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# Ranking sets of interacting objects via semivalues

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## Abstract

In this paper, we address the problem of how to extend a ranking over single objects to another ranking over all possible collections of objects, taking into account the fact that objects grouped together can have mutual interaction.

An answer to this issue is provided by using game theory and, specifically, the fact that an extension (i.e., a total preorder on the set of all subsets of objects) must be *aligned* with some probabilistic value, in the sense that the ranking of the objects (according to some probabilistic value computed on a numerical representation of the extension) must also preserve the primitive preorder on the singletons, no matter which utility function is used to represent the extension.

We characterize families of *aligned* extensions, we focus on their geometric properties and we provide algorithms to verify their alignments. We also show that the framework introduced in this paper may be used to study a new class of extension problems, which integrate some features dealing with *risk* and *complete uncertainty* within the class of preference extension problems known in the literature with the name of *sets as final outcomes*.

## 1 Introduction

Many problems, inspired by individual and collective decision making, deal with preferences over collections of objects (e.g., alternatives, opportunities, candidates, etc.). Consider, for instance, the selection of the candidate members for the formation of evaluation committees, the assessments of equity of sets of rights inside a society, the comparison of assets in portfolio analysis, the comparison of the stability of groups in coalition formation theory, and many others. In practice, however, mainly the information about preferences among single objects is available, and then a central question may arise: given a primitive ranking over the single elements of a set  $N$ , how to derive a compatible ranking over the set of all subsets of  $N$ ?

This question has been carried out in the tradition of the literature on extending an order on a set  $N$  to its *power set* (denoted by  $2^N$ ) with the objective to axiomatically characterize families of ordinal preferences over subsets (see,

for instance, [2, 3, 4, 5, 10, 13, 15, 16]). In this context, an order  $\succsim$  on the power set  $2^N$  is required to be an *extension* of a primitive order  $R$  on  $N$ . This means that the relative ranking of any two singleton sets according to  $\succsim$  must be the same as the relative ranking of the corresponding alternatives according to  $R$  (i.e., for each  $a, b \in N$ ,  $\{a\} \succsim \{b\} \Leftrightarrow aRb$ ). According to this definition, in the remaining of the paper we will simply refer to a total preorder  $\succsim$  on  $2^N$  as an *extension* on  $2^N$ , with the implicit assumption that such a relation  $\succsim$  ranks singletons in  $2^N$  in the same way of some equivalent primitive total preorder  $R$  on  $N$ .

The different axiomatic approaches in the literature are related to the interpretation of the properties used to characterize extensions, that is deeply interconnected to the meaning that is attributed to sets. In fact, according to the survey of [3], the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: 1) *complete uncertainty*, where a decision-maker is asked to rank sets which are considered as formed by mutually exclusive objects (i.e., only one object from a set will materialise), and taking into account that he cannot influence the selection of an object from a set (see, for instance, [2, 15, 21]); 2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, a Decision Maker (DM) compares sets taking into account that he will be able to select a single element from a set (see, for example, [5, 16, 23]); 3) *sets as final outcomes*, where each set contains objects that are assumed to materialise simultaneously (if that set is selected; for instance, see [4, 10, 24]).

In this paper, we focus on the problem of the third class, where sets of elements materialize simultaneously. A standard application of this kind of problems is the *college admissions problem* [24, 12], where colleges need to rank sets of students based on their ranking of individual applicants. In this framework, most of the approaches present in the literature do not take into account possible interactions between objects. For instance, a typical assumption is the property of *responsiveness* (RESP) [24]: an extension  $\succsim$  on  $2^N$  satisfies RESP if for all  $i, j \in N$  and all  $S \in 2^N$ ,  $i, j \notin S$

$$\{i\} \succsim \{j\} \Leftrightarrow S \cup \{i\} \succsim S \cup \{j\}. \quad (1)$$

Clearly, this means that if an object  $i$  is better than another object  $j$ , then it is better to add  $i$  rather than  $j$  to *any* other subcollection of objects. Clearly, the RESP property does not take into account the fact that some objects together can present some form of incompatibility or, on the contrary, of mutual enforcement.

For this reason a recent approach has been developed in [20] with the objective to take into account the possible interactions between objects. The model introduced in [20] relies on the fact that an utility function attached to a total preorder on  $2^N$  represents a coalitional game in characteristic function form (provided we set the utility of the empty set to be equal to zero). Since probabilistic values [26, 6, 7, 17, 22] do provide a natural ranking among the elements of the set  $N$  (the players in the game theoretical context, the objects in this

approach), we can use them in the following sense. According to [20], given a probabilistic value  $\pi$ , an extension  $\succsim$  on  $2^N$  will be a  $\pi$ -aligned extension on  $2^N$ , provided that from  $\{i\} \succsim \{j\}$  it follows that  $\pi$  ranks better  $i$  than  $j$  for every possible choice of the utility function representing  $\succsim$ . In other terms, an extension  $\succsim$  on  $2^N$  is such that it never changes the mutual positions in the rankings of the objects, no matter is the numerical representation we give to the preference relation on  $2^N$ . Here we study in details the problem of characterizing these extensions, both for a single probabilistic value and for a whole family of values.

Another interpretation of the framework used in [20], which further motivates the interest for  $\pi$ -aligned extensions and where some aspects related to the framework of uncertainty are merged with considerations about the problem of ranking sets as final outcomes, is illustrated in the following example.

Suppose you receive an invitation for a dinner and, being considered an expert in food and wine parties, you are asked to bring a bottle of wine to accompany the main course. Further, suppose that for some reason you don't know which main course will be served. In general, drinking between meals, you might prefer to have red wine than white wine. However, as accompanying drink of a meal based on fish, you might prefer to have white wine than the red one. Typically, your preferences over meals (which are collection of food items) do not satisfy the RESP property (you prefer the singleton {'red wine'} to the {'white wine'} one as a between meals drink, but the meal {'fish' and 'white whine'} is preferred to {'fish' and 'red wine'}). On the other hand, under certain conditions concerning both the probability distribution and your preferences over all possible meals, you still might decide to bring with you at dinner a bottle of red wine, preserving your original preference on drinks between meals. This example shows the importance of considering the possibility of mutual enforcement among objects in the context where subsets of objects materialize randomly. But it also shows that the criterion used to compare two different single objects takes into account their respective ability to improve the relative ranking of (stochastically generated) collection of objects. In addition, assuming that a fish-based meal is selected, what really matters in the decision making process specifying your preference is the expectation of how much your choice will marginally contribute to the success of the dinner event and to sustain your good reputation as a food and wine expert. Consistently, you might also prefer the event where you bring a white wine bottle to a fish-based dinner where red wine is already served, than the event where you bring a red wine bottle to a fish-based dinner where white wine is already served, even if in both cases you are going to join a dinner where both white and red wine are served.

The above example of "choosing the wine for a dinner" can be seen as a particular instance of the more general problem of modelling the preferences of a consumer facing a future randomly generated endowment of goods, where a DM (e.g., a consumer) must choose a single object from a set  $N$  (e.g., a set of goods), knowing that after his choice the DM will receive at a second step and at random, a (possibly empty) collection of the other objects in  $N$ . More precisely, for each object  $a \in N$ , a subset  $S \subseteq N \setminus \{a\}$  is randomly

generated (and all of its elements materialize simultaneously) according to a certain probability distribution  $p(S)$  over all subsets  $S$  of  $N \setminus \{a\}$ . Suppose also that the DM is aware of the probability distribution  $p$  over  $2^{N \setminus \{a\}}$ , for each  $a \in N$ . So, we can look at this problem as the one where a DM must choose among different lotteries associated to objects in  $N$ , where each lottery is composed by a deterministic part, the object selected by the DM at the first step of the process, and a stochastic one, the random collection generated with probability  $p$  at the second step. Trying to answer the question of how the DM will compare each pair of object  $a, b$  in  $N$  in such a situation, we might be interested in understanding under which conditions the relative ranking of lotteries associated to objects  $a$  and  $b$  preserves the primitive preference  $\succsim$  over the corresponding singletons  $\{a\}$  and  $\{b\}$ . Clearly, a RESP extension does preserve the primitive preference over the singletons: whatever coalition  $S$  will materialize at the second step (according to the random mechanism governed by probability  $p$ ), having  $S \cup \{a\}$  is preferred to have  $S \cup \{b\}$  whenever  $\{a\}$  is preferred to  $\{b\}$ . Consequently, the lottery associated to objects  $a$  dominates the one associated to  $b$ . But also other extensions on  $2^N$ , that do not necessarily satisfy the RESP property, may guarantee as well the preservation of the original preferences over the singletons. Henceforth, knowing the value function of a DM over the collection of objects of  $N$ , we should focus on the marginal contribution of each element  $a$  to each subset  $S \in 2^N$  not containing  $a$ , and we would apply the classical framework of expected utility theory for comparing the expected marginal contribution of each object in  $N$  to all possible collections of other objects, i.e.  $\pi_a(u) = \sum_{S \subseteq N \setminus \{a\}} p(S)(u(S \cup \{a\}) - u(S))$  of each element  $a$  in  $N$ .

