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Discretization of a dynamic thermoviscoelastic Timoshenko beam

by Christine Bernardi¹ and Maria Inês M. Copetti²

Abstract: We consider a nonlinear model for a thermoelastic beam that can enter in contact with obstacles. We first prove the well-posedness of this problem. Next, we propose a discretization by Euler and Crank-Nicolson schemes in time and finite elements in space and perform the a priori analysis of the discrete problem. Some numerical experiments confirm the interest of this approach.

Résumé: Nous considérons un modèle non linéaire pour une poutre thermoélastique qui peut entrer en contact avec des obstacles. Nous prouvons que ce problème est bien posé. Puis nous écrivons une discrétisation par schémas d'Euler et de Crank-Nicolson en temps et éléments finis en espace et effectuons l'analyse a priori du problème discret. Quelques expériences numériques confirment l'intérêt de cette approche.

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1. Introduction.

In this paper we consider the evolution problem

$$\begin{aligned}
 u_{tt} &= u_{xx} - \phi_x + \zeta(u_{xxt} - \phi_{xt}) + \left(\beta + \rho \int_0^1 u_x^2 dx \right) u_{xx}, \quad x \in I, t > 0, \\
 \phi_{tt} &= \phi_{xx} + u_x - \phi + \zeta(\phi_{xxt} + u_{xt} - \phi_t) - a\theta_x, \quad x \in I, t > 0, \\
 \theta_t &= \theta_{xx} - a\phi_{xt}, \quad x \in I, t > 0,
 \end{aligned} \tag{1.1}$$

where $u(x, t)$, $\phi(x, t)$ and $\theta(x, t)$ denote the vertical displacement, the angular rotation of cross section and the temperature along the transversal direction of a homogeneous, thermoviscoelastic thick beam whose reference configuration is the interval $I =]0, 1[$.

Here, the stress field is given by $\sigma = u_x - \phi + \zeta(u_{xt} - \phi_t) + \left(\beta + \rho \int_0^1 u_x^2 dx \right) u_x$, $\zeta > 0$ is a viscosity coefficient, β accounts for an axial force at rest, $\rho > 0$ is a constant related to the material and a nonlinearity of Kirchhoff type taking into account changes in tension due to variations in the displacement is incorporated in the Timoshenko beam model.

We have the initial conditions

$$\begin{aligned}
 u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in I, \\
 \phi(x, 0) &= \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in I, \\
 \theta(x, 0) &= \theta_0(x), \quad x \in I,
 \end{aligned} \tag{1.2}$$

and the boundary conditions

$$\begin{aligned}
 u(0, t) &= \phi(0, t) = 0, \quad t > 0, \\
 \phi_x(1, t) + \zeta\phi_{xt}(1, t) &= 0, \quad t > 0, \\
 \theta(0, t) &= \theta_A, \quad \theta(1, t) = 0, \quad t > 0, \\
 \sigma(1, t) &= -\frac{1}{\epsilon} \left([u(1, t) - g_2]_+ - [g_1 - u(1, t)]_+ \right), \quad t > 0.
 \end{aligned} \tag{1.3}$$

The condition in the first line of (1.3) is assumed at $x = 0$ on the two unknowns u and ϕ and it means that the beam is clamped at this point. At $x = 1$ the beam is free to come into frictionless contact with two pointed reactive obstacles located at the vertical positions g_1 and g_2 (see Figure 1), $g_1 \leq 0 \leq g_2$. The boundary condition in the fourth line of (1.3) with $\epsilon > 0$ describes the contact and is called normal compliance condition. The symbol $[f]_+ = \max\{f, 0\}$ denotes the positive part of a function f . In the limit $\epsilon \rightarrow 0$ the obstacles become rigid and a Signorini condition is obtained. Moreover, the temperature of the beam is prescribed at both ends.

We remark that the temperature acts on the motion of ϕ directly and indirectly on the displacement u through the coupling of the equations. We refer to the works of Lagnese,

Leugering and Schmidt [10], Sapir and Reiss [16] and Anh and Stewart [1] for details on the modelling.

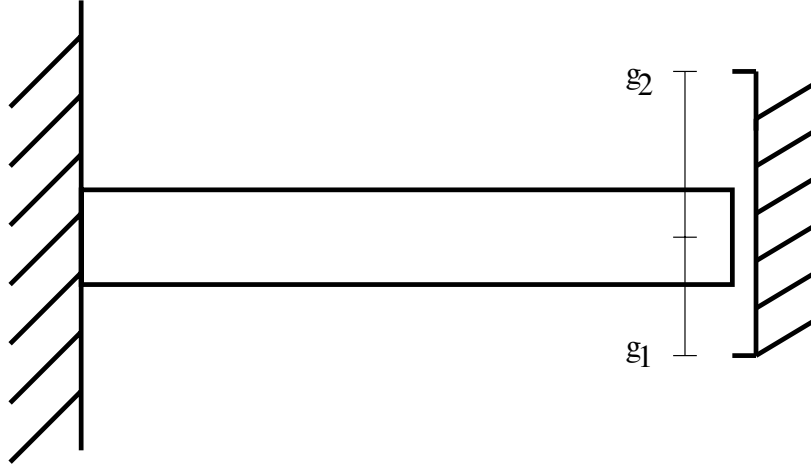


Figure 1. The contact problem.

Problem (1.1) – (1.2) – (1.3) with $\zeta = \beta = \rho = 0$ and Dirichlet and/or Neumann boundary conditions was considered by Rivera and Racke [15] who investigated the decay rate of the energy associated with the system. If longitudinal deformations and temperature variations along the axial direction are taken into account then the thermoelastic Bresse system addressed by Liu and Rao [11] is obtained. A dynamic, nonlinear, elastic, Timoshenko beam problem was studied by Sapir and Reiss [16]. In particular, stationary solutions were described and their stability investigated. In [13] an existence result was given and a numerical method studied for a nonlinear Timoshenko system modelling the dynamic vibrations of an elastic beam. Recently, Anh and Stewart [1] and Copetti and Fernández [8] examined contact problems for an viscoelastic linear Timoshenko beam. In particular, fully discrete approximations were proposed and analysed and the results of numerical simulations presented.

Stationary problems for nonlinear, elastic, Euler–Bernoulli beams were considered previously in [14], [12]. An algorithm based on the Galerkin method was proposed by Peradze [14] to solve an equation for a simply supported beam. In [12], Ma included a nonlinear boundary condition, established existence of solutions and presented a numerical algorithm using the finite difference method. Recently, the quasi-static thermoviscoelastic nonlinear contact problem for an Euler–Bernoulli beam was numerically studied by Copetti and Fernández [7]; in the latter case, a finite element discretization was proposed and analyzed and some numerical experiments were performed. In [3], the nonlinear static Timoshenko model was considered and a full analysis of a finite element discretization was performed.

Thermoviscoelastic linear and nonlinear contact problems involving rods and Euler–Bernoulli beams have been considered by many authors, see for example [2], [5], [6], [7], and the references therein. However, to our knowledge, the present paper is the first work to

consider a dynamic, nonlinear, thermoviscoelastic, Timoshenko beam model with contact boundary conditions.

In this work, we first write a variational formulation of this problem. Despite the complexity of the system, we can exhibit a quantity that we arbitrarily call “energy” and we prove that it decreases in time. Thus, by applying the Cauchy-Lipschitz theorem to a finite-dimensional approximation of the problem and passing to the limit, we prove that it admits a solution. We also investigate the uniqueness of this solution. Next, we propose a finite element discretization of this problem. We then perform its a priori analysis. All this justifies the choice of the discretization that we have made. Finally, we present and analyze an iterative algorithm for solving the nonlinear discrete problem. Some numerical experiments confirm the interest of our approach.

The outline of this article is as follows.

- In Section 2, we prove the well-posedness of the problem.
- Section 3 is devoted to the description and a priori analysis of the discrete problem.
- An iterative algorithm for solving the discrete problem is studied in Section 4.
- In Section 5, we present a few numerical experiments.

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2. The continuous problem and its well-posedness.

We first write a variational formulation of system (1.1) – (1.2) – (1.3). Next, we define and study its so-called energy. Then, we prove successively the uniqueness and the existence of a solution to it.

For brevity, we now work with $\theta_A = 0$ (note that, in any case, the function $\theta - \theta_A(1-x)$ solves the new problem with only a small modification on the second equation)

2.1. The variational formulation.

From now on, we consider the whole scale of Sobolev spaces $H^s(I)$. When s is a positive integer, these spaces are equipped with the semi-norm $|\cdot|_{H^s(I)}$ and norm $\|\cdot\|_{H^s(I)}$. We denote by (\cdot, \cdot) the scalar product of $L^2(I)$ and, in view of the boundary conditions (1.3), we introduce the spaces

$$H_*^1(I) = \{v \in H^1(I); v(0) = 0\}, \quad H_0^1(I) = \{v \in H^1(I); v(0) = v(1) = 0\}. \quad (2.1)$$

We now consider the variational problem:

Find (u, ϕ, θ) in $\mathcal{C}^0(0, T; H_*^1(I)) \times \mathcal{C}^0(0, T; H_*^1(I)) \times \mathcal{C}^0(0, T; H_0^1(I))$ which satisfies (1.2) and such that, for a.e. t in $[0, T]$,

$$\begin{aligned} \forall w \in H_*^1(I), \quad & (u_{tt}, w) + (u_x - \phi, w_x) \\ & + \zeta(u_{xt} - \phi_t, w_x) + (N(u)u_x, w_x) + g(u)(1)w(1) = 0, \\ \forall \chi \in H_*^1(I), \quad & (\phi_{tt}, \chi) + (\phi_x, \chi_x) + (\phi - u_x, \chi) \\ & + \zeta(\phi_t - u_{xt}, \chi) + \zeta(\phi_{xt}, \chi_x) - a(\theta, \chi_x) = 0, \\ \forall \eta \in H_0^1(I), \quad & (\theta_t, \eta) + (\theta_x, \eta_x) + a(\phi_{xt}, \eta) = 0, \end{aligned} \quad (2.2)$$

where the nonlinear quantities $N(u)$ and $g(u)(1)$ are defined by

$$N(u) = \beta + \rho \|u_x\|_{L^2(I)}^2, \quad (2.3)$$

and (we recall that $g_1 \leq 0 \leq g_2$)

$$g(u)(1) = \begin{cases} \frac{1}{\varepsilon}(u(1) - g_2) & \text{if } u(1) \geq g_2, \\ 0 & \text{if } g_1 \leq u(1) \leq g_2, \\ \frac{1}{\varepsilon}(u(1) - g_1) & \text{if } u(1) \leq g_1. \end{cases}$$

