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# Stability properties for quasilinear parabolic equations with measure data and applications

Marie-Françoise BIDAUT-VERON\*

Hung NGUYEN QUOC†

## Abstract

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ , and  $Q = \Omega \times (0, T)$ . We first study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where  $p > 1$ ,  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$ . Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators  $u \mapsto \mathcal{A}(u) = \operatorname{div}(A(x, t, \nabla u))$ .

As an application, we consider perturbed problems *of type*

$$\begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where  $\mathcal{G}(u)$  may be an absorption or a source term. In the model case  $\mathcal{G}(u) = \pm |u|^{q-1} u$  ( $q > p - 1$ ), or  $\mathcal{G}$  has an exponential type. We give existence results when  $q$  is subcritical, or when the measure  $\mu$  is good in time and satisfies suitable capacity conditions.

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# 1 Introduction

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ , and  $Q = \Omega \times (0, T)$ ,  $T > 0$ . We denote by  $\mathcal{M}_b(\Omega)$  and  $\mathcal{M}_b(Q)$  the sets of bounded Radon measures on  $\Omega$  and  $Q$  respectively. We are concerned with the problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\mu \in \mathcal{M}_b(Q)$ ,  $u_0 \in L^1(\Omega)$  and  $A$  is a Caratheodory function on  $Q \times \mathbb{R}^N$ , such that for *a.e.*  $(x, t) \in Q$ , and any  $\xi, \zeta \in \mathbb{R}^N$ ,

$$A(x, t, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, t, \xi)| \leq a(x, t) + c_2 |\xi|^{p-1}, \quad c_1, c_2 > 0, a \in L^{p'}(Q), \quad (1.2)$$

$$(A(x, t, \xi) - A(x, t, \zeta)) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta. \quad (1.3)$$

This includes the model problem

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.4)$$

where  $\Delta_p$  is the  $p$ -Laplacian defined by  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  with  $p > 1$ . As an application, we consider problems with a nonlinear term of order 0:

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.5)$$

where  $A$  is a Caratheodory function on  $\Omega \times \mathbb{R}^N$ , such that, for *a.e.*  $x \in \Omega$ , and any  $\xi, \zeta \in \mathbb{R}^N$ ,

$$A(x, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, \xi)| \leq c_2 |\xi|^{p-1}, \quad c_3, c_4 > 0, \quad (1.6)$$

$$(A(x, \xi) - A(x, \zeta)) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta, \quad (1.7)$$

and  $\mathcal{G}(u)$  may be an absorption or a source term, and possibly depends on  $(x, t) \in Q$ . The model problem is the case where  $\mathcal{G}$  has a power-type  $\mathcal{G}(u) = \pm |u|^{q-1} u$  ( $q > p - 1$ ), or an exponential type.

First make a brief survey of the elliptic associated problem:

$$\begin{cases} -\operatorname{div}(A(x, \nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $\mu \in \mathcal{M}_b(\Omega)$  and assumptions (1.6), (1.7). When  $p = 2$ ,  $A(x, \nabla u) = \nabla u$  existence and uniqueness are proved for general elliptic operators by duality methods in [58]. For  $p > 2 - 1/N$ , the existence of solutions in the sense of distributions is obtained in [23] and [24]. The condition

on  $p$  ensures that the gradient  $\nabla u$  is well defined in  $(L^1(\Omega))^N$ . For general  $p > 1$ , new classes of solutions are introduced, first when  $\mu \in L^1(\Omega)$ , such as *entropy solutions*, and *renormalized solutions*, see [13], and also [57], and existence and uniqueness is obtained. For any  $\mu \in \mathcal{M}_b(\Omega)$  the main work is done in [32, Theorems 3.1, 3.2], where not only existence is proved, but also a stability result, fundamental for applications. Uniqueness is still an open problem.

Next we make a brief survey about problem (1.1).

The first studies concern the case  $\mu \in L^{p'}(Q)$  and  $u_0 \in L^2(\Omega)$ , where existence and uniqueness is obtained by variational methods, see [45]. In the general case  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in \mathcal{M}_b(\Omega)$ , the pionner results come from [23], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \quad (1.8)$$

see also [55], [56], and [26]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces  $u \in L^{p_c, \infty}(Q)$  and  $|\nabla u| \in L^{m_c, \infty}(Q)$ , where

$$p_c = p - 1 + \frac{p}{N}, \quad m_c = p - \frac{N}{N+1}. \quad (1.9)$$

This condition (1.8) ensures that  $u$  and  $|\nabla u|$  belong to  $L^1(Q)$ , since  $m_c > 1$  means  $p > p_1$  and  $p_c > 1$  means  $p > 2N/(N+1)$ . Uniqueness follows in the case  $p = 2$ ,  $A(x, t, \nabla u) = \nabla u$ , by duality methods, see [48].

For  $\mu \in L^1(Q)$ , uniqueness is obtained in new classes of solutions: *entropy solutions*, and *renormalized solutions*, see [19], [54], see also [3] for a semi-group approach.

Then a class of *regular* measures is studied in [33], where a notion of parabolic capacity  $c_p^Q$  is introduced, defined by

$$c_p^Q(E) = \inf_{E \subset U} \inf_{\text{open } \subset Q} \{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \},$$

for any Borel set  $E \subset Q$ , where

$$\begin{aligned} X &= L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega)), \\ W &= \{z : z \in X, z_t \in X'\}, \text{ embedded with the norm } \|u\|_W = \|u\|_X + \|u_t\|_{X'}. \end{aligned}$$

Let  $\mathcal{M}_0(Q)$  be the set of Radon measures  $\mu$  on  $Q$  that do not charge the sets of zero  $c_p^Q$ -capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \implies |\mu(E)| = 0.$$

Then existence and uniqueness of renormalized solutions holds for any measure  $\mu \in \mathcal{M}_b(\Omega) \cap \mathcal{M}_0(Q)$ , called *regular* (or *diffuse*) and  $u_0 \in L^1(\Omega)$ , and  $p > 1$ . The equivalence with the notion of entropy solutions is shown in [34]; see also [20] for more general equations.

Next consider *any* measure  $\mu \in \mathcal{M}_b(Q)$ . Let  $\mathcal{M}_s(Q)$  be the set of all bounded Radon measures on  $Q$  with support on a set of zero  $c_p^Q$  capacity, also called *singular*. Let  $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$

be the positive cones of  $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$ . From [33],  $\mu$  can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \quad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \quad (1.10)$$

and  $\mu_0 \in \mathcal{M}_0(Q)$  admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \quad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \quad (1.11)$$

and we write  $\mu_0 = (f, g, h)$ . The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [19],[49]. In the range (1.8) the existence of a renormalized solution relative to the decomposition (1.11) is proved in [49], using suitable approximations of  $\mu_0$  and  $\mu_s$ . Uniqueness is still open, as well as in the elliptic case.

Next consider the problem (1.5). First we consider the case of an *absorption term*:  $\mathcal{G}(u)u \geq 0$ .

Let us recall the case  $p = 2$ ,  $A(x, \nabla u) = \nabla u$  and  $\mathcal{G}(u) = |u|^{q-1}u$  ( $q > 1$ ). The first results concern the case  $\mu = 0$  and  $u_0$  is a Dirac mass in  $\Omega$ , see [28]: existence holds if and only if  $q < (N + 2)/N$ . Then optimal results are given in [7], for any  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in \mathcal{M}_b(\Omega)$ . Here two capacities are involved: the elliptic Bessel capacity  $C_{\alpha,k}$ , ( $\alpha, k > 1$ ) defined, for any Borel set  $E \subset \mathbb{R}^N$ , by

$$C_{\alpha,k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N), \quad G_\alpha * \varphi \geq \chi_E\},$$

where  $G_\alpha$  is the Bessel kernel of order  $\alpha$ ; and a capacity  $c_{\mathbf{G},k}$  ( $k > 1$ ) adapted to the operator of the heat equation of kernel  $\mathbf{G}(x, t) = \chi_{(0,\infty)}(4\pi t)^{-N/2} e^{-|x|^2/4t}$ : for any Borel set  $E \subset \mathbb{R}^{N+1}$ ,

$$c_{\mathbf{G},k}(E) = \inf\{\|\varphi\|_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}), \quad \mathbf{G} * \varphi \geq \chi_E\}.$$

From [7], there exists a solution if and only if  $\mu$  does not charge the sets of  $c_{\mathbf{G},q'}(E)$  capacity zero and  $u_0$  does not charge the sets of  $C_{2/q,q'}$  capacity zero. Observe that one can reduce to a zero initial data, by considering the measure  $\mu + u_0 \otimes \delta_0^t$  in  $\Omega \times (-T, T)$ , where  $\otimes$  is the tensorial product and  $\delta_0^t$  is the Dirac mass in time at 0.

For  $p \neq 2$  such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case  $\mu = 0$  with  $\Omega$  bounded, or  $\Omega = \mathbb{R}^N$ . The case where  $u_0$  is a Dirac mass in  $\Omega$  is studied in [36], [40] when  $p > 2$ , and [29] when  $p < 2$ . Existence and uniqueness hold in the subcritical case  $q < p_c$ . If  $q \geq p_c$  and  $q > 1$ , there is no solution with an isolated singularity at  $t = 0$ . For  $q < p_c$ , and  $u_0 \in \mathcal{M}_b^+(\Omega)$ , the existence is obtained in the sense of distributions in [60], and for any  $u_0 \in \mathcal{M}_b(\Omega)$  in [16]. The case  $\mu \in L^1(Q)$ ,  $u_0 = 0$  is treated in [30], and  $\mu \in L^1(Q)$ ,  $u_0 = L^1(\Omega)$  in [4] where  $\mathcal{G}$  can be multivalued. The case  $\mu \in \mathcal{M}_0(Q)$  is studied in [50], with a new formulation of the solutions, and existence and uniqueness is obtained for any function  $\mathcal{G} \in C(\mathbb{R})$  such that  $\mathcal{G}(u)u \geq 0$ . Up to our knowledge, up to now no existence results have been obtained for a general measure  $\mu \in \mathcal{M}_b(Q)$ .

The case of a source term  $\mathcal{G}(u) = -u^q$  with  $u \geq 0$  has been treated in [6] for  $p = 2$ , where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general  $p$ .

## 2 Main results

In all the sequel we suppose that  $p$  satisfies (1.8). Then

$$X = L^p(0, T; W_0^{1,p}(\Omega)), \quad X' = L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

We first study problem (1.1). In Section 3 we give some approximations of  $\mu \in \mathcal{M}_b(Q)$ , useful for the applications. In Section 4 we recall the definition of renormalized solutions, that we call R-solutions of (1.1), relative to the decomposition (1.11) of  $\mu_0$ , and study some of their properties.

Our main result is a *stability theorem* for problem (1.1), proved in Section 5, extending to the parabolic case the stability result of [32, Theorem 3.4], and improving the result of [49]:

**Theorem 2.1** *Let  $A : Q \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.2), (1.3). Let  $u_0 \in L^1(\Omega)$ , and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

*with  $f \in L^1(Q)$ ,  $g \in (L^{p'}(Q))^N$ ,  $h \in X$  and  $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$ . Let  $u_{0,n} \in L^1(\Omega)$ ,*

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

*with  $f_n \in L^1(Q)$ ,  $g_n \in (L^{p'}(Q))^N$ ,  $h_n \in X$ , and  $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$ , such that*

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

*with  $\rho_n^1, \eta_n^1 \in L^1(Q)$ ,  $\rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$  and  $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$ . Assume that*

$$\sup_n |\mu_n|(Q) < \infty,$$

*and  $\{u_{0,n}\}$  converges to  $u_0$  strongly in  $L^1(\Omega)$ ,  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ ,  $\{h_n\}$  converges to  $h$  strongly in  $X$ ,  $\{\rho_n\}$  converges to  $\mu_s^+$  and  $\{\eta_n\}$  converges to  $\mu_s^-$  in the narrow topology of measures; and  $\{\rho_n^1\}, \{\eta_n^1\}$  are bounded in  $L^1(Q)$ , and  $\{\rho_n^2\}, \{\eta_n^2\}$  bounded in  $(L^{p'}(Q))^N$ . Let  $\{u_n\}$  be a sequence of R-solutions of*

$$\begin{cases} u_{n,t} - \operatorname{div}(A(x, t, \nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \quad (2.1)$$

*relative to the decomposition  $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$  of  $\mu_{n,0}$ . Let  $v_n = u_n - h_n$ . Then up to a subsequence,  $\{u_n\}$  converges a.e. in  $Q$  to a R-solution  $u$  of (1.1), and  $\{v_n\}$  converges a.e. in  $Q$  to  $v = u - h$ . Moreover,  $\{\nabla u_n\}, \{\nabla v_n\}$  converge respectively to  $\nabla u, \nabla v$  a.e. in  $Q$ , and  $\{T_k(u_n)\}, \{T_k(v_n)\}$  converge to  $T_k(u), T_k(v)$  strongly in  $X$  for any  $k > 0$ .*

In Section 6 we give applications to problems of type (1.5).

We first give an existence result of subcritical type, valid for any measure  $\mu \in \mathcal{M}_b(Q)$  :

**Theorem 2.2** Let  $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfying (1.2), (1.3) with  $a \equiv 0$ . Let  $(x, t, r) \mapsto \mathcal{G}(x, t, r)$  be a Caratheodory function on  $Q \times \mathbb{R}$  and  $G \in C(\mathbb{R}^+)$  be a nondecreasing function with values in  $\mathbb{R}^+$ , such that

$$|\mathcal{G}(x, t, r)| \leq G(|r|) \quad \text{for a.e. } (x, t) \in Q \text{ and any } r \in \mathbb{R}, \quad (2.2)$$

$$\int_1^\infty G(s)s^{-1-p_c} ds < \infty. \quad (2.3)$$

(i) Suppose that  $\mathcal{G}(x, t, r)r \geq 0$ , for a.e.  $(x, t)$  in  $Q$  and any  $r \in \mathbb{R}$ . Then, for any  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$ , there exists a  $R$ -solution  $u$  of problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.4)$$

(ii) Suppose that  $\mathcal{G}(x, t, r)r \leq 0$ , for a.e.  $(x, t) \in Q$  and any  $r \in \mathbb{R}$ , and  $u_0 \geq 0, \mu \geq 0$ . There exists  $\varepsilon > 0$  such that for any  $\lambda > 0$ , any  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$  with  $\lambda + |\mu|(Q) + \|u_0\|_{L^1(\Omega)} \leq \varepsilon$ , problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \lambda \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (2.5)$$

admits a nonnegative  $R$ -solution.

In particular for any  $0 < q < p_c$ , if  $\mathcal{G}(u) = |u|^{q-1}u$ , existence holds for any measure  $\mu \in \mathcal{M}_b(Q)$ ; if  $\mathcal{G}(u) = -|u|^{q-1}u$ , existence holds for  $\mu$  small enough. In the supercritical case  $q \geq p_c$ , the class of "admissible" measures, for which there exist solutions, is not known.

Next we give new results relative to *measures that have a good behaviour in  $t$* , based on recent results of [17] relative to the elliptic case. We recall the notions of (truncated) Wölf potential for any nonnegative measure  $\omega \in \mathcal{M}^+(\mathbb{R}^N)$  any  $R > 0, x_0 \in \mathbb{R}^N$ ,

$$\mathbf{W}_{1,p}^R[\omega](x_0) = \int_0^R (t^{p-N}\omega(B(x_0, t)))^{\frac{1}{p-1}} \frac{dt}{t}.$$

Any measure  $\omega \in \mathcal{M}_b(\Omega)$  is identified with its extension by 0 to  $\mathbb{R}^N$ . In case of absorption, we obtain the following:

**Theorem 2.3** Let  $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfying (1.6), (1.7). Let  $p < N, q > p - 1, \mu \in \mathcal{M}_b(Q), f \in L^1(Q)$  and  $u_0 \in L^1(\Omega)$ . Assume that

$$|\mu| \leq \omega \otimes F, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega), F \in L^1((0, T)), F \geq 0,$$

and  $\omega$  does not charge the sets of  $C_{p, \frac{q}{q+1-p}}$ -capacity zero. Then there exists a  $R$ -solution  $u$  of problem

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + |u|^{q-1}u = f + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.6)$$

We show that some of these measures may not lie in  $\mathcal{M}_0(Q)$ , which improves the existence results of [50], see Proposition 3.3 and Remark 6.7. Otherwise our result can be extended to a more general function  $\mathcal{G}$ , see Remark 6.9. We also consider a source term:

**Theorem 2.4** *Let  $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfying (1.6), (1.7). Let  $p < N$ ,  $q > p-1$ . Let  $\mu \in \mathcal{M}_b^+(Q)$ , and  $u_0 \in L^\infty(\Omega)$ ,  $u_0 \geq 0$ . Assume that*

$$\mu \leq \omega \otimes \chi_{(0,T)}, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega).$$

*Then there exist  $\lambda_0 = \lambda_0(N, p, q, c_3, c_4, \text{diam}\Omega)$  and  $b_0 = b_0(N, p, q, c_3, c_4, \text{diam}\Omega)$  such that, if*

$$\omega(E) \leq \lambda_0 C_{p, \frac{q}{q-p+1}}(E), \quad \forall E \text{ compact } \subset \mathbb{R}^N, \quad \|u_0\|_{\infty, \Omega} \leq b_0, \quad (2.7)$$

*there exists a nonnegative  $R$ -solution  $u$  of problem*

$$\begin{cases} u_t - \text{div}(A(x, \nabla u)) = u^q + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (2.8)$$

*which satisfies, a.e. in  $Q$ ,*

$$u(x, t) \leq C \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega](x) + 2\|u_0\|_{L^\infty}, \quad (2.9)$$

*where  $C = C(N, p, c_3, c_4)$ .*

Corresponding results in case where  $\mathcal{G}$  has exponential type are given at Theorems 6.10 and 6.15.

### 3 Approximations of measures

For any open set  $\varpi$  of  $\mathbb{R}^m$  and  $F \in (L^k(\varpi))^\nu$ ,  $k \in [1, \infty]$ ,  $m, \nu \in \mathbb{N}^*$ , we set  $\|F\|_{k, \varpi} = \|F\|_{(L^k(\varpi))^\nu}$ .

First we give approximations of nonnegative measures in  $\mathcal{M}_0(Q)$ . We recall that any measure  $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$  admits a decomposition under the form  $\mu = (f, g, h)$  given by (1.11). Conversely, any measure of this form, such that  $h \in L^\infty(Q)$ , lies in  $\mathcal{M}_0(Q)$ , see [50, Proposition 3.1].

