Asymptotic of the terms of the Gegenbauer polynomial on the unit circle and applications to the inverse of Toeplitz matrices

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Abstract

Asymptotic of the terms of the Gegenbauer polynomials on the unit circle and applications to the inverse of Toeplitz matrices.
The first part of this paper is devoted to the study of the orthogonal polynomials on the unit circle, with respect of a weight of type $f_\alpha : \theta \mapsto 2^{2\alpha}(\cos\theta - \cos\theta_0)^{2\alpha}c_1$ with $\theta_0 \in]0, \pi[$, $-\frac{1}{2} < \alpha < \frac{1}{2}$ and c_1 a sufficiently smooth function. In a second part of the paper we obtain an asymptotic of the entries $(T_N f_\alpha)^{-1}_{k+1,l+1}$ for $\alpha > 0$ and for sufficiently large values of k, l, with $k \neq l$.

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1 Introduction

The study of the orthogonal polynomials on the unit circle is an old and difficult problem (see [16], [17] or [18]). The Gegenbauer polynomials on the torus are the orthogonal polynomials on the circle with respect to a weight of type $f_\alpha : \theta \mapsto 2^{2\alpha}(\cos\theta - \cos\theta_0)^{2\alpha}c_1$ with $\alpha > -\frac{1}{2}$ and c_1 a positive integrable function. In this paper we assume $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ and c_1 sufficiently smooth regular function. It is said that a function k is regular if $k(\theta) > 0$ for all $\theta \in \mathbb{T}$ and $k \in L^1(\mathbb{T})$. In a first part we are interested in the asymptotic of the coefficients of these polynomials (see Corollary 3). The main tool to compute this is the study of the Toeplitz matrix with symbol f. Given a function h in $L^1(\mathbb{T})$ we denote by $T_N(h)$ the Toeplitz matrix of order N with symbol h the $(N+1) \times (N+1)$ matrix such that

$$(T_N(h))_{i+1,j+1} = \hat{h}(j-i) \quad \forall i, j \quad 0 \leq i, j \leq N$$

where $\hat{m}(s)$ is the Fourier coefficient of order s of the function m (see, for instance [3] and [4]). There is a close connection between Toeplitz matrices and orthogonal polynomials on the complex unit circle. Indeed the coefficients of the orthogonal polynomial of degree N with respect of h are also the coefficients of the last column of $T_N^{-1}(h)$ except for a normalisation (see [11]). Here we give an asymptotic expansion of the entries $(T_N(f_\alpha))^{-1}_{k+1,1}$ (Theorem 4). Using

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the symmetries of the Toeplitz matrix $T_N(f_\alpha)$, we deduce from this last result an asymptotic of $(T_N(f_\alpha))^{-1}_{N-k+1,N+1}$ (corollary 3).
The proof of Theorem 4 often refers to results of [15]. In this last work we have treated the case of the symbols $h_\alpha$defined by $\theta \mapsto (1-\cos\theta)^\alpha c$ whith $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ and the same hypothesis on c as on c_1. We have stated the following Theorem which is an important tool in the demonstration of Theorem 4.

Theorem 1 ([15]) *If $-\frac{1}{2} < \alpha \leq \frac{1}{2}$, $\alpha \neq 0$ we have for $c \in A(\mathbb{T}, \frac{3}{2})$ and $0 < x < 1$*

$$c(1)\,(T_N(h_\alpha))^{-1}_{[Nx]+1,1} = N^{\alpha-1}\frac{1}{\Gamma(\alpha)}x^{\alpha-1}(1-x)^\alpha + o(N^{\alpha-1}).$$

uniformly in x for $x \in [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < 1$,

with the definition

Définition 1 *For all positive real τ we denote by $A(\mathbb{T}, \tau)$ the set*

$$A(\mathbb{T}, \tau) = \{h \in L^2(\mathbb{T}) | \sum_{s\in\mathbb{Z}} |s^\tau \hat{h}(s)| < \infty\}$$

This theorem has also been proved for $\alpha \in \mathbb{N}^*$ in [14] and for $\alpha \in]\frac{1}{2}, +\infty[\backslash\mathbb{N}^*$ in [13].

The second part of the present paper is devoted to the inversion of a class of Toeplitz matrices. We give an asymptotic expansion of $(T_N(f_\alpha))^{-1}_{k+1,l+1}$ for $\alpha \in]0, \frac{1}{2}]$ and $\frac{k}{N} \to x$, $\frac{l}{N} \to y$ and $0 < x \neq y < 1$. First we obtain these entries as a function of $\cos(l-k)\theta_0$ and $(T_N(h_\alpha))^{-1}_{k+1,l+1}$. It is Theorem 6. With the same hypothesis as for Theorem 1 we have stated in [15] the following Theorem

Theorem 2 ([15]) *For $0 < \alpha < \frac{1}{2}$ we have*

$$c(1)\,(T_N(h_\alpha))^{-1}_{[Nx]+1,[Ny]+1} = N^{2\alpha-1}\frac{1}{\Gamma^2(\alpha)}G_\alpha(x,y) + o(N^{2\alpha-1})$$

uniformly in (x, y) for $0 < \delta_1 \leq x \neq y < 1$.

Theorem 2 has been proved for $\alpha \in \mathbb{N}^*$ in [14], for $\alpha = \frac{1}{2}$ in [15] and for $\alpha \in]\frac{1}{2}, +\infty[\backslash\mathbb{N}^*$ in [13]. The quantities $G_\alpha(x, y)$ is the integral kernel on $L^2(0, 1)$ of Corollaries 5.
A direct consequence of theorems 6 and 2 is that, for $\alpha > 0$ the entries of $(T_N(f_\alpha))^{-1}$ are functions of $\cos(l-k)\theta_0$ and the integral kernel $G_\alpha(x, y)$ (see corollaries 5).
The results of this paper are of interest in the analysis of time series. Indeed it is known that the n-th covariance matrix of a time series is a positive Toeplitz matrix. If ϕ is the symbol of this Toeplitz matrix, ϕ is called the spectral density of the time series. The time series with spectral density is the function f_α are also called GARMA processes. For more on this processes we refer the reader to [2, 1, 6] and to [7, 8, 9, 2, 5, 10, 12] for Toeplitz matrices in times series.
Predictor polynomial

Now we have to precise the deep link between the orthogonal polynomials and the inverse of the Toeplitz matrices.
Let $T_n(f)$ a Toeplitz matrix with symbol f and $(\Phi_n)_{n\in\mathbb{N}}$ the orthogonal polynomials with respect to f ([11]). To have the polynomial used for the prediction theory we put

$$\Phi_n^*(z) = \sum_{k=0}^{n} \frac{(T_n(f))^{-1}_{k+1,N+1}}{(T_n(f))^{-1}_{N+1,N+1}} z^k, \quad \mid z \mid = 1. \tag{1}$$

We define the polynomial Φ_n^* (see [16]) as

$$\Phi_n^*(z) = z^n \bar{\Phi}_n(\frac{1}{z}), \tag{2}$$

that implies, with the symmetry of the Toeplitz matrix

$$\Phi_n^*(z) = \sum_{k=0}^{n} \frac{(T_n(f))^{-1}_{k+1,1}}{(T_n(f))^{-1}_{1,1}} z^k, \quad \mid z \mid = 1. \tag{3}$$

The polynomials $P_n = \Phi_n^* \sqrt{(T_n(f))^{-1}_{1,1}}$ are often called predictor polynomials. As we can see in the previous formula their coefficients are, up to a normalisation, the entries of the first column of $T_n(f)^{-1}$.

The proof of Theorem 6 uses the important following theorem ([11]),

Theorem 3 *If h a non negative symbol with a finite set of zeroes, and P_n the predictor polynomial of degree n of h, we have, for all integers s such that $-n \leq s \leq n$,*

$$\widehat{\frac{1}{|P_n|^2}}(s) = \hat{h}(s).$$

It implies

Corollary 1 *For a fonction h as in Theorem 3 we have*

$$T_n\left(\frac{1}{|P_n|^2}\right) = T_n(h).$$

2 Main results

2.1 Main notations

In all the paper we consider the symbol defined by $\theta \mapsto 2^{2\alpha}(\cos\theta - \cos\theta_0)^{2\alpha} c_1$ where $c_1 = \frac{|P|}{|Q|}$ with $P, Q \in \mathbb{R}[X]$, without zeros on the united circle and $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $0 < \theta_0 < \pi$. We have $c_1 = c_{1,1}\bar{c}_{1,1}$ with $c_{1,1} = \frac{P}{Q}$. Obviously $c_{1,1} \in H^{2+}(\mathbb{T})$ since $H^{2+}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u < 0 \implies \hat{h}(u) = 0\}$. If χ is the function $\theta \mapsto e^{i\theta}$ and if $\chi_0 = e^{i\theta_0}$ we put $g_{\alpha,\theta_0,c_1} = (\chi - \chi_0)^\alpha (\chi - \overline{\chi_0})^\alpha c_{1,1}$ and $g_{\alpha,\theta_0} = (\chi - \chi_0)^\alpha (\chi - \overline{\chi_0})^\alpha$ since$(2(\cos\theta - \cos\theta_0))^{2\alpha} = |\chi - \chi_0|^{2\alpha} |\chi - \overline{\chi_0}|^{2\alpha}$. Clearly $g_{\alpha,\theta_0,c_1}, g_{\alpha,\theta_0} \in H^{2+}(\mathbb{T})$ and $2^{2\alpha}(\cos\theta - \cos\theta_0)^{2\alpha} c_1 = g_{\alpha,\theta_0,c_1}\overline{g_{\alpha,\theta_0,c_1}}$, $2^{2\alpha}(\cos\theta - \cos\theta_0)^{2\alpha} = g_{\alpha,\theta_0}\overline{g_{\alpha,\theta_0}}$. Then we denote by $\beta^{(\alpha)}_{k,\theta_0,c_1}$ the Fourier coefficient of $g^{-1}_{\alpha,\theta_0,c_1}$ and by $\beta^{(\alpha)}_{k,\theta_0}$ the one of g^{-1}_{α,θ_0}. Without loss of generality we assume $\beta^{(\alpha)}_{0,\theta_0,c_1} = 1$. We put also $\tilde{\beta}^{(\alpha)}_k = \widehat{\tilde{g}^{-1}_\alpha}(k)$ with $\tilde{g}_\alpha = (1 - \chi)^\alpha$.