Due to the density of the space of infinitely differentiable functions with a compact support in $]0, 1[$ (respectively in $]0, 1[$) in $H_*^1(I)$ (respectively in $H_0^1(I)$) proving the following equivalence result is easy. However it requires a little more regularity of the solution. For this, we introduce the space

$$\mathbb{X} = \left(H^2(0, T; L^2(I)) \cap H^1(0, T; H^1(I)) \right)^2 \times \left(H^1(0, T; L^2(I)) \cap L^2(0, T; H^1(I)) \right). \quad (2.4)$$

Proposition 2.1. *Problems (1.1) – (1.2) – (1.3) and (1.2) – (2.2) are equivalent in the following sense: Any triple (u, ϕ, θ) in \mathbb{X} is a solution of (1.1) – (1.2) – (1.3) in the distribution sense if and only if it is a solution of (1.2) – (2.2).*

2.2. Study of an energy.

For brevity, we denote by $\|\cdot\|$ the norm of $L^2(I)$. We call energy the quantity E defined at each time t by

$$E(t) = \frac{1}{2} (\|u_t\|^2 + \|u_x - \phi\|^2 + \|\phi_t\|^2 + \|\phi_x\|^2 + \|\theta\|^2) + \frac{1}{4\rho} N(u)^2 + G(u)(1), \quad (2.5)$$

where the new term $G(u)(1)$ is defined by

$$G(u)(1) = \begin{cases} \frac{1}{2\varepsilon} (u(1) - g_2)^2 & \text{if } u(1) \geq g_2, \\ 0 & \text{if } g_1 \leq u(1) \leq g_2, \\ \frac{1}{2\varepsilon} (u(1) - g_1)^2 & \text{if } u(1) \leq g_1. \end{cases}$$

The reason for introducing the quantity E is given in the next proposition.

Proposition 2.2. *The energy E satisfies for a.e. t in $[0, T]$*

$$\frac{d}{dt} E(t) \leq 0. \quad (2.6)$$

Proof: When taking in (2.2) w equal to u_t , χ equal to ϕ_t , and summing the two equations, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_t\|^2 + \|u_x - \phi\|^2 + \|\phi_t\|^2 + \|\phi_x\|^2) + (N(u)u_x, u_{xt}) + g(u)(1)u_t(1) \\ = -\zeta \|u_{xt} - \phi_t\|^2 - \zeta \|\phi_{xt}\|^2 + a(\theta, \phi_{xt}). \end{aligned}$$

On the other hand, taking in (2.2) η equal to θ gives

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 = -\|\theta_x\|^2 - a(\theta, \phi_{xt}).$$

To handle the nonlinear terms, we observe that

$$\frac{d}{dt} \left(\frac{1}{4\rho} N(u)^2 \right) = \frac{1}{2\rho} N(u) \frac{d}{dt} N(u) = N(u)(u_x, u_{xt}),$$

and that, for instance when $u(1) \geq g_2$,

$$\frac{d}{dt} G(u)(1) = \frac{1}{\varepsilon} (u(1) - g_2) u_t(1).$$

Thus, by summing the two equations, we obtain

$$\frac{d}{dt}E(t) = -\zeta\|u_{xt} - \phi_t\|^2 - \zeta\|\phi_{xt}\|^2 - \|\theta_x\|^2,$$

and, since ζ is positive, this yields the desired result.

From now on, we assume that the initial data satisfy

$$u_0 \in H_*^1(I), \quad u_1 \in L^2(I), \quad \phi_0 \in H_*^1(I), \quad \phi_1 \in L^2(I), \quad \theta_0 \in L^2(I). \quad (2.7)$$

Then, an important consequence of Proposition 2.2 is that

$$E(t) \leq E(0), \quad (2.8)$$

which gives a stability property of the solution (u, ϕ, θ) .

In view of the last inequality in the previous proof, we have a further estimate, which turns out to be useful in what follows.

Corollary 2.3. *The following estimate holds for a.e. t in $[0, T]$*

$$\zeta \int_0^t (\|u_{xt} - \phi_t\|^2 + \|\phi_{xt}\|^2)(s) ds \leq E(0). \quad (2.9)$$

2.3. Uniqueness of the solution.

To handle the nonlinear terms, we need a preliminary result.

Lemma 2.4. *Any solution u of problem (1.2) – (2.2) satisfies*

$$\|u_x\|^2 \leq cE(0). \quad (2.10)$$

Proof: Let u be a solution of problem (1.2) – (2.2). Using (2.7), we first deduce from (2.8) that

$$\|u_x - \phi\|^2 + \|\phi_x\|^2 \leq 2E(0).$$

Using the Poincaré–Friedrichs and Young-inequalities thus yields

$$2E(0) \geq \|u_x\|^2 + (1+c)\|\phi\|^2 - 2(u_x, \phi) \geq \left(1 - \frac{1}{1+c}\right)\|u_x\|^2,$$

whence the desired result.

The same argument, applied with u replaced by u_t and combined with Corollary 2.3 yields the next result.

Corollary 2.5. *Any solution u of problem (1.2) – (2.2) satisfies*

$$\zeta \int_0^t \|u_{xt}\|^2(s) ds \leq c E(0). \quad (2.11)$$

We are thus in a position to state and prove the uniqueness result.

Theorem 2.6. *Problem (1.2) – (2.2) admits at most a solution (u, ϕ, θ) in \mathbb{X} .*

Proof: Let (u_1, ϕ_1, θ_1) and (u_2, ϕ_2, θ_2) be two solutions of problem (1.2) – (2.2). Setting $v = u_1 - u_2$, $\psi = \phi_1 - \phi_2$ and $\tau = \theta_1 - \theta_2$, we observe that this new triple satisfies

$$v(x, 0) = v_t(x, 0) = \psi(x, 0) = \psi_t(x, 0) = \tau(x, 0) = 0, \quad x \in I, \quad (2.12)$$

and, for a.e. t in $[0, T]$,

$$\begin{aligned} \forall w \in H_*^1(I), \quad & (v_{tt}, w) + (v_x - \psi, w_x) + \zeta(v_{xt} - \psi_t, w_x) \\ & + (N(u_1)u_{1x} - N(u_2)u_{2x}, w_x) + (g(u_1)(1) - g(u_2)(1))w(1) = 0, \\ \forall \chi \in H_*^1(I), \quad & (\psi_{tt}, \chi) + (\psi_x, \chi_x) + (\psi - v_x, \chi) \\ & + \zeta(\psi_t - v_{xt}, \chi) + \zeta(\psi_{xt}, \chi_x) - a(\tau, \chi_x) = 0, \\ \forall \eta \in H_0^1(I), \quad & (\tau_t, \eta) + (\tau_x, \eta_x) + a(\psi_{xt}, \eta) = 0, \end{aligned} \quad (2.13)$$

Using exactly the same arguments as for Proposition 2.2, i.e., taking $w = v_t$, $\chi = \psi_t$ and $\eta = \tau$, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|v_t\|^2 + \|v_x - \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2 + \|\tau\|^2) + \zeta \|v_{xt} - \psi_t\|^2 + \zeta \|\psi_{xt}\|^2 + \|\tau_x\|^2 \\ \leq -(N(u_1)u_{1x} - N(u_2)u_{2x}, v_{xt}) - (g(u_1)(1) - g(u_2)(1))v_t(1). \end{aligned}$$

To evaluate the right-hand side, we first use the definition of N

$$-(N(u_1)u_{1x} - N(u_2)u_{2x}, v_{xt}) \leq -\beta(v_x, v_{xt}) + \rho |(\|u_{1x}\|^2 u_{1x} - \|u_{2x}\|^2 u_{2x}, v_{xt})|.$$

Due to the Lipschitz continuity of the mapping: $v \mapsto \|v_x\|^2 v_x$ (we recall from (2.10) that the norms of $u_1(t)$ and $u_2(t)$ in $H^1(I)$ are bounded) we derive by a Cauchy–Schwarz inequality

$$-(N(u_1)u_{1x} - N(u_2)u_{2x}, v_{xt}) \leq (|\beta| + c\rho) \|v_x\| \|v_{xt}\|.$$

On the other hand, the Lipschitz continuity of g , together with the continuity of the trace yields the same estimate for the second term:

$$-(g(u_1)(1) - g(u_2)(1))v_t(1) \leq \frac{c}{\varepsilon} \|v_x\| \|v_{xt}\|.$$

All this implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v_t\|^2 + \|v_x - \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2 + \|\tau\|^2) \\ & + \zeta \|v_{xt} - \psi_t\|^2 + \zeta \|\psi_{xt}\|^2 + \|\tau_x\|^2 \leq c \max\{|\beta|, \rho, \frac{1}{\varepsilon}\} \|v_x\| \|v_{xt}\|, \end{aligned}$$

whence, since the initial conditions are zero,

$$\begin{aligned} & \|v_t\|^2 + \|v_x - \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2 + \|\tau\|^2 \\ & + \int_0^t (\zeta \|v_{xt} - \psi_t\|^2 + \zeta \|\psi_{xt}\|^2 + \|\tau_x\|^2)(s) ds \leq c \max\{|\beta|, \rho, \frac{1}{\varepsilon}\} \int_0^t \|v_x\|(s) \|v_{xt}\|(s) ds. \end{aligned}$$

Using the same arguments as for Lemma 2.4 and Corollary 2.5 in the left-hand side and a Young's inequality in the right-hand side then yield

$$c \|v_x\|^2 + c' \zeta \int_0^t \|v_{xt}\|^2(s) ds \leq c' \zeta \int_0^t \|v_{xt}\|^2(s) ds + c'' \frac{\max\{|\beta|, \rho, \frac{1}{\varepsilon}\}^2}{\zeta} \int_0^t \|v_x\|^2(s) ds.$$

It thus follows from the Gronwall's lemma, see [17, Lemme 21.9] for instance, that v is zero. The fact that ψ and τ are zero is then derived from the previous inequalities. This concludes the proof.

