hand, from the weak convergence of $\{T_k(v_n)\}$ to $T_k(v)$ in X , we verify that

$$\int_{\{|v_n| \leq k\}} A(x, t, \nabla (T_k(v) + h_n)) \cdot \nabla (T_k(v_n) - T_k(v)) = \omega(n).$$

Thus we get

$$\int_{\{|v_n| \leq k\}} (A(x, t, \nabla u_n) - A(x, t, \nabla(T_k(v) + h_n))) \cdot \nabla (u_n - (T_k(v) + h_n)) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \quad \text{a.e. in } Q. \quad (5.11)$$

Therefore, $\{A(x, t, \nabla u_n)\}$ converges to $A(x, t, \nabla u)$ weakly in $(L^{p'}(Q))^N$; and from (5.10) we find

$$\overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \leq \int_Q A(x, t, \nabla u) \nabla T_k(v).$$

Otherwise, $\{A(x, t, \nabla(T_k(v_n) + h_n))\}$ converges weakly in $(L^{p'}(Q))^N$ to some F_k , from Proposition 4.10, and we obtain that $F_k = A(x, t, \nabla(T_k(v) + h))$. Hence

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(v_n) + h_n)) \cdot \nabla(T_k(v_n) + h_n) &\leq \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \\ &\quad + \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(v_n) + h_n)) \cdot \nabla h_n \\ &\leq \int_Q A(x, t, \nabla(T_k(v) + h)) \cdot \nabla(T_k(v) + h). \end{aligned}$$

As a consequence

$$\{T_k(v_n)\} \text{ converges to } T_k(v), \text{ strongly in } X, \quad \forall k > 0. \quad (5.12)$$

Then to finish the proof we have to check that u is a solution of (1.1). \blacksquare

In order to prove (5.7) we need a first Lemma, inspired of [32, Lemma 6.1], extending [49, Lemma 6 and Lemma 7]:

Lemma 5.3 *Let $\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)$ be uniformly bounded in $W^{1,\infty}(Q)$ with values in $[0, 1]$, such that $\int_Q \psi_{1,\delta} d\mu_s^- \leq \delta$ and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. Then, under the assumptions of Theorem 2.1,*

$$\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (5.13)$$

$$\frac{1}{m} \int_{\{-2m < v_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{-2m < v_n \leq -m\}} |\nabla v_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad (5.14)$$

and for any $k > 0$,

$$\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \int_{\{m \leq v_n < m+k\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (5.15)$$

$$\int_{\{-m-k < v_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \int_{\{-m-k < v_n \leq -m\}} |\nabla v_n|^p \psi_{1,\delta} = \omega(n, m, \delta). \quad (5.16)$$

Proof. (i) Proof of (5.13), (5.14). Set for any $r \in \mathbb{R}$ and any $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left(\frac{-m+\tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m+2h-\tau}{2m+\ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left(\frac{-m+\tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$

Note that $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi, J, S) = (\psi_{2,\delta}, T_1, S_{m,\ell})$ as test functions in (4.15) for u_n , and observe that, from (5.5),

$$\widehat{\mu_{n,0}} = \mu_{n,0} - (h_n)_t = \widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \quad (5.17)$$

Thus we can write $\sum_{i=1}^6 A_i \leq \sum_{i=7}^{12} A_i$, where

$$A_1 = - \int_{\Omega} \psi_{2,\delta}(0) T_1(S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}), \quad A_2 = - \int_Q (\psi_{2,\delta})_t \overline{T_1}(S_{m,\ell}(v_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla \psi_{2,\delta},$$

$$A_4 = \int_Q S'_{m,\ell}(v_n)^2 \psi_{2,\delta} T'_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_5 = \frac{1}{m} \int_{\{m \leq v_n \leq 2m\}} \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_6 = - \frac{1}{2m+\ell} \int_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} A(x, t, \nabla u_n) \nabla v_n,$$

$$A_7 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} f_n, \quad A_8 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) g_n \cdot \nabla \psi_{2,\delta},$$

$$A_9 = \int_Q (S'_{m,\ell}(v_n))^2 T'_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n \cdot \nabla v_n, \quad A_{10} = \frac{1}{m} \int_{m \leq v_n \leq 2m} T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n \cdot \nabla v_n,$$

$$A_{11} = - \frac{1}{2m+\ell} \int_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} g_n \cdot \nabla v_n, \quad A_{12} = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} d(\rho_{n,0} - \eta_{n,0}).$$

Since $\|S_{m,\ell}(u_{0,n})\|_{1,\Omega} \leq \int_{\{m \leq u_{0,n}\}} u_{0,n} dx$, we find $A_1 = \omega(\ell, n, m)$. Otherwise

$$|A_2| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} v_n, \quad |A_3| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} (|a| + c_2 |\nabla u_n|^{p-1}),$$

which implies $A_2 = \omega(\ell, n, m)$ and $A_3 = \omega(\ell, n, m)$. Using (4.3) for u_n , we have

$$A_6 = - \int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence $A_6 = \omega(\ell, n, m, \delta)$, since $(\rho_{n,s} - \eta_{n,s})^+$ converges to μ_s^+ as $n \rightarrow \infty$ in the narrow topology, and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. We also obtain $A_{11} = \omega(\ell)$ from (4.10).

Now $\left\{ S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \right\}_\ell$ converges to $S'_m(v_n) T_1(S_m(v_n))$, $\left\{ S'_m(v_n) T_1(S_m(v_n)) \right\}_n$ converges to $S'_m(v) T_1(S_m(v))$, $\left\{ S'_m(v) T_1(S_m(v)) \right\}_m$ converges to 0, $*$ -weakly in $L^\infty(Q)$, and $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 5.2, we obtain

$$\begin{aligned} A_7 &= \int_Q S'_m(v_n) T_1(S_m(v_n)) \psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(v) T_1(S_m(v)) \psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m), \\ A_8 &= \int_Q S'_m(v_n) T_1(S_m(v_n)) g_n \cdot \nabla \psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(v) T_1(S_m(v)) g \cdot \nabla \psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m). \end{aligned}$$

Otherwise, $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$, and $\left\{ \int_Q \psi_{2,\delta} d\rho_n \right\}$ converges to $\int_Q \psi_{2,\delta} d\mu_s^+$, thus $A_{12} \leq \omega(\ell, n, m, \delta)$. Using Holder inequality and the condition (1.2) we have

$$g_n \cdot \nabla v_n - A(x, t, \nabla u_n) \nabla v_n \leq C_1 \left(|g_n|^{p'} + |\nabla h_n|^p + |a|^{p'} \right)$$

with $C_1 = C_1(p, c_2)$, which implies

$$A_9 - A_4 \leq C_1 \int_Q (S'_{m,\ell}(v_n))^2 T'_1(S_{m,\ell}(v_n)) \psi_{2,\delta} \left(|g_n|^{p'} + |h_n|^p + |a|^{p'} \right) = \omega(\ell, n, m).$$

Similarly we also show that $A_{10} - A_5/2 \leq \omega(\ell, n, m)$. Combining the estimates, we get $A_5/2 \leq \omega(\ell, n, m, \delta)$. Using Holder inequality we have

$$A(x, t, \nabla u_n) \nabla v_n \geq \frac{c_1}{2} |\nabla u_n|^p - C_2 (|a|^{p'} + |\nabla h_n|^p).$$

with $C_2 = C_2(p, c_1, c_2)$, which implies

$$\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that for all $m > 4$, $S_{m,\ell}(r) \geq 1$ for any $r \in [\frac{3}{2}m, 2m]$; hence $T_1(S_{m,\ell}(r)) = 1$. So,

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since $|\nabla v_n|^p \leq 2^{p-1}|\nabla u_n|^p + 2^{p-1}|\nabla h_n|^p$, there also holds

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (5.13) by summing on each set $\{(\frac{4}{3})^\nu m \leq v_n \leq (\frac{4}{3})^{\nu+1} m\}$ for $\nu = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$ as test functions in (4.15) for u_n , where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (5.14).

(ii) Proof of (5.15), (5.16). We set, for any $k, m, \ell \geq 1$,

$$S_{k,m,\ell}(r) = \int_0^r \left(T_k(\tau - T_m(\tau)) \chi_{[m, k+m+\ell]} + k \frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell)]} \right) d\tau$$

$$S_{k,m}(r) = \int_0^s T_k(\tau - T_m(\tau)) \chi_{[m, \infty)} d\tau.$$

We choose $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$ as test functions in (4.15) for u_n . In the same way we also obtain

$$\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that $T_1(S_{k,m,\ell}(r)) = 1$ for any $r \geq m+1$, thus $\int_{\{m+1 \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$, which implies (5.15) by changing m into $m-1$. Similarly, we obtain (5.16). \blacksquare

Next we look at the behaviour near E .

Lemma 5.4 *Estimate (5.7) holds.*

Proof. There holds

$$I_1 = \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) - \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_\nu.$$

From Proposition 4.10, (iv), $\{A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu\}$ converges weakly in $L^1(Q)$ to $F_k \nabla \langle T_k(v) \rangle_\nu$. And $\{\chi_{\{|v_n| \leq k\}}\}$ converges to $\chi_{\{|v| \leq k\}}$, *a.e.* in Q , and $\Phi_{\delta_1, \delta_2}$ converges to 0 *a.e.* in Q as $\delta_1 \rightarrow 0$, and $\Phi_{\delta_1, \delta_2}$ takes its values in $[0, 1]$. Thanks to Remark 5.2, we have

$$\begin{aligned} & \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_\nu \\ &= \int_Q \chi_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu \\ &= \int_Q \chi_{\{|v| \leq k\}} \Phi_{\delta_1, \delta_2} F_k \cdot \nabla \langle T_k(v) \rangle_\nu + \omega(n) = \omega(n, \nu, \delta_1). \end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2), \quad (5.18)$$

then we deduce (5.7). As noticed in [32], [49], it is precisely for this estimate that we need the double cut $\psi_{\delta_1}^+ \psi_{\delta_2}^+$. To do this, we set, for any $m > k > 0$, and any $r \in \mathbb{R}$,

$$\hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau)) H_m(\tau) d\tau,$$

where H_m is defined at (4.14). Hence $\text{supp} \hat{S}_{k,m} \subset [-2m, k]$; and $\hat{S}_{k,m}'' = -\chi_{[-k, k]} + \frac{2k}{m} \chi_{[-2m, -m]}$. We choose $(\varphi, S) = (\psi_{\delta_1}^+ \psi_{\delta_2}^+, \hat{S}_{k,m})$ as test functions in (4.2). From (5.17), we can write

$$A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$\begin{aligned} A_1 &= - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(v_n), & A_2 &= \int_Q (k - T_k(v_n)) H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+), \\ A_3 &= \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla T_k(v_n), & A_4 &= \frac{2k}{m} \int_{\{-2m < v_n \leq -m\}} \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla v_n, \\ A_5 &= - \int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\widehat{\lambda}_{n,0}, & A_6 &= \int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d(\eta_{n,0} - \rho_{n,0}); \end{aligned}$$

and we estimate A_3 . As in [49, p.585], since $\{\hat{S}_{k,m}(v_n)\}$ converges to $\hat{S}_{k,m}(v)$ weakly in X , and $\hat{S}_{k,m}(v) \in L^\infty(Q)$, and from (5.1), there holds

$$A_1 = - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) - \int_Q \psi_{\delta_1}^+ (\psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) + \omega(n) = \omega(n, \delta_1).$$