Unfortunately, in our context, there is no assumption about the risk attitude of a DM, and this makes it impossible to univocally determine the expected utility of an object, face to the successive random selection of a collection. A natural way to avoid this ambiguity is to focus on those extensions  $\succsim$  on  $2^N$  preserving the primitive ranking over singletons whatever utility function representing  $\succsim$  is considered. More precisely, we concentrate our study to extensions such that

$$a \succsim b \Leftrightarrow \pi_a(u) \geq \pi_b(u) \tag{2}$$

for every value function  $u : 2^N \rightarrow \mathbb{R}$  representing  $\succsim$  (and with  $u(\emptyset) = 0$ ), which is precisely the definition of “ $\pi$ -aligned extensions” provided in the next section.

Putting up the argument that in some cases a DM could ignore the probabilistic distribution governing the stochastic generation of collections of objects at the second step of the process, we observe that the framework introduced above can be easily adapted to a situation of (complete) uncertainty by simply requiring that relation (2) holds for every probabilistic value [26]. This is the main reason why in this paper we are also interested in studying families of extensions that are  $\pi$ -aligned to all probabilistic values, or alternatively, to sub-families of these values. In fact, the most interesting results introduced in this paper deal with the well studied family of semivalues [25, 1, 9, 6, 7, 8, 19, 11, 17], where the probability distribution over the subsets  $S$  of  $N$  not containing  $i$  is a

function of the cardinality of  $S$ , and it is the same for all  $i$  in  $N$ . Focusing on the family of semivalues allows to capture some interesting geometric properties of  $\pi$ -aligned extensions that will be presented in Section 4. Moreover, since it is, as we shall argue, useful to characterize preorders which are aligned for a large class of values, we shall see that some restriction on the (probabilistic) values must be imposed, since there exists no total preorder aligned with all probabilistic values. And the class of semivalues naturally emerges in this context.

The outline of the paper is as follows. After some preliminaries and basic definitions, in Section 3 we provide a full analysis of all possible  $\pi$ -aligned extensions on the powerset of a set with three objects, together with the conditions to discerning extensions which are  $\pi$ -aligned with some semivalues from others which are not. This analysis for a “simple” case is preliminary to the geometric characterization of  $\pi$ -aligned extensions in the general case of  $n$  objects, introduced in Section 4. In Section 5, the infinite system of linear inequalities used for the geometric characterization introduced in Section 4 is reduced to a finite system, that works for all semivalues, by means of a decomposition based on dichotomous total preorders. A faster algorithm to check alignment with semivalues based on rational probabilities is then presented in Section 6, together with a characterization inspired to the one for the “Banzhaf-aligned” extensions [1, 20]. Finally, the case of complete uncertainty, where extensions are aligned with all semivalues, is discussed in Section 7. Section 8 draws some conclusions.

## 2 Preliminaries

We are given a *finite* set  $N$ , of cardinality  $n$ . A TU-game on  $N$  is a function  $v : 2^N \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ . A *probabilistic value* (or *probabilistic power index*)  $\pi$  for the game  $v$  is an  $n$ -vector  $\pi^p(v) = (\pi_1^p(v), \pi_2^p(v), \dots, \pi_n^p(v))$ , such that

$$\pi_i^p(v) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S) m_i(S) \quad (3)$$

where  $m_i(S) = v(S \cup \{i\}) - v(S)$  is the *marginal contribution* of  $i$  to  $S \cup \{i\}$ , for each  $i \in N$  and  $S \in 2^{N \setminus \{i\}}$ , and  $p = (p^i : 2^{N \setminus \{i\}} \rightarrow \mathbb{R}^+)_{i \in N}$ , is a collection of non negative real-valued functions fulfilling the condition  $\sum_{S \in 2^{N \setminus \{i\}}} p^i(S) = 1$ . A probabilistic value  $\pi_i^p$  is called *regular* when all the coordinates of  $p$  are strictly positive functions. A particular interesting case is when the probabilistic value  $\pi^p$  is a *semivalue*, which means that non negative weights  $p_0, \dots, p_{n-1}$  are given such that  $p^i(S) = p_s$ , whenever the cardinality of coalition  $S$ , denoted by  $|S|$ , is equal to  $s$  and  $i \in N$ ; furthermore, it is required that  $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ , in order to fulfil the condition  $\sum_{S \in 2^{N \setminus \{i\}}} p_s = 1$ ; thus  $\mathbf{p} = (p_0, \dots, p_{n-1})$  represents a probability distribution on the family of the subsets of  $N$  not containing  $i$ , and it is the same for all  $i \in N$ . We shall denote by  $\mathbf{p}$  a vector  $(p_0, \dots, p_{n-1})$  as above, and, by a slight abuse of notation,  $\pi^{\mathbf{p}}$  is the semivalue engendered by

the vector  $\mathbf{p}$ . Hence

$$\pi_i^{\mathbf{P}}(v) = \sum_{S \in 2^N \setminus \{i\}} p_s m_i(S).^1 \quad (4)$$

The two most famous regular semivalues (i.e., with  $p_s > 0$ , for each  $s = 0, \dots, n-1$ ) are the Shapley value, with  $p_s = \frac{1}{n \binom{n-1}{s}}$ , and the Banzhaf index, with  $p_s = \frac{1}{2^{n-1}}$ , for each  $s = 0, \dots, n-1$ . Other regular semivalues are present in the literature [17, 22, 8].

We shall denote by  $\mathcal{P}$  the set of all probabilistic values and by  $\mathcal{S}$  ( $\mathcal{RS}$ , respectively) the set of all semivalues (regular semivalues, respectively), for the given fixed set  $N$ . From above, we know that  $\mathcal{S}$  is a simplex in an  $n$ -dimensional space, and  $\mathcal{RS}$  is its relative interior.

A *binary relation* on a finite set  $X$  is denoted by  $\succcurlyeq \subseteq X \times X$ . For each  $x, y \in X$ , the more familiar notation  $x \succcurlyeq y$  will be used instead of the more formal  $(x, y) \in \succcurlyeq$ . The following are some standard properties for a binary relation  $\succcurlyeq \subseteq X \times X$ :

- *reflexivity*: for each  $x \in X$ ,  $x \succcurlyeq x$ ;
- *transitivity*: for each  $x, y, z \in X$ ,  $x \succcurlyeq y$  and  $y \succcurlyeq z \Rightarrow x \succcurlyeq z$ ;
- *total*: for each  $x, y \in X$ ,  $x \neq y \Rightarrow x \succcurlyeq y$  or  $x \preccurlyeq y$ ;
- *antisymmetry*: for each  $x, y \in X$ ,  $x \succcurlyeq y$  and  $y \succcurlyeq x \Rightarrow x = y$ .

A *total preorder* on  $2^N$  is a reflexive, transitive and total binary relation  $\succcurlyeq \subseteq 2^N \times 2^N$ . A reflexive, transitive, total and antisymmetric binary relation is called *total order* or *linear order*. We interpret a total preorder  $\succcurlyeq$  on  $2^N$  as a *preference relation* on  $2^N$  (that is, for each  $S, T \in 2^N$ ,  $S \succcurlyeq T$  stands for ‘ $S$  is considered at least as good as  $T$  according to  $\succcurlyeq$ ’), and we will refer to it with the term *extension* on  $2^N$  (of a primitive total preorder  $R$  on  $N$  such that  $iRj \Leftrightarrow \{i\} \succcurlyeq \{j\}$ ).

In this paper, a central property for extensions on  $2^N$  is the *responsiveness* (RESP) property:

**Definition 1** (RESP). *A total preorder  $\succcurlyeq$  on  $2^N$  satisfies the RESP property on  $2^N$  if for all  $i, j \in N$  and all  $S \in 2^N$ ,  $i, j \notin S$  we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow S \cup \{i\} \succcurlyeq S \cup \{j\}. \quad (5)$$

Now, suppose we have an extension  $\succcurlyeq$  on  $2^N$ . This relation naturally induces a TU game for each utility function  $v$  *representing*  $\succcurlyeq$  (such that  $v(\emptyset) = 0$ ), i.e.  $u(S) \geq u(T) \Leftrightarrow S \succcurlyeq T$  for each  $S, T \in 2^N$ . We shall denote by  $V(\succcurlyeq)$  the set of all  $v$  representing the total preorder  $\succcurlyeq$ . Fixed a semivalue, each  $v \in V(\succcurlyeq)$  provides a ranking on the elements of  $N$ . This ranking in general depends on the given utility function selected in  $V(\succcurlyeq)$ .

In this paper we want to investigate several issues related to the following definition.

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<sup>1</sup>We shall use the notation  $\pi_i^q(v)$  also for any  $0 \leq q = (q_0, \dots, q_{n-1})$ .