**Lemma 3.1** *Let  $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$  and  $\varepsilon > 0$ .*

*(i) Then, we can find a decomposition  $\mu = (f, g, h)$  with  $f \in L^1(Q)$ ,  $g \in (L^{p'}(Q))^N$ ,  $h \in X$  such that*

$$\|f\|_{1, Q} + \|g\|_{p', Q} + \|h\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|g\|_{p', Q} + \|h\|_X \leq \varepsilon. \quad (3.1)$$

*(ii) Furthermore, there exists a sequence of measures  $\mu_n = (f_n, g_n, h_n)$ , such that  $f_n, g_n, h_n \in C_c^\infty(Q)$  and strongly converge to  $f, g, h$  in  $L^1(Q)$ ,  $(L^{p'}(Q))^N$  and  $X$  respectively, and  $\mu_n$  converges to  $\mu$  in the narrow topology, and satisfying*

$$\|f_n\|_{1, Q} + \|g_n\|_{p', Q} + \|h_n\|_X \leq (1 + 2\varepsilon)\mu(Q), \quad \|g_n\|_{p', Q} + \|h_n\|_X \leq 2\varepsilon. \quad (3.2)$$



**Proof.** (i) Step 1. Case where  $\mu$  has a compact support in  $Q$ . By [33], we can find a decomposition  $\mu = (f, g, h)$  with  $f, g, h$  have a compact support in  $Q$ . Let  $\{\varphi_n\}$  be sequence of mollifiers in  $\mathbb{R}^{N+1}$ . Then  $\mu_n = \varphi_n * \mu \in C_c^\infty(Q)$  for  $n$  large enough. We see that  $\mu_n(Q) = \mu(Q)$  and  $\mu_n$  admits the decomposition  $\mu_n = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$ . Since  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively, we have for  $n_0$  large enough,

$$\|f - f_{n_0}\|_{1,Q} + \|g - g_{n_0}\|_{p',Q} + \|h - h_{n_0}\|_X \leq \varepsilon \min\{\mu(Q), 1\}.$$

Then we obtain a decomposition  $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$ , such that

$$\|\hat{f}\|_{1,Q} + \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq \varepsilon. \quad (3.3)$$

Step 2. General case. Let  $\{\theta_n\}$  be a nonnegative, nondecreasing sequence in  $C_c^\infty(Q)$  which converges to 1, *a.e.* in  $Q$ . Set  $\tilde{\mu}_0 = \theta_0\mu$ , and  $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$ , for any  $n \geq 1$ . Since  $\tilde{\mu}_n \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$  has compact support, by Step 1, we can find a decomposition  $\tilde{\mu}_n = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$  such that

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon)\mu_n(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2^{-n-1}\varepsilon.$$

Let  $\bar{f}_n = \sum_{k=0}^n \tilde{f}_k$ ,  $\bar{g}_n = \sum_{k=0}^n \tilde{g}_k$  and  $\bar{h}_n = \sum_{k=0}^n \tilde{h}_k$ . Clearly,  $\theta_n\mu = (\bar{f}_n, \bar{g}_n, \bar{h}_n)$ , and  $\{\bar{f}_n\}, \{\bar{g}_n\}, \{\bar{h}_n\}$  converge strongly to some  $f, g, h$ , respectively in  $L^1(Q), (L^{p'}(Q))^N, X$ , with

$$\|\bar{f}_n\|_{1,Q} + \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X \leq (1 + \varepsilon)\mu(Q), \quad \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X \leq \varepsilon.$$

Therefore,  $\mu = (f, g, h)$  and (3.1) holds.

(ii) We take a sequence  $\{m_n\}$  in  $\mathbb{N}$  such that  $f_n = \varphi_{m_n} * \bar{f}_n, g_n = \varphi_{m_n} * \bar{g}_n, h_n = \varphi_{m_n} * \bar{h}_n \in C_c^\infty(Q)$  and

$$\|f_n - \bar{f}_n\|_{1,Q} + \|g_n - \bar{g}_n\|_{p',Q} + \|h_n - \bar{h}_n\|_X \leq \frac{\varepsilon}{n+1} \min\{\mu(Q), 1\}.$$

Let  $\mu_n = \varphi_{m_n} * (\theta_n\mu) = (f_n, g_n, h_n)$ . Therefore,  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively. And (3.2) holds. Furthermore,  $\{\mu_n\}$  converges weak-\* to  $\mu$ , and  $\mu_n(Q) = \int_Q \theta_n d\mu$  converges to  $\mu(Q)$ , thus  $\{\mu_n\}$  converges in the narrow topology.  $\blacksquare$

As a consequence, we get an approximation property for any measure  $\mu \in \mathcal{M}_b^+(Q)$  :

**Proposition 3.2** *Let  $\mu \in \mathcal{M}_b^+(Q)$  and  $\varepsilon > 0$ . Let  $\{\mu_n\}$  be a nondecreasing sequence in  $\mathcal{M}_b^+(Q)$  converging to  $\mu$  in  $\mathcal{M}_b(Q)$ . Then, there exist  $f_n, f \in L^1(Q), g_n, g \in (L^{p'}(Q))^N$  and  $h_n, h \in X, \mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$  such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \quad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively, and  $\{\mu_{n,s}\}$  converges to  $\mu_s$  (strongly) in  $\mathcal{M}_b(Q)$  and

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{n,s}(\Omega) \leq (1 + \varepsilon)\mu(Q), \quad \text{and } \|g_n\|_{p',Q} + \|h_n\|_X \leq \varepsilon. \quad (3.4)$$

**Proof.** Since  $\{\mu_n\}$  is nondecreasing, then  $\{\mu_{n,0}\}, \{\mu_{n,s}\}$  are too. Clearly,  $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$ . Hence,  $\{\mu_{n,s}\}$  converge to  $\mu_s$  and  $\{\mu_{n,0}\}$  converge to  $\mu_0$  (strongly) in  $\mathcal{M}_b(Q)$ . Set  $\tilde{\mu}_{0,0} = \mu_{0,0}$ , and  $\tilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$  for any  $n \geq 1$ . By Lemma 3.1, (i), we can find  $\tilde{f}_n \in L^1(Q)$ ,  $\tilde{g}_n \in (L^{p'}(Q))^N$  and  $\tilde{h}_n \in X$  such that  $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$  and

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon)\tilde{\mu}_{n,0}(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2^{-n-1}\varepsilon.$$

Let  $f_n = \sum_{k=0}^n \tilde{f}_k$ ,  $G_n = \sum_{k=0}^n \tilde{g}_k$  and  $h_n = \sum_{k=0}^n \tilde{h}_k$ . Clearly,  $\mu_{n,0} = (f_n, g_n, h_n)$  and the convergence properties hold with (3.4), since

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X \leq (1 + \varepsilon)\mu_0(Q). \quad \blacksquare$$

In Section 6 we consider some measures  $\mu \in \mathcal{M}_b(Q)$  which satisfy  $|\mu| \leq \omega \otimes F$ , with  $\omega \in \mathcal{M}_b(\Omega)$  and  $F \in L^1((0, T))$ ,  $F \geq 0$ . It is interesting to compare the properties of  $\omega \otimes F$  and  $\omega$  :

Let  $c_p^\Omega$  be the elliptic capacity in  $\Omega$  defined by

$$c_p^\Omega(K) = \inf \left\{ \int_\Omega |\nabla \varphi|^p : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\},$$

for any compact set  $K \subset \Omega$ .

Let  $\mathcal{M}_{0,e}(\Omega)$  be the set of Radon measures  $\omega$  on that do not charge the sets of zero  $c_p^\Omega$ -capacity. Then  $\mathcal{M}_b(\Omega) \cap \mathcal{M}_{0,e}(\Omega)$  is characterised as the set of measures  $\omega \in \mathcal{M}_b(\Omega)$  which can be written under the form  $\tilde{f} - \operatorname{div} \tilde{g}$  with  $\tilde{f} \in L^1(\Omega)$  and  $\tilde{g} \in (L^{p'}(\Omega))^N$ , see [25].

**Proposition 3.3** For any  $F \in L^1((0, T))$  with  $\int_0^T F(t)dt \neq 0$ , and  $\omega \in \mathcal{M}_b(\Omega)$ ,

$$\omega \in \mathcal{M}_{0,e}(\Omega) \iff \omega \otimes F \in \mathcal{M}_0(Q).$$

**Proof.** Assume that  $\omega \otimes F \in \mathcal{M}_0(Q)$ . Then, there exist  $f \in L^1(Q)$ ,  $g \in (L^{p'}(Q))^N$  and  $h \in X$ , such that

$$\int_Q \varphi(x, t)F(t)d\omega(x)dt = \int_Q \varphi(x, t)f(x, t)dxdt + \int_Q g(x, t) \cdot \nabla \varphi(x, t)dxdt - \int_Q h(x, t)\varphi_t(t, x)dxdt, \quad (3.5)$$

for all  $\varphi \in C_c^\infty(\Omega \times [0, T])$ , see [50, Lemma 2.24 and Theorem 2.27]. By choosing  $\varphi(x, t) = \varphi(x) \in C_c^\infty(\Omega)$  and using Fubini's Theorem, (3.5) is rewritten as

$$\int_\Omega \varphi(x)d\omega(x) = \int_\Omega \varphi(x)\tilde{f}(x)dx + \int_\Omega \tilde{g}(x) \cdot \nabla \varphi(x)dx,$$

where  $\tilde{f}(x) = \left( \int_0^T F(t)dt \right)^{-1} \int_0^T f(x, t)dt \in L^1(\Omega)$  and  $\tilde{g}(x) = \left( \int_0^T F(t)dt \right)^{-1} \int_0^T g(x, t)dt \in (L^{p'}(\Omega))^N$ ; hence  $\omega \in \mathcal{M}_{0,e}(\Omega)$ .

Conversely, assume that  $\omega = \tilde{f} - \operatorname{div} \tilde{g} \in \mathcal{M}_{0,\varepsilon}(\Omega)$ , with  $\tilde{f} \in L^1(\Omega)$  and  $\tilde{g} \in \left(L^{p'}(\Omega)\right)^N$ . So  $\omega \otimes T_n(F) = f_n - \operatorname{div} g_n$ , with  $f_n = \tilde{f}T_n(F) \in L^1(Q)$  and  $g_n = \tilde{g}T_n(F) \in \left(L^{p'}(Q)\right)^N$ . Then  $\omega \otimes T_n(F)$  admits the decomposition  $(f_n, g_n, h)$ , with  $h = 0 \in L^\infty(Q)$ , thus  $\omega \otimes T_n(F) \in \mathcal{M}_0(Q)$ . And  $\{\omega \otimes T_n(F)\}$  converges to  $\omega \otimes F$  strongly in  $\mathcal{M}_b(Q)$ , since  $\|\omega \otimes (F - T_n(F))\|_{\mathcal{M}_b(Q)} \leq \|\omega\|_{\mathcal{M}_b(\Omega)} \|F - T_n(F)\|_{L^1((0,T))}$ . Then  $\omega \otimes F \in \mathcal{M}_0(Q)$ , since  $\mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$  is strongly closed in  $\mathcal{M}_b(Q)$ .  $\blacksquare$

## 4 Renormalized solutions of problem (1.1)

### 4.1 Notations and Definition

For any function  $f \in L^1(Q)$ , we write  $\int_Q f$  instead of  $\int_Q f dxdt$ , and for any measurable set  $E \subset Q$ ,  $\int_E f$  instead of  $\int_E f dxdt$ .

We set  $T_k(r) = \max\{\min\{r, k\}, -k\}$ , for any  $k > 0$  and  $r \in \mathbb{R}$ . We recall that if  $u$  is a measurable function defined and finite a.e. in  $Q$ , such that  $T_k(u) \in X$  for any  $k > 0$ , there exists a measurable function  $w$  from  $Q$  into  $\mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{|u| \leq k} w$ , a.e. in  $Q$ , and for any  $k > 0$ . We define the gradient  $\nabla u$  of  $u$  by  $w = \nabla u$ .

Let  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$ , and  $(f, g, h)$  be a decomposition of  $\mu_0$  given by (1.11), and  $\widehat{\mu}_0 = \mu_0 - h_t = f - \operatorname{div} g$ . In the general case  $\widehat{\mu}_0 \notin \mathcal{M}(Q)$ , but we write, for convenience,

$$\int_Q w d\widehat{\mu}_0 := \int_Q (fw + g \cdot \nabla w), \quad \forall w \in X \cap L^\infty(Q).$$

**Definition 4.1** Let  $u_0 \in L^1(\Omega)$ ,  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$ . A measurable function  $u$  is a **renormalized solution**, called **R-solution** of (1.1) if there exists a decomposition  $(f, g, h)$  of  $\mu_0$  such that

$$v = u - h \in L^\sigma(0, T; W_0^{1,\sigma}(\Omega) \cap L^\infty(0, T; L^1(\Omega))), \quad \forall \sigma \in [1, m_c); \quad T_k(v) \in X, \quad \forall k > 0, \quad (4.1)$$

and:

(i) for any  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$ , and  $S(0) = 0$ ,

$$-\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(v)\varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q S'(v)\varphi d\widehat{\mu}_0, \quad (4.2)$$

for any  $\varphi \in X \cap L^\infty(Q)$  such that  $\varphi_t \in X' + L^1(Q)$  and  $\varphi(T, \cdot) = 0$ ;

(ii) for any  $\phi \in C(\overline{Q})$ ,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla v = \int_Q \phi d\mu_s^+ \quad (4.3)$$

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{-m \geq v > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla v = \int_Q \phi d\mu_s^-. \quad (4.4)$$

**Remark 4.2** As a consequence,  $S(v) \in C([0, T]; L^1(\Omega))$  and  $S(v)(0, \cdot) = S(u_0)$  in  $\Omega$ ; and  $u$  satisfies the equation

$$(S(v))_t - \operatorname{div}(S'(v)A(x, t, \nabla u)) + S''(v)A(x, t, \nabla u) \cdot \nabla v = fS'(v) - \operatorname{div}(gS'(v)) + S''(v)g \cdot \nabla v, \quad (4.5)$$

in the sense of distributions in  $Q$ , see [49, Remark 3]. Moreover

$$\begin{aligned} \|S(v)_t\|_{X'+L^1(Q)} &\leq \left\| \operatorname{div}(S'(v)A(x, t, \nabla u)) \right\|_{X'} + \left\| S''(v)A(x, t, \nabla u) \cdot \nabla v \right\|_{1,Q} \\ &\quad + \left\| S'(v)f \right\|_{1,Q} + \left\| g \cdot S''(v)\nabla v \right\|_{1,Q} + \left\| \operatorname{div}(S'(v)g) \right\|_{X'}. \end{aligned}$$

Thus, if  $[-M, M] \supset \operatorname{supp} S'$ ,

$$\begin{aligned} \left\| S''(v)A(x, t, \nabla u) \cdot \nabla v \right\|_{1,Q} &\leq \|S\|_{W^{2,\infty}(\mathbb{R})} \left( \|A(x, t, \nabla u)\chi_{|v| \leq M}\|_{p',Q}^{p'} + \|\nabla T_M(v)\|_{p,Q}^p \right) \\ &\leq C \|S\|_{W^{2,\infty}(\mathbb{R})} \left( \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q} + \|a\|_{p',Q}^{p'} + \|\nabla T_M(v)\|_{p,Q}^p \right) \end{aligned}$$

thus

$$\begin{aligned} \|S(v)_t\|_{X'+L^1(Q)} &\leq C \|S\|_{W^{2,\infty}(\mathbb{R})} \left( \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q}^{1/p'} + \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q} + \|\nabla T_M(v)\|_{p,Q}^p \right. \\ &\quad \left. + \|a\|_{p',Q} + \|a\|_{p',Q}^{p'} + \|f\|_{1,Q} + \|g\|_{p',Q} \|\nabla u\|^p \chi_{|v| \leq M}\|_{1,Q}^{1/p} + \|g\|_{p',Q} \right) \quad (4.6) \end{aligned}$$

We also deduce that, for any  $\varphi \in X \cap L^\infty(Q)$ , such that  $\varphi_t \in X' + L^1(Q)$ ,

$$\begin{aligned} \int_\Omega S(v(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi \\ + \int_Q S''(v)A(x, t, \nabla u) \cdot \nabla v \varphi = \int_Q S'(v)\varphi d\widehat{\mu}_0. \quad (4.7) \end{aligned}$$

**Remark 4.3** Let  $u, v$  satisfying (4.1). It is easy to see that the condition (4.3) ( resp. (4.4) ) is equivalent to

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^+ \quad (4.8)$$

resp.

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \geq v > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^-. \quad (4.9)$$

In particular, for any  $\varphi \in L^{p'}(Q)$  there holds

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla u| \varphi = 0, \quad \lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla v| \varphi = 0. \quad (4.10)$$

**Remark 4.4** (i) Any function  $U \in X$  such that  $U_t \in X' + L^1(Q)$  admits a unique  $c_p^Q$ -quasi continuous representative, defined  $c_p^Q$ -quasi a.e. in  $Q$ , still denoted  $U$ . Furthermore, if  $U \in L^\infty(Q)$ , then for any  $\mu_0 \in \mathcal{M}_0(Q)$ , there holds  $U \in L^\infty(Q, d\mu_0)$ , see [49, Theorem 3 and Corollary 1].

(ii) Let  $u$  be any  $R$ -solution of problem (1.1). Then,  $v = u - h$  admits a  $c_p^Q$ -quasi continuous functions representative which is finite  $c_p^Q$ -quasi a.e. in  $Q$ , and  $u$  satisfies definition 4.1 for every decomposition  $(\tilde{f}, \tilde{g}, \tilde{h})$  such that  $h - \tilde{h} \in L^\infty(Q)$ , see [49, Proposition 3 and Theorem 4].

## 4.2 Steklov and Landes approximations

A main difficulty for proving Theorem 2.1 is the choice of admissible test functions  $(S, \varphi)$  in (4.2), valid for any  $R$ -solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

**Definition 4.5** Let  $\varepsilon \in (0, T)$  and  $z \in L_{loc}^1(Q)$ . For any  $l \in (0, \varepsilon)$  we define the **Steklov time-averages**  $[z]_l, [z]_{-l}$  of  $z$  by

$$[z]_l(x, t) = \frac{1}{l} \int_t^{t+l} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (0, T - \varepsilon),$$

$$[z]_{-l}(x, t) = \frac{1}{l} \int_{t-l}^t z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for  $R$ -solutions can be found in [22]. Recall some properties, given in [50]. Let  $\varepsilon \in (0, T)$ , and  $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0, T])$ ,  $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0, T])$  with  $\text{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T - \varepsilon]$ ,  $\text{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$ . There holds

- (i) If  $z \in X$ , then  $\varphi_1[z]_l$  and  $\varphi_2[z]_{-l} \in W$ .
- (ii) If  $z \in X$  and  $z_t \in X' + L^1(Q)$ , then, as  $l \rightarrow 0$ ,  $(\varphi_1[z]_l)$  and  $(\varphi_2[z]_{-l})$  converge respectively to  $\varphi_1 z$  and  $\varphi_2 z$  in  $X$ , and a.e. in  $Q$ ; and  $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$  converge to  $(\varphi_1 z)_t, (\varphi_2 z)_t$  in  $X' + L^1(Q)$ .
- (iii) If moreover  $z \in L^\infty(Q)$ , then from any sequence  $\{l_n\} \rightarrow 0$ , there exists a subsequence  $\{l_\nu\}$  such that  $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$  converge to  $z$ ,  $c_p^Q$ -quasi everywhere in  $Q$ .