2.2 Orthogonal polynomials

Theorem 4 *Assume* $\theta_0 \in]0,\pi[$ *and* $-\frac{1}{2} < \alpha < \frac{1}{2}$. *Then we have for all integers* k, $\frac{k}{N} \to x$, $0 < x < 1$, *the asymptotic*

$$\begin{aligned}&\left(T_N^{-1}\left(|\chi-\chi_0\ |^{2\alpha}|\chi-\bar{\chi}_0\ |^{2\alpha}c_1\right)\right)_{k+1,1} = \\ =&K_{\alpha,\theta_0,c_1}\cos\left(k\theta_0+\omega_{\alpha,\theta_0}\right)\left(T_N^{-1}\left(|\chi-1|^{2\alpha}\right)\right)_{k+1,1}(1+o(1))\end{aligned}$$

uniformly in k *for* $x \in [\delta_0,\delta_1]$, $0 < \delta_0 < \delta_1 < 1$, *and with* $\omega_{\alpha,\theta_0} = \alpha\theta_0 + \arg\left(c_{1,1}(\theta_0)\right) - \frac{\pi\alpha}{2}$ *and* $K_{\alpha,\theta_0,c_1} = 2^{-\alpha+1}(\sin\theta_0)^{-\alpha}\sqrt{c_1^{-1}(\chi_0)}$.

Then the following statement is an obvious consequence of Theorems 4 and 1.

Corollary 2 *With the same hypotheses as in Theorem 4 we have*

$$\begin{aligned}&\left(T_N^{-1}\left(|\chi-\chi_0\ |^{2\alpha}|\chi-\bar{\chi}_0\ |^{2\alpha}c_1\right)\right)_{k+1,1} = \\ &= \frac{K_{\alpha,\theta_0,c_1}}{\Gamma(\alpha)}\cos\left(k\theta_0+\omega_{\alpha,\theta_0}\right)k^{\alpha-1}(1-\frac{k}{N})^{\alpha}+o(N^{\alpha-1})\end{aligned}$$

uniformly in k *for* $x \in [\delta_0,\delta_1]$ $0 < \delta_0 < \delta_1 < 1$.

Moreover the equalities (2) and (3) provide

Corollary 3 *Let* $\Phi_N = \sum_{j=0}^{N}\delta_j\chi^j$ *be the orthogonal polynomial of degree* N *(Gegenbauer polynomial) with respect to the weight* $\theta \mapsto 2^{2\alpha}(\cos\theta-\cos\theta_0)c_1(\theta)$, *with* $-\frac{1}{2} < \alpha < \frac{1}{2}$. *Then we have, for* $\frac{j}{N} \to x$, $0 < x < 1$,

$$\delta_j = N^{\alpha-1}\frac{K_{\alpha,\theta_0,c_1}}{\Gamma(\alpha)}\cos\left(N-j\theta_0+\omega_{\alpha,\theta_0}\right)j^{\alpha}(1-\frac{j}{N})^{\alpha-1}+o(N^{\alpha-1}).$$

uniformly in j *for* $x \in [\delta_0,\delta_1]$, $0 < \delta_0 < \delta_1 < 1$.

We can also point out the asymptotic of the coefficients of order k of the predictor polynomial when $\frac{k}{N} \to 0$.

Theorem 5 *With the same hypotheses as in Theorem 4 we have, if* $\dfrac{k}{N} \to 0$ *when* N *goes to the infinity*

$$\left(T_N^{-1}\left(|\chi-\chi_0\ |^{2\alpha}|\chi-\bar{\chi}_0\ |^{2\alpha}c_1\right)\right)_{k+1,1} = \beta^{(\alpha)}_{k,\theta_0,c_1}+O(\frac{1}{N}).$$

Lastly when α approaches $\frac{1}{2}$ we obtain the entries of the last column of $T_N\left(2(\cos\theta-\cos\theta_0)c_1\right)$.

Corollary 4 *Assume* $\theta_0 \in]0,\pi[$. *Then for all integers* k *for* $\frac{k}{N} \to x$, $0 < x < 1$, *we have the asymptotic*

$$\begin{aligned}&\left(T_N^{-1}\left(|\chi-\chi_0\ ||\chi-\bar{\chi}_0\ |c_1\right)\right)_{k+1,1} = \\ =&K_{1/2,\theta_0,c_1}\cos\left(k\theta_0+\omega_{1/2,\theta_0}\right)\sqrt{\frac{1}{k}-\frac{1}{N}}+o(\sqrt{N})\end{aligned}$$

uniformly in k *for* $x \in [\delta_0,\delta_1]$, $0 < \delta_0 < \delta_1 < 1$.

Remark 1 *This corollary implies that the coefficient of order* k *of the orthogonal polynomial with respect of* $\theta \mapsto 2(\cos\theta-\cos\theta_0)c_1(\theta)$ *is* $K_{1/2,\theta_0,c_1}\cos\left(k\theta_0+\omega_{1/2,\theta_0}\right)\left(\frac{1}{k}-\frac{1}{N}\right)^{-1}+o(\sqrt{N})$

2.3 Application to Toeplitz matrices

Theorem 6 *Assume $\theta_0 \in]0,\pi[$ and $0<\alpha<\frac{1}{2}$. For $\frac{k}{N}\to x$, $\frac{l}{N}\to y$ and $0<x\neq y<1$, we have asymptotic*

$$\begin{aligned}&\left(T_N^{-1}\left(|\chi-\chi_0\ |^{2\alpha}|\chi-\bar{\chi}_0\ |^{2\alpha}c_1\right)\right)_{k+1,l+1}=\\ =&|K_{\alpha,\theta_0,c_1}|^2\cos\left(\theta_0(k-l)\right)\left(T_N^{-1}\left(|\chi-1|^{2\alpha}\right)\right)_{k+1,l+1}+o(N^{2\alpha-1})\end{aligned}$$

uniformly for k,l such that $0<\delta_1<x\neq y<\delta_2<1$.

At it has been said in the introduction this statement and the results of [15] provides the next corollary.

Corollary 5 *Assume $\alpha\in]0,\frac{1}{2}]$ and $\theta_0\in]0,\pi[$. Let G_α be the function defined on $0<x\neq y<1$ by*

$$G_\alpha(x,y)=\frac{x^\alpha y^\alpha}{\Gamma^2(\alpha)}\int_{\max(x,y)}^1\frac{(t-x)^{\alpha-1}(t-y)^{\alpha-1}}{t^{2\alpha}}dt.$$

With the same hypothesis as in Theorem 6 we have the asymptotic

$$\begin{aligned}&\left(T_N^{-1}\left(|\chi-\chi_0\ |^{2\alpha}|\chi-\overline{\chi_0}\ |^{2\alpha}c_1\right)\right)_{[Nx]+1,[Ny]+1}=\\ =&N^{2\alpha-1}|K_{\alpha,\theta_0,c_1}|^2\cos\left(\theta_0([Nx]-[Ny])\right)G_\alpha(x,y)+o(N^{2\alpha-1})\end{aligned}$$

uniformly in k,l for $0<\delta_1\leq x\neq y\leq\delta_2<1$.

2.4 Jacobi polynomial (in a particular case)

We note that in Theorem 6 one passes from the zeroes χ_0 and $\overline{\chi_0}$ to two zeroes $\chi_1=e^{i\theta_1}$ and $\chi_2=e^{i\theta_2}$ with $|\theta_1-\theta_2|\in]0,\pi[$. Namely it is easy to see that

$$\begin{aligned}&T_N^{-1}(|\chi-\chi_1|^{2\alpha}|\chi-\chi_2|^{2\alpha}c_1)=\\ &\Delta(\chi_1^{1/2}\chi_2^{1/2})T_N^{-1}\left(|\chi_1^{1/2}\chi_2^{-1/2}-\psi|^{2\alpha}|\chi_1^{-1/2}\chi_2^{1/2}-\psi|^{2\alpha}c_{1,\psi}\right)^{-1}\Delta^{-1}(\chi_1^{1/2}\chi_2^{1/2})\end{aligned}$$

with $\Delta(\chi_1^{1/2}\chi^{1/2})$ is the diagonal matrix defined by $\left(\Delta(\chi_1^{1/2}\chi_2^{1/2})\right)_{i,j}=0$ if $i\neq j$ and $(\Delta(\psi))_{j,j}=(\chi_1^{1/2}\chi_2^{1/2})^j$.

From this and Equation (2) we deduce the following proposition

Proposition 1 *Let $\Phi_{1,2}=\sum_{j=0}\tilde{\delta}_j\chi^j$ be the orthogonal polynomial (Jacobi polynomial) with respect to the weight $|\chi-\chi_1|^{2\alpha}|\chi-\chi_2|^{2\alpha}$, with $\alpha\in]-\frac{1}{2},\frac{1}{2}]$. Let $K_{\alpha,\theta_1,\theta_2}$ be the real $2^{-\alpha+1}|\sin(\theta_1-\theta_2)|^{-\alpha}$. Then we have, for $\frac{j}{N}\to x$, $0<x<1$.*

$$\tilde{\delta}_j=N^{\alpha-1}K_{\alpha,\theta_1,\theta_2}(\overline{(\chi_1\chi_2)^{1/2}})^{N-j}\cos\left((\frac{\theta_1-\theta_2}{2})(N-j)+\omega_{\alpha,\theta_1-\theta_2}\right)\frac{j^\alpha(1-\frac{j}{N})^{\alpha-1}}{\Gamma(\alpha)}+o(N^{\alpha-1}),$$

uniformly in j for $x\in[\delta_0,\delta_1]$, $0<\delta_0<\delta_1<1$.

