

Set-Membership Method for Discrete Optimal Control



1. Problem

Considered System

⇒ We consider a control system, defined by the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

$\mathbf{x}(t) \in \mathbb{R}^n$ the state vector

$\mathbf{u}(t) \in \mathbf{U}$ the control vector

⇒ The system is studied over $[t_0, t_f]$

$$t_k = t_0 + k \times \delta_t, t_k \leq t_f, k \in \{1, \dots, m\} \quad (2)$$

It is assumed that $\mathbf{u}(t_k)$ is bounded over $[t_k, t_{k+1}]$ so it is possible to determinate a box $[\mathbf{u}_k]$ such that $\mathbf{u}(t_k) \in [\mathbf{u}_k]$ over $[t_k, t_{k+1}]$

⇒ The flow map of the system is defined as

$$\varphi(t_0, t_k; \mathbf{x}_0, \mathbf{u}(t)) = \mathbf{x}(t) \quad (3)$$

⇒ The reachable set of the system at time t_k is

$$\begin{aligned} \varphi(t_0, t_k; \mathbf{X}_0, \mathbf{U}) = \{ & \varphi(t_0, t_k; \mathbf{x}_0, \mathbf{u}(t)) \mid \varphi(t_0, t_0; \mathbf{x}_0, \mathbf{u}(t)) = \mathbf{x}_0 \\ & \text{and } \varphi : [t_0, t_k] \times \mathbf{X}_0 \times \mathbf{U} \rightarrow \mathbb{R}^n \text{ is a} \\ & \text{solution of (1) for some } \mathbf{u}(t) \in \mathcal{U} \} \end{aligned} \quad (4)$$

where $\mathcal{U} = \{ \mathbf{u} : [t_0, t_{k-1}] \rightarrow \mathbf{U} \mid \mathbf{u} \text{ is continuous over } [t_k, t_{k+1}] \}$ denotes the set of admissible controls and \mathbf{X}_0 a set of possible initial values \mathbf{x}_0

Objective

⇒ Evaluate \mathbf{C}_{t_0, t_f} the subset of initial states of \mathbf{K} (state constraint) from which there exists at least one solution of (1) reaching the target \mathbf{T} in finite time t_f starting at a time t_0 :

$$\mathbf{C}_{t_0, t_f} = \{ \mathbf{x}_0 \in \mathbf{K} \mid \exists \mathbf{u}(t) \in \mathcal{U}, \varphi(t_0, t_f; \mathbf{x}_0, \mathbf{u}(t)) \in \mathbf{T} \} \quad (5)$$