Next we recall the approximation introduced in [42], used in [30], [26], [21]:

**Definition 4.6** Let  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$ . Let  $u$  be a  $R$ -solution of (1.1), and  $v = u - h$  given at (4.1), and  $k > 0$ . For any  $\nu \in \mathbb{N}$ , the **Landes-time approximation**  $\langle T_k(v) \rangle_\nu$  of the truncate function  $T_k(v)$  is defined in the following way:

Let  $\{z_\nu\}$  be a sequence of functions in  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , such that  $\|z_\nu\|_{\infty, \Omega} \leq k$ ,  $\{z_\nu\}$  converges to  $T_k(u_0)$  a.e. in  $\Omega$ , and  $\nu^{-1}\|z_\nu\|_{W_0^{1,p}(\Omega)}^p$  converges to 0. Then,  $\langle T_k(v) \rangle_\nu$  is the unique solution of the problem

$$(\langle T_k(v) \rangle_\nu)_t = \nu(T_k(v) - \langle T_k(v) \rangle_\nu) \quad \text{in the sense of distributions,} \quad \langle T_k(v) \rangle_\nu(0) = z_\nu, \quad \text{in } \Omega.$$

Therefore,  $\langle T_k(v) \rangle_\nu \in X \cap L^\infty(Q)$  and  $(T_k(v))_\nu \in X$ , see [42]. Furthermore, up to subsequences,  $\{\langle T_k(v) \rangle_\nu\}$  converges to  $T_k(v)$  strongly in  $X$  and a.e. in  $Q$ , and  $\|(\langle T_k(v) \rangle_\nu)\|_{L^\infty(Q)} \leq k$ .

### 4.3 First properties

In the sequel we use the following notations: for any function  $J \in W^{1,\infty}(\mathbb{R})$ , nondecreasing with  $J(0) = 0$ , we set

$$\bar{J}(r) = \int_0^r J(\tau) d\tau, \quad \mathcal{J}(r) = \int_0^r J'(\tau) \tau d\tau. \quad (4.11)$$

It is easy to verify that  $\mathcal{J}(r) \geq 0$ ,

$$\mathcal{J}(r) + \bar{J}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geq s(J(r) - J(s)) \quad \forall r, s \in \mathbb{R}. \quad (4.12)$$

In particular we define, for any  $k > 0$ , and any  $r \in \mathbb{R}$ ,

$$\bar{T}_k(r) = \int_0^r T_k(\tau) d\tau, \quad \mathcal{T}_k(r) = \int_0^r T_k'(\tau) \tau d\tau, \quad (4.13)$$

and we use several times a truncature used in [32]:

$$H_m(r) = \chi_{[-m,m]}(r) + \frac{2m - |s|}{m} \chi_{m < |s| \leq 2m}(r), \quad \bar{H}_m(r) = \int_0^r H_m(\tau) d\tau. \quad (4.14)$$

The next Lemma allows to extend the range of the test functions in (4.2). Its proof, given in the Appendix, is obtained by Steklov approximation of the solutions.

**Lemma 4.7** *Let  $u$  be a  $R$ -solution of problem (1.1). Let  $J \in W^{1,\infty}(\mathbb{R})$  be nondecreasing with  $J(0) = 0$ , and  $\bar{J}$  defined by (4.11). Then,*

$$\begin{aligned} & \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla (\xi J(S(v))) + \int_Q S''(v) A(x, t, \nabla u) \cdot \nabla v \xi J(S(v)) \\ & - \int_\Omega \xi(0) J(S(u_0)) S(u_0) - \int_Q \xi_t \bar{J}(S(v)) \\ & \leq \int_Q S'(v) \xi J(S(v)) d\widehat{\mu}_0, \end{aligned} \quad (4.15)$$

for any  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$  and  $S(0) = 0$ , and for any  $\xi \in C^1(Q) \cap W^{1,\infty}(Q)$ ,  $\xi \geq 0$ .

Next we give estimates of the gradient, following the first estimates of [26], see also [33], [49, Proposition 2], [43].

**Proposition 4.8** *If  $u$  is a  $R$ -solution of problem (1.1), then there exists  $c = c(p)$  such that, for any  $k \geq 1$  and  $\ell \geq 0$ ,*

$$\int_{\ell \leq |v| \leq \ell+k} |\nabla u|^p + \int_{\ell \leq |v| \leq \ell+k} |\nabla v|^p \leq ckM \quad (4.16)$$

and

$$\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq c(M + |\Omega|), \quad (4.17)$$

where

$$M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + \|a\|_{p',Q}^{p'}.$$

As a consequence, for any  $k \geq 1$ ,

$$\text{meas}\{|v| > k\} \leq C_1 M_1 k^{-pc}, \quad \text{meas}\{|\nabla v| > k\} \leq C_2 M_2 k^{-mc}, \quad (4.18)$$

$$\text{meas}\{|u| > k\} \leq C_3 M_2 k^{-pc}, \quad \text{meas}\{|\nabla u| > k\} \leq C_4 M_2 k^{-mc}, \quad (4.19)$$

where  $C_i = C_i(N, p, c_1, c_2)$ ,  $i = 1-4$ , and  $M_1 = (M + |\Omega|)^{\frac{p}{N}} M$  and  $M_2 = M_1 + M$ .

**Proof.** Set for any  $r \in \mathbb{R}$ , and  $m, k, \ell > 0$ ,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For  $m > k + \ell$ , we can choose  $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$  as test functions in (4.15), where  $\overline{H_m}$  is defined at (4.14) and  $\xi \in C^1([0, T])$  with values in  $[0, 1]$ , independent on  $x$ . Since  $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$  for all  $r \in \mathbb{R}$ , we obtain

$$\begin{aligned} & - \int_{\Omega} \xi(0) T_{k,\ell}(u_0) \overline{H_m}(u_0) - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(v)) \\ & + \int_{\{\ell \leq |v| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla v - \frac{k}{m} \int_{\{m \leq |v| < 2m\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_Q H_m(v) \xi T_{k,\ell}(v) d\widehat{\mu}_0. \end{aligned}$$

And

$$\int_Q H_m(v) \xi T_{k,\ell}(v) d\widehat{\mu}_0 = \int_Q H_m(v) \xi T_{k,\ell}(v) f + \int_{\{\ell \leq |v| < \ell+k\}} \xi \nabla v \cdot g - \frac{k}{m} \int_{\{m \leq |v| < 2m\}} \xi \nabla v \cdot g.$$

Let  $m \rightarrow \infty$ ; then, for any  $k \geq 1$ , since  $v \in L^1(Q)$  and from (4.3), (4.4), and (4.10), we find

$$- \int_Q \xi_t \overline{T_{k,\ell}}(v) + \int_{\{\ell \leq |v| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_{\{\ell \leq |v| < \ell+k\}} \xi \nabla v \cdot g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \quad (4.20)$$

Next, we take  $\xi \equiv 1$ . We verify that there exists  $c = c(p)$  such that

$$A(x, t, \nabla u) \cdot \nabla v - \nabla v \cdot g \geq \frac{c_1}{4} (|\nabla u|^p + |\nabla v|^p) - c(|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

where  $c_1$  is the constant in (1.2). Hence (4.16) follows. Thus, from (4.20) and the Hölder inequality, we get, with another constant  $c$ , for any  $\xi \in C^1([0, T])$  with values in  $[0, 1]$ ,

$$- \int_Q \xi_t \overline{T_{k,\ell}}(v) \leq ckM$$

Thus  $\int_\Omega \overline{T_{k,\ell}}(v)(t) \leq ckM$ , for a.e.  $t \in (0, T)$ . We deduce (4.17) by taking  $k = 1, \ell = 0$ , since  $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geq |r| - 1$ , for any  $r \in \mathbb{R}$ .

Next, from the Gagliardo-Nirenberg embedding Theorem, we have

$$\int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leq C_1 \|v\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} \int_Q |\nabla T_k(v)|^p,$$

where  $C_1 = C_1(N, p)$ . Then, from (4.16) and (4.17), we get, for any  $k \geq 1$ ,

$$\text{meas} \{|v| > k\} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leq C \|v\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(v)|^p \leq C_2 M_1 k^{-pc},$$

with  $C_2 = C_2(N, p, c_1, c_2)$ . We obtain

$$\begin{aligned} \text{meas} \{|\nabla v| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas} (\{|\nabla v|^p > s\}) ds \\ &\leq \text{meas} \left\{ |v| > k^{\frac{N}{N+1}} \right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas} \left( \left\{ |\nabla v|^p > s, |v| \leq k^{\frac{N}{N+1}} \right\} \right) ds \\ &\leq C_2 M_1 k^{-mc} + \frac{1}{k^p} \int_{|v| \leq k^{\frac{N}{N+1}}} |\nabla v|^p \leq C_2 M_2 k^{-mc}, \end{aligned}$$

with  $C_3 = C_3(N, p, c_1, c_2)$ . Furthermore, for any  $k \geq 1$ ,

$$\text{meas} \{|h| > k\} + \text{meas} \{|\nabla h| > k\} \leq C_4 k^{-p} \|h\|_X^p,$$

where  $C_4 = C_4(N, p, c_1, c_2)$ . Therefore, we easily get (4.19).  $\blacksquare$

**Remark 4.9** If  $\mu \in L^1(Q)$  and  $a \equiv 0$  in (1.2), then (4.16) holds for all  $k > 0$  and the term  $|\Omega|$  in inequality (4.17) can be removed where  $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$ . Furthermore, (4.19) is stated as follows:

$$\text{meas} \{|u| > k\} \leq C_3 M^{\frac{p+N}{N}} k^{-pc}, \quad \text{meas} \{|\nabla u| > k\} \leq C_4 M^{\frac{N+2}{N+1}} k^{-mc}, \forall k > 0. \quad (4.21)$$



To see last inequality, we do in the following way:

$$\begin{aligned}
\text{meas } \{|\nabla v| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas } (\{|\nabla v|^p > s\}) ds \\
&\leq \text{meas } \left\{ |v| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}} \right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas } \left\{ |\nabla v|^p > s, |v| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}} \right\} ds \\
&\leq C_4 M^{\frac{N+2}{N+1}} k^{-m_c}.
\end{aligned}$$

**Proposition 4.10** Let  $\{\mu_n\} \subset \mathcal{M}_b(Q)$ , and  $\{u_{0,n}\} \subset L^1(\Omega)$ , with

$$\sup_n |\mu_n|(Q) < \infty, \text{ and } \sup_n \|u_{0,n}\|_{1,\Omega} < \infty.$$

Let  $u_n$  be a  $R$ -solution of (1.1) with data  $\mu_n = \mu_{n,0} + \mu_{n,s}$  and  $u_{0,n}$ , relative to a decomposition  $(f_n, g_n, h_n)$  of  $\mu_{n,0}$ , and  $v_n = u_n - h_n$ . Assume that  $\{f_n\}$  is bounded in  $L^1(Q)$ ,  $\{g_n\}$  bounded in  $(L^{p'}(Q))^N$  and  $\{h_n\}$  bounded in  $X$ .

Then, up to a subsequence,  $\{v_n\}$  converges a.e. to a function  $v$ , such that  $T_k(v) \in X$  and  $v \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$  for any  $\sigma \in [1, m_c)$ . And

- (i)  $\{v_n\}$  converges to  $v$  strongly in  $L^\sigma(Q)$  for any  $\sigma \in [1, m_c)$ , and  $\sup \|v_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$ ,
- (ii)  $\sup_{k>0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(v_n)|^p < \infty$ ,
- (iii)  $\{T_k(v_n)\}$  converges to  $T_k(v)$  weakly in  $X$ , for any  $k > 0$ ,
- (iv)  $\{A(x, t, \nabla(T_k(v_n) + h_n))\}$  converges to some  $F_k$  weakly in  $(L^{p'}(Q))^N$ .

**Proof.** Take  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$  and  $S(0) = 0$ . We combine (4.6) with (4.16), and deduce that  $\{S(v_n)_t\}$  is bounded in  $X' + L^1(Q)$  and  $\{S(v_n)\}$  bounded in  $X$ . Hence,  $\{S(v_n)\}$  is relatively compact in  $L^1(Q)$ . On the other hand, we choose  $S = S_k$  such that  $S_k(z) = z$ , if  $|z| < k$  and  $S_k(z) = 2k \text{ sign} z$ , if  $|z| > 2k$ . Thanks to (4.17), we obtain

$$\begin{aligned}
\text{meas } \{|v_n - v_m| > \sigma\} &\leq \text{meas } \{|v_n| > k\} + \text{meas } \{|v_m| > k\} + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\} \\
&\leq \frac{1}{k} (\|v_n\|_{1,Q} + \|v_m\|_{1,Q}) + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\} \\
&\leq \frac{C}{k} + \text{meas } \{|S_k(v_n) - S_k(v_m)| > \sigma\}.
\end{aligned} \tag{4.22}$$

Thus, up to a subsequence  $\{u_n\}$  is a Cauchy sequence in measure, and converges a.e. in  $Q$  to a function  $u$ . Thus,  $\{T_k(v_n)\}$  converges to  $T_k(v)$  weakly in  $X$ , since  $\sup_n \|T_k(v_n)\|_X < \infty$  for any  $k > 0$ . And  $\{|\nabla(T_k(v_n) + h_n)|^{p-2} \nabla(T_k(v_n) + h_n)\}$  converges to some  $F_k$  weakly in  $(L^{p'}(Q))^N$ . Furthermore, from (4.18),  $\{v_n\}$  converges to  $v$  strongly in  $L^\sigma(Q)$ , for any  $\sigma < p_c$ . ■

## 5 The convergence theorem

We first recall some properties of the measures, see [49, Lemma 5], [32].

**Proposition 5.1** *Let  $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$ , where  $\mu_s^+$  and  $\mu_s^-$  are concentrated, respectively, on two disjoint sets  $E^+$  and  $E^-$  of zero  $c_p^Q$ -capacity. Then, for any  $\delta > 0$ , there exist two compact sets  $K_\delta^+ \subseteq E^+$  and  $K_\delta^- \subseteq E^-$  such that*

$$\mu_s^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_s^-(E^- \setminus K_\delta^-) \leq \delta,$$

and there exist  $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$  with values in  $[0, 1]$ , such that  $\psi_\delta^+, \psi_\delta^- = 1$  respectively on  $K_\delta^+, K_\delta^-$ , and  $\text{supp}(\psi_\delta^+) \cap \text{supp}(\psi_\delta^-) = \emptyset$ , and

$$\|\psi_\delta^+\|_X + \|(\psi_\delta^+)_t\|_{X'+L^1(Q)} \leq \delta, \quad \|\psi_\delta^-\|_X + \|(\psi_\delta^-)_t\|_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions  $(\psi_\delta^+)_t = (\psi_\delta^+)_t^1 + (\psi_\delta^+)_t^2$  and  $(\psi_\delta^-)_t = (\psi_\delta^-)_t^1 + (\psi_\delta^-)_t^2$  in  $X' + L^1(Q)$ , such that

$$\|(\psi_\delta^+)_t^1\|_{X'} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{1,Q} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^1\|_{X'} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{1,Q} \leq \frac{\delta}{3}. \quad (5.1)$$

Both  $\{\psi_\delta^+\}$  and  $\{\psi_\delta^-\}$  converge to 0, \*-weakly in  $L^\infty(Q)$ , and strongly in  $L^1(Q)$  and up to subsequences, a.e. in  $Q$ , as  $\delta$  tends to 0.

Moreover if  $\rho_n$  and  $\eta_n$  are as in Theorem 2.1, we have, for any  $\delta, \delta_1, \delta_2 > 0$ ,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n, \delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \quad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \quad (5.2)$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \quad (5.3)$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \quad (5.4)$$

Hereafter, if  $n, \varepsilon, \dots, \nu$  are real numbers, and a function  $\phi$  depends on  $n, \varepsilon, \dots, \nu$  and eventual other parameters  $\alpha, \beta, \dots, \gamma$ , and  $n \rightarrow n_0, \varepsilon \rightarrow \varepsilon_0, \dots, \nu \rightarrow \nu_0$ , we write  $\phi = \omega(n, \varepsilon, \dots, \nu)$ , then this means  $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} |\phi| = 0$ , when the parameters  $\alpha, \beta, \dots, \gamma$  are fixed. In the same way,  $\phi \leq \omega(n, \varepsilon, \delta, \dots, \nu)$  means  $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} \phi \leq 0$ , and  $\phi \geq \omega(n, \varepsilon, \dots, \nu)$  means  $-\phi \leq \omega(n, \varepsilon, \dots, \nu)$ .

**Remark 5.2** *In the sequel we use a convergence property, consequence of the Dunford-Pettis theorem, still used in [32]: If  $\{a_n\}$  is a sequence in  $L^1(Q)$  converging to a weakly in  $L^1(Q)$  and  $\{b_n\}$  a bounded sequence in  $L^\infty(Q)$  converging to  $b$ , a.e. in  $Q$ , then  $\lim_{n \rightarrow \infty} \int_Q a_n b_n = \int_Q ab$ .*

Next we prove Theorem 2.1.