**Definition 2** ( $\pi^p$ -alignment). *Given a set  $N$ , a total preorder  $\succsim$  on  $2^N$  and a probabilistic value  $\pi^p \in \mathcal{P}$ , we shall say that  $\succsim$  is  $\pi^p$ -aligned if*

$$\{i\} \succsim \{j\} \Leftrightarrow \pi_i^p(v) \geq \pi_j^p(v)$$

for each  $v \in V(\succsim)$ .

In the sequel, we shall make use of the following notations. We shall denote by  $\Sigma_i$  ( $\Sigma_{ij}$ ) the set of all subsets of  $N$  which do not contain  $i$  (neither  $i$  nor  $j$ ). Also, denote by  $\Sigma_{ij}^s$  the set of the subsets of  $\Sigma_{ij}$  of cardinality  $s$ .

Given a semivalue  $\pi^P \in \mathcal{S}$  and a TU game with  $N$  as set of players, we shall constantly consider the difference  $\pi_i^P(v) - \pi_j^P(v)$ . The next two formulas are useful to this aim. The first one is obvious:

$$\pi_i^P(v) - \pi_j^P(v) = \sum_{s=0}^{n-1} p_s [c_i(s) - c_j(s)] \quad (6)$$

where

$$c_i(s) = \sum_{S \in \Sigma_i: |S|=s} [v(S \cup \{i\}) - v(S)]. \quad (7)$$

For the second, denote by  $d_{ij}^S(v)$  the difference  $d_{ij}^S(v) = v(S \cup \{i\}) - v(S \cup \{j\})$ . Then the following formula holds:

$$\pi_i^P(v) - \pi_j^P(v) = \sum_{s=0}^{n-2} (p_s + p_{s+1}) \left[ \sum_{S \in \Sigma_{ij}^s} d_{ij}^S(v) \right]. \quad (8)$$

For, it is enough to observe that, for every  $h, k \in N$ ,

$$\begin{aligned} \pi_h^P(v) = & \sum_{s=0}^{n-2} \left[ \sum_{S \in \Sigma_{h,k}^s} p_s (v(S \cup \{h\}) - v(S)) + \right. \\ & \left. \sum_{S \in \Sigma_{h,k}^s} p_{s+1} (v(S \cup \{h\} \cup \{k\}) - v(S \cup \{k\})) \right]. \end{aligned}$$

### 3 Lessons from the case $N = \{1, 2, 3\}$

We consider now the case of three objects, thus  $N = \{1, 2, 3\}$ . The set of all total preorders on the subsets of  $N$  is not small, its cardinality being 545835. However the classification of the preorders aligned with some semivalue can be fully characterized with no much difficulty. To perform our analysis, let us make some preliminary consideration. Suppose we are given a preorder  $\succsim$ , and let us denote by  $i, j, k$  the elements of  $N$ . Then the basic formula (8) becomes, for  $i, j$ :

$$\pi_i^P(v) - \pi_j^P(v) = (p_0 + p_1)(v(\{i\}) - v(\{j\})) + (p_1 + p_2)(v(\{i, k\}) - v(\{j, k\})). \quad (9)$$

We have to consider the following cases:

1. suppose  $\{i\} \succ \{j\}$ . There are three possibilities:
  - 1.i)  $\{i, k\} \sim \{j, k\}$ . In this case it is clear that  $\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) > 0$  if and only if  $p_2 \neq 1$ ;
  - 1.ii)  $\{i, k\} \succ \{j, k\}$ . In this case,  $\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) > 0$  for each  $v \in V(\succneq)$ ;
  - 1.iii)  $\{j, k\} \succ \{i, k\}$ . In this case, one must consider two subcases:
    - 1.iii.a) suppose either  $\{j, k\} \succ \{i\}$  or  $\{j\} \succ \{i, k\}$ . In this case the inequality:
 
$$(p_0 + p_1)(v(\{i\}) - v(\{j\})) + (p_1 + p_2)(v(\{i, k\}) - v(\{j, k\})) > 0 \quad (10)$$
 for *any fixed*  $(p_0 + p_1), (p_1 + p_2)$  cannot be verified by *all compatible* functions  $v \in V(\succneq)$ , unless  $p_0 = 1$ ;
    - 1.iii.b) if instead
 
$$\{i\} \succneq \{j, k\} \succ \{i, k\} \succneq \{j\},$$
 then the difference in (9) has the right sign for all functions  $v$  if and only if  $p_0 > p_2$  if both the first and the last relations are actually indifference relations, and if and only if  $p_0 \geq p_2$  otherwise;
2. suppose now two objects are indifferent, say  $\{i\} \sim \{j\}$ . Then, in order to have that (9) is null as needed, either  $p_0 = 1$  or, in case  $p_0 < 1$ , it is clearly necessary and sufficient that  $\{i, k\} \sim \{j, k\}$ .

Summarizing the above considerations, we can state the following.

**Theorem 1.** *Let  $N = \{1, 2, 3\}$  and let  $\succneq$  be a total preorder on  $2^N$ . Then, the following alternatives hold:*

- i)  $\succneq$  is  $\pi^{\mathbf{P}}$ -aligned with the whole set of the semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$ . This happens if and only if for any pair of objects  $i, j \in N$  we have that either

$$\{i\} \succ \{j\} \text{ and } \{i, k\} \succ \{j, k\}$$

or

$$\{i\} \sim \{j\} \text{ and } \{i, k\} \sim \{j, k\};$$

- ii)  $\succneq$  is  $\pi^{\mathbf{P}}$ -aligned with the set of the semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$  such that  $p_0 \geq p_2$ . This happens iff for at least one pair of object  $i, j \in N$  we have that

$$\{i\} \succneq \{j, k\} \succ \{i, k\} \succneq \{j\},$$

and for the remaining pairs of objects in  $N$  the conditions specified at point (i) hold.

- iii)  $\succneq$  is  $\pi^{\mathbf{P}}$ -aligned only if  $p_0 = 1$ . This happens for all other cases not covered by the previous ones.

Notice that, in order to guarantee that  $\succsim$  is  $\pi^{\mathbf{P}}$ -aligned for  $\mathbf{p} = (0, 0, 1)$  it is necessary that, for each  $i, j \in N = \{1, 2, 3\}$ ,  $i \neq j$ , either  $[\{i\} \succ \{j\}]$  and  $\{i, k\} \succ \{j, k\}$  or  $[\{i\} \sim \{j\}]$  and  $\{i, k\} \sim \{j, k\}$ , with  $k \in N \setminus \{i, j\}$ , and therefore by Theorem 1.(i), relation  $\succsim$  is also aligned with all other semivalues in  $\mathcal{S}$ .

Theorem 1.(i) also shows that, for the case with three objects, the RESP property is a *necessary* and sufficient condition for an extension to be  $\pi^{\mathbf{P}}$ -aligned with all semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$ . As we will show in Section 7, this results does not hold for larger set of objects, i.e. the extensions that satisfy the RESP property are  $\pi^{\mathbf{P}}$ -aligned with all semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$  (as it is easy to see), but in general the converse is not true (see Example 4), and many extensions that do not satisfy the RESP property are  $\pi^{\mathbf{P}}$ -aligned with some semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$ .

On the other hand, as already mentioned, it is a nonsense to consider the whole class of probabilistic values. For, we claim that no total preorder can be aligned with all of them. To see this, fix a total preorder  $\succsim$  on  $2^N$  and consider two distinct objects in  $N = \{1, 2, 3\}$ , call them  $i$  and  $j$ , and suppose  $\{i\} \succsim \{j\}$ . Set  $p_i(S) = 0$  unless  $S = \{i\}$ , and  $p_j(S) = 0$  unless  $S = \{\emptyset\}$ . Then it is easy to verify that  $\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) = v(\{i, j\}) - 2v(\{j\})$ , that can clearly be made negative with a suitable choice of  $v$ .

## 4 A geometric description of aligned preorders

Now we start the study of the problem of establishing whether a given total preorder on  $2^N$  is aligned for a given semivalue, or also for a specific subfamily of semivalues.

First, let us make a very simple observation. It is quite possible that a given preorder is aligned with only the trivial semivalue  $(1, 0, \dots, 0)$ : we have already seen this in the case of three objects.

Another obvious remark is that when we want to prove that for some  $i, j \in N$ , it is  $\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) \geq 0$ , in formula (8) we can substitute the coefficients  $p_s + p_{s+1}$ ,  $s = 0, \dots, n-2$ , with coefficients  $a(p_s + p_{s+1})$  for every  $a > 0$ . Thus when the entries  $p_0, \dots, p_{n-1}$  are rational, we shall assume that the associated  $p_s + p_{s+1}$  are actually natural numbers. For instance, in the case of the Banzhaf semivalue, we can assume  $p_s + p_{s+1} = 1$  for all  $s$ .

For easy notation, we shall write  $x_{s+1}$  for  $p_s + p_{s+1}$ :  $x_{s+1} = p_s + p_{s+1}$ .