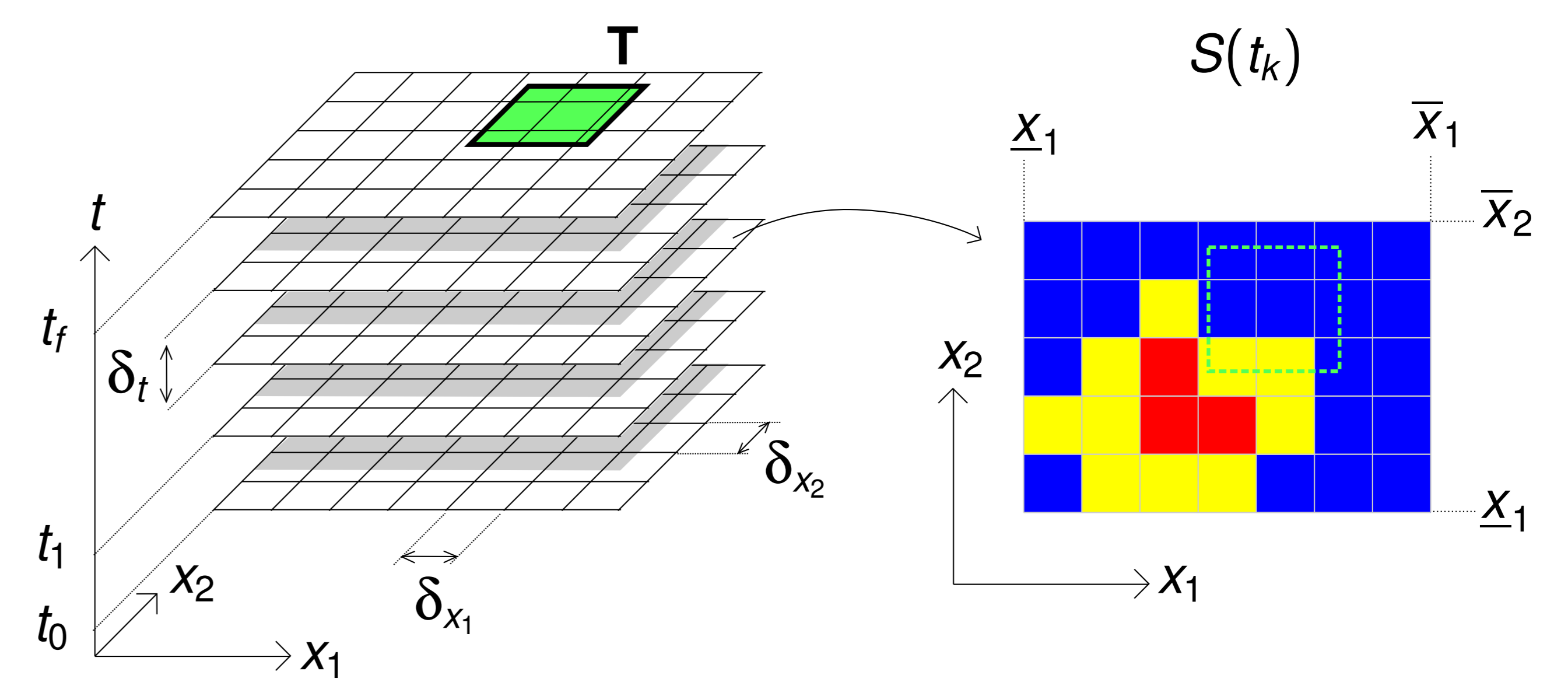
⇒ Using interval analysis to compute an inner and an outer characterisations of \mathbf{C}_{t_0, t_f}

$$\mathbf{C}_{t_0, t_f}^- \subseteq \mathbf{C}_{t_0, t_f} \subseteq \mathbf{C}_{t_0, t_f}^+ \quad (6)$$