3 Inversion formula

3.1 Definitions and notations

Let $H^{2+}(\mathbb{T})$ and $H^{2-}(\mathbb{T})$ the two subspaces of $L^2(\mathbb{T})$ defined by $H^{2+}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u < 0 \implies \hat{h}(u) = 0\}$ and $H^{2-}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u \geq 0 \implies \hat{h}(u) = 0\}$. We denote by π_+ the orthogonal projector on $H^{2+}(\mathbb{T})$ and π_- the orthogonal projector on $H^{2-}(\mathbb{T})$. It is known (see [9]) that if $f \geq 0$ and $\ln f \in L^1(\mathbb{T})$ we have $f = g\bar{g}$ with $g \in H^{2+}(\mathbb{T})$. Put $\Phi_N = \frac{g}{\bar{g}}\chi^{N+1}$. Let H_{Φ_N} and $H^*_{\Phi_N}$ be the two Hankel operators defined respectively on H^{2+} and H^{2-} by

$$H_{\Phi_N} : \quad H^{2+}(\mathbb{T}) \to H^{2-}(\mathbb{T}), \qquad H_{\Phi_N}(\psi) = \pi_-(\Phi_N \psi),$$

and

$$H^*_{\Phi_N} : \quad H^{2-}(\mathbb{T}) \to H^{2+}(\mathbb{T}), \qquad H^*_{\Phi_N}(\psi) = \pi_+(\bar{\Phi}_N \psi).$$

3.2 A generalised inversion formula

We have stated in [15] for a precise class of non regular functions which contains $\cos^\alpha(\theta - \theta_0)c_1$ and $(\cos\theta - \cos\theta_0)^\alpha c_1$ the following lemma (see the appendix of [15] for the demonstration),

Lemma 1 *Let f be an almost everywhere positive function on the torus $\mathbb{T}$ such that $\ln f$, f, and $\frac{1}{f}$ are in $\mathbb{L}^1(\mathbb{T})$. Then $f = g\bar{g}$ with $g \in H^{2+}(\mathbb{T})$. For all trigonometric polynomials P of degree at most N, we define $G_{N,f}(P)$ by*

$$G_{N,f}(P) = \frac{1}{g}\pi_+\left(\frac{P}{\bar{g}}\right) - \frac{1}{g}\pi_+\left(\Phi_N \sum_{s=0}^{\infty} \left(H^*_{\Phi_N} H_{\Phi_N}\right)^s \pi_+ \bar{\Phi}_N \pi_+ \left(\frac{P}{\bar{g}}\right)\right).$$

For all P we have

- *The serie $\sum_{s=0}^{\infty} \left(H^*_{\Phi_N} H_{\Phi_N}\right)^s \pi_+ \bar{\Phi}_N \pi_+ \left(\frac{P}{\bar{g}}\right)$ converges in $L^2(\mathbb{T})$.*
- $\det\left(T_N(f)\right) \neq 0$ *and*
$$\left(T_N(f)\right)^{-1}(P) = G_{N,f}(P).$$

An obvious corollary of Lemma 1 is

Corollary 6 *With the hypotheses of Lemma 1 we have*

$$\left(T_N(f)\right)^{-1}_{l+1,k+1} = \left\langle \pi_+\left(\frac{\chi^k}{\bar{g}}\right) \middle| \left(\frac{\chi^l}{\bar{g}}\right)\right\rangle - \left\langle \sum_{s=0}^{\infty} \left(H^*_{\Phi_N} H_{\Phi_N}\right)^s \pi_+ \bar{\Phi}_N \pi_+ \left(\frac{\chi^k}{\bar{g}}\right) \middle| \bar{\Phi}_N \left(\frac{\chi^l}{\bar{g}}\right)\right\rangle.$$

Lastly if $\gamma_{u,\alpha,\theta} = \widehat{\frac{g_{\alpha,\theta_0}}{g_{\alpha,\theta_0}}}(u)$ we obtain as in [15] the formal result

$$\begin{aligned}\left(H^*_{\Phi_N} H_{\Phi_N}\right)^m \pi_+ \bar{\Phi}_N \pi_+ \left(\frac{\chi^k}{\bar{g}}\right) = \sum_{u=0}^{k} \beta^{(\alpha)}_{u,\theta_0,c_1} \sum_{n_0=0}^{\infty} \Bigg(\sum_{n_1=1}^{\infty} \bar{\gamma}_{-(N+1+n_1+n_0),\alpha,\theta_0} \\ \sum_{n_2=0}^{\infty} \gamma_{-(N+1+n_1+n_2),\alpha,\theta_0} \cdots \sum_{n_{2m-1}=1}^{\infty} \bar{\gamma}_{-(N+1+n_{2m-1}+n_{2m-2}),\alpha,\theta_0} \\ \sum_{n_{2m}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m}),\alpha,\theta_0} \bar{\gamma}_{-(u-(N+1+n_{2m}),\alpha,\theta_0)} \Bigg) \chi^{n_0}\end{aligned}$$

3.3 Application to the orthogonal polynomials

With the corollary 6 and the hypothesis on $\beta^{(\alpha)}_{0,\theta_0,c_1}$ the equality in the corollary 6 becomes, for $l=1$, and for $f=|\chi-\bar{\chi}_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1$

$$(T_N(f))^{-1}_{1,k+1}=\beta^{(\alpha)}_{k,\theta_0,c_1}-\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}H_N(u) \tag{4}$$

with

$$\begin{aligned}H_N(u)=\sum_{m=0}^{+\infty}\Bigg(\sum_{n_0=0}^{\infty}\gamma_{N+1+n_0,\alpha,\theta_0}\Bigg(\sum_{n_1=0}^{\infty}\bar{\gamma}_{-(N+1+n_1+n_0),\alpha,\theta_0}\\ \sum_{n_2=0}^{\infty}\gamma_{-(N+1+n_1+n_2),\alpha,\theta_0}\cdots\sum_{n_{2m-1}=0}^{\infty}\bar{\gamma}_{-(N+1+n_{2m-1}+n_{2m-2}),\alpha,\theta_0}\\ \sum_{n_{2m}=0}^{\infty}\gamma_{-(N+1+n_{2m-1}+n_{2m}),\alpha,\theta_0}\bar{\gamma}_{(u-(N+1+n_{2m}),\alpha,\theta_0}\Bigg)\Bigg)\end{aligned}$$

Our proof consists in the computation of the coefficients $\beta^{(\alpha)}_{u,\theta_0,c_1}$, $\gamma_{u,\alpha,\theta}$ and $H_N(u)$ which appear in the inversion formula. For each step we obtain the corresponding terms for the symbol $2^{\alpha}(1-\cos\theta)c_1$ multiplied by a trigonometric coefficient. That provides the expected link with the formulas in Theorems 1, 2.

4 Demonstration of Theorem 4

4.1 Asymptotic of $\beta^{(\alpha)}_{k,\theta_0,c_1}$

Remark 2 *In the rest of this paper we denote by $c_{1,1}$ the function in $H^{2+}(\mathbb{T})$ such that $c_1=c_{1,1}\overline{c_{1,1}}$. In all this proof we put $\phi_0=\arg(c_{1,1}(\theta_0))$*

Property 1 *For $-\frac{1}{2}<\alpha<\frac{1}{2}$ and $\theta_0\in]0,\pi[$ we have, for sufficiently large k and for the real β defined by $\beta=\alpha-\frac{1}{2}$ if $\alpha<0$ and $\beta=\alpha$ if $\alpha>0$,*

$$\beta^{(\alpha)}_{k,\theta_0,c_1}=K_{\alpha,\theta_0,c_1}\cos(k\theta_0+\omega_{\alpha,\theta_0})\frac{k^{\alpha-1}}{\Gamma(\alpha)}+o(k^{\beta-1})$$

uniformly in k. With $K_{\alpha,\theta_0,c_1}=\frac{1}{\sqrt{c_1(\chi_0)}}2^{-\alpha+1}(\sin\theta_0)^{-\alpha}$ and ω_{α,θ_0} as in the statement of Theorem 4.

First we have to prove the lemma

Lemma 2 *For $-\frac{1}{2}<\alpha<\frac{1}{2}$ and $\theta_0\in]0,\pi[$ we have, for a sufficiently large k.*

$$\beta^{(\alpha)}_{k,\theta_0}=K_{\alpha,\theta_0}\cos((k+\alpha)\theta_0+\omega_{\alpha})\frac{k^{\alpha-1}}{\Gamma(\alpha)}+o(k^{\beta-1}).$$

uniformly in k, with $K_{\alpha,\theta_0}=2^{-\alpha+1}(\sin\theta_0)^{-\alpha}$, $\omega_{\alpha}=-\frac{\pi\alpha}{2}$, and β as in Property 1.

Remark 3 *In these two last statements "uniformly in k " means that for all $\epsilon > 0$ we have an integer k_ϵ such that for all $k \geq k_\epsilon$*

$$\left|\beta_{k,\theta_0}^{(\alpha)} - K_{\alpha,\theta_0}\cos((k+\alpha)\theta_0 + \omega_\alpha)\frac{k^{\alpha-1}}{\Gamma(\alpha)}\right| < \epsilon k^{\beta-1}$$

and

$$\left|\beta_{k,\theta_0,c_1}^{(\alpha)} - K_{\alpha,\theta_0,c_1}\cos((k+\alpha)\theta_0 + \omega_\alpha)\frac{k^{\alpha-1}}{\Gamma(\alpha)}\right| < \epsilon k^{\beta-1}.$$

Proof : With our notations we can write

$$\beta_{k,\theta_0}^{(\alpha)} = \sum_{u=0}^{k} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u}.$$

Put $k_0 = k^\gamma$ with $0 < \gamma < 1$ such that for $u > k_0$ we have

$$\tilde{\beta}_u^{(\alpha)} = \frac{u^{\alpha-1}}{\Gamma(\alpha)} + O(k^{\alpha-2}) \tag{5}$$

uniformly in u (see [19]). Writting for $k \geq k_0$

$$\begin{aligned}\sum_{u=0}^{k} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u} &= \sum_{u=0}^{k_0} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u} \\ &+ \sum_{u=k_0+1}^{k-k_0-1} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u} \\ &+ \sum_{u=k-k_0}^{k} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u}.\end{aligned}$$

The first sum is also

$$\sum_{u=0}^{k_0} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \left(\tilde{\beta}_{k-u}^{(\alpha)} - \tilde{\beta}_k^{(\alpha)} + \tilde{\beta}_k^{(\alpha)}\right)(\overline{\chi_0})^{k-u}.$$