**Scheme of the proof.** Let  $\{\mu_n\}, \{u_{0,n}\}$  and  $\{u_n\}$  satisfying the assumptions of Theorem 2.1. Then we can apply Proposition 4.10. Setting  $v_n = u_n - h_n$ , up to subsequences,  $\{u_n\}$  converges *a.e.* in  $Q$  to some function  $u$ , and  $\{v_n\}$  converges *a.e.* to  $v = u - h$ , such that  $T_k(v) \in X$  and  $v \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$  for every  $\sigma \in [1, m_c)$ . And  $\{v_n\}$  satisfies the conclusions (i) to (iv) of Proposition 4.10. We have

$$\begin{aligned} \mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-, \end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \quad (5.5)$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathcal{M}_b^+(Q) \cap \mathcal{M}_0(Q), \quad \text{and} \quad \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}. \quad (5.6)$$

Let  $E^+, E^-$  be the sets where, respectively,  $\mu_s^+$  and  $\mu_s^-$  are concentrated. For any  $\delta_1, \delta_2 > 0$ , let  $\psi_{\delta_1}^+, \psi_{\delta_2}^+$  and  $\psi_{\delta_1}^-, \psi_{\delta_2}^-$  as in Proposition 5.1 and set

$$\Phi_{\delta_1, \delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near  $E$

$$I_1 := \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2), \quad (5.7)$$

and far from  $E$ ,

$$I_2 := \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2). \quad (5.8)$$

Then it follows that

$$\overline{\lim}_{n, \nu} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - \langle T_k(v) \rangle_\nu) \leq 0, \quad (5.9)$$

which implies

$$\overline{\lim}_{n \rightarrow \infty} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - T_k(v)) \leq 0, \quad (5.10)$$

since  $\{\langle T_k(v) \rangle_\nu\}$  converges to  $T_k(v)$  in  $X$ . On the other hand, from the weak convergence of  $\{T_k(v_n)\}$  to  $T_k(v)$  in  $X$ , we verify that

$$\int_{\{|v_n| \leq k\}} A(x, t, \nabla (T_k(v) + h_n)) \cdot \nabla (T_k(v_n) - T_k(v)) = \omega(n).$$

Thus we get

$$\int_{\{|v_n| \leq k\}} (A(x, t, \nabla u_n) - A(x, t, \nabla(T_k(v) + h_n))) \cdot \nabla (u_n - (T_k(v) + h_n)) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \quad \text{a.e. in } Q. \quad (5.11)$$

Therefore,  $\{A(x, t, \nabla u_n)\}$  converges to  $A(x, t, \nabla u)$  weakly in  $(L^{p'}(Q))^N$ ; and from (5.10) we find

$$\overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \leq \int_Q A(x, t, \nabla u) \nabla T_k(v).$$

Otherwise,  $\{A(x, t, \nabla(T_k(v_n) + h_n))\}$  converges weakly in  $(L^{p'}(Q))^N$  to some  $F_k$ , from Proposition 4.10, and we obtain that  $F_k = A(x, t, \nabla(T_k(v) + h))$ . Hence

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(v_n) + h_n)) \cdot \nabla(T_k(v_n) + h_n) &\leq \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \\ &\quad + \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(v_n) + h_n)) \cdot \nabla h_n \\ &\leq \int_Q A(x, t, \nabla(T_k(v) + h)) \cdot \nabla(T_k(v) + h). \end{aligned}$$

As a consequence

$$\{T_k(v_n)\} \text{ converges to } T_k(v), \text{ strongly in } X, \quad \forall k > 0. \quad (5.12)$$

Then to finish the proof we have to check that  $u$  is a solution of (1.1).  $\blacksquare$

In order to prove (5.7) we need a first Lemma, inspired of [32, Lemma 6.1], extending [49, Lemma 6 and Lemma 7]:

**Lemma 5.3** *Let  $\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)$  be uniformly bounded in  $W^{1,\infty}(Q)$  with values in  $[0, 1]$ , such that  $\int_Q \psi_{1,\delta} d\mu_s^- \leq \delta$  and  $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$ . Then, under the assumptions of Theorem 2.1,*

$$\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (5.13)$$

$$\frac{1}{m} \int_{\{-2m < v_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{-2m < v_n \leq -m\}} |\nabla v_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad (5.14)$$

and for any  $k > 0$ ,

$$\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \int_{\{m \leq v_n < m+k\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (5.15)$$

$$\int_{\{-m-k < v_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \int_{\{-m-k < v_n \leq -m\}} |\nabla v_n|^p \psi_{1,\delta} = \omega(n, m, \delta). \quad (5.16)$$

**Proof.** (i) Proof of (5.13), (5.14). Set for any  $r \in \mathbb{R}$  and any  $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left( \frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m + 2h - \tau}{2m + \ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left( \frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$

Note that  $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$ . We choose  $(\xi, J, S) = (\psi_{2,\delta}, T_1, S_{m,\ell})$  as test functions in (4.15) for  $u_n$ , and observe that, from (5.5),

$$\widehat{\mu}_{n,0} = \mu_{n,0} - (h_n)_t = \widehat{\lambda}_{n,0} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \quad (5.17)$$

Thus we can write  $\sum_{i=1}^6 A_i \leq \sum_{i=7}^{12} A_i$ , where

$$A_1 = - \int_{\Omega} \psi_{2,\delta}(0) T_1(S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}), \quad A_2 = - \int_Q (\psi_{2,\delta})_t \overline{T_1}(S_{m,\ell}(v_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla \psi_{2,\delta},$$

$$A_4 = \int_Q S'_{m,\ell}(v_n)^2 \psi_{2,\delta} T'_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_5 = \frac{1}{m} \int_{\{m \leq v_n \leq 2m\}} \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_6 = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} A(x, t, \nabla u_n) \nabla v_n,$$

$$A_7 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} f_n, \quad A_8 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) g_n \cdot \nabla \psi_{2,\delta},$$

$$A_9 = \int_Q (S'_{m,\ell}(v_n))^2 T'_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n \cdot \nabla v_n, \quad A_{10} = \frac{1}{m} \int_{m \leq v_n \leq 2m} T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n \cdot \nabla v_n,$$

$$A_{11} = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} g_n \cdot \nabla v_n, \quad A_{12} = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} d(\rho_{n,0} - \eta_{n,0}).$$

Since  $\|S_{m,\ell}(u_{0,n})\|_{1,\Omega} \leq \int_{\{m \leq u_{0,n}\}} u_{0,n} dx$ , we find  $A_1 = \omega(\ell, n, m)$ . Otherwise

$$|A_2| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} v_n, \quad |A_3| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} (|a| + c_2 |\nabla u_n|^{p-1}),$$

which implies  $A_2 = \omega(\ell, n, m)$  and  $A_3 = \omega(\ell, n, m)$ . Using (4.3) for  $u_n$ , we have

$$A_6 = - \int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence  $A_6 = \omega(\ell, n, m, \delta)$ , since  $(\rho_{n,s} - \eta_{n,s})^+$  converges to  $\mu_s^+$  as  $n \rightarrow \infty$  in the narrow topology, and  $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$ . We also obtain  $A_{11} = \omega(\ell)$  from (4.10).

Now  $\left\{ S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \right\}_\ell$  converges to  $S'_m(v_n) T_1(S_m(v_n))$ ,  $\left\{ S'_m(v_n) T_1(S_m(v_n)) \right\}_n$  converges to  $S'_m(v) T_1(S_m(v))$ ,  $\left\{ S'_m(v) T_1(S_m(v)) \right\}_m$  converges to 0, \*-weakly in  $L^\infty(Q)$ , and  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ . From Remark 5.2, we obtain

$$\begin{aligned} A_7 &= \int_Q S'_m(v_n) T_1(S_m(v_n)) \psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(v) T_1(S_m(v)) \psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m), \\ A_8 &= \int_Q S'_m(v_n) T_1(S_m(v_n)) g_n \cdot \nabla \psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(v) T_1(S_m(v)) g \cdot \nabla \psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m). \end{aligned}$$

Otherwise,  $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$ , and  $\left\{ \int_Q \psi_{2,\delta} d\rho_n \right\}$  converges to  $\int_Q \psi_{2,\delta} d\mu_s^+$ , thus  $A_{12} \leq \omega(\ell, n, m, \delta)$ . Using Holder inequality and the condition (1.2) we have

$$g_n \cdot \nabla v_n - A(x, t, \nabla u_n) \nabla v_n \leq C_1 \left( |g_n|^{p'} + |\nabla h_n|^p + |a|^{p'} \right)$$

with  $C_1 = C_1(p, c_2)$ , which implies

$$A_9 - A_4 \leq C_1 \int_Q (S'_{m,\ell}(v_n))^2 T'_1(S_{m,\ell}(v_n)) \psi_{2,\delta} \left( |g_n|^{p'} + |h_n|^p + |a|^{p'} \right) = \omega(\ell, n, m).$$

Similarly we also show that  $A_{10} - A_5/2 \leq \omega(\ell, n, m)$ . Combining the estimates, we get  $A_5/2 \leq \omega(\ell, n, m, \delta)$ . Using Holder inequality we have

$$A(x, t, \nabla u_n) \nabla v_n \geq \frac{c_1}{2} |\nabla u_n|^p - C_2 (|a|^{p'} + |\nabla h_n|^p).$$

with  $C_2 = C_2(p, c_1, c_2)$ , which implies

$$\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that for all  $m > 4$ ,  $S_{m,\ell}(r) \geq 1$  for any  $r \in [\frac{3}{2}m, 2m]$ ; hence  $T_1(S_{m,\ell}(r)) = 1$ . So,

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since  $|\nabla v_n|^p \leq 2^{p-1}|\nabla u_n|^p + 2^{p-1}|\nabla h_n|^p$ , there also holds

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (5.13) by summing on each set  $\{(\frac{4}{3})^\nu m \leq v_n \leq (\frac{4}{3})^{\nu+1} m\}$  for  $\nu = 0, 1, 2$ . Similarly, we can choose  $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$  as test functions in (4.15) for  $u_n$ , where  $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$ , and we obtain (5.14).

(ii) Proof of (5.15), (5.16). We set, for any  $k, m, \ell \geq 1$ ,

$$S_{k,m,\ell}(r) = \int_0^r \left( T_k(\tau - T_m(\tau)) \chi_{[m, k+m+\ell]} + k \frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell)]} \right) d\tau$$

$$S_{k,m}(r) = \int_0^s T_k(\tau - T_m(\tau)) \chi_{[m, \infty)} d\tau.$$

We choose  $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$  as test functions in (4.15) for  $u_n$ . In the same way we also obtain

$$\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that  $T_1(S_{k,m,\ell}(r)) = 1$  for any  $r \geq m+1$ , thus  $\int_{\{m+1 \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$ , which implies (5.15) by changing  $m$  into  $m-1$ . Similarly, we obtain (5.16).  $\blacksquare$

Next we look at the behaviour near  $E$ .

**Lemma 5.4** *Estimate (5.7) holds.*

**Proof.** There holds

$$I_1 = \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) - \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_\nu.$$

From Proposition 4.10, (iv),  $\{A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu\}$  converges weakly in  $L^1(Q)$  to  $F_k \nabla \langle T_k(v) \rangle_\nu$ . And  $\{\chi_{\{|v_n| \leq k\}}\}$  converges to  $\chi_{\{|v| \leq k\}}$ , *a.e.* in  $Q$ , and  $\Phi_{\delta_1, \delta_2}$  converges to 0 *a.e.* in  $Q$  as  $\delta_1 \rightarrow 0$ , and  $\Phi_{\delta_1, \delta_2}$  takes its values in  $[0, 1]$ . Thanks to Remark 5.2, we have

$$\begin{aligned} & \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_\nu \\ &= \int_Q \chi_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu \\ &= \int_Q \chi_{\{|v| \leq k\}} \Phi_{\delta_1, \delta_2} F_k \cdot \nabla \langle T_k(v) \rangle_\nu + \omega(n) = \omega(n, \nu, \delta_1). \end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2), \quad (5.18)$$

then we deduce (5.7). As noticed in [32], [49], it is precisely for this estimate that we need the double cut  $\psi_{\delta_1}^+ \psi_{\delta_2}^+$ . To do this, we set, for any  $m > k > 0$ , and any  $r \in \mathbb{R}$ ,

$$\hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau)) H_m(\tau) d\tau,$$

where  $H_m$  is defined at (4.14). Hence  $\text{supp} \hat{S}_{k,m} \subset [-2m, k]$ ; and  $\hat{S}_{k,m}'' = -\chi_{[-k, k]} + \frac{2k}{m} \chi_{[-2m, -m]}$ . We choose  $(\varphi, S) = (\psi_{\delta_1}^+ \psi_{\delta_2}^+, \hat{S}_{k,m})$  as test functions in (4.2). From (5.17), we can write

$$A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$\begin{aligned} A_1 &= - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(v_n), & A_2 &= \int_Q (k - T_k(v_n)) H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+), \\ A_3 &= \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla T_k(v_n), & A_4 &= \frac{2k}{m} \int_{\{-2m < v_n \leq -m\}} \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla v_n, \\ A_5 &= - \int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\widehat{\lambda}_{n,0}, & A_6 &= \int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d(\eta_{n,0} - \rho_{n,0}); \end{aligned}$$

and we estimate  $A_3$ . As in [49, p.585], since  $\{\hat{S}_{k,m}(v_n)\}$  converges to  $\hat{S}_{k,m}(v)$  weakly in  $X$ , and  $\hat{S}_{k,m}(v) \in L^\infty(Q)$ , and from (5.1), there holds

$$A_1 = - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) - \int_Q \psi_{\delta_1}^+ (\psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) + \omega(n) = \omega(n, \delta_1).$$

Next consider  $A_2$ . Notice that  $v_n = T_{2m}(v_n)$  on  $\text{supp}(H_m(v_n))$ . From Proposition 4.10, (iv), the sequence  $\{A(x, t, \nabla (T_{2m}(v_n) + h_n)) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+)\}$  converges to  $F_{2m} \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+)$  weakly in  $L^1(Q)$ . Thanks to Remark 5.2 and the convergence of  $\psi_{\delta_1}^+ \psi_{\delta_2}^+$  in  $X$  to 0 as  $\delta_1$  tends to 0, we find

$$A_2 = \int_Q (k - T_k(v)) H_m(v) F_{2m} \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+) + \omega(n) = \omega(n, \delta_1).$$

Then consider  $A_4$ . Then for some  $C = C(p, c_2)$ ,

$$|A_4| \leq C \frac{2k}{m} \int_{\{-2m < v_n \leq -m\}} (|\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'}) \psi_{\delta_1}^+ \psi_{\delta_2}^+.$$

Since  $\psi_{\delta_1}^+$  takes its values in  $[0, 1]$ , from Lemma 5.3, we get in particular  $A_4 = \omega(n, \delta_1, m, \delta_2)$ .



Now estimate  $A_5$ . The sequence  $\{(k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+\}$  converges weakly in  $X$  to  $(k - T_k(v))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+$ , and  $\{(k - T_k(v_n))H_m(v_n)\}$  converges  $*$ -weakly in  $L^\infty(Q)$  and *a.e.* in  $Q$  to  $(k - T_k(v))H_m(v)$ . Otherwise  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$  and  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ . Thanks to Remark 5.2 and the convergence of  $\psi_{\delta_1}^+\psi_{\delta_2}^+$  to 0 in  $X$  and *a.e.* in  $Q$  as  $\delta_1 \rightarrow 0$ , we deduce that

$$A_5 = - \int_Q (k - T_k(v_n))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\widehat{\nu}_0 + \omega(n) = \omega(n, \delta_1),$$

where  $\widehat{\nu}_0 = f - \operatorname{div} g$ .

Finally  $A_6 \leq 2k \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ d\eta_m$ ; using (5.2) we also find  $A_6 \leq \omega(n, \delta_1, m, \delta_2)$ . By addition, since  $A_3$  does not depend on  $m$ , we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Reasoning as before with  $(\psi_{\delta_1}^-\psi_{\delta_2}^-, \check{S}_{k,m})$  as test function in (4.2), where  $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$ , we get in the same way

$$\int_Q \psi_{\delta_1}^-\psi_{\delta_2}^- A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (5.18) holds. ■

Next we look at the behaviour far from  $E$ .

**Lemma 5.5** . *Estimate (5.8) holds.*

**Proof.** Here we estimate  $I_2$ ; we can write

$$I_2 = \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \nabla (T_k(v_n) - \langle T_k(v) \rangle_\nu).$$

Following the ideas of [51], used also in [49], we define, for any  $r \in \mathbb{R}$  and  $\ell > 2k > 0$ ,

$$R_{n,\nu,\ell} = T_{\ell+k}(v_n - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v_n - T_k(v_n)).$$

Recall that  $\|\langle T_k(v) \rangle_\nu\|_{\infty, Q} \leq k$ , and observe that

$$R_{n,\nu,\ell} = 2k \operatorname{sign}(v_n) \quad \text{in } \{|v_n| \geq \ell + 2k\}, \quad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n, \nu, \ell) \quad \textit{a.e. in } Q, \quad (5.19)$$

$$\lim_{n \rightarrow \infty} R_{n,\nu,\ell} = T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v)), \quad \textit{a.e. in } Q, \quad \text{and weakly in } X. \quad (5.20)$$

Next consider  $\xi_{1,n_1} \in C_c^\infty([0, T])$ ,  $\xi_{2,n_2} \in C_c^\infty((0, T])$  with values in  $[0, 1]$ , such that  $(\xi_{1,n_1})_t \leq 0$  and  $(\xi_{2,n_2})_t \geq 0$ ; and  $\{\xi_{1,n_1}(t)\}$  (resp.  $\{\xi_{1,n_2}(t)\}$ ) converges to 1, for any  $t \in [0, T)$  (resp.  $t \in (0, T]$ );

and moreover, for any  $a \in C([0, T]; L^1(\Omega))$ ,  $\left\{ \int_Q a(\xi_{1, n_1})_t \right\}$  and  $\int_Q a(\xi_{2, n_2})_t$  converge respectively to  $-\int_{\Omega} a(T, \cdot)$  and  $\int_{\Omega} a(0, \cdot)$ . We set

$$\varphi = \varphi_{n, n_1, n_2, l_1, l_2, \ell} = \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} - \xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2}.$$

We can see that

$$\varphi - (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} = \omega(l_1, l_2, n_1, n_2) \quad \text{in norm in } X \text{ and a.e. in } Q. \quad (5.21)$$

We can choose  $(\varphi, S) = (\varphi_{n, n_1, n_2, l_1, l_2, \ell}, \overline{H_m})$  as test functions in (4.7) for  $u_n$ , where  $\overline{H_m}$  is defined at (4.14), with  $m > \ell + 2k$ . We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$\begin{aligned} A_1 &= \int_{\Omega} \varphi(T) \overline{H_m}(v_n(T)) dx, & A_2 &= - \int_{\Omega} \varphi(0) \overline{H_m}(u_{0, n}) dx, \\ A_3 &= - \int_Q \varphi_t \overline{H_m}(v_n), & A_4 &= \int_Q H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla \varphi, \\ A_5 &= \int_Q \varphi H'_m(v_n) A(x, t, \nabla u_n) \cdot \nabla v_n, & A_6 &= \int_Q H_m(v_n) \varphi d\widehat{\lambda}_{n, 0}, \\ A_7 &= \int_Q H_m(v_n) \varphi d(\rho_{n, 0} - \eta_{n, 0}). \end{aligned}$$

**Estimate** of  $A_4$ . This term allows to study  $I_2$ . Indeed,  $\{H_m(v_n)\}$  converges to 1, *a.e.* in  $Q$ ; thanks to (5.21), (5.19) (5.20), we have

$$\begin{aligned} A_4 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} - \int_Q R_{n, \nu, \ell} A(x, t, \nabla u_n) \cdot \nabla \Phi_{\delta_1, \delta_2} + \omega(l_1, l_2, n_1, n_2, m) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + \int_{\{|v_n| > k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n, \nu, \ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell), \end{aligned}$$

where

$$\begin{aligned} B_1 &= \int_{\{|v_n| > k\}} (1 - \Phi_{\delta, \eta}) (\chi_{|v_n - \langle T_k(v) \rangle_{\nu}| \leq \ell+k} - \chi_{|v_n - k| \leq \ell-k}) A(x, t, \nabla u_n) \cdot \nabla v_n, \\ B_2 &= - \int_{\{|v_n| > k\}} (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v_n - \langle T_k(v) \rangle_{\nu}| \leq \ell+k} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(v) \rangle_{\nu}. \end{aligned}$$