2.4. Existence of a solution.

To prove that problem (2.2) admits a solution, we need an intermediary step. For a given function u^* in $\mathcal{C}^0(0, T; H_*^1(I))$, we consider the simpler variational problem

Find (u, ϕ, θ) in $\mathcal{C}^0(0, T; H_*^1(I)) \times \mathcal{C}^0(0, T; H_*^1(I)) \times \mathcal{C}^0(0, T; H_0^1(I))$ which satisfies (1.2) and such that, for a.e. t in $[0, T]$,

$$\begin{aligned} \forall w \in H_*^1(I), \quad & (u_{tt}, w) + (u_x - \phi, w_x) \\ & + \zeta (u_{xt} - \phi_t, w_x) + (N(u^*)u_x, w_x) + g(u)(1)w(1) = 0, \\ \forall \chi \in H_*^1(I), \quad & (\phi_{tt}, \chi) + (\phi_x, \chi_x) + (\phi - u_x, \chi) \\ & + \zeta (\phi_t - u_{xt}, \chi) + \zeta (\phi_{xt}, \chi_x) - a(\theta, \chi_x) = 0, \\ \forall \eta \in H_0^1(I), \quad & (\theta_t, \eta) + (\theta_x, \eta_x) + a(\phi_{xt}, \eta) = 0. \end{aligned} \tag{2.14}$$

The only nonlinearity of this problem comes from the boundary condition in $x = 1$.

This problem can be written in a slightly different form. We introduce the new unknowns $v = u_t$, $\psi = \phi_t$, and defining the vector field $U = (u, v, \phi, \psi, \theta)^T$, we observe that it reads

$$U_t = A(U), \tag{2.15}$$

where the matrix operator $A(U)$ is defined by the twenty-five forms $a_{ij}(\cdot, \cdot)$, $1 \leq i, j \leq 5$, of the product $(A(U), W)$, with $W = (z, w, \omega, \chi, \eta)^T$. All these forms are zero or linear, but the form

$$a_{21}(u, w) = -(u_x, w_x) - (N(u^*)u_x, w_x) - g(u)(1)w(1),$$

which contains all nonlinear terms. It can be noted that it is locally Lipschitz-continuous, i.e. Lipschitz-continuous on any bounded subset of $H_*^1(I)$. We now state and prove the first existence result.

Proposition 2.7. *Assume that the data $(u_0, u_1, \phi_0, \phi_1, \theta_0)$ satisfy (2.7). Then, for any function u^* in $\mathcal{C}^0(0, T; H_*^1(I))$, problem (1.2) – (2.14) admits at least a solution (u, ϕ, θ) in \mathbb{X} .*

Proof: It is performed in several steps.

1) Using the same arguments as for (2.8) and Lemma 2.4, we first derive that any solution (u, ϕ, θ) of problem (1.2) – (2.14) is such that u is bounded in the space $\mathcal{C}^0(0, T; H_*^1(I))$ by a constant C_0 . So, without any restriction, we can replace the operator $A(U)$ by another operator $\tilde{A}(U)$ which coincides with $A(U)$ on all U with first component in the ball with radius C_0 and is globally Lipschitz-continuous.

2) Since $H_*^1(I)$ and $H_0^1(I)$ are separable, there exist two increasing sequences $(\mathbb{V}_n)_n$ and $(\mathbb{W}_n)_n$ of finite-dimensional spaces of $H_*^1(I)$ and $H_0^1(I)$, respectively, such that $\cup_n \mathbb{V}_n$ is dense in $H_*^1(I)$ and $\cup_n \mathbb{W}_n$ is dense in $H_0^1(I)$. We set: $\mathcal{Z}_n = \mathbb{V}_n^4 \times \mathbb{W}_n$ and introduce an appropriate approximation U_0^n of the vector of initial data $(u_0, u_1, \phi_0, \phi_1, \theta_0)^T$ in \mathcal{Z}_n . Thus, when applying the Cauchy–Lipschitz theorem, see [17, Th. 21.1] for instance, there exists a unique U_n in $\mathcal{C}^0(0, T; \mathcal{Z}_n)$ which satisfies $U_n(0) = U_0^n$ and

$$\forall W_n \in \mathcal{Z}_n, \quad (U_{nt}, W_n) = (\tilde{A}(U_n), W_n).$$

3) We now choose U_0^n such that (this requires the density of $H_*^1(I)$ in $L^2(I)$)

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|u_{0x} - u_{0x}^n\| &= 0, & \lim_{n \rightarrow +\infty} \|u_1 - u_1^n\| &= 0, \\ \lim_{n \rightarrow +\infty} \|\phi_{0x} - \phi_{0x}^n\| &= 0, & \lim_{n \rightarrow +\infty} \|\phi_1 - \phi_1^n\| &= 0, & \lim_{n \rightarrow +\infty} \|\theta_0 - \theta_0^n\| &= 0. \end{aligned}$$

With obvious notation, this implies that $E_n(0) \leq cE(0)$ and thus, by exactly the same arguments as for Proposition 2.2, see (2.8), that, for all t , $E_n(t)$ is bounded independently of n .

4) The previous argument yields that there exists a subsequence, still denoted by $(U_n)_n$ for simplicity, which converges to a vector field U weakly in $H^1(I) \times L^2(I) \times H^1(I) \times L^2(I) \times L^2(I)$ and such that $(u_n(1))_n$ converges to $u(1)$.

By using the fact that the sequences $(\mathbb{V}_n)_n$ and $(\mathbb{W}_n)_n$ are increasing, we write the previous problem in the following form: For all $m \leq n$,

$$\forall W_m \in \mathcal{Z}_m^*, \quad (U_{nt}, W_m) = (\tilde{A}^*(U_n), W_m).$$

where \mathcal{Z}_n^* is now equal to $\mathbb{V}_n^4 \times (\mathbb{W}_n \cap H^2(I))$ and $\tilde{A}^*(U_n)$ is constructed by integrating by parts the term (θ_x, η_x) in the last equation. So passing to the limit first on n and second

on m yields that (u, ϕ, θ) is a solution of problem (1.2) – (2.14). It is readily checked that this triple belongs to \mathbb{X} .

Let \mathcal{B} be the ball of $\mathcal{C}^0(0, T; H_*^1(I))$ with radius C_0 and let \mathcal{F} denote the operator which associates with any u^* in \mathcal{B} the part u of the solution of problem (1.2) – (2.14) exhibited in Proposition 2.7. We are in a position to prove the main result of this section by applying a fixed-point theorem.

Theorem 2.8. *There exists a positive real number ρ_0 only depending on $\frac{1}{\varepsilon}$, ζ and T , such that, for all $\rho \leq \rho_0$, problem (1.2) – (2.2) admits at least a solution (u, ϕ, θ) in \mathbb{X} .*

Proof: It is readily checked from the definition of \mathcal{F} that the existence of a solution for problem (1.2) – (2.2) is equivalent to find a fixed-point of \mathcal{F} , i.e. a function u in \mathcal{B} such that $\mathcal{F}(u) = u$. So, we first recall from the previous proof that \mathcal{F} is continuous from \mathcal{B} into itself. Next, let u_1^* and u_2^* be two functions in \mathcal{B} . Denoting by (u_1, ϕ_1, θ_1) and (u_2, ϕ_2, θ_2) the solutions of problem (1.2) – (2.14) associated with them by Proposition 2.7, we observe that, when setting $v = u_1 - u_2$ and so on,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|v_t\|^2 + \|v_x - \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2 + \|\tau\|^2) + \zeta \|v_{xt} - \psi_t\|^2 + \zeta \|\psi_{xt}\|^2 + \|\tau_x\|^2 \\ \leq -(N(u_1^*)u_{1x} - N(u_2^*)u_{2x}, v_{xt}) - (g(u_1)(1) - g(u_2)(1))v_t(1). \end{aligned}$$

Thus, exactly the same arguments as for Theorem 2.6 yield

$$c \|v_x\|^2 \leq c' \frac{\max\{|\beta|, \rho, \frac{1}{\varepsilon}\}^2}{\zeta} \int_0^t \|v_x\|^2(s) ds + c'' \frac{\rho^2}{\zeta} \int_0^t \|u_{1x}^* - u_{2x}^*\|^2(s) ds.$$

Applying once more the Gronwall lemma, see [17, Lemme 21.9], thus implies

$$\int_0^t \|v_x\|^2(s) ds \leq c'' \frac{\rho^2}{\zeta} \int_0^t \left(\int_0^s \|u_{1x}^* - u_{2x}^*\|^2(\xi) d\xi \right) e^{c' \frac{\max\{|\beta|, \rho, \frac{1}{\varepsilon}\}^2}{\zeta} (t-s)} ds,$$

whence

$$c \|v_x\|^2 \leq \frac{\rho^2}{\zeta} T \left(c' \frac{T \max\{|\beta|, \rho, \frac{1}{\varepsilon}\}^2}{2\zeta} e^{c' \frac{\max\{|\beta|, \rho, \frac{1}{\varepsilon}\}^2}{\zeta} T} + c'' \right) \max_{0 \leq s \leq t} \|u_{1x}^* - u_{2x}^*\|^2(s).$$

Thus, we have proven that for an appropriate choice of ρ_0

$$\max_{0 \leq t \leq T} \|u_{1x} - u_{2x}\|^2(t) \leq \kappa \max_{0 \leq t \leq T} \|u_{1x}^* - u_{2x}^*\|^2(t),$$

where the constant κ is < 1 . This means that \mathcal{F} is a contraction of the ball \mathcal{B} . By applying Banach fixed point theorem, we deduce the existence of a u in \mathcal{B} such that $\mathcal{F}(u) = u$, whence the desired result.