We observe that

$$\left|\sum_{u=0}^{k_0} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \left(\tilde{\beta}_{k-u}^{(\alpha)} - \tilde{\beta}_k^{(\alpha)}\right)\right| \leq \frac{1}{\Gamma(\alpha)} \sum_{u=0}^{k_0} |(k-u)^{\alpha-1} - k^{\alpha-1}||\tilde{\beta}_u^{(\alpha)}|. \tag{6}$$

Consequently

$$\begin{aligned}\sum_{u=0}^{k_0} \tilde{\beta}_u^{(\alpha)}(\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)}(\overline{\chi_0})^{k-u} &= \left(\sum_{u=0}^{k_0} \beta_u^{(\alpha)}(\chi_0)^{2u}\right)\bar{\chi}_0^k \frac{k^{\alpha-1}}{\Gamma(\alpha)} + R_1 \\ &= \left(\sum_{u=0}^{+\infty} \tilde{\beta}_u^{(\alpha)}(\chi_0)^{2u} - \sum_{u=k_0}^{+\infty} \tilde{\beta}_u^{(\alpha)}(\chi_0)^{2u}\right)\bar{\chi}_0^k \frac{k^{\alpha-1}}{\Gamma(\alpha)} + R_1\end{aligned}$$

with $R_1 = O(k^{\alpha-2+\gamma})$ if $\alpha < 0$ and $R_1 = O(k^{\alpha-2+\gamma\alpha})$ if $\alpha > 0$. Then Lemma 9 implies

$$|\sum_{u=k_0}^{+\infty} \tilde{\beta}_u^{(\alpha)} (\chi_0)^{2u}| \leq |\tilde{\beta}_{k_0}^{(\alpha)} \chi_0^{2u}| + \sum_{u=k_0}^{\infty} \frac{|\tilde{\beta}_{u+1}^{(\alpha)} - \tilde{\beta}_u^{(\alpha)}|}{\Gamma(\alpha)}, \tag{7}$$

that is

$$\sum_{u=k_0}^{+\infty} \tilde{\beta}_u^{(\alpha)} (\chi_0)^{2u} = O(k_0^{\alpha-1}).$$

Finally we get

$$\sum_{u=0}^{k_0} \tilde{\beta}_u^{(\alpha)} (\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)} (\overline{\chi_0})^{k-u} = \frac{k^{\alpha-1}}{\Gamma(\alpha)} \overline{\chi_0}^k (1-\chi_0^2)^{-\alpha} + O\left(k^{(\alpha-1)(\gamma+1)}\right) + R_1.$$

Analogously we obtain

$$\sum_{u=k-k_0}^{k} \tilde{\beta}_u^{(\alpha)} (\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)} (\overline{\chi_0})^{k-u} = \chi_0^k \frac{k^{\alpha-1}}{\Gamma(\alpha)} (1-\bar{\chi}_0^2)^{-\alpha} + O\left(k^{(\alpha-1)(\gamma+1)}\right) + R_2,$$

with R_2 as R_1.
For the third sum an Abel summation provides

$$\sum_{u=k_0+1}^{k-k_0-1} \tilde{\beta}_u^{(\alpha)} (\chi_0)^u \tilde{\beta}_{k-u}^{(\alpha)} (\overline{\chi_0})^{k-u} = \overline{\chi_0}^k \left(\tilde{\beta}_{k_0}^{(\alpha)} \tilde{\beta}_{k-k_0}^{(\alpha)} \sigma_{k_0-1} \right.$$
$$\left. + \sum_{u=k_0}^{k-k_0-2} (\tilde{\beta}_u^{\alpha} \tilde{\beta}_{k-u}^{(\alpha)} - \tilde{\beta}_{u+1}^{(\alpha)} \tilde{\beta}_{k-u-1}^{(\alpha)}) \sigma_u \right) + \tilde{\beta}_{k-k_0-1}^{(\alpha)} \tilde{\beta}_{k_0}^{(\alpha)} \sigma_{k-k_0}$$

with $\sigma_v = 1 + \chi_0^2 + \cdot + \chi_0^{2v}$. This last sum is also equal to $\overline{\chi_0}^k (A + B)$, with

$$|A| = O\left(\tilde{\beta}_{k_0}^{(\alpha)} \tilde{\beta}_k^{(\alpha)}\right) = O(k_0^{\alpha-1} k^{\alpha-1}) = o(k^{(\alpha-1)(\gamma+1)})$$

and

$$B = -\sum_{u=k_0}^{k-k_0-2} \frac{1}{\Gamma^2(\alpha)} \left(u^{\alpha-1}(k-u)^{\alpha-1} - (u+1)^{\alpha-1}(k-u-1)^{\alpha-1} \right) \frac{\chi_0^{2u+2}}{1-\chi_0^2}.$$

The main value Theorem implies

$$|B| \leq Mk \left(\sum_{v=k_0}^{k-k_0} v^{\alpha-2} (k-v)^{\alpha-2} \right), \tag{8}$$

with M no depending from k. With the Euler and Mac-Laurin formula it is easyly seen that

$$\sum_{v=k_0}^{k-k_0} v^{\alpha-2} (k-v)^{\alpha-1} \sim k_0^{\alpha-2} (k-k_0)^{\alpha-1} + k_0^{\alpha-1} (k-k_0)^{\alpha-2} + \int_{k_0}^{k-k_0} t^{\alpha-2} (k-t)^{\alpha-2} dt.$$

The decomposition

$$\int_{k_0}^{k-k_0} t^{\alpha-2}(k-t)^{\alpha-2}dt = \int_{k_0}^{k/2} t^{\alpha-2}(k-t)^{\alpha-2}dt + \int_{k/2}^{k-k_0} t^{\alpha-2}(k-t)^{\alpha-2}dt$$

provides the estimation $|B| = O(k_0^{\alpha-1}k^{\alpha-1}) = O(k^{(\alpha-1)(\gamma+1)})$. If $\alpha > 0$ and $0 < \gamma < 1$ we have

$$\beta_{k,\theta_0}^{(\alpha)} = \frac{k^{\alpha-1}}{\Gamma(\alpha)}\left(\overline{\chi_0^k}(1-\chi_0^2)^{-\alpha} + \chi_0^k(1-\overline{\chi_0^2})^{-\alpha}\right) + o(k^{\alpha-1})$$

If $\alpha < 0$ and $\gamma = \frac{1}{2}$ we get

$$\beta_{k,\theta_0}^{(\alpha)} = \frac{k^{\alpha-1}}{\Gamma(\alpha)}\left(\overline{\chi_0^k}(1-\chi_0^2)^{-\alpha} + \chi_0^k(1-\overline{\chi_0^2})^{-\alpha}\right) + o(k^{\beta-1})$$

with $\beta = \alpha - \frac{1}{2}$. On the another hand we have

$$\begin{aligned}\beta_{k,\theta_0}^{(\alpha)} &= 2\frac{k^{\alpha-1}}{\Gamma(\alpha)}\Re\left(e^{-ik\theta_0}\left(1-\cos(2\theta_0)-i\sin(2\theta_0)\right)^{-\alpha}\right) + o(k^{\beta-1}) \\ &= 2^{1-\alpha}\frac{k^{\alpha-1}}{\Gamma(\alpha)}\Re\left(e^{-ik\theta_0}\left(\sin(\theta_0)\left(\sin\theta_0 - i\cos\theta_0\right)\right)^{-\alpha}\right) + o(k^{\beta-1})\end{aligned}$$

Since $\theta_0 \in]0,\pi[$ we have $(\sin\theta_0(\sin\theta_0 - i\cos\theta_0))^{-\alpha} = (\sin\theta_0)^{-\alpha}e^{i\alpha(\frac{\pi}{2}-\theta_0)}$
This last remark gives the definition of ω_α. The equations (6), (7), (8), imply the uniformity that completes the proof of the lemma. □

To ends the proof of the property we need to obtain $\beta_{k,\theta_0,c_1}^{(\alpha)}$ from $\beta_{k,\alpha,\theta_0}^{(\alpha)}$ for a sufficiently large k. We can remark that a similar case has been treated in [13] for the function $(1-\chi)^\alpha c_1$. Here we develop the same idea than in this last paper. Let c_m the coefficient of Fourier of order m of the function $c_{1,1}^{-1}$. The hypotheses on $c_{1,1}$ imply that $c_{1,1}^{-1}$ is in $A(\mathbb{T},p) = \{h \in L^2(\mathbb{T}) | \sum_{u\in\mathbb{Z}} u^p|\hat{h}(u)| < \infty\}$ for all positive integer p. We have, $\beta_{m,\theta_0,c_1}^{(\alpha)} = \sum_{s=0}^{m} \beta_{m,\theta_0}^{(\alpha)} c_{m-s}$. For $0 < \nu < 1$ we can write

$$\sum_{s=0}^{m} \beta_{s,\theta_0}^{(\alpha)} c_{m-s} = \sum_{s=0}^{m-m^\nu} \beta_{s,\theta_0}^{(\alpha)} c_{m-s} + \sum_{s=m-m^\nu+1}^{m} \beta_{s,\theta_0}^{(\alpha)} c_{m-s}.$$

Lemma 2 provides

$$\begin{aligned}\sum_{s=m-m^\nu+1}^{m} \beta_{s,\theta_0}^{(\alpha)} c_{m-s} = &\left(K_{\alpha,\theta_0} \sum_{s=m-m^\nu}^{m} \frac{s^{\alpha-1}}{\Gamma(\alpha)}\left(\cos\left((s+\alpha)\theta_0 + \omega_\alpha\right) c_{m-s}\right)\right. \\ &\left. + o(m^{\beta-1}) \sum_{s=m-m^\nu+1}^{m} |c_{m-s}|\right.\end{aligned}$$

and, since $\sum_{s\in\mathbb{Z}} |c_s| < \infty$, we have

$$\sum_{s=m-m^\nu+1}^{m} \beta_{s,\theta_0}^{(\alpha)} c_{m-s} = K_{\alpha,\theta_0}\frac{s^{\alpha-1}}{\Gamma(\alpha)} \sum_{s=m-m^\nu}^{m} \frac{s^{\alpha-1}}{\Gamma(\alpha)}\left(\cos\left((s+\alpha)\theta_0 + \omega_\alpha\right) c_{m-s} + o(m^{\beta-1}).\right.$$