Next consider A_2 . Notice that $v_n = T_{2m}(v_n)$ on $\text{supp}(H_m(v_n))$. From Proposition 4.10, (iv), the sequence $\{A(x, t, \nabla (T_{2m}(v_n) + h_n)) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+)\}$ converges to $F_{2m} \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+)$ weakly in $L^1(Q)$. Thanks to Remark 5.2 and the convergence of $\psi_{\delta_1}^+ \psi_{\delta_2}^+$ in X to 0 as δ_1 tends to 0, we find

$$A_2 = \int_Q (k - T_k(v)) H_m(v) F_{2m} \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+) + \omega(n) = \omega(n, \delta_1).$$

Then consider A_4 . Then for some $C = C(p, c_2)$,

$$|A_4| \leq C \frac{2k}{m} \int_{\{-2m < v_n \leq -m\}} (|\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'}) \psi_{\delta_1}^+ \psi_{\delta_2}^+.$$

Since $\psi_{\delta_1}^+$ takes its values in $[0, 1]$, from Lemma 5.3, we get in particular $A_4 = \omega(n, \delta_1, m, \delta_2)$.

Now estimate A_5 . The sequence $\{(k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+\}$ converges weakly in X to $(k - T_k(v))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+$, and $\{(k - T_k(v_n))H_m(v_n)\}$ converges $*$ -weakly in $L^\infty(Q)$ and *a.e.* in Q to $(k - T_k(v))H_m(v)$. Otherwise $\{f_n\}$ converges to f weakly in $L^1(Q)$ and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. Thanks to Remark 5.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ to 0 in X and *a.e.* in Q as $\delta_1 \rightarrow 0$, we deduce that

$$A_5 = - \int_Q (k - T_k(v_n))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\widehat{\nu}_0 + \omega(n) = \omega(n, \delta_1),$$

where $\widehat{\nu}_0 = f - \operatorname{div} g$.

Finally $A_6 \leq 2k \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ d\eta_m$; using (5.2) we also find $A_6 \leq \omega(n, \delta_1, m, \delta_2)$. By addition, since A_3 does not depend on m , we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Reasoning as before with $(\psi_{\delta_1}^-\psi_{\delta_2}^-, \check{S}_{k,m})$ as test function in (4.2), where $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q \psi_{\delta_1}^-\psi_{\delta_2}^- A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (5.18) holds. ■

Next we look at the behaviour far from E .

Lemma 5.5 . *Estimate (5.8) holds.*

Proof. Here we estimate I_2 ; we can write

$$I_2 = \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \nabla (T_k(v_n) - \langle T_k(v) \rangle_\nu).$$

Following the ideas of [51], used also in [49], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k}(v_n - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v_n - T_k(v)).$$

Recall that $\|\langle T_k(v) \rangle_\nu\|_{\infty, Q} \leq k$, and observe that

$$R_{n,\nu,\ell} = 2k \operatorname{sign}(v_n) \quad \text{in } \{|v_n| \geq \ell + 2k\}, \quad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n, \nu, \ell) \quad \textit{a.e. in } Q, \quad (5.19)$$

$$\lim_{n \rightarrow \infty} R_{n,\nu,\ell} = T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v)), \quad \textit{a.e. in } Q, \quad \text{and weakly in } X. \quad (5.20)$$

Next consider $\xi_{1,n_1} \in C_c^\infty([0, T])$, $\xi_{2,n_2} \in C_c^\infty((0, T])$ with values in $[0, 1]$, such that $(\xi_{1,n_1})_t \leq 0$ and $(\xi_{2,n_2})_t \geq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{1,n_2}(t)\}$) converges to 1, for any $t \in [0, T)$ (resp. $t \in (0, T]$);

and moreover, for any $a \in C([0, T]; L^1(\Omega))$, $\left\{ \int_Q a(\xi_{1, n_1})_t \right\}$ and $\int_Q a(\xi_{2, n_2})_t$ converge respectively to $-\int_{\Omega} a(T, \cdot)$ and $\int_{\Omega} a(0, \cdot)$. We set

$$\varphi = \varphi_{n, n_1, n_2, l_1, l_2, \ell} = \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} - \xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2}.$$

We can see that

$$\varphi - (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} = \omega(l_1, l_2, n_1, n_2) \quad \text{in norm in } X \text{ and a.e. in } Q. \quad (5.21)$$

We can choose $(\varphi, S) = (\varphi_{n, n_1, n_2, l_1, l_2, \ell}, \overline{H_m})$ as test functions in (4.7) for u_n , where $\overline{H_m}$ is defined at (4.14), with $m > \ell + 2k$. We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$\begin{aligned} A_1 &= \int_{\Omega} \varphi(T) \overline{H_m}(v_n(T)) dx, & A_2 &= - \int_{\Omega} \varphi(0) \overline{H_m}(u_{0, n}) dx, \\ A_3 &= - \int_Q \varphi_t \overline{H_m}(v_n), & A_4 &= \int_Q H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla \varphi, \\ A_5 &= \int_Q \varphi H'_m(v_n) A(x, t, \nabla u_n) \cdot \nabla v_n, & A_6 &= \int_Q H_m(v_n) \varphi d\widehat{\lambda}_{n, 0}, \\ A_7 &= \int_Q H_m(v_n) \varphi d(\rho_{n, 0} - \eta_{n, 0}). \end{aligned}$$

Estimate of A_4 . This term allows to study I_2 . Indeed, $\{H_m(v_n)\}$ converges to 1, *a.e.* in Q ; thanks to (5.21), (5.19) (5.20), we have

$$\begin{aligned} A_4 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} - \int_Q R_{n, \nu, \ell} A(x, t, \nabla u_n) \cdot \nabla \Phi_{\delta_1, \delta_2} + \omega(l_1, l_2, n_1, n_2, m) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + \int_{\{|v_n| > k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell), \end{aligned}$$

where

$$\begin{aligned} B_1 &= \int_{\{|v_n| > k\}} (1 - \Phi_{\delta, \eta}) (\chi_{|v_n - \langle T_k(v) \rangle_{\nu}| \leq \ell+k} - \chi_{|v_n - k| \leq \ell-k}) A(x, t, \nabla u_n) \cdot \nabla v_n, \\ B_2 &= - \int_{\{|v_n| > k\}} (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v_n - \langle T_k(v) \rangle_{\nu}| \leq \ell+k} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_{\nu}. \end{aligned}$$

Now $\{A(x, t, \nabla(T_{\ell+2k}(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu\}$ converges to $F_{\ell+2k} \nabla \langle T_k(v) \rangle_\nu$, weakly in $L^1(Q)$. Otherwise $\left\{ \chi_{|v_n|>k} \chi_{|v_n - \langle T_k(v) \rangle_\nu| \leq \ell+k} \right\}$ converges to $\chi_{|v|>k} \chi_{|v - \langle T_k(v) \rangle_\nu| \leq \ell+k}$, *a.e.* in Q . And $\{\langle T_k(v) \rangle_\nu\}$ converges to $T_k(v)$ strongly in X . Thanks to Remark 5.2 we get

$$\begin{aligned} B_2 &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v|>k} \chi_{|v - \langle T_k(v) \rangle_\nu| \leq \ell+k} F_{\ell+2k} \cdot \nabla \langle T_k(v) \rangle_\nu + \omega(n) \\ &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v|>k} \chi_{|v - T_k(v)| \leq \ell+k} F_{\ell+2k} \cdot \nabla T_k(v) + \omega(n, \nu) = \omega(n, \nu), \end{aligned}$$

since $\nabla T_k(v) \chi_{|v|>k} = 0$. Besides, we see that, for some $C = C(p, c_2)$,

$$|B_1| \leq C \int_{\{\ell-2k \leq |v_n| < \ell+2k\}} (1 - \Phi_{\delta_1, \delta_2}) \left(|\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'} \right).$$

Using (5.3) and (5.4) and applying (5.15) and (5.16) to $1 - \Phi_{\delta_1, \delta_2}$, we obtain, for $k > 0$

$$\int_{\{m \leq |v_n| < m+4k\}} (|\nabla u_n|^p + |\nabla v_n|^p) (1 - \Phi_{\delta_1, \delta_2}) = \omega(n, m, \delta_1, \delta_2). \quad (5.22)$$

Thus, $B_1 = \omega(n, \nu, \ell, \delta_1, \delta_2)$, hence $B_1 + B_2 = \omega(n, \nu, \ell, \delta_1, \delta_2)$. Then

$$A_4 = I_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2). \quad (5.23)$$

Estimate of A_5 . For $m > \ell + 2k$, since $|\varphi| \leq 2\ell$, and (5.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned} A_5 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(v_n) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2) \\ &= -\frac{2k}{m} \int_{\{m \leq |v_n| < 2m\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2); \end{aligned}$$

here, the final equality followed from the relation, since $m > \ell + 2k$,

$$R_{n, \nu, \ell} H'_m(v_n) = -\frac{2k}{m} \chi_{m \leq |v_n| \leq 2m}, \quad \textit{a.e. in } Q. \quad (5.24)$$

Next we go to the limit in m , by using (4.3), (4.4) for u_n , with $\phi = (1 - \Phi_{\delta_1, \delta_2})$. There holds

$$A_5 = -2k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d((\rho_{n, s} - \eta_{n, s})^+ + (\rho_{n, s} - \eta_{n, s})^-) + \omega(l_1, l_2, n_1, n_2, m).$$