Now, in order to provide a geometric description of the set of semivalues aligned with a given preorder, we formulate the conditions in Equation (8) with a different notation. Set

$$\left[ \sum_{S \in \Sigma_{ij}^s} d_{ij}^S(v) \right] = a_{s+1}^{ijv}.$$

Denote by  $X$  the set of solutions of the following *semi-infinite system* of

linear inequalities:

$$a_1^{ijv} x_1 + a_2^{ijv} x_2 + \cdots + a_{n-1}^{ijv} x_{n-1} \geq 0, \quad v \in V(\succ), \quad i \succ j, \quad (11)$$

$$x_1 \geq 0, \dots, \quad x_{n-1} \geq 0 \quad (12)$$

Then the set of semivalues aligned with  $\succ$  is the set

$$\mathcal{A} = \{p = (p_0, \dots, p_{n-1}) : Lp \in X\} \cap \mathcal{S},$$

where  $L$  is the  $(n-1) \times n$  matrix

$$L = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}.$$

Clearly  $\mathcal{A}$  is a closed convex containing the vector  $(1, 0, \dots)$ .

Also the following remark is of some interest. The term  $a_1^{ijv} = v(\{i\}) - v(\{j\})$  is positive if and only if  $\{i\} \succ \{j\}$  and null if and only if  $i \sim j$ . This implies that if a nonnegative  $x$  satisfies the system (11), then also  $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_{n-1})$  satisfies the same system, for  $\bar{x}_1 = x_1 + k$ ,  $\bar{x}_i = x_i$  for  $1 < i \leq n-1$ . This in turn implies that when  $\bar{x}_1 < 1$ , then  $\succ$  is aligned with every semivalue  $\tilde{\mathbf{p}}$  fulfilling  $\tilde{p}_0 + \tilde{p}_1 = \bar{x}_1$  and  $\tilde{p}_s + \tilde{p}_{s+1} = \alpha \bar{x}_{s+1}$  for each  $s = 1, \dots, n-2$ , and with  $\alpha$  such that  $\sum_{s=0}^{n-1} \tilde{p}_s \binom{n-1}{s} = 1$ . Finally, it is clear that the system

$$Lp = x$$

has infinite (positive) solutions, for every  $x$ , depending on a free parameter. Summarizing the previous observations we get the following theorem.

**Theorem 2.** *Let  $\succ$  be a total preorder on  $2^N$  which is aligned with the elements of a set of semivalues  $\mathcal{A} \subseteq \mathcal{S}$ . We have that  $\mathcal{A}$  is either reduced to the singleton  $(1, 0, \dots, 0)$  or it is at least a two dimensional closed convex subset of the simplex  $\mathcal{S}$ .*

The next Example shows that the set of semivalues for which a total preorder is aligned can be exactly two dimensional even for  $n > 3$ .

**Example 1.** *Let  $N = \{1, 2, 3, 4, 5\}$  and consider the preference relation for which  $2^N$  is partitioned in three subsets, respectively  $VG$ ,  $G$  and  $B$ , where all elements in the same subset are indifferent and all elements of  $VG$  are strictly preferred to all elements of  $G$  which in turn are strictly preferred to all elements in  $B$ . The sets:*

$$VG = \{\{1\}, \{1, 3\}, \{2\}, \{2, 3, 4\}, \{4, 5\}, \{1, 2, 5\}, \{1, 2, 3, 4\}\}$$

$$G = \{\{3\}, \{1, 3, 5\}, \{4\}, \{1, 2, 4, 5\}, \{2, 4\}, \{3, 5\}\}$$

and

$$B = \{2^N \setminus \{VG \cup G\}\}.$$

Consider the system (11). By comparing the objects 1 and 2 one easily gets  $x_2 = x_3$  and by comparing the objects 3 and 4 it is seen that  $x_4 = x_3$ . A simple calculation shows that no further conditions must be added in order to verify the inequalities. This implies that the preference relation is aligned for every semivalue of the form

$$\mathbf{p} = (p_0, p_1, \frac{1 - p_0 - 2p_1}{2}, p_1, \frac{1 - p_0 - 2p_1}{2})$$

such that  $\pi^{\mathbf{p}} \in \mathcal{S}$ .

To conclude this section, let us observe that another particular case arises when considering the semivalue  $(0, \dots, 0, 1)$ . In this case the partial order is aligned with the semivalue if and only if  $\{i\} \succcurlyeq \{j\} \Rightarrow N \setminus \{j\} \succcurlyeq N \setminus \{i\}$ . In this case the set of semivalues aligned with  $\succcurlyeq$  contains the edge  $[(1, 0, \dots, 0), (0, \dots, 0, 1)]$ .

## 5 Alignment with a finite number of inequalities

In this section we prove that the infinite system of linear inequalities characterizing alignments, as shown in Section 4, can be reduced to a finite system. It will be also apparent that the system is still very heavy to deal with; for this reason in the following section we shall see a faster system to check alignment. However the characterization we prove here is theoretically important since it applies to all semivalues, while the next one is valid only for a (large) subset of semivalues.

Let  $\succcurlyeq$  be a total preorder on  $2^N$ . For each  $A \in 2^N$ , we denote by  $\mathcal{P}_{ij}^s(\succcurlyeq, A)$  the set of all subsets  $S$  containing neither  $i$  nor  $j$  and with cardinality  $s$ , such that  $S \cup \{i\}$  is weakly preferred to  $A$ , i.e.

$$\mathcal{P}_{ij}^s(\succcurlyeq, A) = \{S \in \Sigma_{ij}^s : S \cup \{i\} \succcurlyeq A\}.$$

Moreover, given a set  $T \in 2^N$  such that there exists  $B$  with  $T \succ B$ , we shall denote by  $T^\sigma$  an element of  $2^N$  such that  $T \succ T^\sigma$  and there is no  $C$  such that  $T \succ C \succ T^\sigma$ .

**Definition 3.** A total preorder  $\succcurlyeq$  on  $2^N$  is called *dichotomous* if there is a partition of  $2^N$  in two indifference classes, say  $\mathcal{G}$  and  $\mathcal{B}$ , such that each element in  $\mathcal{G}$  is strictly preferred to each element in  $\mathcal{B}$ .

Given a total preorder  $\succcurlyeq$  on  $2^N$ , for each  $T \in 2^N$  we denote by  $\succcurlyeq_T$  the dichotomous total preorder on  $2^N$  such that  $\mathcal{G} = \{S \in 2^N : S \succcurlyeq T\}$ . We shall say that  $\succcurlyeq_T$  is a dichotomous total preorder associated to  $\succcurlyeq$  on  $T$  for each  $T \in 2^N$ .

We prove now a proposition establishing how a total preorder over  $2^N$  may be decomposed in several dichotomous total preorders.

**Proposition 1.** Let  $\succcurlyeq$  be a total preorder on  $2^N$  and let  $v \in V(\succcurlyeq)$ . For each  $T \in 2^N \setminus \{\emptyset\}$ , consider the game  $v_T^*$  such that for each  $S \in 2^N$

$$\text{if } T \succ \emptyset, \quad v_T^*(S) = \begin{cases} v(T) - v(T^\sigma) & \text{if } S \succcurlyeq T, \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

$$\text{if } \emptyset \succcurlyeq T, \quad v_T^*(S) = \begin{cases} v(T^\sigma) - v(T) & \text{if } T \succ S, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Then,  $v_T^*$  induces the dichotomous total preorder  $\succcurlyeq_T$  associated to  $\succcurlyeq$  on  $T$  for every  $T \in 2^N$  and

$$v = \sum_{T \in 2^N} \frac{1}{n_T} v_T^*$$

where  $n_T$  is the number of elements  $B$  indifferent to  $T$  in the order  $\succcurlyeq$ .

*Proof.* The fact that  $v_T^*$  is dichotomous is obvious by definition. So what we need to prove is that for every  $C \subseteq N$  the following equality holds:

$$v(C) = \sum_{T \in 2^N} \frac{1}{n_T} v_T^*(C).$$

Let us write the ordering given by  $\succcurlyeq$  in the following fashion:

$$[S_1] \succ [S_2] \cdots \succ [S_i] \cdots \succ [S_k] \succ [\emptyset] \succ [W_1] \cdots \succ [W_j],$$

where  $[O]$  represents the indifference class containing the set  $O$ . Now suppose  $C \succ \emptyset$ , and  $C \in [S_i]$ . Then

$$\begin{aligned} v(C) &= \sum_{T \in 2^N} \frac{1}{n_T} v_T^*(C) = \sum_{T: C \succ T \succ \emptyset} \frac{1}{n_T} (v(T) - v(T^\sigma)) = \\ &= (v(C) - v(S_{i+1})) + (v(S_{i+1}) - v(S_{i+2})) + \cdots + (v(S_k) - v(\emptyset)) = v(C). \end{aligned}$$

The case when  $\emptyset \succcurlyeq C$  is handled in the same way.  $\square$

To clarify the above proposition we now provide a simple example.