We have always

$$\Big|\sum_{s=m-m^\nu}^{m}(s^{\alpha-1}-m^{\alpha-1})c_{m-s}\Big|\leq(1-\alpha)m^{\nu+\alpha-2}\sum_{s=m-m^\nu}^{m}|c_{m-s}|. \tag{9}$$

The convergence of (c_s) implies

$$\begin{aligned}&K_{\alpha,\theta_0}\sum_{s=m-m^\nu}^{m}\frac{s^{\alpha-1}-m^{\alpha-1}+m^{\alpha-1}}{\Gamma(\alpha)}(\cos\left((s+\alpha)\theta_0+\omega_\alpha\right)c_{m-s}\\&=K_{\alpha,\theta_0}\frac{m^{\alpha-1}}{\Gamma(\alpha)}\sum_{s=m-m^\nu}^{m}\left(\cos\left((s+\alpha)\theta_0+\omega_\alpha\right)c_{m-s}+O(m^{\alpha-2+\nu}).\right.\end{aligned}$$

For all positive integer p the function $c_{1,1}\ A(p,\mathbb{T}))$. Hence one can prove first

$$\Big|\sum_{v=m^\nu+1}^{\infty}e^{-iv\theta}c_v\Big|\leq(m^{-p\nu})\sum_{s\in\mathbb{Z}}|c_s| \tag{10}$$

and secondly

$$\begin{aligned}\sum_{s=m-m^\nu}^{m}\left(\cos\left((s+\alpha)\theta_0+\omega_\alpha\right)c_{m-s}\right.&=\frac{1}{2}\left(\sum_{s=m-m^\nu}^{m}e^{is\theta_0}c_{m-s}\right)e^{i(\theta_0\alpha+\omega_\alpha)}\\&+\frac{1}{2}\left(\sum_{s=m-m^\nu}^{m}e^{-is\theta_0}c_{m-s}\right)e^{-i(\theta_0\alpha+\omega_\alpha)}\\&=\frac{1}{2}\left(c_{1,1}^{-1}(e^{-i\theta_0})e^{i(m\theta_0+\theta_0\alpha+\omega_\alpha)}+c_{1,1}^{-1}(e^{i\theta_0})e^{-i(m\theta_0+\theta_0\alpha+\omega_\alpha)}\right)\\&+O(m^{-p\nu}).\end{aligned}$$

Since $\overline{c_{1,1}^{-1}(e^{i\theta_0})}=c_{1,1}^{-1}(e^{-i\theta_0})$ that last formula provides

$$\sum_{s=m-m^\nu}^{m}\left(\cos\left((s+\alpha)\theta_0+\omega_\alpha\right)c_{m-s}=\sqrt{c_1^{-1}(\chi_0)}\cos\left((m+\alpha)\theta_0+\omega_\alpha+\phi_0\right)+O(m^{-p\nu})\right. \tag{11}$$

and

$$\begin{aligned}\sum_{s=m-m^\nu+1}^{m}\beta_{s,\theta_0}^{(\alpha)}c_{m-s}&=K_{\alpha,\theta_0}\frac{m^{\alpha-1}}{\Gamma(\alpha)}\sqrt{c_1^{-1}(\chi_0)}\cos\left((m+\alpha)\theta_0+\omega_\alpha+\phi_0\right)\\&+O(m^{\alpha-1-p\nu})+O(m^{\alpha-2+\nu})+o(m^{\beta-1}).\end{aligned}$$

On the other hand we have (because $c_{1,1}^{-1}$ in $A(\mathbb{T},p)$)

$$\Big|\sum_{s=0}^{m-m^\nu}\beta_{s,\alpha}c_{m-s}\Big|\leq\frac{1}{m^{2\nu}}\sum_{v\in\mathbb{Z}}v^p|c_v|\max_{s\in\mathbb{N}}(|\beta_{s,\theta_0}^{(\alpha)}|).$$

For a good choice of p and ν we obtain the expected formula for $\beta_{\alpha,\theta_0,c_1}$. The uniformity is provided by Lemma 2 and the equation (9) and (10).

4.2 Estimation of the Fourier coefficients of $\frac{g_{\alpha,\theta_0}}{\overline{g_{\alpha,\theta_0}}}$

Property 2 *Assume* $-\frac{1}{2} < \alpha < \frac{1}{2}$ *and* $\theta_0 \in]0,\pi[$ *then we have for all integer* $k \geq 0$ *sufficiently large*

$$\widehat{\frac{g_{\alpha,\theta_0}}{\overline{g_{\alpha,\theta_0}}}}(-k) = \frac{2}{k+\alpha}\frac{\sin(\pi\alpha)}{\pi}\cos(\theta_0 k + 2\omega'_{\alpha,\theta_0}) + o(k^{\min(\alpha-1,-1)})$$

uniformly in k *and with* $\omega'_{\alpha,\theta_0} = \phi_\alpha + \phi'_0$ *where* $\phi'_0 = \arg\left(\frac{c_{1,1}}{\bar{c}_{1,1}}\right)(e^{i\theta_0})$ *and* $\phi_\alpha = \arg\left(\frac{\chi_0^2-1}{\bar{\chi}_0^2-1}\right)^\alpha$.

First we have to prove the lemma

Lemma 3 *For* $-\frac{1}{2} < \alpha < \frac{1}{2}$ *and* $\theta_0 \in]0,\pi[$ *we have, for all integer* k *sufficiently large*

$$\gamma_{-k} = \frac{2}{k+\alpha}\frac{\sin(\alpha)}{\pi}\cos\left(\theta_0 k + \phi_\alpha)\right) + o(k^{\min(\alpha-1,-1)}),$$

uniformly in k *and where* γ_k *is the coefficient of order* k *of the function* $\frac{(\chi\chi_0-1)^\alpha(\chi\bar{\chi}_0-1)^\alpha}{(\bar{\chi}\bar{\chi}_0-1)^\alpha(\bar{\chi}\chi_0-1)^\alpha}$.

Proof of Lemma 3: In all this proof we denote respectively by $\tilde{\gamma}_k, \gamma_{1,k}, \gamma_{2,k}$ the Fourier coefficient of order k of $\frac{(\chi-1)^\alpha}{(\bar{\chi}-1)^\alpha}, \frac{(\chi\chi_0-1)^\alpha}{(\bar{\chi}\bar{\chi}_0-1)^\alpha}, \frac{(\chi\bar{\chi}_0-1)^\alpha}{(\bar{\chi}\chi_0-1)^\alpha}$. Clearly $\tilde{\gamma}_k = \frac{\sin(\pi\alpha)}{\pi}\frac{1}{k+\alpha}$ $\gamma_{1,k} = \chi_0^k\tilde{\gamma}_k, \gamma_{2,k} = (\bar{\chi}_0)^k\tilde{\gamma}_k$. Assume also $k \geq 0$. We have $\gamma_{-k} = \sum_{v+u=-k}\gamma_{1,u}\gamma_{2,v}$. For an integer $, k_0$ and $k_0 = k^\tau$, $0 < \tau < 1$ we can split this sum into

$$\sum_{u<-k-k_0}\gamma_{1,u}\gamma_{2,-k-u} + \sum_{u=-k-k_0}^{-k+k_0}\gamma_{1,u}\gamma_{2,-k-u} + \sum_{u=-k+k_0+1}^{-k_0-1}\gamma_{1,u}\gamma_{2,-k-u}$$
$$+ \sum_{u=-k_0}^{k_0}\gamma_{1,u}\gamma_{2,k-u} + \sum_{u>k_0}\gamma_{1,u}\gamma_{2,-k-u}.$$

Write

$$\sum_{u=-k_0}^{k_0}\gamma_{1,u}\gamma_{2,-k-u} = \sum_{u=-k_0}^{k_0}\gamma_{1,u}(\bar{\chi}_0)^{k+u}(\tilde{\gamma}_{-k-u} - \tilde{\gamma}_{-k} + \tilde{\gamma}_{-k}).$$

Since

$$\sum_{u=-k_0}^{k_0}\gamma_{1,u}(\bar{\chi}_0)^{k+u}(\tilde{\gamma}_{-k-u} - \tilde{\gamma}_{-k}) = \frac{\sin(\pi\alpha)}{\pi}\sum_{u=-k_0}^{k_0}\gamma_{1,u}(\bar{\chi}_0)^{k+u}\frac{-u}{(k+u+\alpha)(k+\alpha)} \qquad (12)$$

it follows that

$$\begin{aligned}\sum_{u=-k_0}^{k_0}\gamma_{1,u}\gamma_{2,-k-u} &= \tilde{\gamma}_{-k}\sum_{u=-k_0}^{k_0}\gamma_{1,u}(\bar{\chi}_0)^{-k-u}+O(k_0k^{-2})\\ &= \tilde{\gamma}_{-k}(\chi_0)^k\left(\frac{\chi_0^2-1}{(\bar{\chi}_0)^2-1}\right)^{\alpha}\\ &+\tilde{\gamma}_{-k}(\chi_0)^k\sum_{|u|\geq k_0}\gamma_{1,u}\chi_0^u+O(k_0k^{-2})\\ &= \tilde{\gamma}_{-k}(\chi_0)^k\left(\frac{\chi_0^2-1}{(\bar{\chi}_0)^2-1}\right)^{\alpha}+O\left((k_0k)^{-1}\right)+O(k_0k^{-2})\\ &= \tilde{\gamma}_{-k}(\chi_0)^k\left(\frac{\chi_0^2-1}{(\bar{\chi}_0)^2-1}\right)^{\alpha}+O(k^{\tau-2}).\end{aligned}$$

In the same way we have

$$\sum_{u=-k-k_0}^{-k+k_0}\gamma_{1,u}\gamma_{2,k-u} = \tilde{\gamma}_{-k}(\chi_0)^{-k}\left(\frac{(\bar{\chi}_0)^2-1}{\chi_0^2-1}\right)^{\alpha}+O(k^{\tau-2}).$$

Now using Lemma 9 it is easy to see that

$$\sum_{u<-k-k_0}\gamma_{1,u}\gamma_{2,-k-u}\leq M_1(k_0k)^{-1} \tag{13}$$

$$\sum_{u>k_0}\gamma_{1,u}\gamma_{2,-k-u}\leq M_2(k_0k)^{-1} \tag{14}$$

with M_1 and M_2 no depending from k. For the sum $S=\sum_{u=-k+k_0+1}^{-k_0-1}\gamma_{1,u}\gamma_{2,-k-u}$ we can remark, using an Abel summation, that

$$|S|\leq M_3(k_0k)^{-1}+\sum_{u=-k+k_0+1}^{-k_0-1}\Big|\frac{1}{(u+\alpha)(k-u+\alpha)}-\frac{1}{(u+1+\alpha)(k-u-1+\alpha)}\Big|$$