Now  $\{A(x, t, \nabla(T_{\ell+2k}(v_n) + h_n)) \cdot \nabla \langle T_k(v) \rangle_\nu\}$  converges to  $F_{\ell+2k} \nabla \langle T_k(v) \rangle_\nu$ , weakly in  $L^1(Q)$ . Otherwise  $\left\{ \chi_{|v_n|>k} \chi_{|v_n - \langle T_k(v) \rangle_\nu| \leq \ell+k} \right\}$  converges to  $\chi_{|v|>k} \chi_{|v - \langle T_k(v) \rangle_\nu| \leq \ell+k}$ , *a.e.* in  $Q$ . And  $\{\langle T_k(v) \rangle_\nu\}$  converges to  $T_k(v)$  strongly in  $X$ . Thanks to Remark 5.2 we get

$$\begin{aligned} B_2 &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v|>k} \chi_{|v - \langle T_k(v) \rangle_\nu| \leq \ell+k} F_{\ell+2k} \cdot \nabla \langle T_k(v) \rangle_\nu + \omega(n) \\ &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v|>k} \chi_{|v - T_k(v)| \leq \ell+k} F_{\ell+2k} \cdot \nabla T_k(v) + \omega(n, \nu) = \omega(n, \nu), \end{aligned}$$

since  $\nabla T_k(v) \chi_{|v|>k} = 0$ . Besides, we see that, for some  $C = C(p, c_2)$ ,

$$|B_1| \leq C \int_{\{\ell-2k \leq |v_n| < \ell+2k\}} (1 - \Phi_{\delta_1, \delta_2}) \left( |\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'} \right).$$

Using (5.3) and (5.4) and applying (5.15) and (5.16) to  $1 - \Phi_{\delta_1, \delta_2}$ , we obtain, for  $k > 0$

$$\int_{\{m \leq |v_n| < m+4k\}} (|\nabla u_n|^p + |\nabla v_n|^p) (1 - \Phi_{\delta_1, \delta_2}) = \omega(n, m, \delta_1, \delta_2). \quad (5.22)$$

Thus,  $B_1 = \omega(n, \nu, \ell, \delta_1, \delta_2)$ , hence  $B_1 + B_2 = \omega(n, \nu, \ell, \delta_1, \delta_2)$ . Then

$$A_4 = I_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2). \quad (5.23)$$

**Estimate** of  $A_5$ . For  $m > \ell + 2k$ , since  $|\varphi| \leq 2\ell$ , and (5.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned} A_5 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(v_n) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2) \\ &= -\frac{2k}{m} \int_{\{m \leq |v_n| < 2m\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2); \end{aligned}$$

here, the final equality followed from the relation, since  $m > \ell + 2k$ ,

$$R_{n, \nu, \ell} H'_m(v_n) = -\frac{2k}{m} \chi_{m \leq |v_n| \leq 2m}, \quad \textit{a.e. in } Q. \quad (5.24)$$

Next we go to the limit in  $m$ , by using (4.3), (4.4) for  $u_n$ , with  $\phi = (1 - \Phi_{\delta_1, \delta_2})$ . There holds

$$A_5 = -2k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d((\rho_{n, s} - \eta_{n, s})^+ + (\rho_{n, s} - \eta_{n, s})^-) + \omega(l_1, l_2, n_1, n_2, m).$$

Then, from (5.3) and (5.4), we get  $A_5 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate** of  $A_6$ . Again, from (5.21),

$$\begin{aligned} A_6 &= \int_Q H_m(v_n) \varphi f_n + \int_Q g_n \cdot \nabla (H_m(v_n) \varphi) \\ &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n + \int_Q g_n \cdot \nabla (H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Thus we can write  $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2)$ , where

$$\begin{aligned} D_1 &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n, & D_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(v_n) g_n \cdot \nabla v_n, \\ D_3 &= \int_Q H_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) g_n \cdot \nabla R_{n, \nu, \ell}, & D_4 &= - \int_Q H_m(v_n) R_{n, \nu, \ell} g_n \cdot \nabla \Phi_{\delta_1, \delta_2}. \end{aligned}$$

Since  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ , and (5.19)-(5.20) hold, we get from Remark 5.2,

$$D_1 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) (T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v))) f + \omega(m, n) = \omega(m, n, \nu, \ell).$$

We deduce from (4.10) that  $D_2 = \omega(m)$ . Next consider  $D_3$ . Note that  $H_m(v_n) = 1 + \omega(m)$ , and (5.20) holds, and  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ , and  $\langle T_k(v) \rangle_\nu$  converges to  $T_k(v)$  strongly in  $X$ . Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(v - \langle T_k(v) \rangle_\nu) - T_{\ell-k}(v - T_k(v))) + \omega(m, n) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(v - T_k(v)) - T_{\ell-k}(v - T_k(v))) + \omega(m, n, \nu) \\ &= \omega(m, n, \nu, \ell). \end{aligned}$$

Similarly we also get  $D_4 = \omega(m, n, \nu, \ell)$ . Thus  $A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate** of  $A_7$ . We have

$$\begin{aligned} |A_7| &= \left| \int_Q S'_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} d(\rho_{n,0} - \eta_{m,0}) \right| + \omega(l_1, l_2, n_1, n_2) \\ &\leq 4k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d(\rho_n + \eta_n) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

From (5.3) and (5.4) we get  $A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate** of  $A_1 + A_2 + A_3$ . We set

$$J(r) = T_{\ell-k}(r - T_k(r)), \quad \forall r \in \mathbb{R},$$

and use the notations  $\bar{J}$  and  $\mathcal{J}$  of (4.11). From the definitions of  $\xi_{1,n_1}, \xi_{1,n_2}$ , we can see that

$$\begin{aligned} A_1 + A_2 &= - \int_{\Omega} J(v_n(T)) \overline{H_m}(v_n(T)) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) \overline{H_m}(u_{0,n}) + \omega(l_1, l_2, n_1, n_2) \\ &= - \int_{\Omega} J(v_n(T)) v_n(T) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) u_{0,n} + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \quad (5.25)$$

where  $z_{\nu} = \langle T_k(v) \rangle_{\nu}(0)$ . We can write  $A_3 = F_1 + F_2$ , where

$$\begin{aligned} F_1 &= - \int_Q \left( \xi_{n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t \overline{H_m}(v_n), \\ F_2 &= \int_Q \left( \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \right)_t \overline{H_m}(v_n). \end{aligned}$$

**Estimate of  $F_2$ .** We write  $F_2 = G_1 + G_2 + G_3$ , with

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t \xi_{n_2} [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \overline{H_m}(v_n), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) (\xi_{n_2})_t [T_{\ell-k}(v_n - T_k(v_n))]_{-l_2} \overline{H_m}(v_n), \\ G_3 &= \int_Q \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell-k}(v_n - T_k(v_n))]_{-l_2})_t \overline{H_m}(v_n). \end{aligned}$$

We find easily

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t J(v_n) v_n + \omega(l_1, l_2, n_1, n_2, m), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) (\xi_{n_2})_t J(v_n) \overline{H_m}(v_n) + \omega(l_1, l_2) = \int_{\Omega} J(u_{0,n}) u_{0,n} + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Next consider  $G_3$ . Setting  $b = \overline{H_m}(v_n)$ , there holds from (4.13) and (4.12),

$$([\mathcal{J}(b)]_{-l_2})_t b(\cdot, t) = \frac{b(\cdot, t)}{l_2} (J(b)(\cdot, t) - J(b)(\cdot, t - l_2)).$$

Hence

$$([T_{\ell-k}(v_n - T_k(v_n))]_{-l_2})_t \overline{H_m}(v_n) \geq \left( [\mathcal{J}(\overline{H_m}(v_n))]_{-l_2} \right)_t = ([\mathcal{J}(v_n)]_{-l_2})_t,$$

since  $\mathcal{J}$  is constant in  $\{|r| \geq m + \ell + 2k\}$ . Integrating by parts in  $G_3$ , we find

$$\begin{aligned}
G_3 &\geq \int_Q \xi_{2,n_2} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{J}(v_n)]_{-l_2})_t \\
&= - \int_Q (\xi_{2,n_2} (1 - \Phi_{\delta_1, \delta_2}))_t [\mathcal{J}(v_n)]_{-l_2} + \int_{\Omega} \xi_{2,n_2}(T) [\mathcal{J}(v_n)]_{-l_2}(T) \\
&= - \int_Q (\xi_{2,n_2})_t (1 - \Phi_{\delta_1, \delta_2}) \mathcal{J}(v_n) \\
&\quad + \int_Q \xi_{2,n_2} (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(v_n) + \int_{\Omega} \xi_{2,n_2}(T) \mathcal{J}(v_n(T)) + \omega(l_1, l_2) \\
&= - \int_{\Omega} \mathcal{J}(u_{0,n}) + \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(v_n) + \int_{\Omega} \mathcal{J}(v_n(T)) + \omega(l_1, l_2, n_1, n_2).
\end{aligned}$$

Therefore, since  $\mathcal{J}(v_n) - J(v_n)v_n = -\bar{J}(v_n)$  and  $\bar{J}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$ , we obtain

$$F_2 \geq \int_{\Omega} \bar{J}(u_{0,n}) - \int_Q (\Phi_{\delta_1, \delta_2})_t \bar{J}(v_n) + \int_{\Omega} \mathcal{J}(v_n(T)) + \omega(l_1, l_2, n_1, n_2, m). \quad (5.26)$$

**Estimate of  $F_1$ .** Since  $m > \ell + 2k$ , there holds  $T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) = T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})$  on  $\text{supp} \overline{H_m}(v_n)$ . Hence we can write  $F_1 = L_1 + L_2$ , with

$$\begin{aligned}
L_1 &= - \int_Q \left( \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
L_2 &= - \int_Q \left( \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t \langle T_k(\overline{H_m}(v)) \rangle_{\nu}.
\end{aligned}$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$\begin{aligned}
L_2 &= \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\langle T_k(\overline{H_m}(v)) \rangle_{\nu})_t \\
&\quad + \int_{\Omega} \xi_{1,n_1}(0) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1}(0) \langle T_k(\overline{H_m}(v)) \rangle_{\nu}(0) \\
&= \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} + \omega(l_1, l_2, n_1, n_2).
\end{aligned} \quad (5.27)$$

We decompose  $L_1$  into  $L_1 = K_1 + K_2 + K_3$ , where

$$\begin{aligned}
K_1 &= - \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
K_2 &= \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) \\
K_3 &= - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) \left( [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}).
\end{aligned}$$

Then we check easily that

$$K_1 = \int_{\Omega} T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) (T) (v_n - \langle T_k(v) \rangle_{\nu}) (T) dx + \omega(l_1, l_2, n_1, n_2, m),$$

$$K_2 = \int_Q (\Phi_{\delta_1, \delta_2})_t T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) (v_n - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).$$

Next consider  $K_3$ . Here we use the function  $\mathcal{T}_k$  defined at (4.13). We set  $b = \overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}$ . Hence from (4.12),

$$\begin{aligned} ([T_{\ell+k}(b)]_{l_1})_t b(\cdot, t) &= \frac{b(\cdot, t)}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1) - T_{\ell+k}(b)(\cdot, t)) \\ &\leq \frac{1}{l_1} (\mathcal{T}_{\ell+k}(b)((\cdot, t + l_1)) - \mathcal{T}_{\ell+k}(b)(\cdot, t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t. \end{aligned}$$

Thus

$$\begin{aligned} \left( [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu}) &\leq \left( [\mathcal{T}_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_{\nu})]_{l_1} \right)_t \\ &= \left( [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t. \end{aligned}$$

Then

$$\begin{aligned} K_3 &\geq - \int_Q \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) \left( [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \right)_t \\ &= \int_Q (\xi_{1, n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} - \int_Q \xi_{1, n_1} (\Phi_{\delta_1, \delta_2})_t [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1} \\ &\quad + \int_{\Omega} \xi_{1, n_1}(0) [\mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu})]_{l_1}(0) \\ &= - \int_{\Omega} \mathcal{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) \\ &\quad + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0, n} - z_{\nu}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

We find by addition, since  $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \overline{T}_{\ell+k}(r)$  for any  $r \in \mathbb{R}$ ,

$$\begin{aligned} L_1 &\geq \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0, n} - z_{\nu}) + \int_{\Omega} \overline{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t \overline{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \tag{5.28}$$

We deduce from (5.28), (5.27), (5.26),

$$\begin{aligned}
A_3 &\geq \int_{\Omega} \bar{\mathcal{J}}(u_{0,n}) + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} \\
&+ \int_{\Omega} \bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) + \int_{\Omega} \mathcal{J}(v_n(T)) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v_n)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned} \tag{5.29}$$

Next we add (5.25) and (5.29). Note that  $\mathcal{J}(v_n(T)) - J(v_n(T))v_n(T) = -\bar{\mathcal{J}}(v_n(T))$ , and also  $\mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) - T_{\ell+k}(u_{0,n} - z_{\nu})(z_{\nu} - u_{0,n}) = -\bar{T}_{\ell+k}(u_{0,n} - z_{\nu})$ . Then we find

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_{0,n}) - \bar{T}_{\ell+k}(u_{0,n} - z_{\nu})) + \int_{\Omega} (\bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \bar{\mathcal{J}}(v_n(T))) \\
&+ \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v_n)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v_n - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned}$$

Notice that  $\bar{T}_{\ell+k}(r-s) - \bar{\mathcal{J}}(r) \geq 0$  for any  $r, s \in \mathbb{R}$  such that  $|s| \leq k$ ; thus

$$\int_{\Omega} (\bar{T}_{\ell+k}(v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \bar{\mathcal{J}}(v_n(T))) \geq 0.$$

And  $\{u_{0,n}\}$  converges to  $u_0$  in  $L^1(\Omega)$  and  $\{v_n\}$  converges to  $v$  in  $L^1(Q)$  from Proposition 4.10. Thus we obtain

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v)) \\
&+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) (T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m, n).
\end{aligned}$$

Moreover  $T_{\ell+k}(r-s)(T_k(r) - s) \geq 0$  for any  $r, s \in \mathbb{R}$  such that  $|s| \leq k$ , hence

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - \langle T_k(v) \rangle_{\nu}) - \bar{\mathcal{J}}(v)) \\
&+ \omega(l_1, l_2, n_1, n_2, m, n).
\end{aligned}$$

As  $\nu \rightarrow \infty$ ,  $\{z_{\nu}\}$  converges to  $T_k(u_0)$ , *a.e.* in  $\Omega$ , thus we get

$$\begin{aligned}
A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{\mathcal{J}}(u_0) - \bar{T}_{\ell+k}(u_0 - T_k(u_0))) + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(v - T_k(v)) - \bar{\mathcal{J}}(v)) \\
&+ \omega(l_1, l_2, n_1, n_2, m, n, \nu).
\end{aligned}$$



Finally  $|\overline{T}_{\ell+k}(r-T_k(r)) - \overline{J}(r)| \leq 2k|r|\chi_{\{|r|\geq\ell\}}$  for any  $r \in \mathbb{R}$ , thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain  $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$  which implies (5.8), since  $I_2$  does not depend on  $l_1, l_2, n_1, n_2, m, \ell$ .  $\blacksquare$

Next we conclude the proof of Theorem 2.1:

**Lemma 5.6** *The function  $u$  is a  $R$ -solution of (1.1).*

**Proof.** (i) First show that  $u$  satisfies (4.2). Here we proceed as in [49]. Let  $\varphi \in X \cap L^\infty(Q)$  such  $\varphi_t \in X' + L^1(Q)$ ,  $\varphi(\cdot, T) = 0$ , and  $S \in W^{2,\infty}(\mathbb{R})$ , such that  $S'$  has compact support on  $\mathbb{R}$ ,  $S(0) = 0$ . Let  $M > 0$  such that  $\text{supp} S' \subset [-M, M]$ . Taking successively  $(\varphi, S)$  and  $(\varphi\psi_\delta^\pm, S)$  as test functions in (4.2) applied to  $u_n$ , we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \quad A_{2,\delta,\pm} + A_{3,\delta,\pm} + A_{4,\delta,\pm} = A_{5,\delta,\pm} + A_{6,\delta,\pm} + A_{7,\delta,\pm},$$

where

$$\begin{aligned} A_1 &= - \int_{\Omega} \varphi(0)S(u_{0,n}), \quad A_2 = - \int_Q \varphi_t S(v_n), \quad A_{2,\delta,\pm} = - \int_Q (\varphi\psi_\delta^\pm)_t S(v_n), \\ A_3 &= \int_Q S'(v_n)A(x, t, \nabla u_n) \cdot \nabla \varphi, \quad A_{3,\delta,\pm} = \int_Q S'(v_n)A(x, t, \nabla u_n) \cdot \nabla (\varphi\psi_\delta^\pm), \end{aligned}$$

$$A_4 = \int_Q S''(v_n)\varphi A(x, t, \nabla u_n) \cdot \nabla v_n, \quad A_{4,\delta,\pm} = \int_Q S''(v_n)\varphi\psi_\delta^\pm A(x, t, \nabla u_n) \cdot \nabla v_n,$$

$$A_5 = \int_Q S'(v_n)\varphi d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q S'(v_n)\varphi d\rho_{n,0}, \quad A_7 = - \int_Q S'(v_n)\varphi d\eta_{n,0},$$

$$A_{5,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\widehat{\lambda}_{n,0}, \quad A_{6,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7,\delta,\pm} = - \int_Q S'(v_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since  $\{u_{0,n}\}$  converges to  $u_0$  in  $L^1(\Omega)$ , and  $\{S(v_n)\}$  converges to  $S(v)$  strongly in  $X$  and weak\* in  $L^\infty(Q)$ , there holds, from (5.2),

$$A_1 = - \int_{\Omega} \varphi(0)S(u_0) + \omega(n), \quad A_2 = - \int_Q \varphi_t S(v) + \omega(n), \quad A_{2,\delta,\psi_\delta^\pm} = \omega(n, \delta).$$