**Example 2.** Let  $N = \{1, 2, 3\}$  and let  $\succcurlyeq$  be the total preorder on  $2^N$  such that  $\{1, 2, 3\} \sim \{2\} \succ \{3\} \succ \{1, 3\} \sim \{2, 3\} \succ \emptyset \succ \{1\} \succ \{1, 2\}$ . Consider the following representation  $v \in V(\succcurlyeq)$ :

$$\begin{aligned} v(\{1, 2, 3\}) &= v(\{2\}) = 7, \quad v(\{3\}) = 5, \quad v(\{1, 3\}) = v(\{2, 3\}) = 3, \quad v(\emptyset) = 0, \\ v(\{1\}) &= -2, \quad v(\{1, 2\}) = -5. \end{aligned}$$

The functions  $v_T^*$  are displayed in Table 2.

Now, we focus on the class of dichotomous total preorders to show a nice property of  $\pi^{\mathbf{P}}$ -aligned extensions in this class.

**Proposition 2.** Let  $\succcurlyeq$  be a dichotomous preorder and let  $\pi^{\mathbf{P}}$  be a semivalue  $\mathcal{S}$ . Then  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned if and only if for all  $i, j \in N$  and all  $A \in 2^N$

$$\{i\} \succcurlyeq \{j\} \Rightarrow \sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succcurlyeq, A)| - |\mathcal{P}_{ji}^s(\succcurlyeq, A)|) \geq 0 \quad (15)$$

where  $|A|$  denotes the cardinality of the set  $A$  and with the usual notation  $x_{s+1} = p_s + p_{s+1}$  for each  $s \in \{0, 1, \dots, n-2\}$ .

Table 1: A decomposition of the game defined in Example 2 by  $v_T^*$  functions. Note that every function  $v_{*T} \in V(\succ_T)$ , where  $\succ_T$  is the dichotomous total preorder associated to  $\succ$  on  $T \in 2^N$ .

	$\{1, 2\}$	$\{1\}$	$\emptyset$	$\{2, 3\}$	$\{1, 3\}$	$\{3\}$	$\{2\}$	$\{1, 2, 3\}$
$v_{\{1,2\}}^*$	0	0	0	0	0	0	0	0
$v_{\{1\}}^*$	-3	0	0	0	0	0	0	0
$v_{\emptyset}^*$	-2	-2	0	0	0	0	0	0
$v_{\{2,3\}}^*$	0	0	0	3	3	3	3	3
$v_{\{1,3\}}^*$	0	0	0	3	3	3	3	3
$v_{\{3\}}^*$	0	0	0	0	0	2	2	2
$v_{\{2\}}^*$	0	0	0	0	0	0	2	2
$v_{\{1,2,3\}}^*$	0	0	0	0	0	0	2	2
$v = \sum_{T \in 2^N} \frac{1}{n_T} v_T^*$	-5	-2	0	3	3	5	7	7

*Proof.* The if part. Fix  $A \in \mathcal{G}$  and suppose  $\emptyset \in \mathcal{B}$  (the proof of the case  $\emptyset \in \mathcal{G}$  is analogous). Set  $\alpha > 0$  to be  $\alpha = v(G)$  for each  $G \in \mathcal{G}$ .

$$\begin{aligned}
\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) &= \\
&\sum_{s=0}^{n-2} x_{s+1} \left[ \sum_{S \in \Sigma_{ij}^s} v(S \cup \{i\}) - \sum_{S \in \Sigma_{ij}^s} v(S \cup \{j\}) \right] = \\
&\alpha \sum_{s=0}^{n-2} x_{s+1} \left( \sum_{S \in \Sigma_{ij}^s: S \cup \{i\} \in \mathcal{G}} 1 - \sum_{S \in \Sigma_{ij}^s: S \cup \{j\} \in \mathcal{G}} 1 \right) = \\
&= \alpha \left[ \sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succ, A)| - |\mathcal{P}_{ji}^s(\succ, A)|) \right].
\end{aligned} \tag{16}$$

For the only if part, we need only to consider two cases: either when  $A \in \mathcal{G}$  or when  $A \in \mathcal{B}$ . But in the first case the above calculation holds, while if  $A \in \mathcal{B}$  the inequality is an obvious equality.  $\square$

In order to put in relation the set  $\mathcal{P}_{ji}^s(\succ, A)$  for a total preorder  $\succ$  on  $2^N$  with the corresponding sets in the associated dichotomous total preorders, we need the following.

**Lemma 1.** *Let  $\succ$  be a total preorder on  $2^N$  and let  $x \in \mathbb{R}_+^{n-1}$ . Then, for every  $i, j \in N$  and every  $A \in 2^N$ , we have that*

$$\sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succ_T, A)| - |\mathcal{P}_{ji}^s(\succ_T, A)|) = \sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succ, A)| - |\mathcal{P}_{ji}^s(\succ, A)|) \tag{17}$$

for every  $T \in 2^N$  such that  $A \succ_T T$  (where  $\succ_T$  is the dichotomous preorder associated to  $\succ$  on  $T$ ).

*Proof.* The proof follows immediately from the fact that for every set  $A \in 2^N$  and for all  $s = 0, \dots, n-2$  the following equality holds

$$|\mathcal{P}_{ij}^s(\succ_T, A)| = |\mathcal{P}_{ij}^s(\succ, A)| \tag{18}$$

for every set  $T \in 2^N$  such that  $A \succ_T T$ .  $\square$

**Proposition 3.** *Let  $\succcurlyeq$  be a total preorder on  $2^N$  and let  $x \in \mathbb{R}_+^{n-1}$ . Then, for each  $i, j \in N$  and every  $A, T \in 2^N$*

$$\sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succcurlyeq, A)| - |\mathcal{P}_{ji}^s(\succcurlyeq, A)|) \geq 0 \Leftrightarrow \sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succcurlyeq_T, A)| - |\mathcal{P}_{ji}^s(\succcurlyeq_T, A)|) \geq 0. \quad (19)$$

*Proof.* ( $\Rightarrow$ ) First, note that for every  $A \in 2^N$  such that  $T \succcurlyeq_T A$  we have that  $|\mathcal{P}_{ij}^s(A)| = |\mathcal{P}_{ji}^s(A)| = 2^{|N|-1}$  (since all sets of  $2^N$  are weakly preferred to  $A$  w.r.t.  $\succcurlyeq_T$ ). So,

$$\sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succcurlyeq_T, A)| - |\mathcal{P}_{ji}^s(\succcurlyeq_T, A)|) = 0$$

for every dichotomous preorder  $\succcurlyeq_T$  with  $T \succcurlyeq_T A$ .

Second, for every set  $A \succcurlyeq_T T$ , the inequality on the right in relation (19) follows by Lemma 1.

( $\Leftarrow$ ) For every  $A \in 2^N$ , the implication follows immediately by Lemma 1 considering the dichotomous preorder  $\succcurlyeq_A$  associated to  $\succcurlyeq$ .  $\square$

Now we can prove the main result of this section.

**Theorem 3.** *Let  $\succcurlyeq$  be a total preorder on  $2^N$ . Then  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned for a given semivalue  $\pi^{\mathbf{P}}$  if and only if for all  $i, j \in N$  and all  $A \in 2^N$*

$$\sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succcurlyeq, A)| - |\mathcal{P}_{ji}^s(\succcurlyeq, A)|) \geq 0 \Leftrightarrow \{i\} \succcurlyeq \{j\}. \quad (20)$$

*Proof.* Suppose first that (20) holds. Then by Proposition 3, also every  $\succcurlyeq_T$  fulfills (20), and, by Proposition 2, each  $\succcurlyeq_T$  is  $\pi^{\mathbf{P}}$ -aligned. Now, using Proposition 1 and the additivity of the semivalues, we have that

$$\pi_i^{\mathbf{P}}(v) = \sum_{T \in 2^N} \pi_i^{\mathbf{P}}(v_T^*) \geq \sum_{T \in 2^N} \pi_j^{\mathbf{P}}(v_T^*) = \pi_j^{\mathbf{P}}(v), \quad (21)$$

for every  $v \in V(\succcurlyeq)$ ; this proves that  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned.