2. Characterisation computation

Proposed approach

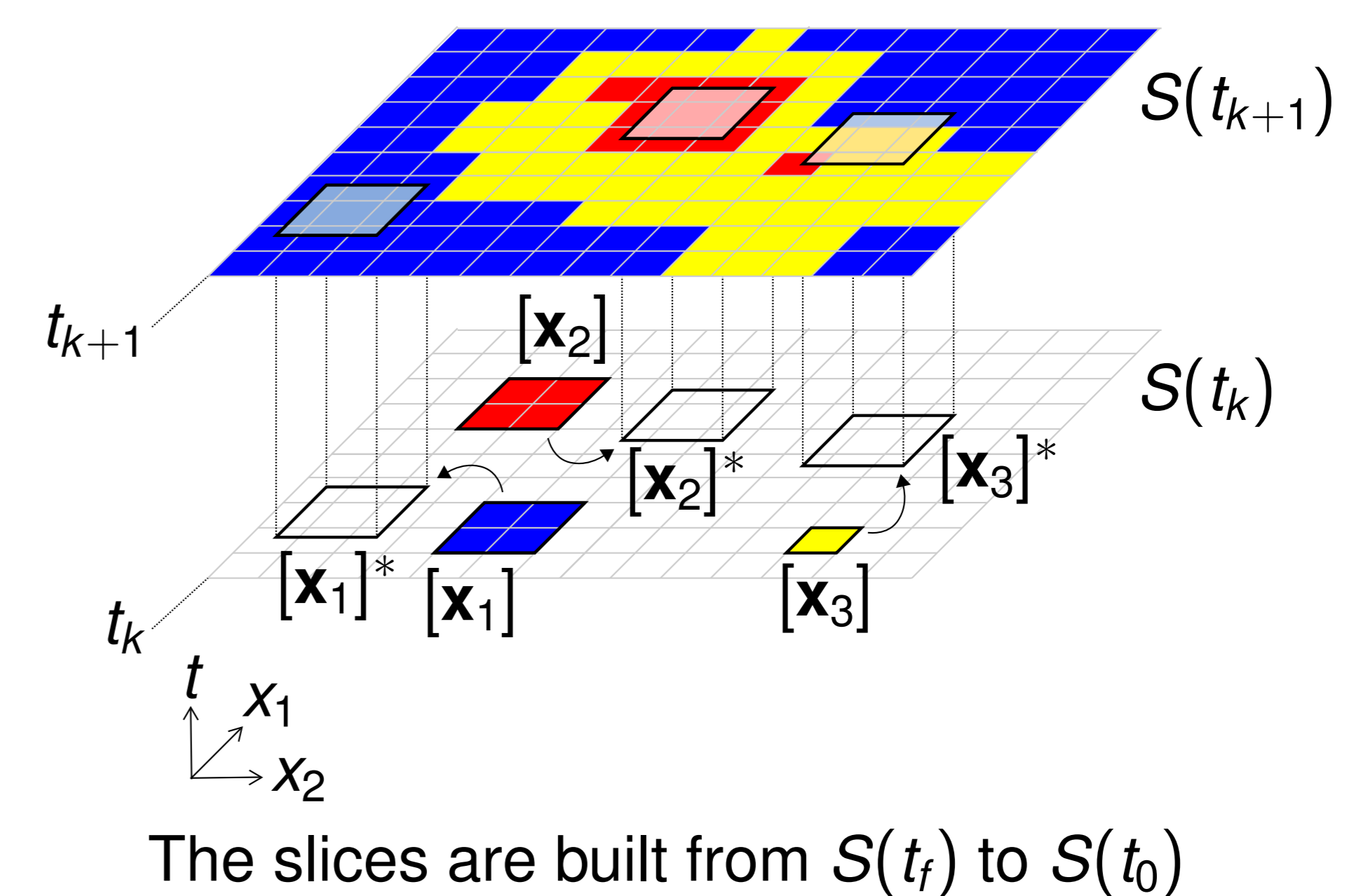
For each time t_k the algorithm computes a gridding of \mathbf{K} (a slice), noted $S(t_k)$. The resolution of the gridding is $\delta_{\mathbf{K}} = (\delta_{x_1}, \dots, \delta_{x_i}, \dots, \delta_{x_n})$ where δ_{x_i} corresponds to the i^{th} dimension of \mathbf{K}



Slice computation

We propose an iterative algorithm that classifies the cells of each slice in three categories:

- *unreachable* (blue), no state inside the cell allows the system to reach the target at time t_f
- *reachable* (red), all the states inside the cell allow the system to reach the target at time t_f
- *indeterminate* (yellow), neither *reachable* nor *unreachable*