Moreover  $T_M(v_n)$  converges to  $T_M(v)$ , then  $T_M(v_n) + h_n$  converges to  $T_k(v) + h$  strongly in  $X$ , thus

$$\begin{aligned} A_3 &= \int_Q S'(v_n)A(x, t, \nabla (T_M(v_n) + h_n)) \cdot \nabla \varphi \\ &= \int_Q S'(v)A(x, t, \nabla (T_M(v) + h)) \cdot \nabla \varphi + \omega(n) \\ &= \int_Q S'(v)A(x, t, \nabla u) \cdot \nabla \varphi + \omega(n); \end{aligned}$$

and

$$\begin{aligned}
A_4 &= \int_Q S''(v_n) \varphi A(x, t, \nabla (T_M(v_n) + h_n)) \cdot \nabla T_M(v_n) \\
&= \int_Q S''(v) \varphi A(x, t, \nabla (T_M(v) + h)) \cdot \nabla T_M(v) + \omega(n) \\
&= \int_Q S''(v) \varphi A(x, t, \nabla u) \cdot \nabla v + \omega(n).
\end{aligned}$$

In the same way, since  $\psi_\delta^\pm$  converges to 0 in  $X$ ,

$$\begin{aligned}
A_{3,\delta,\pm} &= \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla (\varphi \psi_\delta^\pm) + \omega(n) = \omega(n, \delta), \\
A_{4,\delta,\pm} &= \int_Q S''(v) \varphi \psi_\delta^\pm A(x, t, \nabla u) \cdot \nabla v + \omega(n) = \omega(n, \delta).
\end{aligned}$$

And  $\{g_n\}$  converges strongly in  $(L^{p'}(\Omega))^N$ , thus

$$\begin{aligned}
A_5 &= \int_Q S'(v_n) \varphi f_n + \int_Q S'(v_n) g_n \cdot \nabla \varphi + \int_Q S''(v_n) \varphi g_n \cdot \nabla T_M(v_n) \\
&= \int_Q S'(v) \varphi f + \int_Q S'(v) g \cdot \nabla \varphi + \int_Q S''(v) \varphi g \cdot \nabla T_M(v) + \omega(n) \\
&= \int_Q S'(v) \varphi d\widehat{\mu}_0 + \omega(n).
\end{aligned}$$

and  $A_{5,\delta,\pm} = \int_Q S'(v) \varphi \psi_\delta^\pm d\widehat{\lambda}_{n,0} + \omega(n) = \omega(n, \delta)$ . Then  $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n, \delta)$ . From (5.2) we verify that  $A_{7,\delta,+} = \omega(n, \delta)$  and  $A_{6,\delta,-} = \omega(n, \delta)$ . Moreover, from (5.6) and (5.2), we find

$$|A_6 - A_{6,\delta,+}| \leq \int_Q |S'(v_n) \varphi| (1 - \psi_\delta^+) d\rho_{n,0} \leq \|S\|_{W^{2,\infty}(\mathbb{R})} \|\varphi\|_{L^\infty(Q)} \int_Q (1 - \psi_\delta^+) d\rho_n = \omega(n, \delta).$$

Similarly we also have  $|A_7 - A_{7,\delta,-}| \leq \omega(n, \delta)$ . Hence  $A_6 = \omega(n)$  and  $A_7 = \omega(n)$ . Therefore, we finally obtain (4.2):

$$-\int_\Omega \varphi(0) S(u_0) - \int_Q \varphi_t S(v) + \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(v) \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q S'(v) \varphi d\widehat{\mu}_0. \tag{5.30}$$

(ii) Next, we prove (4.3) and (4.4). We take  $\varphi \in C_c^\infty(Q)$  and take  $((1 - \psi_\delta^-)\varphi, \overline{H_m})$  as test functions in (5.30), with  $\overline{H_m}$  as in (4.14). We can write  $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$ , where

$$\begin{aligned} D_{1,m} &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t \overline{H_m}(v), & D_{2,m} &= \int_Q H_m(v) A(x, t, \nabla u) \cdot \nabla ((1 - \psi_\delta^-)\varphi), \\ D_{3,m} &= \int_Q H_m(v) (1 - \psi_\delta^-) \varphi d\widehat{\mu}_0, & D_{4,m} &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla v, \\ D_{5,m} &= - \frac{1}{m} \int_{-2m \leq v \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla v. \end{aligned} \quad (5.31)$$

Taking the same test functions in (4.2) applied to  $u_n$ , there holds  $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$ , where

$$\begin{aligned} D_{1,m}^n &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t \overline{H_m}(v_n), & D_{2,m}^n &= \int_Q H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-)\varphi), \\ D_{3,m}^n &= \int_Q H_m(v_n) (1 - \psi_\delta^-) \varphi d(\widehat{\lambda}_{n,0} + \rho_{n,0} - \eta_{n,0}), & D_{4,m}^n &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla v_n, \\ D_{5,m}^n &= - \frac{1}{m} \int_{-2m \leq v_n \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla v_n \end{aligned} \quad (5.32)$$

In (5.32), we go to the limit as  $m \rightarrow \infty$ . Since  $\{\overline{H_m}(v_n)\}$  converges to  $v_n$  and  $\{H_m(v_n)\}$  converges to 1, *a.e.* in  $Q$ , and  $\{\nabla H_m(v_n)\}$  converges to 0, weakly in  $(L^p(Q))^N$ , we obtain the relation  $D_1^n + D_2^n = D_3^n + D^n$ , where

$$\begin{aligned} D_1^n &= - \int_Q ((1 - \psi_\delta^-)\varphi)_t v_n, & D_2^n &= \int_Q A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-)\varphi), & D_3^n &= \int_Q (1 - \psi_\delta^-) \varphi d\widehat{\lambda}_{n,0} \\ D^n &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1 - \psi_\delta^-) \varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\ &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_n - \eta_n). \end{aligned}$$

Clearly,  $D_{i,m} - D_i^n = \omega(n, m)$  for  $i = 1, 2, 3$ . From Lemma (5.3) and (5.2)-(5.4), we obtain  $D_{5,m} = \omega(n, m, \delta)$ , and

$$\frac{1}{m} \int_{\{m \leq v < 2m\}} \psi_\delta^- \varphi A(x, t, \nabla u) \cdot \nabla v = \omega(n, m, \delta),$$

thus,

$$D_{4,m} = \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v + \omega(n, m, \delta).$$

Since  $\left| \int_Q (1 - \psi_\delta^-) \varphi d\eta_n \right| \leq \|\varphi\|_{L^\infty} \int_Q (1 - \psi_\delta^-) d\eta_n$ , it follows that  $\int_Q (1 - \psi_\delta^-) \varphi d\eta_n = \omega(n, m, \delta)$  from (5.4). And  $\left| \int_Q \psi_\delta^- \varphi d\rho_n \right| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$ , thus, from (5.2),  $\int_Q (1 - \psi_\delta^-) \varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$ . Then  $D^n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$ . Therefore by subtraction, we get

$$\frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta),$$

hence

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+, \quad (5.33)$$

which proves (4.3) when  $\varphi \in C_c^\infty(Q)$ . Next assume only  $\varphi \in C^\infty(\overline{Q})$ . Then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v \\ &= \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi \psi_\delta^+ A(x, t, \nabla u) \cdot \nabla v + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v \\ &= \int_Q \varphi \psi_\delta^+ d\mu_s^+ + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi d\mu_s^+ + D, \end{aligned}$$

where,

$$D = \int_Q \varphi (1 - \psi_\delta^+) d\mu_s^+ + \lim_{n \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla v = \omega(\delta).$$

Therefore, (5.33) still holds for  $\varphi \in C^\infty(\overline{Q})$ , and we deduce (4.3) by density, and similarly, (4.4). This completes the proof of Theorem 2.1.  $\blacksquare$

As a consequence of Theorem 2.1, we get the following:

**Corollary 5.7** *Let  $u_0 \in L^1(\Omega)$  and  $\mu \in \mathcal{M}_b(Q)$ . Then there exists a R-solution  $u$  to the problem 1.1 with data  $(\mu, u_0)$ . Furthermore, if  $v_0 \in L^1(\Omega)$  and  $\omega \in \mathcal{M}_b(Q)$  such that  $u_0 \leq v_0$  and  $\mu \leq \omega$ , then one can find R-solution  $v$  to the problem 1.1 with data  $(\omega, v_0)$  such that  $u \leq v$ .*

*In particular, if  $a \equiv 0$  in (1.2), then  $u$  satisfies (4.21) and  $\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq M$  with  $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$ .*

## 6 Equations with perturbation terms

Let  $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfying (1.2), (1.3) with  $a \equiv 0$ . Let  $\mathcal{G} : \Omega \times (0, T) \times \mathbb{R} \mapsto \mathbb{R}$  be a Caratheodory function. If  $U$  is a function defined in  $Q$  we define the function  $\mathcal{G}(U)$  in  $Q$  by

$$\mathcal{G}(U)(x, t) = \mathcal{G}(x, t, U(x, t)) \quad \text{for a.e. } (x, t) \in Q.$$

We consider the problem (1.5):

$$\begin{cases} u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where  $\mu \in \mathcal{M}_b(Q)$ ,  $u_0 \in L^1(\Omega)$ . We say that  $u$  is a R-solution of problem (1.5) if  $\mathcal{G}(u) \in L^1(Q)$  and  $u$  is a R-solution of (1.1) with data  $(\mu - \mathcal{G}(u), u_0)$ .

## 6.1 Subcritical type results

For proving Theorem 2.2, we begin by an integration Lemma:

**Lemma 6.1** *Let  $G$  satisfying (2.3). If a measurable function  $V$  in  $Q$  satisfies*

$$\operatorname{meas} \{|V| \geq t\} \leq Mt^{-p_c}, \quad \forall t \geq 1,$$

for some  $M > 0$ , then for any  $L > 1$ ,

$$\int_{\{|V| \geq L\}} G(|V|) \leq p_c M \int_L^\infty G(s) s^{-1-p_c} ds. \quad (6.1)$$

**Proof.** Indeed, setting  $G_L(s) = \chi_{[L, \infty)}(s)G(s)$ , we have

$$\int_{\{|V| \geq L\}} G(|V|) dx dt = \int_Q G_L(|V|) dx dt \leq \int_0^\infty G_L(|V|^*(s)) ds$$

where  $|V|^*$  is and the rearrangement of  $|V|$ , defined by

$$|V|^*(s) = \inf\{a > 0 : \operatorname{meas} \{|V| > a\} \leq s\}, \quad \forall s \geq 0.$$

From the assumption, we get  $|V|^*(s) \leq \sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)$ . Thus, for any  $L > 1$ ,

$$\int_{\{|V| \geq L\}} G(|V|) dx dt \leq \int_0^\infty G_L\left(\sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)\right) ds = p_c M \int_L^\infty G(s) s^{-1-p_c} ds,$$

which implies (6.1). ■

**Proof of Theorem 2.2. Proof of (i)** Let  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$ , with  $\mu_0 \in \mathcal{M}_0(Q)$ ,  $\mu_s \in \mathcal{M}_s(Q)$ , and  $u_0 \in L^1(\Omega)$ . Then  $\mu_0^+, \mu_0^-$  can be decomposed as  $\mu_0^+ = (f_1, g_1, h_1)$ ,  $\mu_0^- = (f_2, g_2, h_2)$ . Let  $\mu_{n,s,i} \in C_c^\infty(Q)$ ,  $\mu_{n,s,i} \geq 0$ , converging respectively to  $\mu_s^+, \mu_s^-$  in the narrow topology. By Lemma 3.1, we can find  $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$  which strongly converge to  $f_i, g_i, h_i$  in  $L^1(Q)$ ,  $(L^{p'}(Q))^N$  and

$X$  respectively,  $i = 1, 2$ , such that  $\mu_0^+ = (f_1, g_1, h_1)$ ,  $\mu_0^- = (f_2, g_2, h_2)$ , and  $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ , converging respectively to  $\mu_0^+$ ,  $\mu_0^-$  in the narrow topology. Furthermore, if we set

$$\mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2},$$

then  $|\mu_n|(Q) \leq |\mu|(Q)$ . Consider a sequence  $\{u_{0,n}\} \subset C_c^\infty(\Omega)$  which strongly converges to  $u_0$  in  $L^1(\Omega)$  and satisfies  $\|u_{0,n}\|_{1,\Omega} \leq \|u_0\|_{1,\Omega}$ .

Let  $u_n$  be a solution of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x, t, \nabla u_n)) + \mathcal{G}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases}$$

We can choose  $\varphi = \varepsilon^{-1}T_\varepsilon(u_n)$  as test function of above problem. Then we find

$$\int_Q (\varepsilon^{-1}\overline{T_\varepsilon}(u_n))_t + \int_Q \varepsilon^{-1}A(x, t, \nabla T_\varepsilon(u_n)) \cdot \nabla T_\varepsilon(u_n) + \int_Q \mathcal{G}(x, t, u_n)\varepsilon^{-1}T_\varepsilon(u_n) = \int_Q \varepsilon^{-1}T_\varepsilon(u_n)d\mu_n.$$

Since

$$\int_Q (\varepsilon^{-1}\overline{T_\varepsilon}(u_n))_t = \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_n(T))dx - \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_{0,n})dx \geq -\|u_{0,n}\|_{L^1(\Omega)},$$

there holds

$$\int_Q \mathcal{G}(x, t, u_n)\varepsilon^{-1}T_\varepsilon(u_n) \leq |\mu_n|(Q) + \|u_{0,n}\|_{L^1(\Omega)} \leq |\mu|(Q) + \|u_0\|_{1,\Omega}.$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\int_Q |\mathcal{G}(x, t, u_n)| \leq |\mu|(Q) + \|u_0\|_{1,\Omega}. \quad (6.2)$$

Next apply Proposition 4.8 and Remark 4.9 to  $u_n$  with initial data  $u_{0,n}$  and measure data  $\mu_n - \mathcal{G}(u_n) \in L^1(Q)$ , we get

$$\operatorname{meas} \{|u_n| \geq s\} \leq C(|\mu|(Q) + \|u_0\|_{L^1(\Omega)})^{\frac{p+N}{N}} s^{-pc}, \quad \forall s > 0, \forall n \in \mathbb{N},$$

for some  $C = C(N, p, c_1, c_2)$ . Since  $|\mathcal{G}(x, t, u_n)| \leq G(|u_n|)$ , we deduce from (6.1) that  $\{|\mathcal{G}(u_n)|\}$  is equi-integrable. Then, thanks to Proposition 4.10, up to a subsequence,  $\{u_n\}$  converges to some function  $u$ , *a.e.* in  $Q$ , and  $\{\mathcal{G}(u_n)\}$  converges to  $\mathcal{G}(u)$  in  $L^1(Q)$ . Therefore, by Theorem 2.1,  $u$  is a R-solution of (2.4).

**Proof of (ii).** Let  $\{u_n\}_{n \geq 1}$  be defined by induction as nonnegative R-solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x, t, \nabla u_1)) = \mu & \text{in } Q, \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ u_1(0) = u_0 & \text{in } \Omega, \end{cases} \quad \begin{cases} (u_{n+1})_t - \operatorname{div}(A(x, t, \nabla u_{n+1})) = \mu - \lambda\mathcal{G}(u_n) & \text{in } Q, \\ u_{n+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{n+1}(0) = u_0 & \text{in } \Omega, \end{cases}$$

Thanks to Corollary 5.7 we can assume that  $\{u_n\}$  is nondecreasing and satisfies for any  $s > 0$  and  $n \in \mathbb{N}$

$$\text{meas} \{|u_n| \geq s\} \leq C_1 K_n s^{-p_c}, \quad (6.3)$$

where  $C_1$  does not depend on  $s, n$ , and

$$K_1 = (\|u_0\|_{1,\Omega} + |\mu|(Q))^{\frac{p+N}{N}},$$

$$K_{n+1} = (\|u_0\|_{1,\Omega} + |\mu|(Q) + \lambda \|\mathcal{G}(u_n)\|_{1,Q})^{\frac{p+N}{N}},$$

for any  $n \geq 1$ . Take  $\varepsilon = \lambda + |\mu|(Q) + \|u_0\|_{L^1(\Omega)} \leq 1$ . Denoting by  $C_i$  some constants independent on  $n, \varepsilon$ , there holds  $K_1 \leq C_2 \varepsilon$ , and for  $n \geq 1$ ,

$$K_{n+1} \leq C_3 \varepsilon (\|\mathcal{G}(u_n)\|_{1,Q}^{1+\frac{p}{N}} + 1).$$

From (6.1) and (6.3), we find

$$\|\mathcal{G}(u_n)\|_{L^1(Q)} \leq |Q| G(2) + \int_{\{|u_n| \geq 2\}} G(|u_n|) dx dt \leq |Q| G(2) + C_4 K_n \int_2^\infty G(s) s^{-1-p_c} ds.$$

Thus,  $K_{n+1} \leq C_5 \varepsilon (K_n^{1+\frac{p}{N}} + 1)$ . Therefore, if  $\varepsilon$  is small enough,  $\{K_n\}$  is bounded. Then, again from (6.1) and the relation  $|\mathcal{G}(x, t, u_n)| \leq G(|u_n|)$  we verify that  $\{\mathcal{G}(u_n)\}$  converges. Then by Theorem 2.1, up to a subsequence,  $\{u_n\}$  converges to a R-solution  $u$  of (2.5).  $\blacksquare$

## 6.2 General case with absorption terms

In the sequel we assume that  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  does not depend on  $t$ . We recall a result obtained in [53],[17] in the elliptic case:

**Theorem 6.2** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ . Let  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6),(1.7). Then there exists a constant  $\kappa$  depending on  $N, p, c_3, c_4$  such that, if  $\omega \in \mathcal{M}_b(\Omega)$  and  $u$  is a R-solution of problem*

$$\begin{cases} -\text{div}(A(x, \nabla u)) = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there holds

$$-\kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^-] \leq u \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^+]. \quad (6.4)$$

Next we give a general result in case of absorption terms:

**Theorem 6.3** *Let  $p < N$ ,  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6),(1.7), and  $\mathcal{G} : Q \times \mathbb{R} \mapsto \mathbb{R}$  be a Caratheodory function such that the map  $s \mapsto \mathcal{G}(x, t, s)$  is nondecreasing and odd, for a.e.  $(x, t)$  in  $Q$ .*

Let  $\mu_1, \mu_2 \in \mathcal{M}_b^+(Q)$  such that there exist  $\omega_n \in \mathcal{M}_b^+(\Omega)$  and nondecreasing sequences  $\{\mu_{1,n}\}, \{\mu_{2,n}\}$  in  $\mathcal{M}_b^+(Q)$  with compact support in  $Q$ , converging to  $\mu_1, \mu_2$ , respectively in the narrow topology, and

$$\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi_{(0,T)}, \quad \mathcal{G}((n + \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega_n])) \in L^1(Q),$$

where the constant  $c$  is given at Theorem 6.2. Let  $u_0 \in L^1(\Omega)$ , and  $\mu = \mu_1 - \mu_2$ . Then there exists a  $R$ -solution  $u$  of problem (1.5).