M_3 no depending from k. Consequently the main values theorem provides

$$|S|\leq M_3(k_0k)^{-1}+\sum_{u=-k+k_0+1}^{-k_0-1}\frac{k-2u}{(k-u)^2u^2}. \tag{15}$$

with M_3 no depending from k. Then Euler and Mac-Laurin formula provides the upper bound

$$|S|\leq O\left((k_0k)^{-1}\right)+\int_{-k+k_0+1}^{-k_0-1}\frac{k-2u}{(k-u)^2u^2}du.$$

Since

$$\int_{-k+k_0+1}^{-k_0-1}\frac{k-2u}{(k-u)^2u^2}du\leq\frac{3k}{(k+k_0)^2}\int_{-k+k_0+1}^{-k_0-1}\frac{1}{u^2}du$$

we get

$$\sum_{u<-k+k_0+1}^{-k_0-1} \gamma_{1,u}\gamma_{2,k-u} = O\left((k_0 k)^{-1}\right)$$

and

$$\gamma_k = \frac{2}{k+\alpha}\frac{\sin(\alpha)}{\pi}\cos\left(\theta_0 k + \phi_\alpha\right) + O\left((k_0 k)^{-1}\right) + O(k^{\alpha-2}).$$

Then with a good choice of τ we obtain the expected formula. The uniformity is a direct consequence of the equations (12), (13), (14), (15). □

The rest of the proof of Lemma 3 can be treated as the end of the proof of property 1.

4.3 Expression of $\left(T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha}c_1\right)\right)_{k+1,1}$.

First we have to prove the next lemma

Lemma 4 *For $\alpha \in]-\frac{1}{2},\frac{1}{2}[$ we have a function $F_{N,\alpha} \in C^1[0,\delta]$ for all $\delta \in]0,1[$, such that*

i)

$$\forall z \in [0,\delta[\quad |F_{N,\alpha}(z)| \leq K_0(1+|\ln(1-z+\frac{1+\alpha}{N})|)$$

where K_0 is a constant no depending from N.

ii) F_N and F'_N have a modulus of continuity no depending from N.

iii)

$$\left(T_N^{-1}\left(|\chi-\chi_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1\right)\right)_{k+1,1} =$$
$$= \left(\beta^{(\alpha)}_{k,\theta_0,c_1} - \frac{2}{N}\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{\alpha,N}(\frac{u}{N})\cos(u\theta_0)\right) + R_{N,\alpha}$$

uniformly in k, $0 \leq k \leq N$, with

$$R_{N,\alpha} = o\left(N^{-1}\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{\alpha,N}(\frac{u}{N})\right) \quad \text{if} \quad \alpha > 0$$

and

$$R_{N,\alpha} = o\left(N^{\alpha-1}\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{\alpha,N}(\frac{u}{N})\right) \quad \text{if} \quad \alpha < 0$$

Remark 4 *This lemma and the continuity of $F_{N,\alpha}$ in zero imply directly Theorem 5.*

Remark 5 *Lemma 4 and the continuity of the function F_α imply that*

$$\left(T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha}c_1\right)\right)_{1,1} = \beta^{(\alpha)}_{0,\theta_0,c_1} + \frac{1}{N}\beta^{(\alpha)}_{0,\theta_0,c_1}F_{N,\alpha}(0)\left(1+o(1)\right).$$

Since $F_{N,\alpha}(0) = \alpha^2 + o(1)$ (see [15]) the hypothesis $\beta_{0,\theta_0,c_1} = 1$ means that the coefficients of the predictor polynomial are $\left(T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha}c_1\right)\right)_{k+1,1}(1+o(1))$ uniformly in k (it is a direct consequence of the equality (3).Indeed these of the orthogonal polynomial are

$$\overline{\left(T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha}c_1\right)\right)_{N-k+1,1}}(1+o(1))$$

(we can refer to the equations 3 and 2).

Proof of the lemma 4: In the rest of the paper we slighty change of notation and denote by γ_k the Fourier coefficient of order k of the function $\frac{(\chi\chi_0-1)^\alpha(\chi\bar{\chi}_0-1)^\alpha c_{1,1}}{(\bar{\chi}\bar{\chi}_0-1)^\alpha(\bar{\chi}\chi_0-1)^\alpha \bar{c}_{1,1}}$ by γ_k. As for [15] and using the inversion formula and Corollary 6 we have to consider the sums

$$H_{m,N}(u) = \left(\sum_{n_0=0}^{\infty} \gamma_{-(N+1+n_0)} \sum_{n_1=0}^{\infty} \overline{\gamma_{-(N+1+n_1+n_0)}} \sum_{n_2=0}^{\infty} \gamma_{-(N+1+n_1+n_2)} \times \cdots \right.$$
$$\left. \times \sum_{n_{2m-1}=0}^{\infty} \overline{\gamma_{-(N+1+n_{2m-2}+n_{2m-1})}} \sum_{n_{2m}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m})} \overline{\gamma_{u-(N+1+n_{2m})}} \right).$$

If

$$S_{2m} = \sum_{n_{2m}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m})} \overline{\gamma_{u-(N+1+n_{2m})}}$$

we can write, following the previous Lemma, $S_{2m} = S_{2m,0} + S_{2m,1}$ with

$$S_{2m,0} = 4 \left(\frac{\sin(\pi\alpha)}{\pi} \right)^2$$
$$\sum_{n_{2m}=0}^{\infty} \cos\left((N+1+n_{2m-1}+n_{2m})\theta_0 + 2\omega'_{\alpha,\theta_0}\right) \cos\left((N+1+n_{2m})-u)\theta_0 + 2\omega'_{\alpha,\theta_0}\right)$$
$$\times \frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha} \frac{1}{N+1+n_{2m}-u+\alpha}$$
$$= 2 \left(\frac{\sin(\pi\alpha)}{\pi} \right)^2 \left(\sum_{n_{2m}=0}^{\infty} \frac{\cos\left(n_{2m-1}+u)\theta_0\right)}{N+1+n_{2m-1}+n_{2m}+\alpha} \frac{1}{N+1+n_{2m}-u+\alpha} \right.$$
$$+ \sum_{n_{2m}=0}^{\infty} \cos\left(\left(2(N+1+n_{2m}+4\omega'_\alpha)+n_{2m-1}-u)\right)\theta_0\right)$$
$$\left. \times \frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha} \frac{1}{N+1+wn_{2m}-u+\alpha} \right)$$

Let us study the order of the second sum. To do this we can evaluate the order of the expression

$$\sum_{j=0}^{M} \chi_0^j \frac{1}{N+1+n_{2m-1}+j+\alpha} \frac{1}{N+1+j-u+\alpha}$$

where M goes to the infinity and $N = o(M)$. As for the previous proofs it is clear that this sum is bounded by

$$\sum_{j=0}^{M} \Big| \frac{1}{N+2+n_{2m-1}+j} \frac{1}{N+2+j-u} - \frac{1}{N+1+n_{2m-1}+j} \frac{1}{N+1+j-u} . \Big|$$

Obviously

$$\Big| \frac{1}{N+2+n_{2m-1}+j} \frac{1}{N+2+j-u} - \frac{1}{N+1+n_{2m-1}+j} \frac{1}{N+1+j-u} \Big|$$
$$\leq \Big| \frac{2N+2+2j+n_{2m-1}-u}{(N+1+n_{2m-1}+j)^2(N+1+j-u)^2} \Big|$$

and

$$\Big|\frac{2N+2+2j+n_{2m-1}-u}{(N+1+n_{2m-1}+j)^2(N+1+j-u)^2}\Big|$$
$$=\Big|\frac{1}{N+1+j+n_{2m-1}}+\frac{1}{N+1+j-u}\Big|\frac{1}{(N+1+j+n_{2m-1})(N+1+j-u)}$$
$$\leq \frac{1}{N}\frac{1}{(N+1+j+n_{2m-1})(N+1+j-u)}.$$

In the other hand we have, for $\alpha\in]0,\frac{1}{2}[$

$$S_{2m,1}=o\left(\sum_{n_{2m}=0}^{\infty}\frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha}\frac{1}{N+1+n_{2m}-u+\alpha}\right)$$

and for $\alpha\in]-\frac{1}{2},0[$.

$$S_{2m,1}=o\left(N^{\alpha}\sum_{n_{2m}=0}^{\infty}\frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha}\frac{1}{N+1+n_{2m}-u+\alpha}\right).$$

Hence we can write

$$S_{2m}=S'_{2m}\left(\cos\left(\theta_0(n_{2m-1}+u)\right)+r_{m,\alpha}\right),$$

with

$$S'_{2m}=\sum_{n_{2m}=0}^{+\infty}\frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha}\frac{1}{N+1+n_{2m}-u+\alpha}.$$

and

$$\begin{cases} r_{m,\alpha}= & o(1) \quad \text{if} \quad \alpha\in]0,\frac{1}{2}[\\ r_{m,\alpha}= & o(N^{\alpha}) \quad \text{if} \quad \alpha\in]-\frac{1}{2},0[. \end{cases}$$

For $z\in[0,1]$ we define $F_{m,N,\alpha}(z)$ by

$$F_{m,N,\alpha}(z)=\sum_{n_0=0}^{\infty}\frac{1}{N+1+n_0}\sum_{n_1=0}^{\infty}\frac{1}{N+1+w_1+w_0}\times\cdots$$
$$\times\sum_{n_{2m-1}=0}^{\infty}\frac{1}{N+1+n_{2m-2}+n_{2m-1}+\alpha}$$
$$\times\sum_{n_{2m}=0}^{\infty}\frac{1}{N+1+n_{2m-1}+n_{2m}+\alpha}\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}.$$

Repeating the same idea as previously for the sums on $n_{2m-1},\cdots,n_0$ we finally obtain

$$H_{m,N}(u)=\frac{2}{N}\left(\frac{\sin(\pi\alpha)}{\pi}\right)^{2m+2}F_{m,N,\alpha}(\frac{u}{N})(\cos(u\theta_0)+R_{N,\alpha}).$$

with $R_{N,\alpha}$ as announced previously.
We established in [15] the continuity of the function $F_{m,N,\alpha}$ and the uniform convergence

in $[0,1]$ of the sequence $\sum_{m=0}^{\infty}\left(\frac{\sin(\pi\alpha)}{\pi}\right)^{2m}F_{m,N,\alpha}(z)$. Let us denote by $F_{N,\alpha}(z)$ the sum $\sum_{m=0}^{+\infty}\left(\frac{\sin\pi\alpha}{\pi}\right)^{2m}F_{m,N,\alpha}(z)$. The function $F_{N,\alpha}$ is defined, continuous and derivable on $[0,1[$ (see [15] Lemma 4). Moreover for all $z\in[0,\delta]$, $0<\delta<1$ we have the inequality

$$\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}\leq\frac{1}{1+\frac{1+\alpha}{N}-\delta}.$$

Hence

$$\left(\frac{1+\frac{1+\alpha}{N}-\delta}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}\right)^2\leq\frac{1+\frac{1+\alpha}{N}-\delta}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}$$

and

$$\left(\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}\right)^2\leq\frac{1}{1+\frac{1+\alpha}{N}-\delta}\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z}.$$

These last inequalities and the proof of Lemma 4 in [15] prove that $F_{N,\alpha}$ is in $C^1[0,1[$.