Then, from (5.3) and (5.4), we get $A_5 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of A_6 . Again, from (5.21),

$$\begin{aligned} A_6 &= \int_Q H_m(v_n) \varphi f_n + \int_Q g_n \cdot \nabla (H_m(v_n) \varphi) \\ &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n + \int_Q g_n \cdot \nabla (H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Thus we can write $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2)$, where

$$\begin{aligned} D_1 &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n, & D_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(v_n) g_n \cdot \nabla v_n, \\ D_3 &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) g_n \cdot \nabla R_{n, \nu, \ell}, & D_4 &= - \int_Q H_m(v_n) R_{n, \nu, \ell} g_n \cdot \nabla \Phi_{\delta_1, \delta_2}. \end{aligned}$$

Since $\{f_n\}$ converges to f weakly in $L^1(Q)$, and (5.19)-(5.20) hold, we get from Remark 5.2,

$$D_1 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) (T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v))) f + \omega(m, n) = \omega(m, n, \nu, \ell).$$

We deduce from (4.10) that $D_2 = \omega(m)$. Next consider D_3 . Note that $H_m(v_n) = 1 + \omega(m)$, and (5.20) holds, and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, and $\langle T_k(v) \rangle_\nu$ converges to $T_k(v)$ strongly in X . Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v))) + \omega(m, n) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(v - T_k(v)) - T_{\ell-k}(v - T_k(v))) + \omega(m, n, \nu) \\ &= \omega(m, n, \nu, \ell). \end{aligned}$$

Similarly we also get $D_4 = \omega(m, n, \nu, \ell)$. Thus $A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of A_7 . We have

$$\begin{aligned} |A_7| &= \left| \int_Q S'_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} d(\rho_{n,0} - \eta_{m,0}) \right| + \omega(l_1, l_2, n_1, n_2) \\ &\leq 4k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d(\rho_n + \eta_n) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

From (5.3) and (5.4) we get $A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of $A_1 + A_2 + A_3$. We set

$$J(r) = T_{\ell-k}(r - T_k(r)), \quad \forall r \in \mathbb{R},$$

and use the notations \bar{J} and \mathcal{J} of (4.11). From the definitions of ξ_{1,n_1}, ξ_{1,n_2} , we can see that

$$\begin{aligned} A_1 + A_2 &= - \int_{\Omega} J(v_n(T)) \overline{H_m}(v_n(T)) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) \overline{H_m}(u_{0,n}) + \omega(l_1, l_2, n_1, n_2) \\ &= - \int_{\Omega} J(v_n(T)) v_n(T) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) u_{0,n} + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \quad (5.25)$$

where $z_{\nu} = \langle T_k(v) \rangle_{\nu}(0)$. We can write $A_3 = F_1 + F_2$, where

$$\begin{aligned} F_1 &= - \int_Q \left(\xi_{n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t \overline{H_m}(v_n), \\ F_2 &= \int_Q \left(\xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \right)_t \overline{H_m}(v_n). \end{aligned}$$

Estimate of F_2 . We write $F_2 = G_1 + G_2 + G_3$, with

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t \xi_{n_2} [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \overline{H_m}(v_n), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) (\xi_{n_2})_t [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \overline{H_m}(v_n), \\ G_3 &= \int_Q \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell-k}(v_n - T_k(v_n))]_{-l_2})_t \overline{H_m}(v_n). \end{aligned}$$

We find easily

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t J(v_n) v_n + \omega(l_1, l_2, n_1, n_2, m), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) (\xi_{n_2})_t J(v_n) \overline{H_m}(v_n) + \omega(l_1, l_2) = \int_{\Omega} J(u_{0,n}) u_{0,n} + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Next consider G_3 . Setting $b = \overline{H_m}(v_n)$, there holds from (4.13) and (4.12),

$$([\mathcal{J}(b)]_{-l_2})_t b(\cdot, t) = \frac{b(\cdot, t)}{l_2} (J(b)(\cdot, t) - J(b)(\cdot, t - l_2)).$$

Hence

$$([T_{\ell-k}(v_n - T_k(v_n))]_{-l_2})_t \overline{H_m}(v_n) \geq \left([\mathcal{J}(\overline{H_m}(v_n))]_{-l_2} \right)_t = ([\mathcal{J}(v_n)]_{-l_2})_t,$$

since \mathcal{J} is constant in $\{|r| \geq m + \ell + 2k\}$. Integrating by parts in G_3 , we find

$$\begin{aligned}
G_3 &\geq \int_Q \xi_{2,n_2} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{J}(v_n)]_{-l_2})_t \\
&= - \int_Q (\xi_{2,n_2} (1 - \Phi_{\delta_1, \delta_2}))_t [\mathcal{J}(v_n)]_{-l_2} + \int_{\Omega} \xi_{2,n_2}(T) [\mathcal{J}(v_n)]_{-l_2}(T) \\
&= - \int_Q (\xi_{2,n_2})_t (1 - \Phi_{\delta_1, \delta_2}) \mathcal{J}(v_n) \\
&\quad + \int_Q \xi_{2,n_2} (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(v_n) + \int_{\Omega} \xi_{2,n_2}(T) \mathcal{J}(v_n(T)) + \omega(l_1, l_2) \\
&= - \int_{\Omega} \mathcal{J}(u_{0,n}) + \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(v_n) + \int_{\Omega} \mathcal{J}(v_n(T)) + \omega(l_1, l_2, n_1, n_2).
\end{aligned}$$

Therefore, since $\mathcal{J}(v_n) - J(v_n)v_n = -\bar{J}(v_n)$ and $\bar{J}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$, we obtain

$$F_2 \geq \int_{\Omega} \bar{J}(u_{0,n}) - \int_Q (\Phi_{\delta_1, \delta_2})_t \bar{J}(v_n) + \int_{\Omega} \mathcal{J}(v_n(T)) + \omega(l_1, l_2, n_1, n_2, m). \quad (5.26)$$

Estimate of F_1 . Since $m > \ell + 2k$, there holds $T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) = T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})$ on $\text{supp} \overline{H_m}(v_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$\begin{aligned}
L_1 &= - \int_Q \left(\xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
L_2 &= - \int_Q \left(\xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t \langle T_k(\overline{H_m}(v)) \rangle_{\nu}.
\end{aligned}$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$\begin{aligned}
L_2 &= \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\langle T_k(\overline{H_m}(v)) \rangle_{\nu})_t \\
&\quad + \int_{\Omega} \xi_{1,n_1}(0) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1}(0) \langle T_k(\overline{H_m}(v)) \rangle_{\nu}(0) \\
&= \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} + \omega(l_1, l_2, n_1, n_2).
\end{aligned} \quad (5.27)$$

We decompose L_1 into $L_1 = K_1 + K_2 + K_3$, where

$$\begin{aligned}
K_1 &= - \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
K_2 &= \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
K_3 &= - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) \left([T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}).
\end{aligned}$$

Then we check easily that

$$K_1 = \int_{\Omega} T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) (T) (v_n - \langle T_k(v) \rangle_{\nu}) (T) dx + \omega(l_1, l_2, n_1, n_2, m),$$

$$K_2 = \int_Q (\Phi_{\delta_1, \delta_2})_t T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) (v_n - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).$$

Next consider K_3 . Here we use the function \mathcal{T}_k defined at (4.13). We set $b = \overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}$. Hence from (4.12),

$$\begin{aligned} ([T_{\ell+k}(b)]_{l_1})_t b(\cdot, t) &= \frac{b(\cdot, t)}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1) - T_{\ell+k}(b)(\cdot, t)) \\ &\leq \frac{1}{l_1} (\mathcal{T}_{\ell+k}(b)((\cdot, t + l_1)) - \mathcal{T}_{\ell+k}(b)(\cdot, t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t. \end{aligned}$$

Thus

$$\begin{aligned} \left([T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) &\leq \left([\mathcal{T}_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t \\ &= \left([\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t. \end{aligned}$$

Then

$$\begin{aligned} K_3 &\geq - \int_Q \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) \left([\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t \\ &= \int_Q (\xi_{1, n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} - \int_Q \xi_{1, n_1} (\Phi_{\delta_1, \delta_2})_t [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \\ &\quad + \int_{\Omega} \xi_{1, n_1}(0) [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1}(0) \\ &= - \int_{\Omega} \mathcal{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) \\ &\quad + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0, n} - z_{\nu}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

We find by addition, since $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \overline{T}_{\ell+k}(r)$ for any $r \in \mathbb{R}$,

$$\begin{aligned} L_1 &\geq \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0, n} - z_{\nu}) + \int_{\Omega} \overline{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t \overline{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \tag{5.28}$$

We deduce from (5.28), (5.27), (5.26),

$$\begin{aligned}
A_3 &\geq \int_{\Omega} \bar{\mathcal{J}}(u_{0,n}) + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} \\
&+ \int_{\Omega} \bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) + \int_{\Omega} \mathcal{J}(v_n(T)) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v_n)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned} \tag{5.29}$$

Next we add (5.25) and (5.29). Note that $\mathcal{J}(v_n(T)) - J(v_n(T))v_n(T) = -\bar{\mathcal{J}}(v_n(T))$, and also $\mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) - T_{\ell+k}(u_{0,n} - z_{\nu})(z_{\nu} - u_{0,n}) = -\bar{T}_{\ell+k}(u_{0,n} - z_{\nu})$. Then we find

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_{0,n}) - \bar{T}_{\ell+k}(u_{0,n} - z_{\nu})) + \int_{\Omega} (\bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \bar{\mathcal{J}}(v_n(T))) \\
&+ \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v_n)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned}$$

Notice that $\bar{T}_{\ell+k}(r-s) - \bar{\mathcal{J}}(r) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$; thus

$$\int_{\Omega} (\bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \bar{\mathcal{J}}(v_n(T))) \geq 0.$$

And $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$ and $\{v_n\}$ converges to v in $L^1(Q)$ from Proposition 4.10. Thus we obtain

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m, n).
\end{aligned}$$

Moreover $T_{\ell+k}(r-s)(T_k(r) - s) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$, hence

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v)) \\
&+ \omega(l_1, l_2, n_1, n_2, m, n).
\end{aligned}$$

As $\nu \rightarrow \infty$, $\{z_{\nu}\}$ converges to $T_k(u_0)$, *a.e.* in Ω , thus we get

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - T_k(u_0))) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - T_k(v)) - \bar{\mathcal{J}}(v)) \\
&+ \omega(l_1, l_2, n_1, n_2, m, n, \nu).
\end{aligned}$$