Moreover if  $u_0 \in L^\infty(\Omega)$ , and  $\omega_n \leq \gamma$  for any  $n \in \mathbb{N}$ , for some  $\gamma \in \mathcal{M}_b^+(\Omega)$ , then a.e. in  $Q$ ,

$$|u(x, t)| \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) + \|u_0\|_{\infty, \Omega}. \quad (6.5)$$

For proving this result, we need two Lemmas:

**Lemma 6.4** Let  $\mathcal{G}$  satisfy the assumptions of Theorem 6.3 and  $\mathcal{G} \in L^\infty(Q \times \mathbb{R})$ . For  $i = 1, 2$ , let  $u_{0,i} \in L^\infty(\Omega)$  be nonnegative, and  $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in \mathcal{M}_b^+(Q)$  with compact support in  $Q$ ,  $\gamma \in \mathcal{M}_b^+(\Omega)$  with compact support in  $\Omega$  such that  $\lambda_i \leq \gamma \otimes \chi_{(0,T)}$ . Let  $\lambda_{i,0} = (f_i, g_i, h_i)$  be a decomposition of  $\lambda_{i,0}$  into functions with compact support in  $Q$ . Then, there exist  $R$ -solutions  $u, u_1, u_2$ , to problems

$$u_t - \text{div}(A(x, \nabla u)) + \mathcal{G}(u) = \lambda_1 - \lambda_2 \quad \text{in } Q, \quad u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(0) = u_{0,1} - u_{0,2}, \quad (6.6)$$

$$(u_i)_t - \text{div}(A(x, \nabla u_i)) + \mathcal{G}(u_i) = \lambda_i \quad \text{in } Q, \quad u_i = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_i(0) = u_{0,i}, \quad (6.7)$$

relative to decompositions  $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}), (f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$ , such that a.e. in  $Q$ ,

$$-\|u_{0,2}\|_{\infty, \Omega} - \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) \leq -u_2(x, t) \leq u(x, t) \leq u_1(x, t) \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega} \gamma(x) + \|u_{0,1}\|_{\infty, \Omega}, \quad (6.8)$$

and

$$\int_Q |\mathcal{G}(u)| \leq \sum_{i=1,2} (\lambda_i(Q) + \|u_{0,i}\|_{L^1(\Omega)}), \quad \text{and} \quad \int_Q \mathcal{G}(u_i) \leq \lambda_i(Q) + \|u_{0,i}\|_{L^1(\Omega)}, \quad i = 1, 2. \quad (6.9)$$

Furthermore, assume that  $\mathcal{H}, \mathcal{K}$  have the same properties as  $\mathcal{G}$ , and  $\mathcal{H}(x, t, s) \leq \mathcal{G}(x, t, s) \leq \mathcal{K}(x, t, s)$  for any  $s \in (0, +\infty)$  and a.e. in  $Q$ . Then, one can find solutions  $u_i(\mathcal{H}), u_i(\mathcal{K})$ , corresponding to  $\mathcal{H}, \mathcal{K}$  with data  $\lambda_i$ , such that  $u_i(\mathcal{H}) \geq u_i \geq u_i(\mathcal{K})$ ,  $i = 1, 2$ .

Assume that  $\omega_i, \theta_i$  have the same properties as  $\lambda_i$  and  $\omega_i \leq \lambda_i \leq \theta_i$ ,  $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$ ,  $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$ . Then one can find solutions  $u_i(\omega_i), u_i(\theta_i)$ , corresponding to  $(\omega_i, u_{0,i,2}), (\theta_i, u_{0,i,1})$ , such that  $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$ .

**Proof.** Let  $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$  be sequences of mollifiers in  $\mathbb{R}$  and  $\mathbb{R}^N$ , and  $\varphi_n = \varphi_{1,n} \varphi_{2,n}$ . Set  $\gamma_n = \varphi_{2,n} * \gamma$ , and for  $i = 1, 2$ ,  $u_{0,i,n} = \varphi_{2,n} * u_{0,i}$ ,

$$\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \text{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},$$

where  $f_{i,n} = \varphi_n * f_i$ ,  $g_{i,n} = \varphi_n * g_i$ ,  $h_{i,n} = \varphi_n * h_i$ ,  $\lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$ , and

$$\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \text{div}(g_n) + (h_n)_t + \lambda_{s,n},$$



where  $f_n = f_{1,n} - f_{2,n}$ ,  $g_n = g_{1,n} - g_{2,n}$ ,  $h_n = h_{1,n} - h_{2,n}$ ,  $\lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$ . Then for  $n$  large enough,  $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q)$ ,  $\gamma_n \in C_c^\infty(\Omega)$ . Thus there exist unique solutions  $u_n, u_{i,n}, v_{i,n}$ ,  $i = 1, 2$ , of problems

$$\begin{aligned} (u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) &= \lambda_{1,n} - \lambda_{2,n} \quad \text{in } Q, & u_n &= 0 \quad \text{on } \partial\Omega \times (0, T), & u_n(0) &= u_{0,1,n} - u_{0,2,n} \quad \text{in } \Omega, \\ (u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) &= \lambda_{i,n} \quad \text{in } Q, & u_{i,n} &= 0 \quad \text{on } \partial\Omega \times (0, T), & u_{i,n}(0) &= u_{0,i,n} \quad \text{in } \Omega, \\ -\operatorname{div}(A(x, \nabla w_n)) &= \gamma_n \quad \text{in } \Omega, & w_n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

such that

$$-||u_{0,2}||_{\infty, \Omega} - w_n(x) \leq -u_{2,n}(x, t) \leq u_n(x, t) \leq u_{1,n}(x, t) \leq w_n(x) + ||u_{0,1}||_{\infty, \Omega}, \quad a.e. \text{ in } Q.$$

Moreover, as in the Proof of Theorem 2.2, (i), there holds

$$\int_Q |\mathcal{G}(u_n)| \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}), \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}, \quad i = 1, 2.$$

By Proposition 4.10, up to a common subsequence,  $\{u_n, u_{1,n}, u_{2,n}\}$  converge to some  $(u, u_1, u_2)$ , *a.e.* in  $Q$ . Since  $\mathcal{G}$  is bounded, in particular,  $\{\mathcal{G}(u_n)\}$  converges to  $\mathcal{G}(u)$  and  $\{\mathcal{G}(u_{i,n})\}$  converges to  $\mathcal{G}(u_i)$  in  $L^1(Q)$ . Thus, (6.9) is satisfied. Moreover  $\{\lambda_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$  and  $\{\lambda_n - \mathcal{G}(u_n), f_n - \mathcal{G}(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$  are approximations of  $(\lambda_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i})$  and  $(\lambda - \mathcal{G}(u), f - \mathcal{G}(u), g, h, \lambda_s, u_{0,1} - u_{0,2})$ , in the sense of Theorem 2.1. Thus, we can find (different) subsequences converging *a.e.* to  $u, u_1, u_2$ , R-solutions of (6.6) and (6.7). Furthermore, from [47, Corollary 3.4], up to a subsequence,  $\{w_n\}$  converges *a.e.* in  $Q$  to a R-solution

$$-\operatorname{div}(A(x, \nabla w)) = \gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

such that  $w \leq c\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma$  *a.e.* in  $\Omega$ . Hence, we get the inequality (6.8). The other conclusions follow in the same way.  $\blacksquare$

**Lemma 6.5** *Let  $\mathcal{G}$  satisfy the assumptions of Theorem 6.3. For  $i = 1, 2$ , let  $u_{0,i} \in L^\infty(\Omega)$  be nonnegative,  $\lambda_i \in \mathcal{M}_b^+(Q)$  with compact support in  $Q$ , and  $\gamma \in \mathcal{M}_b^+(\Omega)$  with compact support in  $\Omega$ , such that*

$$\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \quad \mathcal{G}((||u_{0,i}||_{\infty, \Omega} + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}(\Omega)} \gamma)) \in L^1(Q). \quad (6.10)$$

*Then, there exist R-solutions  $u, u_1, u_2$  of the problems (6.6) and (6.7), respectively relative to the decompositions  $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$ ,  $(f_i - \mathcal{G}(u_i), g_i, h_i)$ , satisfying (6.8) and (6.9).*

*Moreover, assume that  $\omega_i, \theta_i$  have the same properties as  $\lambda_i$  and  $\omega_i \leq \lambda_i \leq \theta_i$ ,  $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$ ,  $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$ . Then, one can find solutions  $u_i(\omega_i, u_{0,i,2})$ ,  $u_i(\theta_i, u_{0,i,1})$ , corresponding with  $(\omega_i, u_{0,i,2})$ ,  $(\theta_i, u_{0,i,1})$ , such that  $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$ .*

**Proof.** From Lemma 6.4 there exist R-solutions  $u_n, u_{i,n}$  to problems

$$(u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_n(0) = u_{0,1} - u_{0,2}$$

$$(u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i \quad \text{in } Q, \quad u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_{i,n}(0) = u_{0,i},$$

relative to the decompositions  $(f_1 - f_2 - T_n(\mathcal{G}(u_n)), g_1 - g_2, h_1 - h_2)$ ,  $(f_i - T_n(\mathcal{G}(u_{i,n})), g_i, h_i)$ ; and they satisfy

$$\begin{aligned} -\|u_{0,2}\|_{\infty, \Omega} - \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma(x) &\leq -u_{2,n}(x, t) \leq u_n(x, t) \\ &\leq u_{1,n}(x, t) \leq \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega} \gamma(x) + \|u_{0,1}\|_{\infty, \Omega}, \end{aligned} \quad (6.11)$$

$$\int_Q |T_n(\mathcal{G}(u_n))| \leq \sum_{i=1,2} (\lambda_i(Q) + \|u_{0,i}\|_{1,\Omega}), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) \leq \lambda_i(Q) + \|u_{0,i}\|_{1,\Omega}.$$

As in Lemma 6.4, up to a common subsequence,  $\{u_n, u_{1,n}, u_{2,n}\}$  converges *a.e.* in  $Q$  to  $\{u, u_1, u_2\}$  for which (6.8) is satisfied *a.e.* in  $Q$ . From (6.10), (6.11) and the dominated convergence Theorem, we deduce that  $\{T_n(\mathcal{G}(u_n))\}$  converges to  $\mathcal{G}(u)$  and  $\{T_n(\mathcal{G}(u_{i,n}))\}$  converges to  $\mathcal{G}(u_i)$  in  $L^1(Q)$ . Thus, from Theorem 2.1,  $u$  and  $u_i$  are respective R-solutions of (6.6) and (6.7) relative to the decompositions  $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$ ,  $(f_i - \mathcal{G}(u_i), g_i, h_i)$ , and (6.8) and (6.9) hold. The last statement follows from the same assertion in Lemma 6.4.  $\blacksquare$

**Proof of Theorem 6.3.** By Proposition 3.2, for  $i = 1, 2$ , there exist  $f_{i,n}, f_i \in L^1(Q)$ ,  $g_{i,n}, g_i \in (L^{p'}(Q))^N$  and  $h_{i,n}, h_i \in X$ ,  $\mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}_s^+(Q)$  such that

$$\mu_i = f_i - \operatorname{div} g_i + (h_i)_t + \mu_{i,s}, \quad \mu_{i,n} = f_{i,n} - \operatorname{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},$$

and  $\{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\}$  strongly converge to  $f_i, g_i, h_i$  in  $L^1(Q)$ ,  $(L^{p'}(Q))^N$  and  $X$  respectively, and  $\{\mu_{i,n}\}, \{\mu_{i,n,s}\}$  converge to  $\mu_i, \mu_{i,s}$  (strongly) in  $\mathcal{M}_b(Q)$ , and

$$\|f_{i,n}\|_{1,Q} + \|g_{i,n}\|_{p',Q} + \|h_{i,n}\|_X + \mu_{i,n,s}(\Omega) \leq 2\mu(Q).$$

By Lemma 6.5, there exist R-solutions  $u_n, u_{i,n}$  to problems

$$(u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_n(0) = T_n(u_0) \quad (6.12)$$

$$(u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \mu_{i,n} \quad \text{in } \Omega, \quad u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u_{i,n}(0) = T_n(u_0^\pm), \quad (6.13)$$

for  $i = 1, 2$ , relative to the decompositions  $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$ ,  $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$ , such that  $\{u_{i,n}\}$  is nonnegative and nondecreasing, and  $-u_{2,n} \leq u_n \leq u_{1,n}$ ; and

$$\int_Q |\mathcal{G}(u_n)| \leq \mu_1(Q) + \mu_2(Q) + \|u_0\|_{1,\Omega}, \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \mu_i(Q) + \|u_0\|_{1,\Omega}, \quad i = 1, 2. \quad (6.14)$$

As in the proof of Lemma 6.5, up to a common subsequence  $\{u_n, u_{1,n}, u_{2,n}\}$  converge *a.e.* in  $Q$  to  $\{u, u_1, u_2\}$ . Since  $\{\mathcal{G}(u_{i,n})\}$  is nondecreasing, and nonnegative, from the monotone convergence

Theorem and (6.14), we obtain that  $\{\mathcal{G}(u_{i,n})\}$  converges to  $\mathcal{G}(u_i)$  in  $L^1(Q)$ ,  $i = 1, 2$ . Finally,  $\{\mathcal{G}(u_n)\}$  converges to  $\mathcal{G}(u)$  in  $L^1(Q)$ , since  $|\mathcal{G}(u_n)| \leq \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n})$ . Thus, we can see that

$$\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_0^+) - T_n(u_0^-)\}$$

is an approximation of  $(\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)$ , in the sense of Theorem 2.1; and

$$\{\mu_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \mu_{i,s,n}, T_n(u_0^\pm)\}$$

is an approximation of  $(\mu_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \mu_{i,s}, u_0^\pm)$ . Therefore,  $u$  is a R-solution of (1.5), and (6.5) holds if  $u_0 \in L^\infty(\Omega)$  and  $\omega_n \leq \gamma$  for any  $n \in \mathbb{N}$  and some  $\gamma \in \mathcal{M}_b^+(\Omega)$ .  $\blacksquare$

As a consequence we prove Theorem 2.3. We use the following result of [17]:

**Proposition 6.6** ( see [17]) *Let  $q > p - 1$ ,  $\alpha \in \left(0, \frac{N(q+1-p)}{pq}\right)$ ,  $r > 0$  and  $\nu \in \mathcal{M}_b^+(\Omega)$ . If  $\nu$  does not charge the sets of  $C_{\alpha p, \frac{q}{q+1-p}}$ -capacity zero, there exists a nondecreasing sequence  $\{\nu_n\} \subset \mathcal{M}_b^+(\Omega)$  with compact support in  $\Omega$  which converges to  $\nu$  strongly in  $\mathcal{M}_b(\Omega)$  and such that  $\mathbf{W}_{\alpha,p}^r[\nu_n] \in L^q(\mathbb{R}^N)$ , for any  $n \in \mathbb{N}$ .*

**Proof of Theorem 2.3.** Let  $f \in L^1(Q)$ ,  $u_0 \in L^1(\Omega)$ , and  $\mu \in \mathcal{M}_b(Q)$  such that  $|\mu| \leq \omega \otimes F$ , where  $F \in L^1((0, T))$  and  $\omega$  does not charge the sets of  $C_{p, \frac{q}{q+1-p}}$ -capacity zero. From Proposition 6.6, there exists a nondecreasing sequence  $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$  with compact support in  $\Omega$  which converges to  $\omega$ , strongly in  $\mathcal{M}_b(\Omega)$ , such that  $\mathbf{W}_{1,p}^{2diam\Omega}[\omega_n] \in L^q(\mathbb{R}^N)$ . We can write

$$f + \mu = \mu_1 - \mu_2, \quad \mu_1 = f^+ + \mu^+, \quad \mu_2 = f^- + \mu^-, \quad (6.15)$$

and  $\mu^+, \mu^- \leq \omega \otimes F$ . We set

$$Q_n = \{(x, t) \in \Omega \times \left(\frac{1}{n}, T - \frac{1}{n}\right) : d(x, \partial\Omega) > \frac{1}{n}\}, \quad F_n = T_n(\chi_{(\frac{1}{n}T - \frac{1}{n})} F), \quad (6.16)$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \quad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}. \quad (6.17)$$

Then  $\{\mu_{1,n}\}, \{\mu_{2,n}\}$  are nondecreasing sequences with compact support in  $Q$ , and  $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$ , with  $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$  and  $(n + \kappa \mathbf{W}_{1,p}^{2diam\Omega}[\omega_n])^q \in L^1(Q)$ . Besides,  $\omega_n \otimes F_n$  converges to  $\omega \otimes F$  strongly in  $\mathcal{M}_b(Q)$ : indeed we easily check that

$$\|\omega_n \otimes F_n - \omega \otimes F\|_{\mathcal{M}_b(Q)} \leq \|F_n\|_{L^1((0,T))} \|\omega_n - \omega\|_{\mathcal{M}_b(\Omega)} + \|\omega\|_{\mathcal{M}_b(\Omega)} \|F_n - F\|_{L^1((0,T))}$$

Observe that for any measures  $\nu, \theta, \eta \in \mathcal{M}_b(Q)$ , there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leq |\theta - \eta|,$$

hence  $\{\mu_{1,n}\}, \{\mu_{2,n}\}$  converge to  $\mu_1, \mu_2$  respectively in  $\mathcal{M}_b(Q)$ . Therefore, the result follows from Theorem 6.3.  $\blacksquare$

**Remark 6.7** Our result improves the existence results of [50], where  $\mu \in \mathcal{M}_0(Q)$ . Indeed, let  $p_e = N(p-1)/(N-p)$  be the critical exponent for the elliptic problem

$$-\Delta_p w + |w|^{q-1} w = \omega \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

Notice that  $p_c < p_e$ , since  $p > p_1$ . If  $q \geq p_e$ , there exist measures  $\omega \in \mathcal{M}_b^+(\Omega)$  which do not charge the sets of  $C_{p, \frac{q}{q+1-p}}$ -capacity zero, such that  $\omega \notin \mathcal{M}_{0,e}(\Omega)$ . Then for any  $F \in L^1((0, T))$ ,  $F \geq 0, F \not\equiv 0$ , we have  $\omega \otimes F \notin \mathcal{M}_0(Q)$ .

**Remark 6.8** Let  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6), (1.7). Let  $\mathcal{G} : Q \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that the map  $s \mapsto \mathcal{G}(x, t, s)$  is nondecreasing and odd, for a.e.  $(x, t)$  in  $Q$ . Assume that  $\omega \in \mathcal{M}_{0,e}(\Omega)$ . Thus, we have  $\omega(\{x : W_{1,p}^{2\text{diam}(\Omega)}[\omega](x) = \infty\}) = 0$ . As in the proof of Theorem 2.3 with  $\omega_n = \chi_{W_{1,p}^{2\text{diam}\Omega}[\omega] \leq n} \omega$ , we get that (1.5) has a  $R$ -solution.