Now we prove the opposite implication. Let  $\succcurlyeq$  be  $\pi^{\mathbf{P}}$ -aligned and take  $i, j \in N$  such that  $\{i\} \succcurlyeq \{j\}$ . First, we prove that this implies

$$\pi_i^{\mathbf{P}}(v_T^*) \geq \pi_j^{\mathbf{P}}(v_T^*) \quad (22)$$

for all  $v_T^* \in V(\succcurlyeq_T)$ . Suppose by contradiction that there exists some  $T \subset N$  and some  $v_T^* \in V(\succcurlyeq_T)$  such that  $\pi_j^{\mathbf{P}}(v_T^*) - \pi_i^{\mathbf{P}}(v_T^*) = 7\alpha$ , with  $\alpha > 0$ . It is easy to construct a function  $\bar{v}$  representing  $\succcurlyeq$  such that  $\|v_T^* - \bar{v}\|_\infty < \alpha$ . Since for every semivalue  $\pi^{\mathbf{P}}$  the map  $\pi^{\mathbf{P}} : \mathbb{R}^{2^n-1} \rightarrow \mathbb{R}^N$ , is 2-Lipschitz (endowing both spaces with  $\|\cdot\|_\infty$ ), then it is easily seen that  $\pi_j^{\mathbf{P}}(\bar{v}) - \pi_i^{\mathbf{P}}(\bar{v}) > \alpha$ , which yields a contradiction with the  $\pi^{\mathbf{P}}$ -alignment of  $\succcurlyeq$ .



Then, we have proved that relation (22) holds for all  $v_T^*$  and, by Proposition 2, we have that

$$\sum_{s=0}^{n-2} x_{s+1} (|\mathcal{P}_{ij}^s(\succ_T, A)| - |\mathcal{P}_{ji}^s(\succ_T, A)|) \geq 0$$

for every dichotomous total preorder  $\succ_T$  associated to  $\succ$  on  $T \in 2^N$ . This concludes the proof in view of Proposition 3.  $\square$

Theorem 3 shows that in order to verify that an extension  $\succ$  is  $\pi^{\mathbf{P}}$ -aligned with some semivalue  $\pi^{\mathbf{P}}$ , one can compare, for each pair of objects  $i$  and  $j$  and every subset  $S$  of  $N$ , the respective number of times that  $i$  and  $j$  are contained in sets which are (weakly) preferred to  $S$  according to  $\succ$ . Notice that in such a comparison, characteristic functions  $v \in V(\succ)$  do not play any role, and only the probability distribution  $\mathbf{p}$  is considered.

## 6 A faster algorithm to check $\pi^{\mathbf{P}}$ -alignment

In this section we want to produce another condition equivalent to  $\pi^{\mathbf{P}}$ -alignment, for a fixed semivalue  $\pi^{\mathbf{P}}$ . Differently from that one given in the previous section, it does not apply to all semivalues, but to the large class of semivalues such that the quantities  $x_{s+1} = p_s + p_{s+1}$  are rational. This clearly suffices for practical purposes. On the other hand, the condition provided here is much more easily handleable w.r.t. the one provided in Theorem 3. We remind that it is possible to substitute, in all inequalities, the quantity  $x_{s+1}$ , for  $s = 1, \dots, n-1$ , with  $ax_{s+1}$ , for any  $a > 0$ . Thus, to simplify notations, we shall always assume, in this section, that the quantities  $x_{s+1}$  are actually *natural numbers*.

In order to introduce our condition, we need a bit more of notation.

Let us then fix a preference relation  $\succ$  on  $2^N$ , suppose  $\pi^{\mathbf{P}}$  is a given semivalue and fix the associated natural number  $x_1, \dots, x_{n-1}$ . For a given  $i \in N$  and a subfamily  $\mathcal{F}$  of  $2^N$ , we write  $\theta^{\mathbf{P}}(\mathcal{F}, i)$  for the vector constructed in the following way. Order in decreasing order of preference the sets  $S \cup \{i\}$ , where  $S \in \mathcal{F}$ :

$$S_1 \cup \{i\} \succ S_2 \cup \{i\} \succ \dots \succ S_l \cup \{i\} \succ \dots,$$

then replicate each coalition  $S_k$   $x_{s_k}$  times, if  $|S_k| = s_k$ , and form the vector

$$\theta^{\mathbf{P}}(\mathcal{F}, i) = (\underbrace{S_1 \cup \{i\}, \dots, S_1 \cup \{i\}}_{x_{s_1} \text{ times}}, \underbrace{S_2 \cup \{i\}, \dots, S_2 \cup \{i\}}_{x_{s_2} \text{ times}}, \dots).$$

**Example 3.** Let  $N = \{1, 2, 3\}$ , let

$$N \succ \{1\} \succ \{2, 3\} \succ \{1, 3\} \succ \{2\} \succ \{1, 2\} \succ \{3\} \succ \emptyset.$$

Let  $\mathcal{F} = \Sigma_{12} = \{\emptyset, \{3\}\}$ . Let  $\mathbf{p}$  be the semivalue with  $x = (1, 1, 2)$ . Then

$$\theta^{\mathbf{P}}(\Sigma_{12}, 1) = (\{1\}, \{1\}, \{1, 3\}, \{1, 3\}, \{1, 3\})$$

and

$$\theta^{\mathbf{P}}(\Sigma_{12}, 2) = (\{2, 3\}, \{2, 3\}, \{2, 3\}, \{2\}, \{2\}).$$

With a little abuse of notation, we shall write

$$\theta^{\mathbf{P}}(\Sigma_{ij}, i) \succcurlyeq \theta^{\mathbf{P}}(\Sigma_{ij}, j)$$

if

$$(\theta^{\mathbf{P}}(\Sigma_{ij}, i))_k \succcurlyeq (\theta^{\mathbf{P}}(\Sigma_{ij}, j))_k, \quad \forall k \in \{1, \dots, \sum_{s=0}^{n-2} \binom{n-2}{s} x_{s+1}\}$$

and we define a relation  $\succeq_p$  over  $N$  such that  $\{i\} \succeq_p \{j\} :\Leftrightarrow \theta^{\mathbf{P}}(\Sigma_{ij}, i) \succcurlyeq \theta^{\mathbf{P}}(\Sigma_{ij}, j)$ . It is clear that  $\succeq_p$  induces a partial order on the set of singletons in  $N$ .

**Definition 4 (p-WPR).** *We say that the total preorder  $\succcurlyeq$  on  $2^N$  satisfies the p-weighted permutational responsiveness (p-WPR) property if for each  $i, j \in N$  we have that*

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow i \succeq_p j. \quad (23)$$

In other words, a total preorder on  $2^N$  has the p-WPR property provided the partial order defined by  $\succeq$  actually *coincides* with the original total preorder on  $N$ . The easiest case is when  $x_i = 1$  for  $i = 1, \dots, n-1$ , which corresponds to the case where the Banzhaf semivalue is considered.

It is easy to check that the p-WPR property directly implies condition (20): if a total preorder  $\succcurlyeq$  on  $2^N$  satisfies the p-WPR property, and  $i \succeq_p j$ , then the number of coalitions  $S$  in  $\theta^{\mathbf{P}}(\Sigma_{ij}, i)$  with  $S \succcurlyeq A$  is not smaller than the number of coalitions  $T$  in  $\theta^{\mathbf{P}}(\Sigma_{ij}, j)$  with  $T \succcurlyeq A$ , for each coalition  $A \in 2^N$ . The converse follows from Theorem 3 and the following Theorem 4, whose proof requires Lemma 2.

**Lemma 2.** *Let  $x_1 \geq x_2 \geq \dots \geq x_l$  and  $y_1 \geq y_2 \geq \dots \geq y_l$  be real numbers such that*

$$\sum_{r=1}^l f(x_r) - f(y_r) \geq 0$$

*for every strictly increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $x_r \geq y_r$  for all  $r \in \{1, \dots, l\}$ .*

*Proof.* Suppose not. Take the first index  $h$  such that  $y_h - x_h > 0$ , and suppose, w.l.o.g.  $y_h - x_h = 1$ . Consider any strictly increasing continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that:

$$f(m) = 0, \quad f(x_h) = \frac{1}{2l}, \quad f(t) = t + \frac{1}{2l} - x_h, t \in [x_h, y_h],$$

where  $m = \min\{x_l, y_l\}$  and

$$\lim_{t \rightarrow \infty} f(t) = y_h - x_h + \frac{1}{l}.$$

Notice that, for every  $r = 1, \dots, l$ , with  $r \neq h$ , such that  $x_r \geq y_r$ , we have that  $f(x_r) - f(y_r) < \frac{1}{l}$  (precisely, if  $r \leq h$ , then we always have that  $x_r \geq y_r$ ). Then, we have the following contradiction

$$\begin{aligned} & \sum_{r=1}^l (f(x_r) - f(y_r)) \\ &= \sum_{r=1}^{h-1} (f(x_r) - f(y_r)) + (f(x_h) - f(y_h)) + \sum_{r=h+1}^l (f(x_r) - f(y_r)) \\ &\leq \sum_{r=1}^{h-1} (f(x_r) - f(y_r)) + (x_h - y_h) + \sum_{l \geq r \geq h: y_r \leq x_r} (f(x_r) - f(y_r)) \\ &\leq (h-1)\frac{1}{l} - 1 + (l-h)\frac{1}{l} = \frac{l-1}{l} - 1 < 0. \end{aligned}$$

□

We are ready to state and prove the main result of the section.