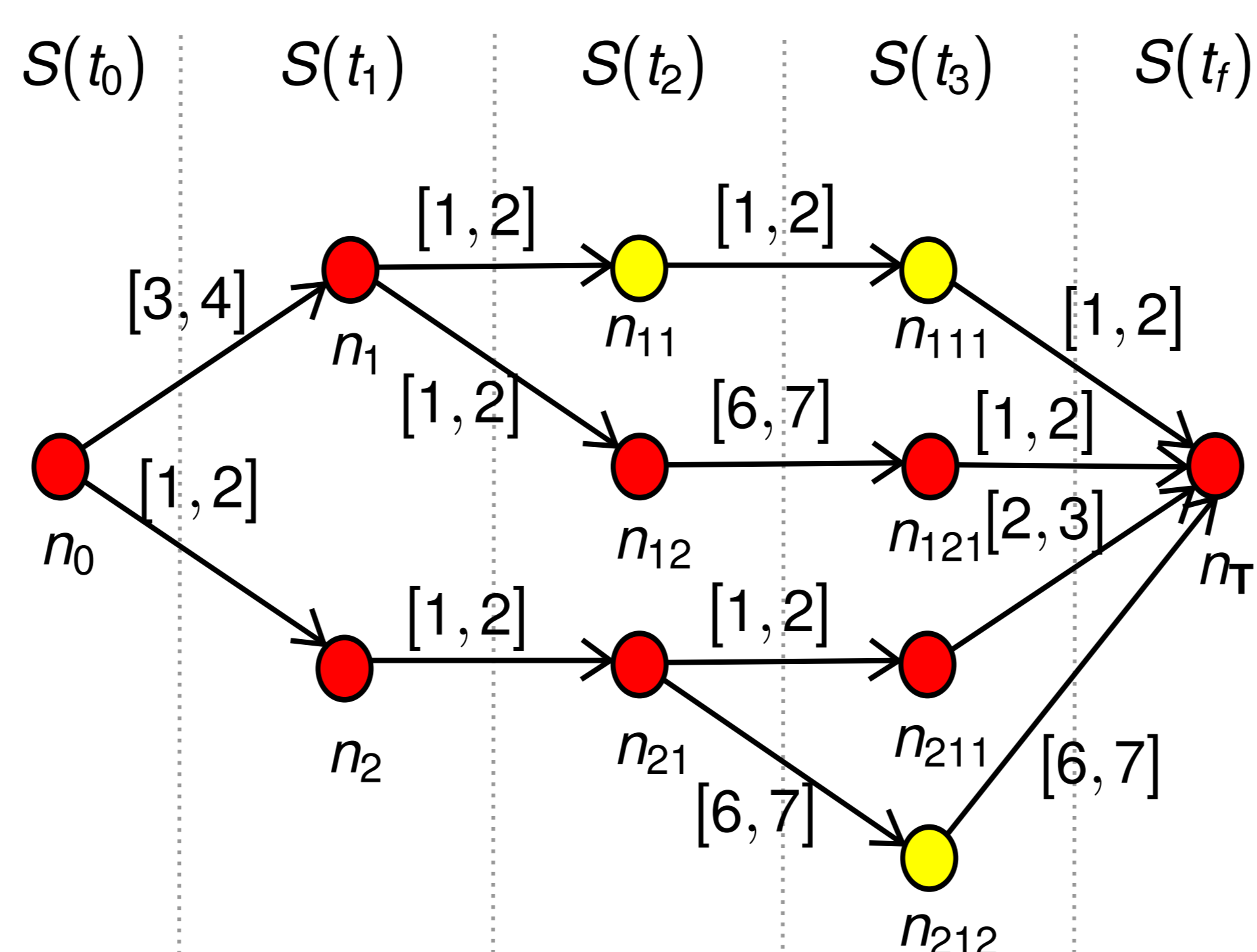
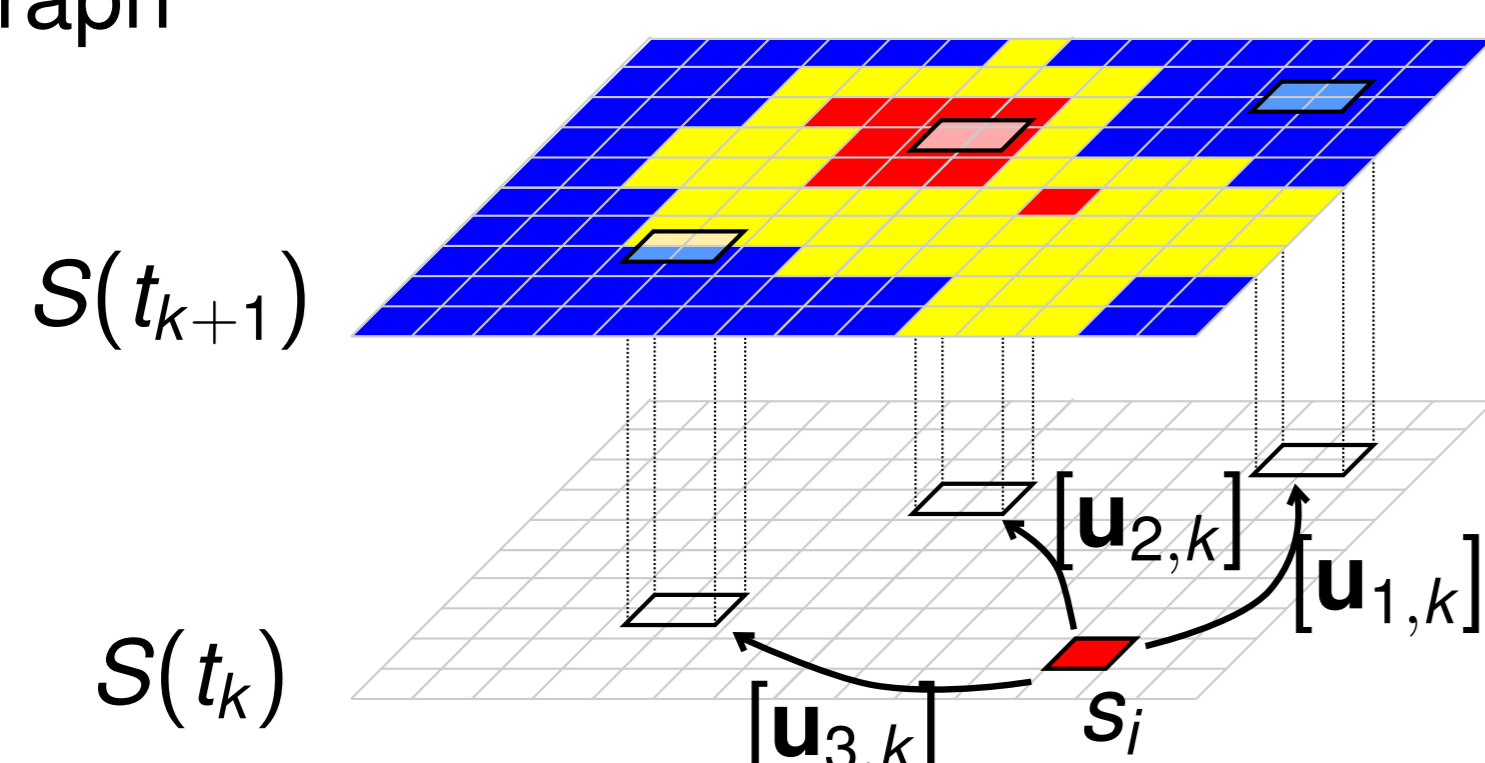