Always in [15] we have obtained that, for all z in $[0,1]$,

$$\left|F_{N,\alpha}(z)\right|\leq K_0\left(1+\left|\ln(1-z+\frac{1+\alpha}{N})\right|\right)\tag{16}$$

where K_0 is a constant no depending from N.
Now we have to prove the point ii) of the statement. For $z,z'\in[0,\delta]$

$$\begin{aligned}&\left|\frac{z-z'}{(1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z)(1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z')}\right|\\&\leq\frac{|z-z'|}{1-\delta}\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-\delta}\end{aligned}$$

that implies, with the inequality (16)

$$|F_{N,\alpha}(z)-F_{N,\alpha}(z')\mid\leq|z-z'|\frac{K_0\left(1+\left|\ln(1-\delta+\frac{1+\alpha}{N})\right|\right)}{1-\delta}.\tag{17}$$

In the same way we have

$$\begin{aligned}&|z-z'|\left|\frac{((1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z)+(1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z')}{(1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z)^2(1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-z')^2}\right|\\&\leq 2|z-z'|\frac{1}{(1-\delta)^2}\frac{1}{1+\frac{1+\alpha}{N}+\frac{n_{2m}}{N}-\delta}\end{aligned}$$

and always with the inequality (16)

$$|F'_{N,\alpha}(z)-F'_{N,\alpha}(z')\mid\leq 2|z-z'|\frac{K_0\left(1+\left|\ln(1-\delta+\frac{1+\alpha}{N})\right|\right)}{(1-\delta)^2}.\tag{18}$$

Using (17) and (18) we get the point *ii*).
To achieve the proof we have to remark that the uniformity in k in the point *iii*) is a direct consequence of Property 2. □

We have now to state the following lemma

Lemma 5 *for* $\frac{k}{N} \to x$, $0 < x < 1$ *we have*

$$2\sum_{u=0}^{k} \overline{\beta^{(\alpha)}_{k-u,\theta_0,c_1}} \cos(u\theta_0) F_{N,\alpha}(\frac{u}{N}) = K_{\alpha,\theta_0,c_1} \cos(k\theta_0 + \omega_{\alpha,\theta_0}) \sum_{u=0}^{k} \tilde{\beta}^{(\alpha)}_{k-u} F_{N\alpha}(\frac{u}{N}) + o(k^{\alpha-1}),$$

uniformly in k *for* x *in all compact of* $]0,1[$

Remark 6 *This Lemma and Lemma 4 imply the equality*

$$T_N^{-1}\left(|\chi-\chi_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1\right)_{k+1,1} = K_{\alpha,\theta_0,c_1}\cos(k\theta_0+\omega_{\alpha,\theta_0})T_N^{-1}\left(|1-\chi|^{2\alpha}\right)_{k+1,1} + o(k^{\alpha-1})$$

with (see [15] Lemma 3)

$$T_N^{-1}\left(|1-\chi|^{2\alpha}\right)_{k+1,1} = \left(\tilde{\beta}^{(\alpha)}_k - \frac{1}{N}\sum_{u=0}^{k}\tilde{\beta}^{(\alpha)}_{k-u}F_{N,\alpha}(\frac{u}{N})\right).$$

Proof of lemma 5: With our notation assume $x \in [0,\delta]$, $0 < \delta < 1$. Put $k_0 = N^\gamma$ with $\gamma \in]\max(\frac{\alpha}{\beta}, \frac{-\alpha}{1-\alpha}), 1[$ if $\alpha < 0$, and $\gamma \in]0,1[$ if $\alpha > 0$. We can splite the sum $2\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)$ into $2\sum_{u=k-k_0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)$ and $2\sum_{u=0}^{k-k_0}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)$. Property 1 and the assumption on β show that

$$\begin{aligned}
2\sum_{u=0}^{k-k_0}\overline{\beta^{(\alpha)}_{k-u,\theta_0,c_1}}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0) &= 2K_{\alpha,\theta_0,c_1} \\
&\times \sum_{u=0}^{k-k_0}\tilde{\beta}^{(\alpha)}_{k-u}\cos((k-u)\theta_0+\omega_{\alpha,\theta_0})\cos(u\theta_0)F_{N,\alpha}(\frac{u}{N}) + o(k^\alpha) \\
&= K_{\alpha,\theta_0,c_1}\left(\sum_{u=0}^{k-k_0}\tilde{\beta}^{(\alpha)}_{k-u}\cos(k\theta_0+\omega_{\alpha,\theta_0})F_\alpha(\frac{u}{N})\right. \\
&+ \left.\sum_{u=0}^{k-k_0}\tilde{\beta}^{(\alpha)}_{k-u}\cos((k-2u)\theta_0)+\omega_{\alpha,\theta_0})F_{N,\alpha}(\frac{u}{N})\right) + o(k^\alpha),
\end{aligned}$$

uniformly in k. It is known that the second sum is also

$$\sum_{u=0}^{k-k_0}\frac{(k-u)^{\alpha-1}}{\Gamma(\alpha)}\cos((k-2u)\theta_0)+\omega_{\alpha,\theta_0})F_{N,\alpha}(\frac{u}{N}) + o(k^{\alpha-1}),$$

uniformly in k with the equation (5). Then an Abel summation provides that the quantity $\Big|\sum_{u=0}^{k-k_0}(k-u)^{\alpha-1}\cos((k-2u)\theta_0+\omega_{\alpha,\phi_0})F_{N,\alpha}(\frac{u}{N})\Big|$ is bounded by
$M_1k_0^{\alpha-1}+\sum_{u=0}^{k-k_0}|(k-u-1)^{\alpha-1}F_{N,\alpha}(\frac{u+1}{N})-(k-u)^{\alpha-1}F_{N,\alpha}(\frac{u}{N})|$ with M_1 no depending from k. Moreover

$$\begin{aligned}&\sum_{u=0}^{k-k_0}|(k-u-1)^{\alpha-1}F_{N,\alpha}(\frac{u+1}{N})-(k-u)^{\alpha-1}F_{N,\alpha}(\frac{u}{N})|\\&\leq\sum_{u=0}^{k-k_0}|(k-u-1)^{\alpha-1}-(k-u)^{\alpha-1}||F_{N,\alpha}(\frac{u}{N})|\\&+\sum_{u=0}^{k-k_0}|F_{N,\alpha}(\frac{u+1}{N})-F_{N,\alpha}(\frac{u}{N})||(k-u-1)^{\alpha-1}|\end{aligned}$$

From the inequality 16 (we have assumed $0<\frac{k}{N}<\delta$) we infer

$$\sum_{u=0}^{k-k_0}|(k-u-1)^{\alpha-1}-(k-u)^{\alpha-1}||F_{N,\alpha}(u)|\leq M_2\sum_{w=k_0}^{k}v^{\alpha-2}$$

with M_2 no depending from k. We finally get

$$\begin{aligned}\sum_{u=0}^{k-k_0}|(k-u-1)^{\alpha-1}-(k-u)^{\alpha-1}||F_{N,\alpha}(u)|&=O\left(\sum_{w=k_0}^{k}v^{\alpha-2}\right)\\&=O\left(k_0^{\alpha-1}\right)=o(k^\alpha)\end{aligned}$$

Identically Lemma 4 and the main value theorem provides

$$\sum_{u=0}^{k-k_0}|F_{N,\alpha}(\frac{u+1}{N})-F_{N,\alpha}(\frac{u}{N})||(k-u-1)^{\alpha-1}|\leq M_3\frac{k^\alpha}{N}=o(k^\alpha)$$

with M_3 no depending from N. By definition of k_0 and with Property 1 we have easily the existence of a constant M_4, always no depending from k, such that for $\alpha>0$

$$\Big|\sum_{u=k-k_0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)\Big|\leq M_4k_0^\alpha.$$

Consequently for $\alpha>0$

$$\begin{aligned}&2\sum_{u=0}^{k}\beta^{(\alpha)}_{k-u,\theta_0,c_1}F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)\\&\qquad=K_{\alpha,\theta_0,c_1}\cos((k-\alpha)\theta_0+\omega_{\alpha,\theta_0})\sum_{u=0}^{k}\tilde{\beta}^{(\alpha)}_{k-u}F_{N,\alpha}(\frac{u}{N})+o(k^\alpha).\end{aligned}$$

uniformly in k with the definition of the constants M_i, $1 \leq i \leq 4$ and we get the Lemma for $\alpha > 0$.
Since we have the result for the positive case we assume in the rest of the demonstration that $\alpha \in]-\frac{1}{2}, 0[$. Recall that now $\gamma \in]\max(\frac{\alpha}{\beta}, \frac{-\alpha}{1-\alpha}), 1[$.