Remark 2.9. It follows from the previous proof that the limitation on ρ is more precisely a limitation on $\rho^2 T$. So, on a very small time interval $]0, T[$, the existence of a solution holds with high values of ρ .

3. The discrete problem and its a priori analysis.

We first describe the discrete problem which is constructed from the Euler and Crank–Nicolson schemes in time and standard finite elements in space. Next, we prove the stability of its solution by bounding a discrete energy. We conclude with a priori error estimates on its solution.

3.1. Description of the discrete problem.

In view of the time discretization, for a given final time $T > 0$ and a given positive integer N , we define the time step $\delta t = T/N$ and the nodes $t_n = n\delta t$, $n = 0, 1, \dots, N$. For any sequence $(y^n)_{0 \leq n \leq N}$, we use the notation

$$y^{n-1/2} = \frac{y^n + y^{n-1}}{2}. \quad (3.1)$$

For the space discretization, we introduce a regular family $(\mathcal{T}_h)_h$ of triangulations of I by closed intervals in the sense that, for each h ,

- The closure $\bar{I} = [0, 1]$ of I is the union of all elements of \mathcal{T}_h ;
- The intersection of two different elements of \mathcal{T}_h is either empty or an endpoint of both of them;
- If two intervals K and K' of \mathcal{T}_h are adjacent, i.e. share an endpoint, their lengths h_K and $h_{K'}$ satisfy

$$h_K \leq \tau h_{K'},$$

for a constant τ independent of h .

As usual, h stands for the maximum of the lengths h_K , $K \in \mathcal{T}_h$. In what follows, c, c', \dots stand for generic constants that may vary from line to line but are always independent of the parameter h .

For a positive integer k and each K in \mathcal{T}_h , we introduce the space $\mathcal{P}_k(K)$ of restrictions to K of polynomials with one variable and degree $\leq k$. Then, we define the discrete space

$$\mathbb{S}_h = \{\mu_h \in H^1(I); \forall K \in \mathcal{T}_h, \mu_h|_K \in \mathcal{P}_1(K)\}, \quad (3.2)$$

and its subspaces

$$\mathbb{S}_h^* = \mathbb{S}_h \cap H_*^1(I), \quad \mathbb{S}_h^0 = \mathbb{S}_h \cap H_0^1(I). \quad (3.3)$$

We also introduce the projection operators $P_h^* : H_*^1(I) \rightarrow \mathbb{S}_h^*$ and $P_h^0 : H_0^1(I) \rightarrow \mathbb{S}_h^0$, defined respectively by

$$\forall \chi_h \in \mathbb{S}_h^*, ((P_h^* \eta - \eta)_x, \chi_{hx}) = 0 \quad \text{and} \quad \forall \chi_h \in \mathbb{S}_h^0, ((P_h^0 \eta - \eta)_x, \chi_{hx}) = 0.$$

The operator P_h^* (see [9, Lemma 2.1] or [4, Section IX.2]) preserves the values at all endpoints of the elements in \mathcal{T}_h , satisfies for all η in $H_*^1(I)$

$$\|P_h^* \eta - \eta\| \leq ch \|\eta_x\| \quad (3.4)$$

and also, for more regular functions η ,

$$\|P_h^* \eta - \eta\| + h \|(P_h^* \eta - \eta)_x\| \leq ch^2 \|\eta_{xx}\|. \quad (3.5)$$

Similar properties hold for the operator P_h^0 .

In order to define approximations of the initial data, assuming that they are smooth enough, we set

$$u_h^0 = P_h^* u_0, \quad \hat{u}_h^0 = P_h^* u_1, \quad \phi_h^0 = P_h^* \phi_0, \quad \hat{\phi}_h^0 = P_h^* \phi_1 \quad \text{and} \quad \theta_h^0 = P_h^0 \theta_0. \quad (3.6)$$

When using a combination of backward Euler and Crank-Nicolson schemes, the finite element approximation to the variational problem (1.2) – (2.2) is written as follows: For $n = 1, \dots, N$,

Find $(\hat{u}_h^n, \hat{\phi}_h^n, \theta_h^n)$ in $\mathbb{S}_h^* \times \mathbb{S}_h^* \times \mathbb{S}_h^0$ satisfying

$$\begin{aligned} \forall w_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t} (\hat{u}_h^n - \hat{u}_h^{n-1}, w_h) + (u_{hx}^{n-1/2} - \phi_h^{n-1/2}, w_{hx}) + \zeta(\hat{u}_{hx}^n - \hat{\phi}_h^n, w_{hx}) \\ & + \left(\frac{N(u_h^n) + N(u_h^{n-1})}{2} u_{hx}^{n-1/2}, w_{hx} \right) + g(u_h^n)(1) w_h(1) = 0, \\ \forall \chi_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t} (\hat{\phi}_h^n - \hat{\phi}_h^{n-1}, \chi_h) + (\phi_{hx}^{n-1/2}, \chi_{hx}) + (\phi_h^{n-1/2} - u_{hx}^{n-1/2}, \chi_h) \\ & + \zeta(\hat{\phi}_h^n - \hat{u}_{hx}^n, \chi_h) + \zeta(\hat{\phi}_{hx}^n, \chi_{hx}) - a(\theta^{n-1/2}, \chi_{hx}) = 0, \\ \forall \eta_h \in \mathbb{S}_h^0, \quad & \frac{1}{\delta t} (\theta_h^n - \theta_h^{n-1}, \eta_h) + (\theta_{hx}^{n-1/2}, \eta_{hx}) + a(\hat{\phi}_{hx}^n, \eta_h) = 0. \end{aligned} \quad (3.7)$$

The discrete displacement u_h^n and rotation angle ϕ_h^n are then given by

$$u_h^n = u_h^{n-1} + \delta t \hat{u}_h^n, \quad \phi_h^n = \phi_h^{n-1} + \delta t \hat{\phi}_h^n, \quad (3.8)$$

while θ_h^n is the discrete temperature.

Owing to (3.1), by inserting (3.8) into (3.7), it can be checked that the only unknowns of problem (3.7) are $(\hat{u}_h^n, \hat{\phi}_h^n, \theta_h^n)$. The well-posedness of this discrete problem is not at all obvious and requires further arguments.

3.2. Study of a discrete energy.

Let E^n be the discrete energy function defined by

$$E^n = \frac{1}{2} \left(\|\hat{u}_h^n\|^2 + \|u_{hx}^n - \phi_h^n\|^2 + \|\hat{\phi}_h^n\|^2 + \|\phi_{hx}^n\|^2 + \|\theta_h^n\|^2 \right) + \frac{1}{4\rho} N(u_h^n)^2 + G(u_h^n)(1), \quad (3.9)$$

where the function G is introduced in Section 2.2. This energy satisfies the discrete analogue of (2.6).

Proposition 3.1. *The solution $(\hat{u}_h^n, \hat{\phi}_h^n, \theta_h^n)$ to the discrete problem (3.6) – (3.7) satisfies the energy decay property*

$$\frac{E^n - E^{n-1}}{\delta t} \leq 0. \quad (3.10)$$

Proof: For simplicity, we suppress all the indices h in this proof. We take w equal to \hat{u}^n , χ equal to $\hat{\phi}^n$ and η equal to $\theta^{n-1/2}$ in (3.7), multiply everything by δt and sum up the three equations. Thanks to the formulas

$$\begin{aligned} (\hat{u}^n - \hat{u}^{n-1}, \hat{u}^n) &= \frac{1}{2} \left(\|\hat{u}^n - \hat{u}^{n-1}\|^2 + \|\hat{u}^n\|^2 - \|\hat{u}^{n-1}\|^2 \right), \\ (\hat{\phi}^n - \hat{\phi}^{n-1}, \hat{\phi}^n) &= \frac{1}{2} \left(\|\hat{\phi}^n - \hat{\phi}^{n-1}\|^2 + \|\hat{\phi}^n\|^2 - \|\hat{\phi}^{n-1}\|^2 \right), \\ (\theta^n - \theta^{n-1}, \theta^{n-1/2}) &= \frac{1}{2} \left(\|\theta^n\|^2 - \|\theta^{n-1}\|^2 \right) \\ \delta t(u_x^{n-1/2}, \hat{u}_x^n) &= \frac{1}{2} \left(\|u_x^n\|^2 - \|u_x^{n-1}\|^2 \right), \\ \delta t(\phi_x^{n-1/2}, \hat{\phi}_x^n) &= \frac{1}{2} \left(\|\phi_x^n\|^2 - \|\phi_x^{n-1}\|^2 \right), \\ \delta t(u_x^{n-1/2} - \phi^{n-1/2}, \hat{u}_x^n - \hat{\phi}_x^n) &= \frac{1}{2} \left(\|u_x^n - \phi^n\|^2 - \|u_x^{n-1} - \phi^{n-1}\|^2 \right), \end{aligned}$$

we thus derive

$$\begin{aligned} &\frac{1}{2} \left(\|\hat{u}^n\|^2 - \|\hat{u}^{n-1}\|^2 + \|u_x^n - \phi^n\|^2 - \|u_x^{n-1} - \phi^{n-1}\|^2 + \|\hat{\phi}^n\|^2 - \|\hat{\phi}^{n-1}\|^2 \right) \\ &+ \frac{1}{2} \left(\|\phi_x^n\|^2 - \|\phi_x^{n-1}\|^2 + \|\theta^n\|^2 - \|\theta^{n-1}\|^2 \right) + \delta t \zeta (\|\hat{u}_x^n - \hat{\phi}_x^n\|^2 + \|\hat{\phi}_x^n\|^2) \\ &+ \frac{\beta}{2} \left(\|u_x^n\|^2 - \|u_x^{n-1}\|^2 \right) + \frac{\rho}{4} \left(\|u_x^n\|^4 - \|u_x^{n-1}\|^4 \right) + \delta t \|\theta_x^{n-1/2}\|^2 \\ &+ \frac{1}{\varepsilon} \left([u^n(1) - g_2]_+ - [g_1 - u^n(1)]_+ \right) (u^n(1) - u^{n-1}(1)) \\ &= -\frac{1}{2} \left(\|\hat{u}^n - \hat{u}^{n-1}\|^2 + \|\hat{\phi}^n - \hat{\phi}^{n-1}\|^2 \right) \leq 0. \end{aligned}$$