Finally $|\overline{T}_{\ell+k}(r-T_k(r)) - \overline{J}(r)| \leq 2k|r|\chi_{\{|r|\geq\ell\}}$ for any $r \in \mathbb{R}$, thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ which implies (5.8), since I_2 does not depend on $l_1, l_2, n_1, n_2, m, \ell$. \blacksquare

Next we conclude the proof of Theorem 2.1:

Lemma 5.6 *The function u is a R -solution of (1.1).*

Proof. (i) First show that u satisfies (4.2). Here we proceed as in [49]. Let $\varphi \in X \cap L^\infty(Q)$ such $\varphi_t \in X' + L^1(Q)$, $\varphi(\cdot, T) = 0$, and $S \in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on \mathbb{R} , $S(0) = 0$. Let $M > 0$ such that $\text{supp} S' \subset [-M, M]$. Taking successively (φ, S) and $(\varphi\psi_\delta^\pm, S)$ as test functions in (4.2) applied to u_n , we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \quad A_{2,\delta,\pm} + A_{3,\delta,\pm} + A_{4,\delta,\pm} = A_{5,\delta,\pm} + A_{6,\delta,\pm} + A_{7,\delta,\pm},$$

where

$$\begin{aligned} A_1 &= - \int_{\Omega} \varphi(0)S(u_{0,n}), \quad A_2 = - \int_Q \varphi_t S(v_n), \quad A_{2,\delta,\pm} = - \int_Q (\varphi\psi_\delta^\pm)_t S(v_n), \\ A_3 &= \int_Q S'(v_n)A(x, t, \nabla u_n) \cdot \nabla \varphi, \quad A_{3,\delta,\pm} = \int_Q S'(v_n)A(x, t, \nabla u_n) \cdot \nabla (\varphi\psi_\delta^\pm), \end{aligned}$$

$$A_4 = \int_Q S''(v_n)\varphi A(x, t, \nabla u_n) \cdot \nabla v_n, \quad A_{4,\delta,\pm} = \int_Q S''(v_n)\varphi\psi_\delta^\pm A(x, t, \nabla u_n) \cdot \nabla v_n,$$

$$A_5 = \int_Q S'(v_n)\varphi d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q S'(v_n)\varphi d\rho_{n,0}, \quad A_7 = - \int_Q S'(v_n)\varphi d\eta_{n,0},$$

$$A_{5,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\widehat{\lambda}_{n,0}, \quad A_{6,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7,\delta,\pm} = - \int_Q S'(v_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$, and $\{S(v_n)\}$ converges to $S(v)$ strongly in X and weak* in $L^\infty(Q)$, there holds, from (5.2),

$$A_1 = - \int_{\Omega} \varphi(0)S(u_0) + \omega(n), \quad A_2 = - \int_Q \varphi_t S(v) + \omega(n), \quad A_{2,\delta,\psi_\delta^\pm} = \omega(n, \delta).$$

Moreover $T_M(v_n)$ converges to $T_M(v)$, then $T_M(v_n) + h_n$ converges to $T_k(v) + h$ strongly in X , thus

$$\begin{aligned} A_3 &= \int_Q S'(v_n)A(x, t, \nabla (T_M(v_n) + h_n)) \cdot \nabla \varphi \\ &= \int_Q S'(v)A(x, t, \nabla (T_M(v) + h)) \cdot \nabla \varphi + \omega(n) \\ &= \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi + \omega(n); \end{aligned}$$

and

$$\begin{aligned}
A_4 &= \int_Q S''(v_n) \varphi A(x, t, \nabla (T_M(v_n) + h_n)) \cdot \nabla T_M(v_n) \\
&= \int_Q S''(v) \varphi A(x, t, \nabla (T_M(v) + h)) \cdot \nabla T_M(v) + \omega(n) \\
&= \int_Q S''(v) \varphi A(x, t, \nabla u) \cdot \nabla v + \omega(n).
\end{aligned}$$

In the same way, since ψ_δ^\pm converges to 0 in X ,

$$\begin{aligned}
A_{3,\delta,\pm} &= \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla (\varphi \psi_\delta^\pm) + \omega(n) = \omega(n, \delta), \\
A_{4,\delta,\pm} &= \int_Q S''(v) \varphi \psi_\delta^\pm A(x, t, \nabla u) \cdot \nabla v + \omega(n) = \omega(n, \delta).
\end{aligned}$$

And $\{g_n\}$ converges strongly in $(L^{p'}(\Omega))^N$, thus

$$\begin{aligned}
A_5 &= \int_Q S'(v_n) \varphi f_n + \int_Q S'(v_n) g_n \cdot \nabla \varphi + \int_Q S''(v_n) \varphi g_n \cdot \nabla T_M(v_n) \\
&= \int_Q S'(v) \varphi f + \int_Q S'(v) g \cdot \nabla \varphi + \int_Q S''(v) \varphi g \cdot \nabla T_M(v) + \omega(n) \\
&= \int_Q S'(v) \varphi d\widehat{\mu}_0 + \omega(n).
\end{aligned}$$

and $A_{5,\delta,\pm} = \int_Q S'(v) \varphi \psi_\delta^\pm d\widehat{\lambda}_{n,0} + \omega(n) = \omega(n, \delta)$. Then $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n, \delta)$. From (5.2) we verify that $A_{7,\delta,+} = \omega(n, \delta)$ and $A_{6,\delta,-} = \omega(n, \delta)$. Moreover, from (5.6) and (5.2), we find

$$|A_6 - A_{6,\delta,+}| \leq \int_Q |S'(v_n) \varphi| (1 - \psi_\delta^+) d\rho_{n,0} \leq \|S\|_{W^{2,\infty}(\mathbb{R})} \|\varphi\|_{L^\infty(Q)} \int_Q (1 - \psi_\delta^+) d\rho_n = \omega(n, \delta).$$

Similarly we also have $|A_7 - A_{7,\delta,-}| \leq \omega(n, \delta)$. Hence $A_6 = \omega(n)$ and $A_7 = \omega(n)$. Therefore, we finally obtain (4.2):

$$-\int_\Omega \varphi(0) S(u_0) - \int_Q \varphi_t S(v) + \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(v) \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q S'(v) \varphi d\widehat{\mu}_0. \tag{5.30}$$

(ii) Next, we prove (4.3) and (4.4). We take $\varphi \in C_c^\infty(Q)$ and take $((1 - \psi_\delta^-)\varphi, \overline{H_m})$ as test functions in (5.30), with $\overline{H_m}$ as in (4.14). We can write $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$, where

$$\begin{aligned} D_{1,m} &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t \overline{H_m}(v), & D_{2,m} &= \int_Q H_m(v) A(x, t, \nabla u) \cdot \nabla ((1 - \psi_\delta^-)\varphi), \\ D_{3,m} &= \int_Q H_m(v) (1 - \psi_\delta^-) \varphi d\widehat{\mu}_0, & D_{4,m} &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla v, \\ D_{5,m} &= - \frac{1}{m} \int_{-2m \leq v \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla v. \end{aligned} \quad (5.31)$$

Taking the same test functions in (4.2) applied to u_n , there holds $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$, where

$$\begin{aligned} D_{1,m}^n &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t \overline{H_m}(v_n), & D_{2,m}^n &= \int_Q H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-)\varphi), \\ D_{3,m}^n &= \int_Q H_m(v_n) (1 - \psi_\delta^-) \varphi d(\widehat{\lambda}_{n,0} + \rho_{n,0} - \eta_{n,0}), & D_{4,m}^n &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla v_n, \\ D_{5,m}^n &= - \frac{1}{m} \int_{-2m \leq v_n \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla v_n \end{aligned} \quad (5.32)$$

In (5.32), we go to the limit as $m \rightarrow \infty$. Since $\{\overline{H_m}(v_n)\}$ converges to v_n and $\{H_m(v_n)\}$ converges to 1, *a.e.* in Q , and $\{\nabla H_m(v_n)\}$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation $D_1^n + D_2^n = D_3^n + D^n$, where

$$\begin{aligned} D_1^n &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t v_n, & D_2^n &= \int_Q A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-)\varphi), & D_3^n &= \int_Q (1 - \psi_\delta^-) \varphi d\widehat{\lambda}_{n,0} \\ D^n &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1 - \psi_\delta^-) \varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\ &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_n - \eta_n). \end{aligned}$$

Clearly, $D_{i,m} - D_i^n = \omega(n, m)$ for $i = 1, 2, 3$. From Lemma (5.3) and (5.2)-(5.4), we obtain $D_{5,m} = \omega(n, m, \delta)$, and

$$\frac{1}{m} \int_{\{m \leq v < 2m\}} \psi_\delta^- \varphi A(x, t, \nabla u) \cdot \nabla v = \omega(n, m, \delta),$$

thus,

$$D_{4,m} = \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v + \omega(n, m, \delta).$$

Since $\left| \int_Q (1 - \psi_\delta^-) \varphi d\eta_n \right| \leq \|\varphi\|_{L^\infty} \int_Q (1 - \psi_\delta^-) d\eta_n$, it follows that $\int_Q (1 - \psi_\delta^-) \varphi d\eta_n = \omega(n, m, \delta)$ from (5.4). And $\left| \int_Q \psi_\delta^- \varphi d\rho_n \right| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$, thus, from (5.2), $\int_Q (1 - \psi_\delta^-) \varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$. Then $D^n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$. Therefore by subtraction, we get

$$\frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta),$$

hence

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+, \quad (5.33)$$

which proves (4.3) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v \\ &= \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi \psi_\delta^+ A(x, t, \nabla u) \cdot \nabla v + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v \\ &= \int_Q \varphi \psi_\delta^+ d\mu_s^+ + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+ + D, \end{aligned}$$

where,

$$D = \int_Q \varphi (1 - \psi_\delta^+) d\mu_s^+ + \lim_{n \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v = \omega(\delta).$$

Therefore, (5.33) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (4.3) by density, and similarly, (4.4). This completes the proof of Theorem 2.1. \blacksquare

As a consequence of Theorem 2.1, we get the following:

Corollary 5.7 *Let $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q)$. Then there exists a R-solution u to the problem 1.1 with data (μ, u_0) . Furthermore, if $v_0 \in L^1(\Omega)$ and $\omega \in \mathcal{M}_b(Q)$ such that $u_0 \leq v_0$ and $\mu \leq \omega$, then one can find R-solution v to the problem 1.1 with data (ω, v_0) such that $u \leq v$.*

In particular, if $a \equiv 0$ in (1.2), then u satisfies (4.21) and $\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq M$ with $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$.