**Theorem 4.** *Let  $\succcurlyeq$  be a total preorder on  $2^N$  and let  $\pi^{\mathbf{P}}$  be a semivalue with rational probabilities. The following two statements are equivalent:*

- 1)  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned;
- 2)  $\succcurlyeq$  satisfies the  $\mathbf{p}$ -WPR property.

*Proof.* Take two elements  $i, j \in N$ . For a function  $v \in V(\succcurlyeq)$  representing the order  $\succcurlyeq$ , we have that

$$\begin{aligned} & \pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) \\ &= \sum_{s=0}^{n-2} x_{s+1} \left[ \sum_{S \in \Sigma_{ij}^s} v(S \cup \{i\}) - \sum_{S \in \Sigma_{ij}^s} v(S \cup \{j\}) \right] \\ &= \sum_{k=1}^l v(\theta^{\mathbf{P}}(\Sigma_{ij}^s, i)_k) - v(\theta^{\mathbf{P}}(\Sigma_{ij}^s, j)_k), \end{aligned} \quad (24)$$

where the second equality is obtained by rearranging terms in the sum, and  $l = \sum_{s=0}^{n-2} x_{s+1} \binom{n-2}{s}$ .

Now suppose  $\{i\} \succcurlyeq \{j\}$ .

Equation (24) shows that the  $\mathbf{p}$ -WPR property implies  $\pi_i^{\mathbf{P}}(v) - \pi_j^{\mathbf{P}}(v) \geq 0$  for all  $v$  representing  $V(\succcurlyeq)$ , i.e. the  $\mathbf{p}$ -WPR property implies that  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned.

To see that the  $\mathbf{p}$ -WPR property is also necessary to have  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned, it is enough to fix a  $\bar{v} \in V(\succcurlyeq)$  and to notice that every transformation of  $\bar{v}$  by a strictly increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is still an element of  $V(\succcurlyeq)$ . Then, if  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned, by equation (24) we have that

$$\sum_{k=1}^l f(\bar{v}(\theta^{\mathbf{P}}(\Sigma_{ij}^s, i)_k)) - f(\bar{v}(\theta^{\mathbf{P}}(\Sigma_{ij}^s, j)_k)) \geq 0, \quad (25)$$

for every strictly increasing real-valued function  $f$ . The proof is concluded by relation (25) and using Lemma 2 with  $\bar{v}(\theta^{\mathbf{P}}(\Sigma_{ij}^s, i)_k)$  and  $\bar{v}(\theta^{\mathbf{P}}(\Sigma_{ij}^s, j)_k)$  in the role of  $x_r$  and  $y_r$ , respectively. □

**Remark 1.** *Given a total preorder  $\succcurlyeq$  on  $2^N$  that satisfies the  $\mathbf{p}$ -WPR property, an alternative way to prove Theorem 4 is by noticing that relation (24) implies that, for each  $v \in V(\succcurlyeq)$  and for each  $i, j \in N$  such that  $\{i\} \succcurlyeq \{j\}$ , the gamble  $I = (v(S \cup \{i\}); x_{|S|+1})_{S \in \Sigma_{ij}}$  first-order stochastically dominates (FSD) ([14];*

see also the book [18]) the gamble  $J = (v(S \cup \{j\}); x_{|S|+1})_{S \in \Sigma_{ij}}$ , which means that the cumulative distribution function (cdf)  $F_I(w) = \sum_{S \in \Sigma_{ij}: v(S \cup \{i\}) \leq w} x_{|S|+1}$  is smaller or equal to the cdf  $F_J(w) = \sum_{S \in \Sigma_{ij}: v(S \cup \{j\}) \leq w} x_{|S|+1}$ , for each  $w \in \mathbb{R}$ .

It is a well known fact that a gamble  $I$  first-order stochastically dominates  $J$  if and only if the expected value  $E_I(u(v))$  of gamble  $I$  is larger or equal than the expected value  $E_J(u(v))$  of gamble  $J$  for all increasing functions  $u$ . Since  $u(v)$  is still an element of  $V(\succ)$  and, obviously,  $E_I(u(v)) = \pi_i^{\mathbf{P}}(u(v))$  and  $E_J(u(v)) = \pi_j^{\mathbf{P}}(u(v))$ , the statement of Theorem 4 follows.

## 7 $\pi^{\mathbf{P}}$ -alignment with all semivalues

In this section we want to characterize those total preorders  $\succ$  on  $2^N$  that are  $\pi^{\mathbf{P}}$ -aligned for all semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$ . In the literature it is possible to find some sufficient condition, no one of them being necessary.

For example, the RESP property, introduced in Section 2 and further discussed in Section 3, is a sufficient condition for a total preorder to be  $\pi^{\mathbf{P}}$ -aligned with all regular semivalues [20]. Another condition, known in literature as *permutational responsiveness* (PR)[20], generalizes the notion of RESP property and also guarantees the alignment of a total preorder with any semivalue. To define it, for a given subfamily  $\mathcal{F}$  of  $2^N$ , we use the notation of the previous section, denoting by  $\theta(\mathcal{F}, i)$  the vector  $\theta^{\mathbf{P}}(\mathcal{F}, i)$  when  $\mathbf{p} = (1, \dots, 1)$ , that is each coalition  $S \in \mathcal{F}$  is replicated in  $\theta(\mathcal{F}, i)$  precisely once.

**Definition 5** (PR). *We say that a total preorder  $\succ$  on  $2^N$  satisfies the permutational responsiveness (PR) property if for each  $i, j \in N$  we have that*

$$\{i\} \succ \{j\} \Leftrightarrow \theta(\Sigma_{ij}^s, i)_k \succ \theta(\Sigma_{ij}^s, j)_k \quad (26)$$

for every  $k = 1, \dots, |\Sigma_{ij}^s|$  and every  $s = 0, \dots, n - 2$ .

One more time we see that the condition  $\theta(\Sigma_{ij}^s, i)_k \succ \theta(\Sigma_{ij}^s, j)_k$  for every  $k = 1, \dots, |\Sigma_{ij}^s|$  and every  $s = 0, \dots, n - 2$ , induces a partial order on the elements of  $N$ , and the PR condition, analogously to the  $\mathbf{p}$ -WPR given in the previous section, requires that this partial order actually coincides with the total preorder on the singletons.

In other terms, for each  $i, j \in N$  such that  $\{i\} \succ \{j\}$  and for each  $s = 0, \dots, n - 2$ , the PR property admits the possibility of relative rankings which violate the conditions imposed by the RESP property (i.e.,  $S \cup \{j\}$  is preferred to  $S \cup \{i\}$ ) due to the effect of mutual interaction within the objects in  $S$ . Nevertheless, such an interaction should be compatible with the requirement that, between sets of the same cardinality, the original relative ranking between  $\{i\}$  and  $\{j\}$  should be preserved with respect to the position of subsets in  $\Sigma_{ij}^s$  and  $\Sigma_{ji}^s$ , when they are arranged in descending order of preference (i.e., the most preferred subsets in  $\Sigma_{ij}^s$  should be preferred to the most preferred subsets in  $\Sigma_{ji}^s$ , the second most preferred subsets in  $\Sigma_{ij}^k$  should be preferred to the second most preferred subsets in  $\Sigma_{ji}^k$ , etc.).

In the context of the introductory example of “choosing the wine for a dinner”, the PR property says that comparing meals of the same size (cardinality), and assuming they have the same probability to realize (like we do assume, using semivalues), even if the meal ‘fish and white wine’ is the top meal among the ones served with white wine, and it is preferred to ‘fish and red wine’, then the choice to bring at dinner ‘red wine’ instead of ‘white wine’ should be a consequence of the fact that the top meal among the one served with ‘red wine’ (e.g., ‘beef and red wine’) is in turn preferred to ‘fish and white wine’.

In [20] it is shown that the PR condition is sufficient to guarantee that a total preorder is aligned with all semivalues. The following example, instead, displays a total preorder on  $2^{\{1,2,3,4\}}$  which is  $\pi^{\mathbf{P}}$ -aligned for all  $\pi^{\mathbf{P}} \in \mathcal{S}$  and that does not satisfy the PR property.

**Example 4.** Let  $X = \{1, 2, 3, 4\}$  and let  $\succcurlyeq$  be a total preorder such that  $\{1, 2, 3, 4\} \succ \{2, 3, 4\} \succ \{3, 4\} \succ \{4\} \succ \{3\} \succ \{2\} \succ \{2, 4\} \succ \{1, 4\} \succ \{1, 3\} \succ \{2, 3\} \succ \{1, 3, 4\} \succ \{1, 2, 4\} \succ \{1, 2, 3\} \succ \{1, 2\} \succ \{1\} \succ \emptyset$ .

Note that  $\succcurlyeq$  does not satisfy PR because  $\{2\} \succ \{1\}$ ,  $\{2, 4\}$  is strictly preferred to  $\{1, 4\}$  and  $\{1, 3\}$  is strictly preferred to  $\{2, 3\}$ . However,  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned for all semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$  (see the Appendix for the details of the proof).