For handling the term on the boundary, we observe from [8, Proof of Thm 3] that

$$\begin{aligned} &\left([u^n(1) - g_2]_+ - [g_1 - u^n(1)]_+ \right) (u^n(1) - u^{n-1}(1)) \\ &\geq \frac{1}{2} [u^n(1) - g_2]_+^2 - \frac{1}{2} [u^{n-1}(1) - g_2]_+^2 + \frac{1}{2} [g_1 - u^n(1)]_+^2 - \frac{1}{2} [g_1 - u^{n-1}(1)]_+^2. \end{aligned}$$

This yields

$$\begin{aligned}
& \frac{1}{2} \left(\|\hat{u}^n\|^2 - \|\hat{u}^{n-1}\|^2 + \|u_x^n - \phi^n\|^2 - \|u_x^{n-1} - \phi^{n-1}\|^2 + \|\hat{\phi}^n\|^2 - \|\hat{\phi}^{n-1}\|^2 \right) \\
& + \frac{1}{2} \left(\|\phi_x^n\|^2 - \|\phi_x^{n-1}\|^2 + \|\theta^n\|^2 - \|\theta^{n-1}\|^2 \right) \\
& + \frac{\beta}{2} \left(\|u_x^n\|^2 - \|u_x^{n-1}\|^2 \right) + \frac{\rho}{4} \left(\|u_x^n\|^4 - \|u_x^{n-1}\|^4 \right) \\
& + \frac{1}{2\varepsilon} \left([u^n(1) - g_2]_+^2 - [u^{n-1}(1) - g_2]_+^2 \right) \\
& + \frac{1}{2\varepsilon} \left([g_1 - u^n(1)]_+^2 - [g_1 - u^{n-1}(1)]_+^2 \right) \\
& \leq -\delta t \zeta \|\hat{u}_x^n - \hat{\phi}^n\|^2 - \delta t \zeta \|\hat{\phi}_x^n\|^2 - \delta t \|\theta_x^{n+1/2}\|^2,
\end{aligned}$$

whence the desired result.

As a consequence we have the stability result:

Corollary 3.2. *Assume that the initial data satisfy*

$$u_0 \in H_*^1(I), \quad u_1 \in H_*^1(I), \quad \phi_0 \in H_*^1(I), \quad \phi_1 \in H_*^1(I), \quad \theta_0 \in H_0^1(I). \quad (3.11)$$

Then, any solution $(\hat{u}_h^n, \hat{\phi}_h^n, \theta_h^n)$ to the discrete problem (3.6) – (3.7) satisfies, for a constant $c > 0$, independent of h and δt ,

$$\begin{aligned}
& \|\hat{u}_h^n\|^2 + \|u_{hx}^n - \phi_h^n\|^2 + \|\hat{\phi}_h^n\|^2 + \|\phi_{hx}^n\|^2 + \|\theta_h^n\|^2 + \|u_{hx}^n\|^2 \\
& + \left([u^n(1) - g_2]_+^2 + [g_1 - u^n(1)]_+^2 \right) \leq c,
\end{aligned} \quad (3.12)$$

where u_h^n and ϕ_h^n are defined in (3.8).

Proof: We derive by applying iteratively Proposition 3.1 that

$$E^n \leq E^0,$$

and all the terms in E^0 are bounded owing to (3.4) thanks to the assumptions on the initial data. On the other hand, all the terms in the left-hand side of (3.12) which do not appear explicitly in the definition (3.9) of E^n can be bounded by exactly the same arguments as for Lemma 2.4.

Remark 3.3. In the case where $N(u)$ is equal to a constant N and $g(u)(1)$ is equal to a constant g , problem (3.7) is linear, hence results into a square linear system. It follows from the previous proof that, when all data are zero, each solution $(\hat{u}_h^n, \hat{\phi}_h^n, \theta_h^n)$ is zero. So, in this case, problem (3.7) admits a unique solution. But the arguments for proving the existence and uniqueness in the nonlinear case are much more complex, similar to those for Proposition 2.7 and Theorem 2.8, and we prefer to skip them.

3.3. A priori error estimates.

We now state and prove the main result of this paper.

Theorem 3.4. *Assume that the solution (u, ϕ, θ) to the continuous problem (1.2) – (2.2) belongs to the space*

$$\begin{aligned} \mathbb{X}_* = & \left(H^4(0, T; L^2(I)) \cap H^3(0, T; H^1(I)) \cap H^1(0, T; H^2(I)) \right)^2 \\ & \times \left(H^3(0, T; L^2(I)) \cap H^2(0, T; H^1(I)) \right). \end{aligned} \quad (3.13)$$

Then, there exists a constant c independent of h and δt such that

$$\begin{aligned} \|\hat{u}_h^n - u_t(t_n)\|^2 + \|u_{hx}^n - \phi_h^n - (u_x(t_n) - \phi(t_n))\|^2 + \|\hat{\phi}_h^n - \phi_t(t_n)\|^2 \\ + \|\phi_{hx}^n - \phi_x(t_n)\|^2 + \|\theta_h^n - \theta(t_n)\|^2 \leq c(h^2 + (\delta t)^2). \end{aligned} \quad (3.14)$$

The proof of this theorem is very technical, see the proof of the much simpler result stated in Proposition 3.1, and we prefer to give only a sketch of it.

Sketch of proof: We only bring to light the main steps of such an estimate. Let us set:

$$\begin{aligned} e^n = u_h^n - P_h^* u(t_n), \quad y^n = \hat{u}_h^n - P_h^* u_t(t_n), \quad q^n = \phi_h^n - P_h^* \phi(t_n), \\ p^n = \hat{\phi}_h^n - P_h^* \phi_t(t_n) \quad r^n = \theta_h^n - P_h^0 \theta(t_n). \end{aligned} \quad (3.15)$$

1) We write the first equation in problem (3.7) with u_h^n replaced by e^n , \hat{u}_h^n replaced by y^n and so on. Next, we take w_h equal to y^n . We combine the result with the first equation in (2.2) at time $t_{n-1/2}$, again with $w = y^n$. All this gives

$$\begin{aligned} & \frac{1}{2\delta t} (\|y^n - y^{n-1}\|^2 + \|y^n\|^2 - \|y^{n-1}\|^2) + (e_x^{n-1/2} - q^{n-1/2}, y_x^n) \\ & + \zeta(y_x^n - p^n, y_x^n) + \left(\beta + \rho \frac{\|u_{hx}^n\|^2 + \|u_{hx}^{n-1}\|^2}{2} \right) (u_{hx}^{n-1/2}, y_x^n) \\ & + \frac{1}{\varepsilon} \left([u_h^n(1) - g_2]_+ - [g_1 - u_h^n(1)]_+ \right) y^n(1) \\ & \quad - \frac{1}{\varepsilon} \left([u(1, t_{n-1/2}) - g_2]_+ - [g_1 - u(1, t_{n-1/2})(1)]_+ \right) y^n(1) \\ = & \left(u_{tt}(t_{n-1/2}) - \frac{P_h^* u_t(t_n) - P_h^* u_t(t_{n-1})}{\delta t}, y^n \right) \\ & + \left(u_x(t_{n-1/2}) - \frac{(P_h^* u(t_n))_x + (P_h^* u(t_{n-1}))_x}{2}, y_x^n \right) \\ & - \left(\phi(t_{n-1/2}) - \frac{P_h^* \phi(t_n) + P_h^* \phi(t_{n-1})}{2}, y_x^n \right) + (\beta + \rho \|u_x(t_{n-1/2})\|^2) (u_x(t_{n-1/2}), y_x^n) \\ & + \zeta(u_{xt}(t_{n-1/2}) - \phi_t(t_{n-1/2}) - (P_h^* u_t(t_n))_x + P_h^* \phi_t(t_n), y_x^n). \end{aligned}$$

2) The same arguments applied to the second equations in (3.7) and (2.2) together with the definition of the operator P_h^* yield

$$\begin{aligned}
& \frac{1}{2\delta t} (\|p^n - p^{n-1}\|^2 + \|p^n\|^2 - \|p^{n-1}\|^2) + (q_x^{n-1/2}, p_x^n) \\
& \quad - (e_x^{n-1/2} - q^{n-1/2}, p^n) - \zeta(y_x^n - p^n, p^n) + \zeta\|p_x^n\|^2 - a(r^{n-1/2}, p_x^n) \\
& = \left(\phi_{tt}(t_{n-1/2}) - \frac{P_h^* \phi_t(t_n) - P_h^* \phi_t(t_{n-1})}{\delta t}, p^n \right) \\
& \quad + \left(\phi_x(t_{n-1/2}) - \frac{\phi_x(t_n) + \phi_x(t_{n-1})}{2}, p_x^n \right) \\
& \quad - \left(u_x(t_{n-1/2}) - \frac{(P_h^* u(t_n))_x + (P_h^* u(t_{n-1}))_x}{2}, p^n \right) \\
& \quad + \left(\phi(t_{n-1/2}) - \frac{P_h^* \phi(t_n) + P_h^* \phi(t_{n-1})}{2}, p^n \right) \\
& \quad - \zeta(u_{xt}(t_{n-1/2}) - \phi_t(t_{n-1/2}) - (P_h^* u_t(t_n))_x + P_h^* \phi_t(t_n)), p^n) \\
& \quad + \zeta(\phi_{xt}(t_{n-1/2}) - \phi_{xt}(t_n), p_x^n) \\
& \quad - a \left(\theta(t_{n-1/2}) - \frac{P_h^0 \theta(t_n) + P_h^0 \theta(t_{n-1})}{2}, p_x^n \right).
\end{aligned}$$