6 Equations with perturbation terms

Let $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying (1.2), (1.3) with $a \equiv 0$. Let $\mathcal{G} : \Omega \times (0, T) \times \mathbb{R} \mapsto \mathbb{R}$ be a Caratheodory function. If U is a function defined in Q we define the function $\mathcal{G}(U)$ in Q by

$$\mathcal{G}(U)(x, t) = \mathcal{G}(x, t, U(x, t)) \quad \text{for a.e. } (x, t) \in Q.$$

We consider the problem (1.5):

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$. We say that u is a R-solution of problem (1.5) if $\mathcal{G}(u) \in L^1(Q)$ and u is a R-solution of (1.1) with data $(\mu - \mathcal{G}(u), u_0)$.

6.1 Subcritical type results

For proving Theorem 2.2, we begin by an integration Lemma:

Lemma 6.1 *Let G satisfying (2.3). If a measurable function V in Q satisfies*

$$\operatorname{meas} \{|V| \geq t\} \leq Mt^{-p_c}, \quad \forall t \geq 1,$$

for some $M > 0$, then for any $L > 1$,

$$\int_{\{|V| \geq L\}} G(|V|) \leq p_c M \int_L^\infty G(s) s^{-1-p_c} ds. \quad (6.1)$$

Proof. Indeed, setting $G_L(s) = \chi_{[L, \infty)}(s)G(s)$, we have

$$\int_{\{|V| \geq L\}} G(|V|) dx dt = \int_Q G_L(|V|) dx dt \leq \int_0^\infty G_L(|V|^*(s)) ds$$

where $|V|^*$ is and the rearrangement of $|V|$, defined by

$$|V|^*(s) = \inf\{a > 0 : \operatorname{meas} \{|V| > a\} \leq s\}, \quad \forall s \geq 0.$$

From the assumption, we get $|V|^*(s) \leq \sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)$. Thus, for any $L > 1$,

$$\int_{\{|V| \geq L\}} G(|V|) dx dt \leq \int_0^\infty G_L\left(\sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)\right) ds = p_c M \int_L^\infty G(s) s^{-1-p_c} ds,$$

which implies (6.1). ■

Proof of Theorem 2.2. Proof of (i) Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, with $\mu_0 \in \mathcal{M}_0(Q)$, $\mu_s \in \mathcal{M}_s(Q)$, and $u_0 \in L^1(\Omega)$. Then μ_0^+, μ_0^- can be decomposed as $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$. Let $\mu_{n,s,i} \in C_c^\infty(Q)$, $\mu_{n,s,i} \geq 0$, converging respectively to μ_s^+, μ_s^- in the narrow topology. By Lemma 3.1, we can find $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$ which strongly converge to f_i, g_i, h_i in $L^1(Q)$, $(L^{p'}(Q))^N$ and

X respectively, $i = 1, 2$, such that $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$, and $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$, converging respectively to μ_0^+ , μ_0^- in the narrow topology. Furthermore, if we set

$$\mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2},$$

then $|\mu_n|(Q) \leq |\mu|(Q)$. Consider a sequence $\{u_{0,n}\} \subset C_c^\infty(\Omega)$ which strongly converges to u_0 in $L^1(\Omega)$ and satisfies $\|u_{0,n}\|_{1,\Omega} \leq \|u_0\|_{1,\Omega}$.

Let u_n be a solution of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x, t, \nabla u_n)) + \mathcal{G}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases}$$

We can choose $\varphi = \varepsilon^{-1}T_\varepsilon(u_n)$ as test function of above problem. Then we find

$$\int_Q (\varepsilon^{-1}\overline{T_\varepsilon}(u_n))_t + \int_Q \varepsilon^{-1}A(x, t, \nabla T_\varepsilon(u_n)) \cdot \nabla T_\varepsilon(u_n) + \int_Q \mathcal{G}(x, t, u_n)\varepsilon^{-1}T_\varepsilon(u_n) = \int_Q \varepsilon^{-1}T_\varepsilon(u_n)d\mu_n.$$

Since

$$\int_Q (\varepsilon^{-1}\overline{T_\varepsilon}(u_n))_t = \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_n(T))dx - \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_{0,n})dx \geq -\|u_{0,n}\|_{L^1(\Omega)},$$

there holds

$$\int_Q \mathcal{G}(x, t, u_n)\varepsilon^{-1}T_\varepsilon(u_n) \leq |\mu_n|(Q) + \|u_{0,n}\|_{L^1(\Omega)} \leq |\mu|(Q) + \|u_0\|_{1,\Omega}.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\int_Q |\mathcal{G}(x, t, u_n)| \leq |\mu|(Q) + \|u_0\|_{1,\Omega}. \quad (6.2)$$

Next apply Proposition 4.8 and Remark 4.9 to u_n with initial data $u_{0,n}$ and measure data $\mu_n - \mathcal{G}(u_n) \in L^1(Q)$, we get

$$\operatorname{meas}\{|u_n| \geq s\} \leq C(|\mu|(Q) + \|u_0\|_{L^1(\Omega)})^{\frac{p+N}{N}} s^{-pc}, \quad \forall s > 0, \forall n \in \mathbb{N},$$

for some $C = C(N, p, c_1, c_2)$. Since $|\mathcal{G}(x, t, u_n)| \leq G(|u_n|)$, we deduce from (6.1) that $\{|\mathcal{G}(u_n)|\}$ is equi-integrable. Then, thanks to Proposition 4.10, up to a subsequence, $\{u_n\}$ converges to some function u , *a.e.* in Q , and $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$. Therefore, by Theorem 2.1, u is a R-solution of (2.4).

Proof of (ii). Let $\{u_n\}_{n \geq 1}$ be defined by induction as nonnegative R-solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x, t, \nabla u_1)) = \mu & \text{in } Q, \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ u_1(0) = u_0 & \text{in } \Omega, \end{cases} \quad \begin{cases} (u_{n+1})_t - \operatorname{div}(A(x, t, \nabla u_{n+1})) = \mu - \lambda\mathcal{G}(u_n) & \text{in } Q, \\ u_{n+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{n+1}(0) = u_0 & \text{in } \Omega, \end{cases}$$

Thanks to Corollary 5.7 we can assume that $\{u_n\}$ is nondecreasing and satisfies for any $s > 0$ and $n \in \mathbb{N}$

$$\text{meas} \{|u_n| \geq s\} \leq C_1 K_n s^{-p_c}, \quad (6.3)$$

where C_1 does not depend on s, n , and

$$K_1 = (\|u_0\|_{1,\Omega} + |\mu|(Q))^{\frac{p+N}{N}},$$

$$K_{n+1} = (\|u_0\|_{1,\Omega} + |\mu|(Q) + \lambda \|\mathcal{G}(u_n)\|_{1,Q})^{\frac{p+N}{N}},$$

for any $n \geq 1$. Take $\varepsilon = \lambda + |\mu|(Q) + \|u_0\|_{L^1(\Omega)} \leq 1$. Denoting by C_i some constants independent on n, ε , there holds $K_1 \leq C_2 \varepsilon$, and for $n \geq 1$,

$$K_{n+1} \leq C_3 \varepsilon (\|\mathcal{G}(u_n)\|_{1,Q}^{1+\frac{p}{N}} + 1).$$

From (6.1) and (6.3), we find

$$\|\mathcal{G}(u_n)\|_{L^1(Q)} \leq |Q| G(2) + \int_{\{|u_n| \geq 2\}} G(|u_n|) dx dt \leq |Q| G(2) + C_4 K_n \int_2^\infty G(s) s^{-1-p_c} ds.$$

Thus, $K_{n+1} \leq C_5 \varepsilon (K_n^{1+\frac{p}{N}} + 1)$. Therefore, if ε is small enough, $\{K_n\}$ is bounded. Then, again from (6.1) and the relation $|\mathcal{G}(x, t, u_n)| \leq G(|u_n|)$ we verify that $\{\mathcal{G}(u_n)\}$ converges. Then by Theorem 2.1, up to a subsequence, $\{u_n\}$ converges to a R-solution u of (2.5). \blacksquare

6.2 General case with absorption terms

In the sequel we assume that $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ does not depend on t . We recall a result obtained in [53],[17] in the elliptic case:

Theorem 6.2 *Let Ω be a bounded domain of \mathbb{R}^N . Let $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6),(1.7). Then there exists a constant κ depending on N, p, c_3, c_4 such that, if $\omega \in \mathcal{M}_b(\Omega)$ and u is a R-solution of problem*

$$\begin{cases} -\text{div}(A(x, \nabla u)) = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there holds

$$-\kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^-] \leq u \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^+]. \quad (6.4)$$

Next we give a general result in case of absorption terms:

Theorem 6.3 *Let $p < N$, $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6),(1.7), and $\mathcal{G} : Q \times \mathbb{R} \mapsto \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto \mathcal{G}(x, t, s)$ is nondecreasing and odd, for a.e. (x, t) in Q .*

Let $\mu_1, \mu_2 \in \mathcal{M}_b^+(Q)$ such that there exist $\omega_n \in \mathcal{M}_b^+(\Omega)$ and nondecreasing sequences $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ in $\mathcal{M}_b^+(Q)$ with compact support in Q , converging to μ_1, μ_2 , respectively in the narrow topology, and

$$\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi_{(0,T)}, \quad \mathcal{G}((n + \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega_n])) \in L^1(Q),$$

where the constant c is given at Theorem 6.2. Let $u_0 \in L^1(\Omega)$, and $\mu = \mu_1 - \mu_2$. Then there exists a R -solution u of problem (1.5).