We are ready to introduce the property that we shall show to be necessary and sufficient for the alignment of the preorder to all semivalues. One more time we introduce a partial order on  $N$ , and the condition will require that it coincides with the initial total preorder.

For every  $i, j \in N$ , we set  $\mathcal{D}_{ij}^s$  to be the set  $\mathcal{D}_{ij}^s = \Sigma_{ij}^s \cup \Sigma_{ij}^{s+1}$  for  $s = 0, \dots, n-3$ . With a little abuse of notation, set  $\mathcal{D}_{ij}^{n-2} = \Sigma_{ij}^{n-1}$ .

**Definition 6 (DPR).** We say that a total preorder on  $2^N$  satisfies the double permutational responsiveness (DPR) property if for each  $i, j \in N$  we have that

$$\{i\} \succcurlyeq \{j\} \Leftrightarrow \theta(\mathcal{D}_{ij}^s, i)_k \succcurlyeq \theta(\mathcal{D}_{ij}^s, j)_k \quad (27)$$

for every  $l = 1, \dots, |\mathcal{D}_{ij}^s|$  and every  $s = 0, \dots, n-2$ .

We are ready to prove the main theorem of this section.

**Theorem 5.** Let  $\succcurlyeq$  be a total preorder on  $2^N$  and let  $\pi^{\mathbf{P}}$  be a semivalue. The following statements are equivalent:

- 1)  $\succcurlyeq$  fulfills the DPR property;
- 2)  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned for all semivalues.

*Proof.* It is clear that 2) can be equivalently expressed by saying that  $\succcurlyeq$  is  $\pi^{\mathbf{P}}$ -aligned when  $\mathbf{p}$  ranges on the extreme points of the simplex of the semivalues. Now observe that for the extreme semivalue  $(0, \dots, 1, \dots, 0)$ , where 1 is at the  $s$ -th place, it can be set  $x_i = 0$  for  $i \neq s$ ,  $i \neq s+1$ , while  $x_s = x_{s+1} = 1$ . Observing that to be aligned with  $(0, \dots, 1)$  is equivalent to

$$\{i\} \succcurlyeq \{j\} \Rightarrow \theta(\mathcal{D}_{ij}^{n-1}, i) \succcurlyeq \theta(\mathcal{D}_{ij}^{n-1}, j)$$

the thesis is a consequence of Theorem 4. □

We conclude with the following observation. In [11], a game  $v$  is defined weakly complete provided the relation

$$i \geq j \Rightarrow c_i(s) \geq c_j(s) \tag{28}$$

for every  $s = 0, \dots, n - 1$  provides a total preorder on the set  $N$  ( $c_i(s), c_j(s)$  were defined in Equation (7)). Then one important result of [11] is that for a game to be weakly complete is equivalent to the fact that all semivalues provide the same ranking on the elements of  $N$ . Our result shows in particular that for a total preorder fulfilling DPR every numerical representation by a utility function (with  $v(\emptyset) = 0$ ) generates a weakly complete game. However it is possible that a given game is weakly complete without the underlying preorder fulfilling DPR. In such a case another utility function representing the same preorder will not be weakly complete. A simple example is the following game with  $N = \{1, 2, 3, 4\}$ .

$$v(\{1, 4\}) = 100, v(\{1\}) = 4, v(\{2, 3\}) = 3, v(\{2, 4\}) = 2, v(\{2\}) = 1, \\ v(\{4\}) = -1, v(\{A\}) = 0 \text{ otherwise.}$$

It is easy to see that the game  $v$  is weakly complete but the total preorder  $\succsim$  on  $2^N$  such that  $S \succsim T \Leftrightarrow v(S) \geq v(T)$ , for each  $S, T \in 2^N$ , does not satisfy the DPR property (to check this, it is sufficient to make the comparison between objects 1 and 2).

## 8 Conclusions

In this paper, the idea of *alignment* with a semivalue was developed in order to have meaningful extensions of a total preorder on a finite set  $N$ , to the set of its subsets, and keeping into account the possibility that objects within each subset may interact. Specifically, the fact that an extension must be *aligned* with a semivalue means that the ranking of the objects according to the semivalue must be the same and must preserve the primitive total preorder on the singletons, no matter which utility function is used to describe the preorder. In the most favourable situation it remains the same for the whole simplex of the semivalues (if and only if the total preorder on  $2^N$  fulfills the DPR property).

We want also to stress the fact that one can think of the possibility to do a kind a inverse process: from a ranking  $\succsim$  on the power set of a given set of objects how to derive a ranking on the objects, that will take into account the interactions, and thus not necessarily preserving the ranking  $\succsim$  restricted to the singletons. Again, this makes sense in the setting of considering interactions between objects: as a striking example we can think that *any* reasonable ranking of top soccer players will *never* give a good team just by taking the first 11 of the list! In this case one can consider as nice the total preorders that are aligned (according to a semivalue), in the sense that the final ranking on the objects will *not* depend from the utility function used to represent the preorder  $\succsim$ . However the analysis of ordinality, which presents analogies but also some meaningful differences, will be pursued on a subsequent paper.

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## Appendix

We provide the details of the calculations proving that the relation  $\succ$  of Example 4 is  $\pi^{\mathbf{P}}$ -aligned with all semivalues  $\pi^{\mathbf{P}} \in \mathcal{S}$ .

Take a semivalue  $\pi^{\mathbf{P}} \in \mathcal{S}$ , for some collection of probability distribution  $\mathbf{p}$ , and consider a numerical representation  $v \in V(\succ)$ . We have that

$$\begin{aligned}
& \pi_2^{\mathbf{P}}(v) - \pi_1^{\mathbf{P}}(v) = \\
& (p^0 + p^1)(v(2) - v(1)) + (p^1 + p^2)(v(2, 3) - v(1, 3)) + \\
& (p^1 + p^2)(v(2, 4) - v(1, 4)) + (p^2 + p^3)(v(2, 3, 4) - v(1, 3, 4)) > \quad (29) \\
& (p^0 + p^1)(v(2) - v(1)) + (p^1 + p^2)(v(2, 3) - v(1, 3)) \\
& + (p^2 + p^3)(v(2, 3, 4) - v(1, 3, 4)).
\end{aligned}$$



where the inequality follows from the fact that  $v(2, 4) - v(1, 4) > 0$  for every semivalue  $\pi^{\mathbf{P}}$  and every  $v \in V(\succsim)$ . Note that, due to the relative ranking of coalitions, we have that

$$v(2) - v(1) > v(1, 3) - v(2, 3), \quad (30)$$

and

$$v(2, 3, 4) - v(1, 3, 4) > v(1, 3) - v(2, 3). \quad (31)$$

Moreover, we have that

$$\begin{aligned} & (p^0 + p^1)(v(2) - v(1)) + (p^2 + p^3)(v(2, 3, 4) - v(1, 3, 4)) \geq \\ & (p^0 + p^1 + p^2 + p^3) \min\{[v(2) - v(1)], [v(2, 3, 4) - v(1, 3, 4)]\} \geq \\ & (p^1 + p^2) \min\{[v(2) - v(1)], [v(2, 3, 4) - v(1, 3, 4)]\} > \\ & (p^1 + p^2)(v(1, 3) - v(2, 3)), \end{aligned} \quad (32)$$

where the strict inequality follows from relations (30) and (31). By relation (29), it immediately follows that  $\pi_2^{\mathbf{P}}(v) - \pi_1^{\mathbf{P}}(v) > 0$  for every semivalue  $\pi^{\mathbf{P}}$  and every  $v \in V(\succsim)$ .

In a similar way, for every semivalue  $\pi^{\mathbf{P}}$  and every  $v \in V(\succsim)$ , we have immediately that

$$\begin{aligned} & \pi_3^{\mathbf{P}}(v) - \pi_2^{\mathbf{P}}(v) = \\ & (p^0 + p^1)(v(3) - v(2)) + (p^1 + p^2)(v(1, 3) - v(1, 2)) + \\ & (p^1 + p^2)(v(3, 4) - v(2, 4)) + (p^2 + p^3)(v(1, 3, 4) - v(1, 2, 4)) > 0. \end{aligned} \quad (33)$$

Finally,

$$\begin{aligned} & \pi_4^{\mathbf{P}}(v) - \pi_3^{\mathbf{P}}(v) = \\ & (p^0 + p^1)(v(4) - v(3)) + (p^1 + p^2)(v(1, 4) - v(1, 3)) + \\ & (p^1 + p^2)(v(2, 4) - v(2, 3)) + (p^2 + p^3)(v(1, 2, 4) - v(1, 2, 3)) = \\ & (p^0 + p^1)(v(4) - v(3)) + \\ & (p^1 + p^2)(v(2, 4) - v(2, 3)) + (p^2 + p^3)(v(1, 2, 4) - v(1, 2, 3)) > 0. \end{aligned} \quad (34)$$

Of course one can verify as well that the preorder fulfills the DPR property.