3. Optimal discrete path evaluation

Slice modification and graph building

⇒ For each cell $s_i \in S(t_k)$ is defined a set of input vectors $\mathbf{U}(s_i)$ that leads s_i to reachable or indeterminate cells of $S(t_{k+1})$

⇒ Gather the cells into nodes and build a graph



$$\begin{aligned} J(P(n_0, n_1, n_{11}, n_{111}, n_T)) &= [6, 10] \\ J(P(n_0, n_1, n_{12}, n_{21}, n_T)) &= [11, 15] \\ J(P(n_0, n_2, n_{21}, n_{211}, n_T)) &= [5, 9] \\ J(P(n_0, n_2, n_{21}, n_{212}, n_T)) &= [14, 18] \end{aligned}$$

Obtained paths

⇒ Using the graph and a shortest path algorithm (e.g. Interval Dijkstra) it is possible to compute:

- an enclosure (\mathbf{P}^*) of the optimal discrete control vector to reach the target from an initial state $[\mathbf{x}_0] \in \mathbf{C}_{t_0, t_f}$
- an evaluation of the cost ($J(\mathbf{P}^*)$) of this control vector

⇒ For instance

$$\begin{aligned} \mathbf{P}^* &= \{ P(n_0, n_1, n_{11}, n_{111}, n_T), \\ & P(n_0, n_2, n_{21}, n_{211}, n_T) \} \\ J(\mathbf{P}^*) &= [6, 10] \cup [5, 9] = [5, 9] \end{aligned}$$