3) Again the same arguments applied to the third equations in (3.7) and (2.2) lead to the simpler inequality

$$\begin{aligned}
& \frac{1}{2\delta t} (\|r^n\|^2 - \|r^{n-1}\|^2) + \|r_x^{n-1/2}\|^2 + a(p_x^n, r^{n-1/2}) \\
& = \left(\theta_t(t_n) - \frac{P_h^0 \theta(t_n) - P_h^0 \theta(t_{n-1})}{\delta t}, r^{n-1/2} \right) \\
& \quad + \left(\theta_x(t_{n-1/2}) - \frac{\theta_x(t_n) + \theta_x(t_{n-1})}{2}, r_x^{n-1/2} \right) \\
& \quad - a(\phi_t - P_h^* \phi_t(t_n), r_x^{n-1/2}).
\end{aligned}$$

4) We sum up the last three inequalities. To handle the boundary term, keeping in mind that $u(1, t_n) = (P_h^* u(\cdot, t_n))(1)$, we have

$$\begin{aligned}
& \left([u(1, t_n) - g_2]_+ - [g_1 - u(1, t_n)]_+ - [u_h^n(1) - g_2]_+ + [g_1 - u_h^n(1)]_+ \right) y^n(1) \\
& \leq c |u(1, t_n) - u_h^n(1)| |y^n(1)| \\
& \leq c \left(\|e_x^n - q^n\|^2 + \|q^n\|^2 + \|p^n\|^2 \right) + \frac{\zeta}{4} \|y_x^n - p^n\|^2.
\end{aligned}$$

5) We now set

$$Z_n = \|y^n\|^2 + \|p^n\|^2 + \|r^n\|^2 + \|q_x^n\|^2 + \|e_x^n - q^n\|^2.$$

Collecting all these estimates, we observe that

$$(1 - 2c\delta t)Z_n \leq Z_{n-1} + 2\delta t R_n, \quad (3.16)$$

where the residual R_n is the sum of the two approximation errors

$$\|(u_t(t_n) - P_h^* u_t(t_n))_x\|^2, \quad \|\phi_t(t_n) - P_h^* \phi_t(t_n)\|^2,$$

and 15 more complex terms.

6) All these terms combine time error which is estimated by using Taylor's expansion in time of the exact solution u and space error which can be bounded from (3.5). This yields

$$\delta t \sum_{k=0}^{n-1} R_k \leq c \left(h^2 + (\delta t)^2 \right),$$

where the constant c only depends on appropriate norms of the solution u , see the definition of the space \mathbb{X}_* .

7) To conclude, we apply the discrete Gronwall lemma to the inequality (3.16), see [17, Lemme 22.7]:

$$Z_n \leq Z_0 e^{cn\delta t} + \delta t \sum_{k=0}^{n-1} R_k e^{c'(n-k-1)\delta t}.$$

It is readily checked that $Z_0 = 0$. Since $n\delta t \leq T$, this gives the right bound for each Z_n and applying a triangle inequality and (3.5) gives the desired result.

Estimate (3.14) is fully optimal and the regularity of the solution which is required for it seems reasonable. In any case, combining this bound with the stability property stated in Corollary 3.2 implies the convergence of the discretization.

4. An iterative algorithm.

As standard, to solve the nonlinear problem (3.7), we use an iterative algorithm that we now describe. Assuming that $(\hat{u}^{n-1}, \hat{\phi}^{n-1}, \theta^{n-1})$ is known, we first set:

$$u_h^{n,0} = u_h^{n-1}, \quad \hat{u}_h^{n,0} = \hat{u}_h^{n-1}, \quad \phi_h^{n,0} = \phi_h^{n-1}, \quad \hat{\phi}_h^{n,0} = \hat{\phi}_h^{n-1} \quad \text{and} \quad \theta_h^{n,0} = \theta_h^{n-1}. \quad (4.1)$$

Next, we solve iteratively the problem:

$$\begin{aligned} & \text{Find } (\hat{u}_h^{n,\ell}, \hat{\phi}_h^{n,\ell}, \theta_h^{n,\ell}) \text{ in } \mathbb{S}_h^* \times \mathbb{S}_h^* \times \mathbb{S}_h^0 \text{ satisfying} \\ \forall w_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t} (\hat{u}_h^{n,\ell} - \hat{u}_h^{n-1}, w_h) + (u_{hx}^{n-1/2,\ell} - \phi_h^{n-1/2,\ell}, w_{hx}) + \zeta(\hat{u}_{hx}^{n,\ell} - \hat{\phi}_h^{n,\ell}, w_{hx}) \\ & + \left(\frac{N(u_h^{n,\ell-1}) + N(u_h^{n-1})}{2} u_{hx}^{n-1/2,\ell}, w_{hx} \right) + g(u_h^{n,\ell-1})(1)w_h(1) = 0, \\ \forall \chi_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t} (\hat{\phi}_h^{n,\ell} - \hat{\phi}_h^{n-1}, \chi_h) + (\phi_{hx}^{n-1/2,\ell}, \chi_{hx}) + (\phi_h^{n-1/2,\ell} - u_{hx}^{n-1/2,\ell-1}, \chi_h) \\ & + \zeta(\hat{\phi}_h^{n,\ell} - \hat{u}_{hx}^{n,\ell-1}, \chi_h) + \zeta(\hat{\phi}_{hx}^{n,\ell}, \chi_{hx}) - a(\theta^{n-1/2,\ell}, \chi_{hx}) = 0, \\ \forall \eta_h \in \mathbb{S}_h^0, \quad & \frac{1}{\delta t} (\theta_h^{n,\ell} - \theta_h^{n-1}, \eta_h) + (\theta_{hx}^{n-1/2,\ell}, \eta_{hx}) + a(\hat{\phi}_{hx}^{n,\ell-1}, \eta_h) = 0, \end{aligned} \quad (4.2)$$

where the following notation, valid for any sequence $(y^{n,\ell})$,

$$y^{n-1/2,\ell} = \frac{y^{n,\ell} + y^{n-1}}{2}, \quad y^{n,\ell} = y^{n-1} + \delta t \hat{y}^{n,\ell} \quad (4.3)$$

Despite its complex form, problem (4.2) is easy to solve. We now prove its well-posedness.

Proposition 4.1. *For any data $(u_0, u_1, \phi_0, \phi_1, \theta_0)$ satisfying (3.11), for $n = 1, \dots, N$ and any positive integer ℓ , problem (4.2) has a unique solution when δt is small enough.*

Proof: We proceed by induction on n and ℓ . Indeed, the initial conditions are given by (3.6) and (4.1). At each step (n, ℓ) , problem (4.2) results into a square finite-dimensional linear system. So, assume that all data $(\hat{u}_h^{n-1}, \hat{\phi}_h^{n-1}, \theta_h^{n-1})$ and $(\hat{u}_h^{n,\ell-1}, \hat{\phi}_h^{n,\ell-1}, \theta_h^{n,\ell-1})$ and also $(u_h^{n-1}, \phi_h^{n-1})$ are zero and note from (4.3) that $y^{n,\ell} = \delta t \hat{y}^{n,\ell}$. It can thus be re-written

$$\begin{aligned} \forall w_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t^2} (u_h^{n,\ell}, w_h) + \frac{1}{2} (u_{hx}^{n,\ell} - \phi_h^{n,\ell}, w_{hx}) + \frac{\zeta}{\delta t} (u_{hx}^{n,\ell} - \phi_h^{n,\ell}, w_{hx}), \\ & + \left(\frac{\beta}{2} u_{hx}^{n,\ell}, w_{hx} \right) = 0, \\ \forall \chi_h \in \mathbb{S}_h^*, \quad & \frac{1}{\delta t^2} (\phi_h^{n,\ell}, \chi_h) + \frac{1}{2} (\phi_{hx}^{n,\ell}, \chi_{hx}) + \frac{1}{2} (\phi_h^{n,\ell}, \chi_h) \\ & + \frac{\zeta}{\delta t} (\phi_h^{n,\ell}, \chi_h) + \frac{\zeta}{\delta t} (\phi_{hx}^{n,\ell}, \chi_{hx}) - \frac{1}{2} a(\theta^{n,\ell}, \chi_{hx}) = 0, \\ \forall \eta_h \in \mathbb{S}_h^0, \quad & \frac{1}{\delta t} (\theta_h^{n,\ell}, \eta_h) + \frac{1}{2} (\theta_{hx}^{n,\ell}, \eta_{hx}) = 0. \end{aligned} \quad (4.4)$$

By taking η_h equal to $\theta_h^{n,\ell}$ in the last line, we immediately derive that $\theta_h^{n,\ell}$ is zero. On the other hand, taking χ_h equal to $\phi_h^{n,\ell}$ in the second equation thus yields that $\phi_h^{n,\ell}$ is zero. Finally, taking w_h equal to $u_h^{n,\ell}$ in the first equation implies that $u_h^{n,\ell}$ is zero (even when β is negative, $|\beta|\delta t$ is smaller than $\frac{1}{\delta t}$ or ζ). As a consequence, problem (4.2) has at most a solution, hence has a unique solution.

It follows from the previous proof that problem (4.2) results into three uncoupled equations: Solve first the equation on $\theta_h^{n,\ell}$, next the equation on $\hat{\phi}_h^{n,\ell}$ and finally the equation on $\hat{u}_h^{n,\ell}$. This brings to light the interest of the algorithm.