Moreover if $u_0 \in L^\infty(\Omega)$, and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$, for some $\gamma \in \mathcal{M}_b^+(\Omega)$, then a.e. in Q ,

$$|u(x, t)| \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) + \|u_0\|_{\infty, \Omega}. \quad (6.5)$$

For proving this result, we need two Lemmas:

Lemma 6.4 Let \mathcal{G} satisfy the assumptions of Theorem 6.3 and $\mathcal{G} \in L^\infty(Q \times \mathbb{R})$. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, and $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in \mathcal{M}_b^+(Q)$ with compact support in Q , $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω such that $\lambda_i \leq \gamma \otimes \chi_{(0,T)}$. Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in Q . Then, there exist R -solutions u, u_1, u_2 , to problems

$$u_t - \text{div}(A(x, \nabla u)) + \mathcal{G}(u) = \lambda_1 - \lambda_2 \quad \text{in } Q, \quad u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(0) = u_{0,1} - u_{0,2}, \quad (6.6)$$

$$(u_i)_t - \text{div}(A(x, \nabla u_i)) + \mathcal{G}(u_i) = \lambda_i \quad \text{in } Q, \quad u_i = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_i(0) = u_{0,i}, \quad (6.7)$$

relative to decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that a.e. in Q ,

$$-\|u_{0,2}\|_{\infty, \Omega} - \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) \leq -u_2(x, t) \leq u(x, t) \leq u_1(x, t) \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) + \|u_{0,1}\|_{\infty, \Omega}, \quad (6.8)$$

and

$$\int_Q |\mathcal{G}(u)| \leq \sum_{i=1,2} (\lambda_i(Q) + \|u_{0,i}\|_{L^1(\Omega)}), \quad \text{and} \quad \int_Q \mathcal{G}(u_i) \leq \lambda_i(Q) + \|u_{0,i}\|_{L^1(\Omega)}, \quad i = 1, 2. \quad (6.9)$$

Furthermore, assume that \mathcal{H}, \mathcal{K} have the same properties as \mathcal{G} , and $\mathcal{H}(x, t, s) \leq \mathcal{G}(x, t, s) \leq \mathcal{K}(x, t, s)$ for any $s \in (0, +\infty)$ and a.e. in Q . Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to \mathcal{H}, \mathcal{K} with data λ_i , such that $u_i(\mathcal{H}) \geq u_i \geq u_i(\mathcal{K})$, $i = 1, 2$.

Assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega_i, u_{0,i,2})$, $(\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in \mathbb{R} and \mathbb{R}^N , and $\varphi_n = \varphi_{1,n} \varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} * \gamma$, and for $i = 1, 2$, $u_{0,i,n} = \varphi_{2,n} * u_{0,i}$,

$$\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \text{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},$$

where $f_{i,n} = \varphi_n * f_i$, $g_{i,n} = \varphi_n * g_i$, $h_{i,n} = \varphi_n * h_i$, $\lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$, and

$$\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \text{div}(g_n) + (h_n)_t + \lambda_{s,n},$$

where $f_n = f_{1,n} - f_{2,n}$, $g_n = g_{1,n} - g_{2,n}$, $h_n = h_{1,n} - h_{2,n}$, $\lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$. Then for n large enough, $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q)$, $\gamma_n \in C_c^\infty(\Omega)$. Thus there exist unique solutions $u_n, u_{i,n}, v_{i,n}$, $i = 1, 2$, of problems

$$\begin{aligned} (u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) &= \lambda_{1,n} - \lambda_{2,n} \quad \text{in } Q, & u_n &= 0 \quad \text{on } \partial\Omega \times (0, T), & u_n(0) &= u_{0,1,n} - u_{0,2,n} \quad \text{in } \Omega, \\ (u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) &= \lambda_{i,n} \quad \text{in } Q, & u_{i,n} &= 0 \quad \text{on } \partial\Omega \times (0, T), & u_{i,n}(0) &= u_{0,i,n} \quad \text{in } \Omega, \\ -\operatorname{div}(A(x, \nabla w_n)) &= \gamma_n \quad \text{in } \Omega, & w_n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

such that

$$-||u_{0,2}||_{\infty, \Omega} - w_n(x) \leq -u_{2,n}(x, t) \leq u_n(x, t) \leq u_{1,n}(x, t) \leq w_n(x) + ||u_{0,1}||_{\infty, \Omega}, \quad a.e. \text{ in } Q.$$

Moreover, as in the Proof of Theorem 2.2, (i), there holds

$$\int_Q |\mathcal{G}(u_n)| \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}), \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}, \quad i = 1, 2.$$

By Proposition 4.10, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converge to some (u, u_1, u_2) , *a.e.* in Q . Since \mathcal{G} is bounded, in particular, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ and $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, (6.9) is satisfied. Moreover $\{\lambda_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$ and $\{\lambda_n - \mathcal{G}(u_n), f_n - \mathcal{G}(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$ are approximations of $(\lambda_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i})$ and $(\lambda - \mathcal{G}(u), f - \mathcal{G}(u), g, h, \lambda_s, u_{0,1} - u_{0,2})$, in the sense of Theorem 2.1. Thus, we can find (different) subsequences converging *a.e.* to u, u_1, u_2 , R-solutions of (6.6) and (6.7). Furthermore, from [47, Corollary 3.4], up to a subsequence, $\{w_n\}$ converges *a.e.* in Q to a R-solution

$$-\operatorname{div}(A(x, \nabla w)) = \gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

such that $w \leq c\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma$ *a.e.* in Ω . Hence, we get the inequality (6.8). The other conclusions follow in the same way. \blacksquare

Lemma 6.5 *Let \mathcal{G} satisfy the assumptions of Theorem 6.3. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, $\lambda_i \in \mathcal{M}_b^+(Q)$ with compact support in Q , and $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω , such that*

$$\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \quad \mathcal{G}((||u_{0,i}||_{\infty, \Omega} + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}(\Omega)} \gamma)) \in L^1(Q). \quad (6.10)$$

Then, there exist R-solutions u, u_1, u_2 of the problems (6.6) and (6.7), respectively relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, satisfying (6.8) and (6.9).

Moreover, assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then, one can find solutions $u_i(\omega_i, u_{0,i,2})$, $u_i(\theta_i, u_{0,i,1})$, corresponding with $(\omega_i, u_{0,i,2})$, $(\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. From Lemma 6.4 there exist R-solutions $u_n, u_{i,n}$ to problems

$$(u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_n(0) = u_{0,1} - u_{0,2}$$

$$(u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i \quad \text{in } Q, \quad u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_{i,n}(0) = u_{0,i},$$

relative to the decompositions $(f_1 - f_2 - T_n(\mathcal{G}(u_n)), g_1 - g_2, h_1 - h_2)$, $(f_i - T_n(\mathcal{G}(u_{i,n})), g_i, h_i)$; and they satisfy

$$\begin{aligned} -\|u_{0,2}\|_{\infty, \Omega} - \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma(x) &\leq -u_{2,n}(x, t) \leq u_n(x, t) \\ &\leq u_{1,n}(x, t) \leq \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma(x) + \|u_{0,1}\|_{\infty, \Omega}, \end{aligned} \quad (6.11)$$

$$\int_Q |T_n(\mathcal{G}(u_n))| \leq \sum_{i=1,2} (\lambda_i(Q) + \|u_{0,i}\|_{1,\Omega}), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) \leq \lambda_i(Q) + \|u_{0,i}\|_{1,\Omega}.$$

As in Lemma 6.4, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converges *a.e.* in Q to $\{u, u_1, u_2\}$ for which (6.8) is satisfied *a.e.* in Q . From (6.10), (6.11) and the dominated convergence Theorem, we deduce that $\{T_n(\mathcal{G}(u_n))\}$ converges to $\mathcal{G}(u)$ and $\{T_n(\mathcal{G}(u_{i,n}))\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, from Theorem 2.1, u and u_i are respective R-solutions of (6.6) and (6.7) relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, and (6.8) and (6.9) hold. The last statement follows from the same assertion in Lemma 6.4. \blacksquare

Proof of Theorem 6.3. By Proposition 3.2, for $i = 1, 2$, there exist $f_{i,n}, f_i \in L^1(Q)$, $g_{i,n}, g_i \in (L^{p'}(Q))^N$ and $h_{i,n}, h_i \in X$, $\mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}_s^+(Q)$ such that

$$\mu_i = f_i - \operatorname{div} g_i + (h_i)_t + \mu_{i,s}, \quad \mu_{i,n} = f_{i,n} - \operatorname{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},$$

and $\{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\}$ strongly converge to f_i, g_i, h_i in $L^1(Q)$, $(L^{p'}(Q))^N$ and X respectively, and $\{\mu_{i,n}\}, \{\mu_{i,n,s}\}$ converge to $\mu_i, \mu_{i,s}$ (strongly) in $\mathcal{M}_b(Q)$, and

$$\|f_{i,n}\|_{1,Q} + \|g_{i,n}\|_{p',Q} + \|h_{i,n}\|_X + \mu_{i,n,s}(\Omega) \leq 2\mu(Q).$$

By Lemma 6.5, there exist R-solutions $u_n, u_{i,n}$ to problems

$$(u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_n(0) = T_n(u_0) \quad (6.12)$$

$$(u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \mu_{i,n} \quad \text{in } \Omega, \quad u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_{i,n}(0) = T_n(u_0^\pm), \quad (6.13)$$

for $i = 1, 2$, relative to the decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that $\{u_{i,n}\}$ is nonnegative and nondecreasing, and $-u_{2,n} \leq u_n \leq u_{1,n}$; and

$$\int_Q |\mathcal{G}(u_n)| \leq \mu_1(Q) + \mu_2(Q) + \|u_0\|_{1,\Omega}, \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \mu_i(Q) + \|u_0\|_{1,\Omega}, \quad i = 1, 2. \quad (6.14)$$

As in the proof of Lemma 6.5, up to a common subsequence $\{u_n, u_{1,n}, u_{2,n}\}$ converge *a.e.* in Q to $\{u, u_1, u_2\}$. Since $\{\mathcal{G}(u_{i,n})\}$ is nondecreasing, and nonnegative, from the monotone convergence

Theorem and (6.14), we obtain that $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$, $i = 1, 2$. Finally, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$, since $|\mathcal{G}(u_n)| \leq \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n})$. Thus, we can see that

$$\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_0^+) - T_n(u_0^-)\}$$

is an approximation of $(\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)$, in the sense of Theorem 2.1; and

$$\{\mu_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \mu_{i,s,n}, T_n(u_0^\pm)\}$$

is an approximation of $(\mu_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \mu_{i,s}, u_0^\pm)$. Therefore, u is a R-solution of (1.5), and (6.5) holds if $u_0 \in L^\infty(\Omega)$ and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$ and some $\gamma \in \mathcal{M}_b^+(\Omega)$. \blacksquare

As a consequence we prove Theorem 2.3. We use the following result of [17]:

Proposition 6.6 (see [17]) *Let $q > p - 1$, $\alpha \in \left(0, \frac{N(q+1-p)}{pq}\right)$, $r > 0$ and $\nu \in \mathcal{M}_b^+(\Omega)$. If ν does not charge the sets of $C_{\alpha p, \frac{q}{q+1-p}}$ -capacity zero, there exists a nondecreasing sequence $\{\nu_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ν strongly in $\mathcal{M}_b(\Omega)$ and such that $\mathbf{W}_{\alpha,p}^r[\nu_n] \in L^q(\mathbb{R}^N)$, for any $n \in \mathbb{N}$.*

Proof of Theorem 2.3. Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \leq \omega \otimes F$, where $F \in L^1((0, T))$ and ω does not charge the sets of $C_{p, \frac{q}{q+1-p}}$ -capacity zero. From Proposition 6.6, there exists a nondecreasing sequence $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ω , strongly in $\mathcal{M}_b(\Omega)$, such that $\mathbf{W}_{1,p}^{2diam\Omega}[\omega_n] \in L^q(\mathbb{R}^N)$. We can write

$$f + \mu = \mu_1 - \mu_2, \quad \mu_1 = f^+ + \mu^+, \quad \mu_2 = f^- + \mu^-, \quad (6.15)$$

and $\mu^+, \mu^- \leq \omega \otimes F$. We set

$$Q_n = \{(x, t) \in \Omega \times \left(\frac{1}{n}, T - \frac{1}{n}\right) : d(x, \partial\Omega) > \frac{1}{n}\}, \quad F_n = T_n(\chi_{(\frac{1}{n}T - \frac{1}{n})} F), \quad (6.16)$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \quad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}. \quad (6.17)$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences with compact support in Q , and $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $(n + \kappa \mathbf{W}_{1,p}^{2diam\Omega}[\omega_n])^q \in L^1(Q)$. Besides, $\omega_n \otimes F_n$ converges to $\omega \otimes F$ strongly in $\mathcal{M}_b(Q)$: indeed we easily check that

$$\|\omega_n \otimes F_n - \omega \otimes F\|_{\mathcal{M}_b(Q)} \leq \|F_n\|_{L^1((0,T))} \|\omega_n - \omega\|_{\mathcal{M}_b(\Omega)} + \|\omega\|_{\mathcal{M}_b(\Omega)} \|F_n - F\|_{L^1((0,T))}$$

Observe that for any measures $\nu, \theta, \eta \in \mathcal{M}_b(Q)$, there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leq |\theta - \eta|,$$

hence $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ converge to μ_1, μ_2 respectively in $\mathcal{M}_b(Q)$. Therefore, the result follows from Theorem 6.3. \blacksquare

Remark 6.7 Our result improves the existence results of [50], where $\mu \in \mathcal{M}_0(Q)$. Indeed, let $p_e = N(p-1)/(N-p)$ be the critical exponent for the elliptic problem

$$-\Delta_p w + |w|^{q-1} w = \omega \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

Notice that $p_c < p_e$, since $p > p_1$. If $q \geq p_e$, there exist measures $\omega \in \mathcal{M}_b^+(\Omega)$ which do not charge the sets of $C_{p, \frac{q}{q+1-p}}$ -capacity zero, such that $\omega \notin \mathcal{M}_{0,e}(\Omega)$. Then for any $F \in L^1((0, T))$, $F \geq 0, F \not\equiv 0$, we have $\omega \otimes F \notin \mathcal{M}_0(Q)$.

Remark 6.8 Let $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6), (1.7). Let $\mathcal{G} : Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto \mathcal{G}(x, t, s)$ is nondecreasing and odd, for a.e. (x, t) in Q . Assume that $\omega \in \mathcal{M}_{0,e}(\Omega)$. Thus, we have $\omega(\{x : W_{1,p}^{2\text{diam}(\Omega)}[\omega](x) = \infty\}) = 0$. As in the proof of Theorem 2.3 with $\omega_n = \chi_{W_{1,p}^{2\text{diam}\Omega}[\omega] \leq n} \omega$, we get that (1.5) has a R -solution.

Remark 6.9 As in [17], from Theorem 6.3, we can extend Theorem 2.3 given for $\mathcal{G}(u) = |u|^{q-1} u$, to the case of a function $\mathcal{G}(x, t, \cdot)$, odd for a.e. $(x, t) \in Q$, such that

$$|\mathcal{G}(x, t, u)| \leq G(|u|), \quad \int_1^\infty G(s) s^{-q-1} ds < \infty,$$

where G is a nondecreasing continuous, under the condition that ω does not charge the sets of zero $C_{p, \frac{q}{q-p+1}, 1}$ -capacity, where for any Borel set $E \subset \mathbb{R}^N$,

$$C_{p, \frac{q}{q-p+1}, 1}(E) = \inf\{\|\varphi\|_{L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N), \quad G_p * \varphi \geq \chi_E\}$$

where $L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)$ is the Lorentz space of order $(q/(q-p+1), 1)$.

In case \mathcal{G} is of exponential type, we introduce the notion of maximal fractional operator, defined for any $\eta \geq 0, R > 0, x_0 \in \mathbb{R}^N$ by

$$\mathbf{M}_{p,R}^\eta[\omega](x_0) = \sup_{t \in (0,R)} \frac{\omega(B(x_0, t))}{t^{N-p} h_\eta(t)}, \quad \text{where } h_\eta(t) = \inf((- \ln t)^{-\eta}, (\ln 2)^{-\eta}).$$

We obtain the following:

Theorem 6.10 Let $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N$ and $\tau > 0, \beta > 1, \mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Assume that $|\mu| \leq \omega \otimes F$, with $\omega \in \mathcal{M}_b^+(\Omega)$, $F \in L^1((0, T))$ be nonnegative. Assume that one of the following assumptions is satisfied:

(i) $\|F\|_{L^\infty((0,T))} \leq 1$ and for some $M_0 = M_0(N, p, \beta, \tau, c_3, c_4, \text{diam}\Omega)$,

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{p-1}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} < M_0, \tag{6.18}$$

(ii) there exists $\beta_0 > \beta$ such that $\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{p-1}{\beta_0}}[\omega] \in L^\infty(\mathbb{R}^N)$.

Then there exists a R -solution to the problem

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + (e^{\tau|u|^\beta} - 1)\operatorname{sign}u = F + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

For the proof we use the following result of [17]:

Proposition 6.11 (see [17], Theorem 2.4) *Suppose $1 < p < N$. Let $\nu \in \mathcal{M}_b^+(\Omega)$, $\beta > 1$, and $\delta_0 = ((12\beta)^{-1})^\beta p \ln 2$. There exists $C = C(N, p, \beta, \operatorname{diam}\Omega)$ such that, for any $\delta \in (0, \delta_0)$,*

$$\int_{\Omega} \exp \left(\delta \frac{(\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\nu])^\beta}{\|\mathbf{M}_{p,2\operatorname{diam}\Omega}^{\frac{p-1}{\beta'}}[\nu]\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}}} \right) \leq \frac{C}{\delta_0 - \delta}.$$

Proof of Theorem 6.10. Let Q_n be defined at (6.16), and $\omega_n = \omega\chi_{\Omega_n}$, where $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$. We still consider $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$ as in (6.15), (6.17).

Case 1: Assume that $\|F\|_{L^\infty((0,T))} \leq 1$ and (6.18) holds. We have $\mu_{1,n}, \mu_{2,n} \leq n\chi_\Omega + \omega$. For any $\varepsilon > 0$, there exists $c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, \operatorname{diam}\Omega) > 0$ such that

$$(n + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta \leq c_\varepsilon n^{\frac{\beta p}{p-1}} + (1 + \varepsilon)\kappa^\beta (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta$$

a.e. in Ω . Thus,

$$\exp \left(\tau(n + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta \right) \leq \exp \left(\tau c_\varepsilon n^{\frac{\beta p}{p-1}} \right) \exp \left(\tau(1 + \varepsilon)\kappa^\beta (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta \right)$$

If (6.18) holds with $M_0 = (\delta_0/\tau\kappa^\beta)^{(p-1)/\beta}$ then we can chose ε such that

$$\tau(1 + \varepsilon)\kappa^\beta \|\mathbf{M}_{p,2\operatorname{diam}\Omega}^{\frac{p-1}{\beta'}}[\nu]\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}} < \delta_0.$$

From Proposition 6.11, we get $\exp(\tau(1 + \varepsilon)\kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta \in L^1(\Omega)$, which implies $\exp(\tau(n + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta) \in L^1(\Omega)$ for all n . We conclude from Theorem 6.3.

Case 2: Assume that there exists $\varepsilon > 0$ such that $\mathbf{M}_{p,2\operatorname{diam}\Omega}^{(p-1)/(\beta+\varepsilon)' }[\omega] \in L^\infty(\mathbb{R}^N)$. Now we use the inequality $\mu_{1,n}, \mu_{2,n} \leq n(\chi_\Omega + \omega)$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $c_{\varepsilon,n} > 0$ such that

$$(n + \kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta \leq c_{\varepsilon,n} + \varepsilon (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^{\beta_0}$$

Thus, from Proposition 6.11 we get $\exp(\tau(n + \kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta) \in L^1(\Omega)$ for all n . We conclude from Theorem 6.3. \blacksquare

6.3 Equations with source term

As a consequence of Theorem 6.3, we get a first result for problem (1.1):

Corollary 6.12 *Let $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6)(1.7). Let $u_0 \in L^\infty(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \leq \omega \otimes \chi_{(0,T)}$ for some $\omega \in \mathcal{M}_b^+(\Omega)$. Then there exist a R -solution u of (1.1), such that*

$$|u(x, t)| \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega](x) + \|u_0\|_{\infty, \Omega}, \quad \text{for a.e. } (x, t) \in Q, \quad (6.19)$$

where κ is defined at Theorem 6.2.