**Remark 6.9** As in [17], from Theorem 6.3, we can extend Theorem 2.3 given for  $\mathcal{G}(u) = |u|^{q-1} u$ , to the case of a function  $\mathcal{G}(x, t, \cdot)$ , odd for a.e.  $(x, t) \in Q$ , such that

$$|\mathcal{G}(x, t, u)| \leq G(|u|), \quad \int_1^\infty G(s) s^{-q-1} ds < \infty,$$

where  $G$  is a nondecreasing continuous, under the condition that  $\omega$  does not charge the sets of zero  $C_{p, \frac{q}{q-p+1}, 1}$ -capacity, where for any Borel set  $E \subset \mathbb{R}^N$ ,

$$C_{p, \frac{q}{q-p+1}, 1}(E) = \inf\{ \|\varphi\|_{L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N), \quad G_p * \varphi \geq \chi_E \}$$

where  $L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)$  is the Lorentz space of order  $(q/(q-p+1), 1)$ .

In case  $\mathcal{G}$  is of exponential type, we introduce the notion of maximal fractional operator, defined for any  $\eta \geq 0, R > 0, x_0 \in \mathbb{R}^N$  by

$$\mathbf{M}_{p,R}^\eta[\omega](x_0) = \sup_{t \in (0,R)} \frac{\omega(B(x_0, t))}{t^{N-p} h_\eta(t)}, \quad \text{where } h_\eta(t) = \inf((- \ln t)^{-\eta}, (\ln 2)^{-\eta}).$$

We obtain the following:

**Theorem 6.10** Let  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6), (1.7). Let  $p < N$  and  $\tau > 0, \beta > 1, \mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$ . Assume that  $|\mu| \leq \omega \otimes F$ , with  $\omega \in \mathcal{M}_b^+(\Omega)$ ,  $F \in L^1((0, T))$  be nonnegative. Assume that one of the following assumptions is satisfied:

(i)  $\|F\|_{L^\infty((0,T))} \leq 1$  and for some  $M_0 = M_0(N, p, \beta, \tau, c_3, c_4, \text{diam}\Omega)$ ,

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{p-1}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} < M_0, \tag{6.18}$$

(ii) there exists  $\beta_0 > \beta$  such that  $\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{p-1}{\beta_0}}[\omega] \in L^\infty(\mathbb{R}^N)$ .

Then there exists a  $R$ -solution to the problem

$$\begin{cases} u_t - \operatorname{div}(A(x, \nabla u)) + (e^{\tau|u|^\beta} - 1)\operatorname{sign}u = F + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

For the proof we use the following result of [17]:

**Proposition 6.11** (see [17], Theorem 2.4) *Suppose  $1 < p < N$ . Let  $\nu \in \mathcal{M}_b^+(\Omega)$ ,  $\beta > 1$ , and  $\delta_0 = ((12\beta)^{-1})^\beta p \ln 2$ . There exists  $C = C(N, p, \beta, \operatorname{diam}\Omega)$  such that, for any  $\delta \in (0, \delta_0)$ ,*

$$\int_{\Omega} \exp \left( \delta \frac{(\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\nu])^\beta}{\|\mathbf{M}_{p,2\operatorname{diam}\Omega}^{\frac{p-1}{\beta'}}[\nu]\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}}} \right) \leq \frac{C}{\delta_0 - \delta}.$$

**Proof of Theorem 6.10.** Let  $Q_n$  be defined at (6.16), and  $\omega_n = \omega\chi_{\Omega_n}$ , where  $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$ . We still consider  $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$  as in (6.15), (6.17).

Case 1: Assume that  $\|F\|_{L^\infty((0,T))} \leq 1$  and (6.18) holds. We have  $\mu_{1,n}, \mu_{2,n} \leq n\chi_\Omega + \omega$ . For any  $\varepsilon > 0$ , there exists  $c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, \operatorname{diam}\Omega) > 0$  such that

$$(n + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta \leq c_\varepsilon n^{\frac{\beta p}{p-1}} + (1 + \varepsilon)\kappa^\beta (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta$$

a.e. in  $\Omega$ . Thus,

$$\exp \left( \tau(n + \kappa \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta \right) \leq \exp \left( \tau c_\varepsilon n^{\frac{\beta p}{p-1}} \right) \exp \left( \tau(1 + \varepsilon)\kappa^\beta (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta \right)$$

If (6.18) holds with  $M_0 = (\delta_0/\tau\kappa^\beta)^{(p-1)/\beta}$  then we can chose  $\varepsilon$  such that

$$\tau(1 + \varepsilon)\kappa^\beta \|\mathbf{M}_{p,2\operatorname{diam}\Omega}^{\frac{p-1}{\beta'}}[\nu]\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}} < \delta_0.$$

From Proposition 6.11, we get  $\exp(\tau(1 + \varepsilon)\kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^\beta \in L^1(\Omega)$ , which implies  $\exp(\tau(n + \kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n\chi_\Omega + \omega])^\beta) \in L^1(\Omega)$  for all  $n$ . We conclude from Theorem 6.3.

Case 2: Assume that there exists  $\varepsilon > 0$  such that  $\mathbf{M}_{p,2\operatorname{diam}\Omega}^{(p-1)/(\beta+\varepsilon)' }[\omega] \in L^\infty(\mathbb{R}^N)$ . Now we use the inequality  $\mu_{1,n}, \mu_{2,n} \leq n(\chi_\Omega + \omega)$ . For any  $\varepsilon > 0$  and  $n \in \mathbb{N}$  there exists  $c_{\varepsilon,n} > 0$  such that

$$(n + \kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta \leq c_{\varepsilon,n} + \varepsilon (\mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[\omega])^{\beta_0}$$

Thus, from Proposition 6.11 we get  $\exp(\tau(n + \kappa^\beta \mathbf{W}_{1,p}^{2\operatorname{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta) \in L^1(\Omega)$  for all  $n$ . We conclude from Theorem 6.3.  $\blacksquare$

### 6.3 Equations with source term

As a consequence of Theorem 6.3, we get a first result for problem (1.1):

**Corollary 6.12** *Let  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6)(1.7). Let  $u_0 \in L^\infty(\Omega)$ , and  $\mu \in \mathcal{M}_b(Q)$  such that  $|\mu| \leq \omega \otimes \chi_{(0,T)}$  for some  $\omega \in \mathcal{M}_b^+(\Omega)$ . Then there exist a  $R$ -solution  $u$  of (1.1), such that*

$$|u(x, t)| \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega](x) + \|u_0\|_{\infty, \Omega}, \quad \text{for a.e. } (x, t) \in Q, \quad (6.19)$$

where  $\kappa$  is defined at Theorem 6.2.

**Proof.** Let  $\{\phi_n\}$  be a nonnegative, nondecreasing sequence in  $C_c^\infty(Q)$  which converges to 1, a.e. in  $Q$ . Since  $\{\phi_n \mu^+\}, \{\phi_n \mu^-\}$  are nondecreasing sequences, the result follows from Theorem 6.3. ■

Our proof of Theorem 2.4 is based on a property of Wölf potentials:

**Theorem 6.13** (see [53]) *Let  $q > p - 1$ ,  $0 < p < N$ ,  $\omega \in \mathcal{M}_b^+(\Omega)$ . If for some  $\lambda > 0$ ,*

$$\omega(E) \leq \lambda C_{p, \frac{q}{p-q+1}}(E) \quad \text{for any compact set } E \subset \mathbb{R}^N, \quad (6.20)$$

then  $(\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q \in L^1(\Omega)$ , and there exists  $M = M(N, p, q, \text{diam}(\Omega))$  such that, a.e. in  $\Omega$ ,

$$\mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[ \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] \right]^q \leq M \lambda^{\frac{q-p+1}{(p-1)^2}} \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] < \infty. \quad (6.21)$$

We deduce the following:

**Lemma 6.14** *Let  $\omega \in \mathcal{M}_b^+(\Omega)$ , and  $b \geq 0$  and  $K > 0$ . Suppose that  $\{u_m\}_{m \geq 1}$  is a sequence of nonnegative functions in  $\Omega$  that satisfies*

$$u_1 \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + b, \quad u_{m+1} \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[u_m^q + \omega] + b \quad \forall m \geq 1.$$

Assume that  $\omega$  satisfies (6.20) for some  $\lambda > 0$ . Then there exist  $\lambda_0$  and  $b_0$ , depending on  $N, p, q, K$ , and  $\text{diam}\Omega$ , such that, if  $\lambda \leq \lambda_0$  and  $b \leq b_0$ , then  $\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] \in L^q(\Omega)$  and for any  $m \geq 1$ ,

$$u_m \leq 2\beta_p K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + 2b, \quad \beta_p = \max(1, 3^{\frac{2-p}{p-1}}). \quad (6.22)$$

**Proof.** Clearly, (6.22) holds for  $m = 1$ . Now, assume that it holds at the order  $m$ . Then

$$u_m^q \leq 2^{q-1} (2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1} (2b)^q$$

Using (6.21) we get

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[ 2^{q-1} (2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1} (2b)^q + \omega \right] + b \\ &\leq \beta_p K \left( A_1 \mathbf{W}_{1,p}^{2\text{diam}\Omega} \left[ (\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega])^q \right] + \mathbf{W}_{1,p}^{2\text{diam}\Omega} [(2b)^q] + \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] \right) + b \\ &\leq \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1) \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + \beta_p K \mathbf{W}_{1,p}^{2\text{diam}\Omega} [(2b)^q] + b \\ &= \beta_p K (A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1) \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + A_2 b^{\frac{q}{p-1}} + b, \end{aligned}$$

where  $M$  is as in (6.21) and  $A_1 = (2^{q-1}(2\beta_p)^q K^q)^{1/(p-1)}$ ,  $A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1} (2\text{diam}\Omega)^{p'}$ . Thus, (6.22) holds for  $m = n + 1$  if we prove that

$$A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} \leq 1 \text{ and } A_2 b^{\frac{q}{p-1}} \leq b,$$

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}} \text{ and } b \leq A_2^{-\frac{p-1}{q-p+1}}.$$

Therefore, we obtain the result with  $\lambda_0 = (A_1 M)^{-(p-1)^2/(q-p+1)}$  and  $b_0 = A_2^{-(p-1)/(q-p+1)}$ .  $\blacksquare$

**Proof of Theorem 2.4.** From Corollary 5.7 and 6.12, we can construct a sequence of nonnegative nondecreasing R-solutions  $\{u_m\}_{m \geq 1}$  defined in the following way:  $u_1$  is a R-solution of (1.1), and  $u_{m+1}$  is a nonnegative R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = u_m^q + \mu & \text{in } Q, \\ u_{m+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 & \text{in } \Omega. \end{cases}$$

Setting  $\bar{u}_m = \sup_{t \in (0, T)} u_m(t)$  for all  $m \geq 1$ , there holds

$$\bar{u}_1 \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + \|u_0\|_{\infty, \Omega}, \quad \bar{u}_{m+1} \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\bar{u}_m^q + \omega] + \|u_0\|_{\infty, \Omega} \quad \forall m \geq 1.$$

From Lemma 6.14, we can find  $\lambda_0 = \lambda_0(N, p, q, \text{diam}\Omega)$  and  $b_0 = b_0(N, p, q, \text{diam}\Omega)$  such that if (2.7) is satisfied with  $\lambda_0$  and  $b_0$ , then

$$u_m \leq \bar{u}_m \leq 2\beta_p \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + 2\|u_0\|_{\infty, \Omega} \quad \forall m \geq 1. \quad (6.23)$$

Thus  $\{u_m\}$  converges *a.e.* in  $Q$  and in  $L^1(Q)$  to some function  $u$ , for which (2.9) is satisfied in  $\Omega$  with  $c = 2\beta_p \kappa$ . Finally, one can apply Theorem 2.1 to the sequence of measures  $\{u_m^q + \mu\}$ , and obtain that  $u$  is a R-solution of (2.8).  $\blacksquare$

Next we consider the exponential case.

**Theorem 6.15** *Let  $A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  satisfying (1.6), (1.7). Let  $\tau > 0, l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > p - 1$ . Set*

$$\mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \quad \forall s \in \mathbb{R}. \quad (6.24)$$

*Let  $\mu \in \mathcal{M}_b^+(Q)$ ,  $\omega \in \mathcal{M}_b^+(\Omega)$  such that  $\mu \leq \chi_{(0, T)} \otimes \omega$ . Then, there exist  $b_0$  and  $M_0$  depending on  $N, p, \beta, \tau, l$  and  $\text{diam}\Omega$ , such that if*

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_0, \quad \|u_0\|_{\infty, \Omega} \leq b_0,$$

*the problem*

$$\begin{cases} u_t - \text{div}(A(x, \nabla u)) = \mathcal{E}(\tau u^\beta) + \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega \end{cases} \quad (6.25)$$

admits nonnegative R- solution  $u$ , which satisfies, a.e. in  $Q$ , for some  $c$ , depending on  $N, p, c_3, c_4$

$$u(x, t) \leq c \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega](x) + 2b_0. \quad (6.26)$$

For the proof we first recall an approximation property, which is a consequence of [47, Theorem 2.5]:

**Theorem 6.16** *Let  $\tau > 0$ ,  $b \geq 0$ ,  $K > 0$ ,  $l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > p - 1$ . Let  $\mathcal{E}$  be defined by (6.24). Let  $\{v_m\}$  be a sequence of nonnegative functions in  $\Omega$  such that, for some  $K > 0$ ,*

$$v_1 \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + b, \quad v_{m+1} \leq K \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mathcal{E}(\tau u_m^\beta) + \mu] + b, \quad \forall m \geq 1.$$

*Then, there exist  $b_0$  and  $M_0$ , depending on  $N, p, \beta, \tau, l, K$  and  $\text{diam}\Omega$  such that if  $b \leq b_0$  and*

$$\|\mathbf{M}_{p, 2\text{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{\infty, \mathbb{R}^N} \leq M_0, \quad (6.27)$$

*then, setting  $c_p = 2\max(1, 2^{\frac{2-p}{p-1}})$ ,*

$$\begin{aligned} \exp(\tau(Kc_p \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + 2b_0)^\beta) &\in L^1(\Omega), \\ v_m &\leq Kc_p \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\mu] + 2b_0, \quad \forall m \geq 1. \end{aligned} \quad (6.28)$$

**Proof of Theorem 6.15.** From Corollary 5.7 and 6.12 we can construct a sequence of nonnegative nondecreasing R-solutions  $\{u_m\}_{m \geq 1}$  defined in the following way:  $u_1$  is a R-solution of problem (1.1), and by induction,  $u_{m+1}$  is a R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\ u_{m+1} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 & \text{in } \Omega. \end{cases}$$

And, setting  $\bar{u}_m = \sup_{t \in (0, T)} u_m(t)$ , there holds

$$\bar{u}_1 \leq \kappa W_{1,p}^{2\text{diam}\Omega}[\omega] + \|u_0\|_{\infty, \Omega}, \quad \bar{u}_{m+1} \leq \kappa W_{1,p}^{2\text{diam}\Omega}[\mathcal{E}(\tau \bar{u}_m^\beta) + \omega] + \|u_0\|_{\infty, \Omega}, \quad \forall m \geq 1.$$

Thus, from Theorem 6.16, there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, p, \beta, \tau, l$  and  $\text{diam}\Omega$  such that, if (6.27) holds, then (6.28) is satisfied with  $v_m = \bar{u}_m$ . As a consequence,  $u_m$  is well defined. Thus,  $\{u_m\}$  converges a.e. in  $Q$  to some function  $u$ , for which (6.26) is satisfied in  $\Omega$ . Furthermore,  $\{\mathcal{E}(\tau u_m^\beta)\}$  converges to  $\mathcal{E}(\tau u^\beta)$  in  $L^1(Q)$ . Finally, one can apply Theorem 2.1 to the sequence of measures  $\{\mathcal{E}(\tau u_m^\beta) + \mu\}$ , and obtain that  $u$  is a R-solution of (6.25).  $\blacksquare$



## 7 Appendix

**Proof of Lemma 4.7.** Let  $\mathcal{J}$  be defined by (4.11). Let  $\zeta \in C_c^1([0, T])$  with values in  $[0, 1]$ , such that  $\zeta_t \leq 0$ , and  $\varphi = \zeta \xi [j(S(v))]_l$ . Clearly,  $\varphi \in X \cap L^\infty(Q)$ ; we choose the pair of functions  $(\varphi, S)$  as test function in (4.2). Thanks to convergence properties of Steklov time-averages, we easily will obtain (4.15) if we prove that

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left( - \int_Q (\zeta \xi [j(S(v))]_l)_t S(v) \right) \geq - \int_Q \xi_t J(S(v)).$$

We can write  $-\int_Q (\zeta \xi [j(S(v))]_l)_t S(v) = F + G$ , with

$$F = - \int_Q (\zeta \xi)_t [j(S(v))]_l S(v), \quad G = - \int_Q \zeta \xi S(v) \frac{1}{l} (j(S(v))(x, t+l) - j(S(v))(x, t)).$$

Using (4.12) and integrating by parts we have

$$\begin{aligned} G &\geq - \int_Q \zeta \xi \frac{1}{l} (\mathcal{J}(S(v))(x, t+l) - \mathcal{J}(S(v))(x, t)) \\ &= - \int_Q \zeta \xi \frac{\partial}{\partial t} ([\mathcal{J}(S(v))]_l) = \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l + \int_\Omega \zeta(0) \xi(0) [\mathcal{J}(S(v))]_l(0) \\ &\geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l, \end{aligned}$$

since  $\mathcal{J}(S(v)) \geq 0$ . Hence,

$$- \int_Q (\zeta \xi [j(S(v))]_l)_t S(v) \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(v))]_l + F = \int_Q (\zeta \xi)_t ([\mathcal{J}(S(v))]_l - [J(S(v))]_l S(v))$$

Otherwise,  $\mathcal{J}(S(v))$  and  $J(S(v)) \in C([0, T]; L^1(\Omega))$ , thus  $\{(\zeta \xi)_t ([\mathcal{J}(S(u))]_l - [J(S(u))]_l S(u))\}$  converges to  $-(\zeta \xi)_t J(S(u))$  in  $L^1(Q)$  as  $l \rightarrow 0$ . Therefore,

$$\begin{aligned} \lim_{l \rightarrow 0, \zeta \rightarrow 1} \left( - \int_Q (\zeta \xi [J(S(v))]_l)_t S(v) \right) &\geq \lim_{\zeta \rightarrow 1} \left( - \int_Q (\zeta \xi)_t J(S(v)) \right) \\ &\geq \lim_{\zeta \rightarrow 1} \left( - \int_Q \zeta \xi_t J(S(v)) \right) = - \int_Q \xi_t J(S(v)), \end{aligned}$$

which achieves the proof. ■

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