The consistency of problem (4.2) with problem (3.7) is readily checked (suppress all the ℓ in (4.2)). On the other hand, its convergence can be proved by the same technical arguments as for Theorem 3.4. So, we prefer to skip it. In any case, we solve the well-posed problem (4.2) either a finite number of times, say L times, or until the smaller L such that the difference between $(u_h^{n,L}, \phi_h^{n,L}, \theta_h^{n,L})$ and $(u_h^{n,L-1}, \phi_h^{n,L-1}, \theta_h^{n,L-1})$ in an appropriate norm becomes smaller than a given tolerance η^* . We finally set:

$$\hat{u}_h^n = \hat{u}_h^{n,L}, \quad \hat{\phi}_h^n = \hat{\phi}_h^{n,L}, \quad \theta_h^n = \theta_h^{n,L}. \quad (4.5)$$

The numerical experiments in the next section confirm the efficiency of this approach.

5. Numerical experiments.

In our simulations we take

$$g_1 = -0.01, \quad g_2 = 0.1, \quad \zeta = 0.1, \quad a = 0.017, \quad \rho = 1, \quad \epsilon = 0.001. \quad (5.1)$$

The discretization parameters are fixed equal to

$$h = 10^{-2}, \quad \delta t = 0.0001, \quad (5.2)$$

and the triangulation \mathcal{T}_h is uniform. A tolerance $\eta^* = 10^{-7}$ is used to stop the iterative procedure.

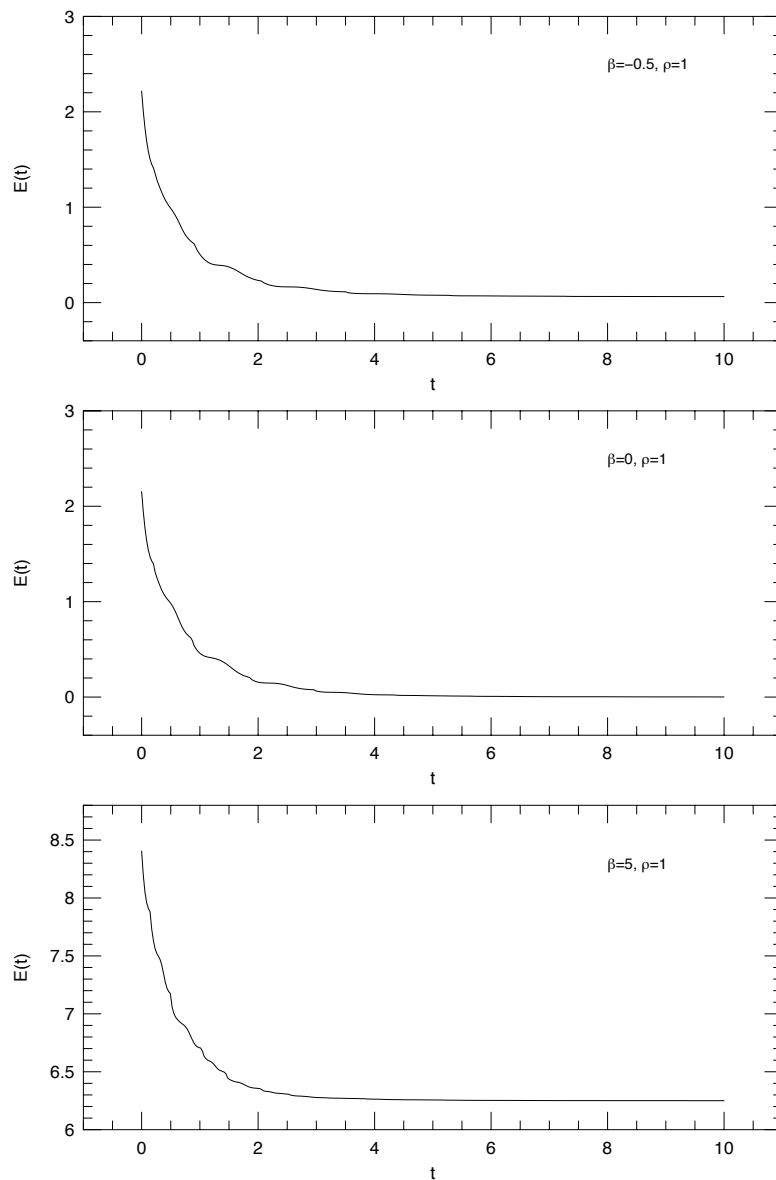


Figure 2. The energy as a function of t , for $\beta = -0.5, 0$ and 5 .

We work with the initial values

$$u_0(x) = 0.01x(x - 2), \quad u_1(x) = 20x(x - 1)^2, \quad \phi_0(x) = 0, \quad \phi_1(x) = 0. \quad (5.3)$$

We now investigate the influence of the initial temperature, for three different values of β .

In the first experiment, we choose

$$\theta_0(x) = \sin(\pi x), \quad (5.4)$$

and we work with the three values $\beta = -0.5, 0$ and 5 . The evolution of the system is towards the state $u = \theta = \phi = 0$ and the energy decays to the limit $\beta^2/4$, (see Figure 2). Figures 3 and 4 show the displacement and the angular rotation at the end $x = 1$. In Figure 3, we observe that, at the beginning, the beam gets in contact with both obstacles, with the frequency of oscillations increasing with the size of β . The amplitude of oscillations present in the profiles of ϕ are larger when $\beta = -0.5$ (Figure 4).

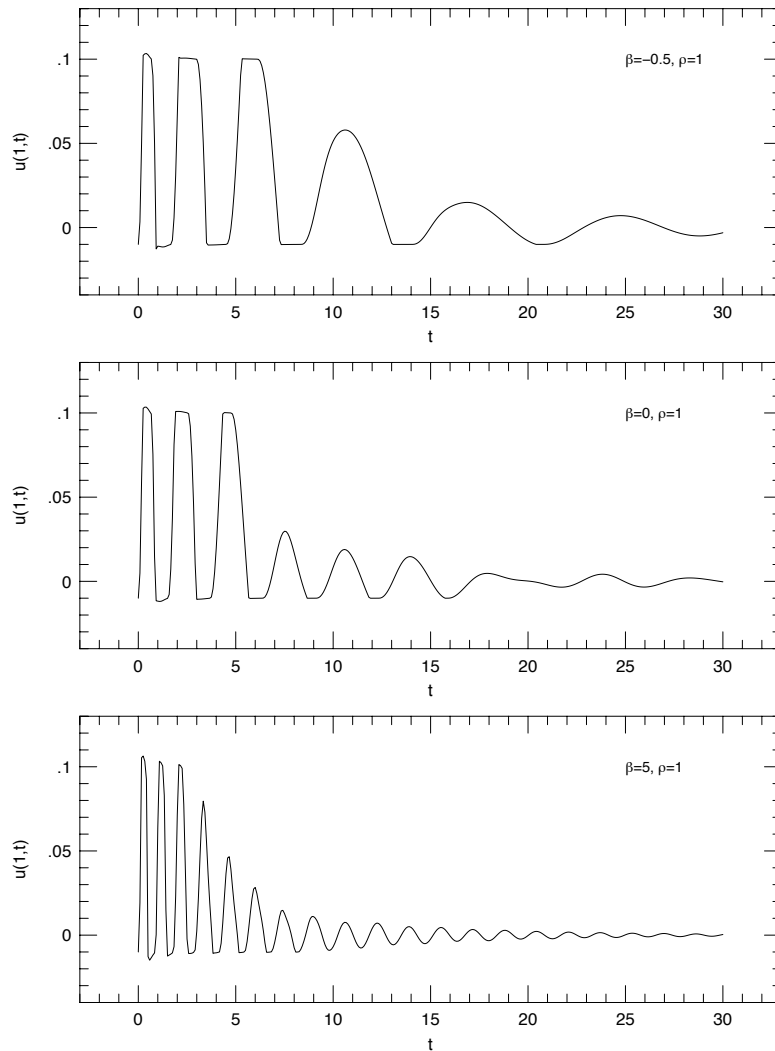


Figure 3. The displacement at $x = 1$ as a function of t , for $\beta = -0.5, 0$ and 5 .

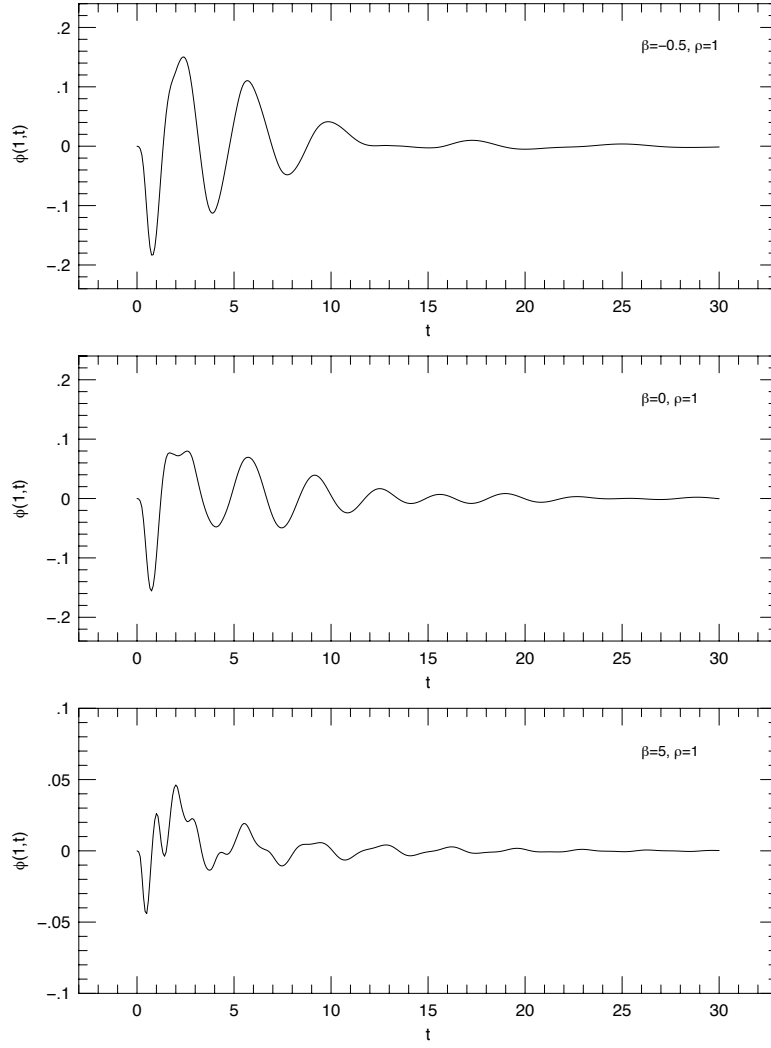


Figure 4. The angular rotation at $x = 1$ as a function of t , for $\beta = -0.5, 0$ and 5 .