Proof. Let $\{\phi_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, a.e. in Q . Since $\{\phi_n \mu^+\}, \{\phi_n \mu^-\}$ are nondecreasing sequences, the result follows from Theorem 6.3. ■

Our proof of Theorem 2.4 is based on a property of Wölf potentials:

Theorem 6.13 (see [53]) *Let $q > p - 1$, $0 < p < N$, $\omega \in \mathcal{M}_b^+(\Omega)$. If for some $\lambda > 0$,*

$$\omega(E) \leq \lambda C_{p, \frac{q}{p-q+1}}(E) \quad \text{for any compact set } E \subset \mathbb{R}^N, \quad (6.20)$$

then $(\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q \in L^1(\Omega)$, and there exists $M = M(N, p, q, \text{diam}(\Omega))$ such that, a.e. in Ω ,

$$\mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] \right]^q \leq M \lambda^{\frac{q-p+1}{(p-1)^2}} \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] < \infty. \quad (6.21)$$

We deduce the following:

Lemma 6.14 *Let $\omega \in \mathcal{M}_b^+(\Omega)$, and $b \geq 0$ and $K > 0$. Suppose that $\{u_m\}_{m \geq 1}$ is a sequence of nonnegative functions in Ω that satisfies*

$$u_1 \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + b, \quad u_{m+1} \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[u_m^q + \omega] + b \quad \forall m \geq 1.$$

Assume that ω satisfies (6.20) for some $\lambda > 0$. Then there exist λ_0 and b_0 , depending on N, p, q, K , and $\text{diam}\Omega$, such that, if $\lambda \leq \lambda_0$ and $b \leq b_0$, then $\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] \in L^q(\Omega)$ and for any $m \geq 1$,

$$u_m \leq 2\beta_p K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + 2b, \quad \beta_p = \max(1, 3^{\frac{2-p}{p-1}}). \quad (6.22)$$

Proof. Clearly, (6.22) holds for $m = 1$. Now, assume that it holds at the order m . Then

$$u_m^q \leq 2^{q-1} (2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1} (2b)^q$$

Using (6.21) we get

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[2^{q-1} (2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1} (2b)^q + \omega \right] + b \\ &\leq \beta_p K \left(A_1 \mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[(\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q \right] + \mathbf{W}_{1,p}^{2\text{diam}\Omega} [(2b)^q] + \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] \right) + b \\ &\leq \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1) \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + \beta_p K \mathbf{W}_{1,p}^{2\text{diam}\Omega} [(2b)^q] + b \\ &= \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1) \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + A_2 b^{\frac{q}{p-1}} + b, \end{aligned}$$

where M is as in (6.21) and $A_1 = (2^{q-1}(2\beta_p)^q K^q)^{1/(p-1)}$, $A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1} (2\text{diam}\Omega)^{p'}$. Thus, (6.22) holds for $m = n + 1$ if we prove that

$$A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} \leq 1 \text{ and } A_2 b^{\frac{q}{p-1}} \leq b,$$

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}} \text{ and } b \leq A_2^{-\frac{p-1}{q-p+1}}.$$

Therefore, we obtain the result with $\lambda_0 = (A_1 M)^{-(p-1)^2/(q-p+1)}$ and $b_0 = A_2^{-(p-1)/(q-p+1)}$. \blacksquare

Proof of Theorem 2.4. From Corollary 5.7 and 6.12, we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m \geq 1}$ defined in the following way: u_1 is a R-solution of (1.1), and u_{m+1} is a nonnegative R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = u_m^q + \mu & \text{in } Q, \\ u_{m+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 & \text{in } \Omega. \end{cases}$$

Setting $\bar{u}_m = \sup_{t \in (0, T)} u_m(t)$ for all $m \geq 1$, there holds

$$\bar{u}_1 \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + \|u_0\|_{\infty, \Omega}, \quad \bar{u}_{m+1} \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\bar{u}_m^q + \omega] + \|u_0\|_{\infty, \Omega} \quad \forall m \geq 1.$$

From Lemma 6.14, we can find $\lambda_0 = \lambda_0(N, p, q, \text{diam}\Omega)$ and $b_0 = b_0(N, p, q, \text{diam}\Omega)$ such that if (2.7) is satisfied with λ_0 and b_0 , then

$$u_m \leq \bar{u}_m \leq 2\beta_p \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + 2\|u_0\|_{\infty, \Omega} \quad \forall m \geq 1. \quad (6.23)$$

Thus $\{u_m\}$ converges *a.e.* in Q and in $L^1(Q)$ to some function u , for which (2.9) is satisfied in Ω with $c = 2\beta_p \kappa$. Finally, one can apply Theorem 2.1 to the sequence of measures $\{u_m^q + \mu\}$, and obtain that u is a R-solution of (2.8). \blacksquare

Next we consider the exponential case.

Theorem 6.15 *Let $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying (1.6), (1.7). Let $\tau > 0, l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Set*

$$\mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \quad \forall s \in \mathbb{R}. \quad (6.24)$$

Let $\mu \in \mathcal{M}_b^+(Q)$, $\omega \in \mathcal{M}_b^+(\Omega)$ such that $\mu \leq \chi_{(0, T)} \otimes \omega$. Then, there exist b_0 and M_0 depending on N, p, β, τ, l and $\text{diam}\Omega$, such that if

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_0, \quad \|u_0\|_{\infty, \Omega} \leq b_0,$$

the problem

$$\begin{cases} u_t - \text{div}(A(x, \nabla u)) = \mathcal{E}(\tau u^\beta) + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega \end{cases} \quad (6.25)$$

admits nonnegative R- solution u , which satisfies, a.e. in Q , for some c , depending on N, p, c_3, c_4

$$u(x, t) \leq c \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega](x) + 2b_0. \quad (6.26)$$

For the proof we first recall an approximation property, which is a consequence of [47, Theorem 2.5]:

Theorem 6.16 *Let $\tau > 0$, $b \geq 0$, $K > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Let \mathcal{E} be defined by (6.24). Let $\{v_m\}$ be a sequence of nonnegative functions in Ω such that, for some $K > 0$,*

$$v_1 \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + b, \quad v_{m+1} \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mathcal{E}(\tau u_m^\beta) + \mu] + b, \quad \forall m \geq 1.$$

Then, there exist b_0 and M_0 , depending on N, p, β, τ, l, K and $\text{diam}\Omega$ such that if $b \leq b_0$ and

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{\infty, \mathbb{R}^N} \leq M_0, \quad (6.27)$$

then, setting $c_p = 2\max(1, 2^{\frac{2-p}{p-1}})$,

$$\begin{aligned} \exp(\tau(Kc_p \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + 2b_0)^\beta) &\in L^1(\Omega), \\ v_m &\leq Kc_p \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + 2b_0, \quad \forall m \geq 1. \end{aligned} \quad (6.28)$$

Proof of Theorem 6.15. From Corollary 5.7 and 6.12 we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m \geq 1}$ defined in the following way: u_1 is a R-solution of problem (1.1), and by induction, u_{m+1} is a R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\ u_{m+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 & \text{in } \Omega. \end{cases}$$

And, setting $\bar{u}_m = \sup_{t \in (0, T)} u_m(t)$, there holds

$$\bar{u}_1 \leq \kappa W_{1,p}^{2\text{diam}\Omega}[\omega] + \|u_0\|_{\infty, \Omega}, \quad \bar{u}_{m+1} \leq \kappa W_{1,p}^{2\text{diam}\Omega}[\mathcal{E}(\tau \bar{u}_m^\beta) + \omega] + \|u_0\|_{\infty, \Omega}, \quad \forall m \geq 1.$$

Thus, from Theorem 6.16, there exist $b_0 \in (0, 1]$ and $M_0 > 0$ depending on N, p, β, τ, l and $\text{diam}\Omega$ such that, if (6.27) holds, then (6.28) is satisfied with $v_m = \bar{u}_m$. As a consequence, u_m is well defined. Thus, $\{u_m\}$ converges a.e. in Q to some function u , for which (6.26) is satisfied in Ω . Furthermore, $\{\mathcal{E}(\tau u_m^\beta)\}$ converges to $\mathcal{E}(\tau u^\beta)$ in $L^1(Q)$. Finally, one can apply Theorem 2.1 to the sequence of measures $\{\mathcal{E}(\tau u_m^\beta) + \mu\}$, and obtain that u is a R-solution of (6.25). \blacksquare

7 Appendix

Proof of Lemma 4.7. Let \mathcal{J} be defined by (4.11). Let $\zeta \in C_c^1([0, T])$ with values in $[0, 1]$, such that $\zeta_t \leq 0$, and $\varphi = \zeta \xi [j(S(v))]_l$. Clearly, $\varphi \in X \cap L^\infty(Q)$; we choose the pair of functions (φ, S) as test function in (4.2). Thanks to convergence properties of Steklov time-averages, we easily will obtain (4.15) if we prove that

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left(- \int_Q (\zeta \xi [j(S(v))]_l)_t S(v) \right) \geq - \int_Q \xi_t J(S(v)).$$

We can write $-\int_Q (\zeta \xi [j(S(v))]_l)_t S(v) = F + G$, with

$$F = - \int_Q (\zeta \xi)_t [j(S(v))]_l S(v), \quad G = - \int_Q \zeta \xi S(v) \frac{1}{l} (j(S(v))(x, t+l) - j(S(v))(x, t)).$$

Using (4.12) and integrating by parts we have

$$\begin{aligned} G &\geq - \int_Q \zeta \xi \frac{1}{l} (\mathcal{J}(S(v))(x, t+l) - \mathcal{J}(S(v))(x, t)) \\ &= - \int_Q \zeta \xi \frac{\partial}{\partial t} ([\mathcal{J}(S(v))]_l) = \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l + \int_\Omega \zeta(0) \xi(0) [\mathcal{J}(S(v))]_l(0) \\ &\geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l, \end{aligned}$$

since $\mathcal{J}(S(v)) \geq 0$. Hence,

$$- \int_Q (\zeta \xi [j(S(v))]_l)_t S(v) \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l + F = \int_Q (\zeta \xi)_t ([\mathcal{J}(S(v))]_l - [J(S(v))]_l S(v))$$

Otherwise, $\mathcal{J}(S(v))$ and $J(S(v)) \in C([0, T]; L^1(\Omega))$, thus $\{(\zeta \xi)_t ([\mathcal{J}(S(u))]_l - [J(S(u))]_l S(u))\}$ converges to $-(\zeta \xi)_t J(S(u))$ in $L^1(Q)$ as $l \rightarrow 0$. Therefore,

$$\begin{aligned} \lim_{l \rightarrow 0, \zeta \rightarrow 1} \left(- \int_Q (\zeta \xi [J(S(v))]_l)_t S(v) \right) &\geq \lim_{\zeta \rightarrow 1} \left(- \int_Q (\zeta \xi)_t J(S(v)) \right) \\ &\geq \lim_{\zeta \rightarrow 1} \left(- \int_Q \zeta \xi_t J(S(v)) \right) = - \int_Q \xi_t J(S(v)), \end{aligned}$$

which achieves the proof. ■

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