Next, we change the initial temperature to

$$\theta_0(x) = 20 \cos(5\pi x/2). \quad (5.5)$$

In this case, when $\beta = -0.5$ and $\beta = 0$, at the steady-state, the beam is in contact with the upper obstacle. The results are presented in Figures 5 and 6.

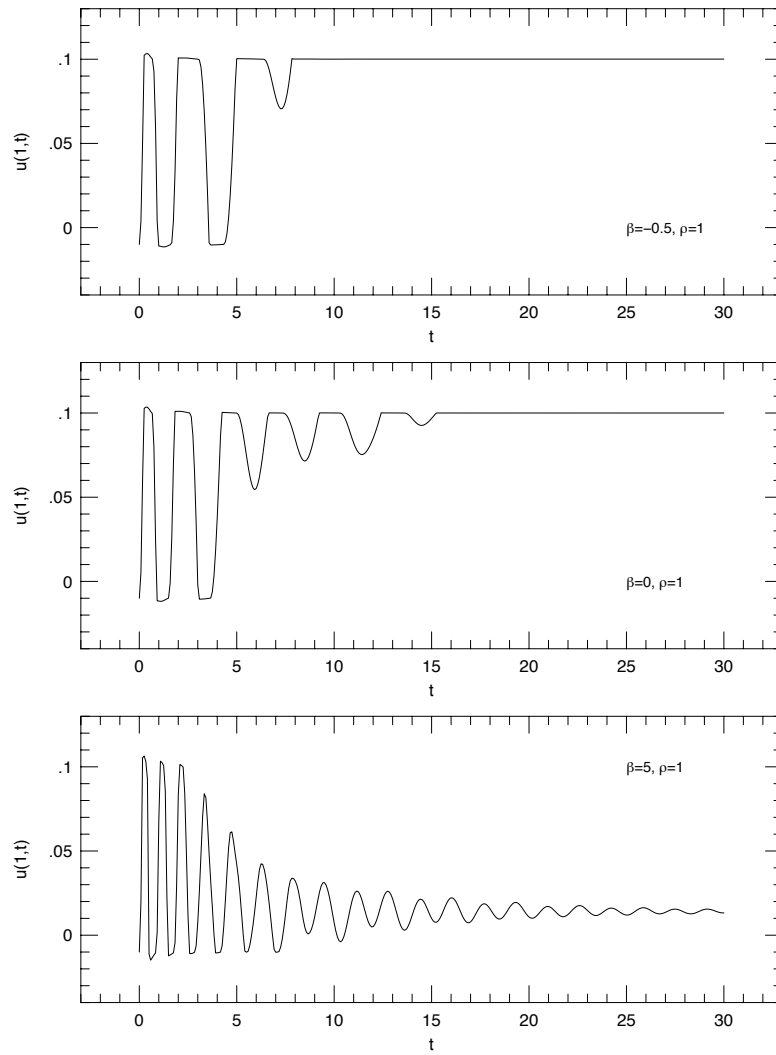


Figure 5. The displacement at $x = 1$ as a function of t , for $\beta = -0.5, 0$ and 5 .

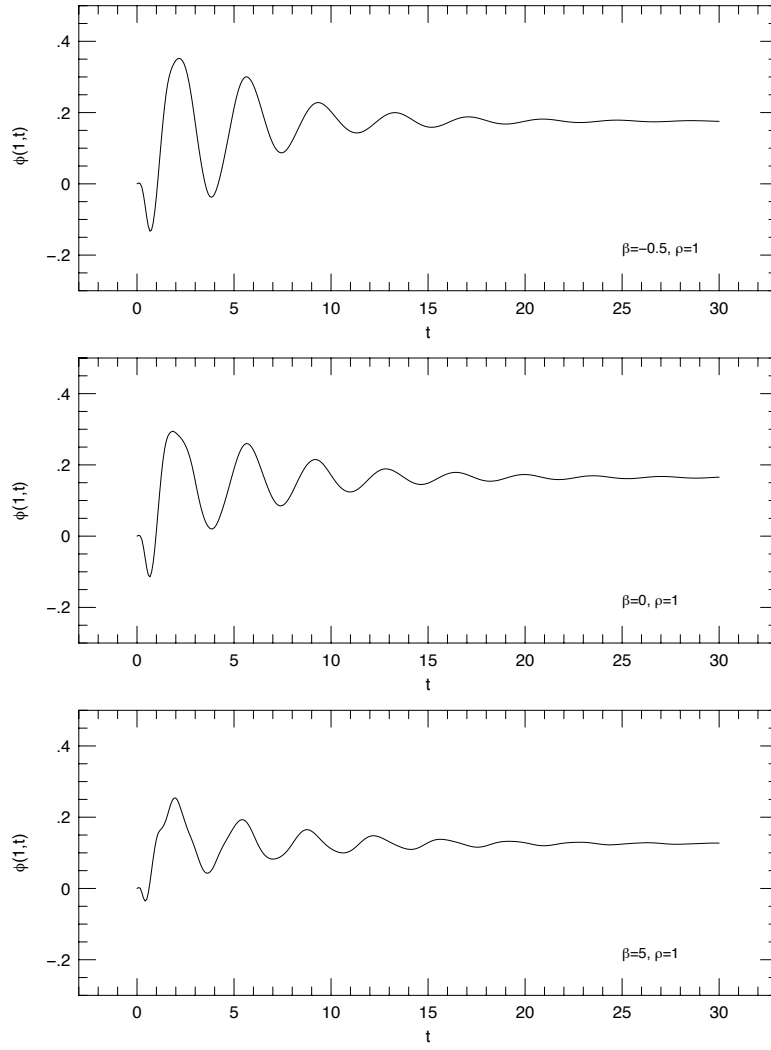


Figure 6. The angular rotation at $x = 1$ as a function of t , for $\beta = -0.5, 0$ and 5 .

In the last experiment, we still work with the initial value of θ given in (5.4). At least for smooth solutions, this means that

$$\theta_A = 0,$$

and in this case $u = \phi = \theta = 0$ is a solution to the stationary problem. We run an experiment with $\beta = -0.8$ and obtain a stationary solution which is not zero. On the other hand, the uniqueness of the solution of the stationary problem is proven in [3, Thm 2.8] only with the condition $\beta > -\frac{1}{2}$, and the numerical experiment gives evidence that, for $\beta < -\frac{1}{2}$, there are multiple steady-state solutions.

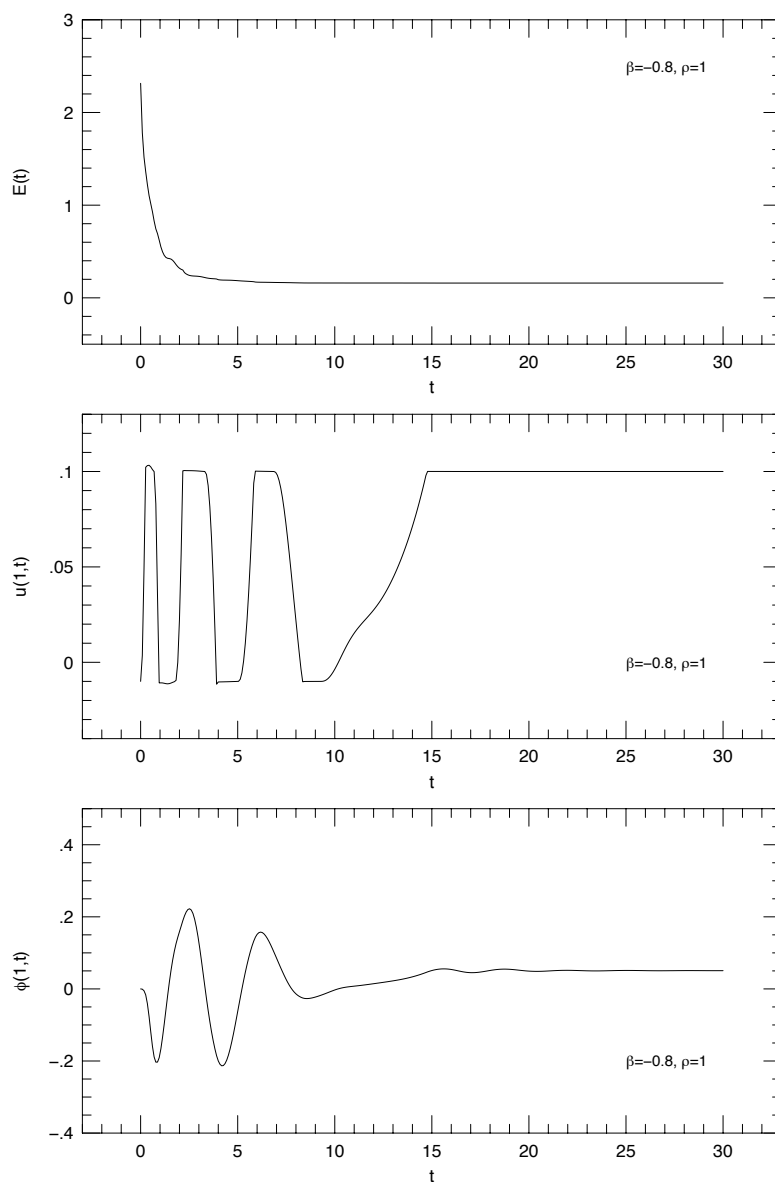


Figure 7. The energy and the solution (u_h, ϕ_h) at $x = 1$ as a function of t for $\beta = -0.8$.

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