First we have to evaluate the sum $\sum_{u=k-k_0}^{k} \overline{\beta^{(\alpha)}_{k-u,\theta_0,c_1}} \cos(u\theta_0) F_\alpha(u)$. Since $F_{N,\alpha} \in C^1[0,\delta]$ we have for $\frac{k-k_0}{N} \leq \frac{u}{N} \leq \frac{k}{N} \leq \delta <$ the formula $F_{\alpha,N}(\frac{u}{N}) - F_{\alpha,N}(\frac{k}{N}) + F_{\alpha,N}(\frac{k}{N}) = F_{\alpha,N}(\frac{k}{N}) + O(\frac{k_0}{N}) = F_{\alpha,N}(\frac{k}{N}) + o(k^\alpha)$ uniformly in k (see once a more the definition of γ).
Property 1 provides $\overline{\beta^{(\alpha)}_{k-u,\theta_0,c_1}} = \beta^{(\alpha)}_{k-u,\theta_0,c_1} + o(k^{\beta-1})$. Hence we can write, uniformly in k,

$$\begin{aligned}
2\sum_{u=k-k_0}^{k} \beta^{(\alpha)}_{k-u,\theta_0,c_1} \cos(u\theta_0) F_\alpha(\frac{u}{N}) &= 2\Re\left(\chi_0^k \sum_{u=k-k_0}^{k} \beta^{(\alpha)}_{k-u,\theta_0,c_1} (\bar{\chi}_0)^{k-u} F_\alpha(\frac{k}{N})\right) + o(k^\alpha) \\
&= 2\Re\left(\chi_0^k \sum_{v=0}^{k_0} \beta^{(\alpha)}_{v,\theta_0,c_1} (\bar{\chi}_0)^{v} F_\alpha(\frac{k}{N})\right) + o(k^\alpha) \\
&= -2\Re\left(\chi_0^k \sum_{v=k_0+1}^{\infty} \beta^{(\alpha)}_{v,\theta_0,c_1} (\bar{\chi}_0)^{v} F_\alpha(\frac{k}{N})\right) + o(k^\alpha).
\end{aligned}$$

Moreover we have, uniformly with Property 1,

$$2\sum_{v=k_0+1}^{\infty} \beta^{(\alpha)}_{v,\theta_0,c_1} (\bar{\chi}_0)^v = K_{\alpha,\theta_0,c_1} \sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v \left(e^{i(v\theta_0+\omega_{\alpha,\theta_0})} + e^{-i(v\theta_0+\omega_{\alpha,\theta_0})}\right) e^{-iv\theta_0} + o(k_0^\beta).$$

Consequently $\gamma \in]\max(\frac{\alpha}{\beta}, \frac{-\alpha}{1-\alpha}), 1[$ infer that

$$2\sum_{v=k_0+1}^{\infty} \beta^{(\alpha)}_{v,\theta_0,c_1} (\bar{\chi}_0)^v = K_{\alpha,\theta_0,c_1} \sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v \left(e^{i(v\theta_0+\omega_{\alpha,\theta_0})} + e^{-i(v\theta_0+\omega_{\alpha,\theta_0})}\right) e^{-iv\theta_0} + o(k^\alpha).$$

We have

$$\begin{aligned}
&\sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v (e^{i(v\theta_0+\omega_{\alpha,\theta_0})} + e^{-i(v\theta_0+\omega_{\alpha,\theta_0})}) e^{-iv\theta_0} \\
&= \sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v (e^{i(\omega_{\alpha,\theta_0})} + e^{-i(2v\theta_0+\omega_{\alpha,\theta_0})}) \\
&= \sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v e^{i(\omega_{\alpha,\theta_0})} + R.
\end{aligned}$$

An Abdel summation provides $|R| \leq M_4 k_0^{\alpha-1} = o(k^\alpha)$ uniformly in k.

Hence we have

$$\begin{aligned}
2\sum_{u=k-k_0}^{k} \beta^{(\alpha)}_{k-u,\theta_0,c_1} \cos(u\theta_0) F_\alpha(\frac{u}{N}) &= -K_{\alpha,\theta_0,c_1} \cos(k\theta_0 + \omega_{\alpha,\theta_0}) \sum_{v=k_0+1}^{\infty} \tilde{\beta}^{(\alpha)}_v F_{N,\alpha}(\frac{k}{N}) + o(k^\alpha) \\
&= K_{\alpha,\theta_0,c_1} \cos(k\theta_0 + \omega_{\alpha,\theta_0}) \sum_{v=0}^{k_0} \tilde{\beta}^{(\alpha)}_v F_{N,\alpha}(\frac{k}{N}) + o(k^\alpha).
\end{aligned}$$

With Lemma 4 we obtain, as previously

$$\sum_{u=k-k_0}^{k} \tilde{\beta}_{k-u}^{(\alpha)} F_\alpha(\frac{u}{N}) = \sum_{v=0}^{k_0} \tilde{\beta}_v^{(\alpha)} F_{N,\alpha}(\frac{k}{N}) + o(k^\alpha)$$

uniformly in k. Since we have seen that the sum

$$2\sum_{u=0}^{k-k_0} \beta_{k-u,\theta_0,c_1}^{(\alpha)} F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)$$

is equal to

$$K_{\alpha,\theta_0,c_1}\cos(k\theta_0+\omega_{\alpha,\theta_0})\sum_{u=0}^{k-k_0} \tilde{\beta}_{k-u}^{(\alpha)} F_{N,\alpha}(\frac{u}{N}) + o(k^\alpha)$$

we can also conclude, as for $\alpha > 0$

$$2\sum_{u=0}^{k} \beta_{k-u,\theta_0,c_1}^{(\alpha)} F_{N,\alpha}(\frac{u}{N})\cos(u\theta_0)$$

$$= K_{\alpha,\theta_0,c_1}\cos(k\theta_0+\omega_{\alpha,\theta_0})\sum_{u=0}^{k} \tilde{\beta}_{k-u}^{(\alpha)} F_{N,\alpha}(\frac{u}{N}) + o(k^\alpha).$$

The uniformity is clearly provided by the uniformity in Lemma 4 and by the previous remarks. This last remark is sufficient to prove Lemma 5. □

Then Theorem 4 is a direct consequence of the inversion formula and of Lemma 5.

5 Proof of Theorem 6

Let us recall the following formula, which can be related with the Gobberg-Semencul formula.

Lemma 6 *If* $P = \sum_{u=0}^{N} \delta_u \chi^u$ *a trigonometric polynomial of degree* N*. Then we have, if* $k \le l$

$$\left(T_N^{-1}\left(\frac{1}{|P|^2}\right)\right)_{k+1,l+1} = \sum_{u=0}^{k} \bar{\delta}_u \delta_{l-k+u} - \sum_{u=0}^{k} \delta_{N-k+u}\bar{\delta}_{N-l+u}.$$

Let P_{N,α,θ_0} and $P_{N,\alpha}$ be the predictor polynomials of $|\chi-\chi_0|^{2\alpha}|\chi\bar{\chi}_0|c_1$ and $|1-\chi|^{2\alpha}$. We put $P_{N,\alpha,\theta_0} = \sum_{u=0}^{N} \delta_{u,\theta_0}^{(\alpha)}\chi^u$ and $P_{N,\alpha} = \sum_{u=0}^{N} \tilde{\delta}_u^{(\alpha)}\chi^u$. Following Formula (3) we have

$$\delta_{u,\theta_0}^{(\alpha)} = \frac{T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^\alpha c_1\right)_{u+1,1}}{\sqrt{T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^\alpha c_1\right)_{1,1}}}$$

and

$$\tilde{\delta}_u^{(\alpha)} = \frac{T_N^{-1}\left((1-\cos\theta)^\alpha c_1\right)_{u+1,1}}{\sqrt{T_N^{-1}\left((1-\cos\theta)^\alpha c_1\right)_{1,1}}}.$$

Then Remark 5 and the hypothesis β^0_{0,θ_0,c_1} give the equalities

$$\delta^{(\alpha)}_{u,\theta_0} = T_N^{-1}\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^\alpha c_1\right)_{u+1,1}(1+o(1))$$

and for the same reasons

$$\tilde{\delta}^{(\alpha)}_{u} = T_N^{-1}\left(1-\cos\theta)^\alpha c_1\right)_{u+1,1}(1+o(1)).$$

According to Lemma 6 we have to treat the two sums (with the hypothesis $x<y$) $S_{1,\alpha} = \sum_{u=0}^{k}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0}$ and $S_{2,\alpha} = \sum_{u=0}^{k}\delta^{(\alpha)}_{N-k+u,\theta_0}\overline{\delta^{(\alpha)}_{N-l+u,\theta_0}}$. For a sufficiently large integer k_0 we can split the sum $S_{1,\alpha}$ into $\sum_{u=0}^{k_0}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0}$ and $\sum_{u=k_0+1}^{k}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0}$. We have

$$\sum_{u=0}^{k_0}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0} \le \frac{K_{\alpha,\theta_0,c_1}}{\Gamma(\alpha)}(l-k)^{\alpha-1}(1-\frac{l-k}{N})^\alpha M k_0.$$

with $M=\max\{\delta^{(\alpha)}_{u,\theta_0}\}$. Assume now $k_0=N^\gamma$ with $0<\gamma<\alpha$. We get

$$\sum_{u=0}^{k_0}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0} = o(N^{2\alpha-1}) \tag{19}$$

In the other hand we have, following Theorem 4

$$\sum_{u=k_0+1}^{k}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0} = |K_{\alpha,\theta_0}|^2\sum_{u=k_0+1}^{k}\cos\left(u\theta_0+\omega_{\alpha,\theta_0}\right)\cos\left((l-k)\theta_0+\omega_{\alpha,\theta_0}\right)$$
$$u^{\alpha-1}(1-\frac{u}{N})^\alpha(l-k+u)^{\alpha-1}(1-\frac{l-k+u}{N})^\alpha + o(k^{2\alpha-1})$$

As previously we obtain, with an Abel summation, that

$$\sum_{u=k_0+1}^{k}\overline{\delta^{(\alpha)}_{u,\theta_0}}\delta^{(\alpha)}_{l-k+u,\theta_0}$$
$$= \frac{|K_{\alpha,\theta_0,c_1}|^2}{\Gamma^2(\alpha)}\cos\left((l-k)\theta_0\right)\sum_{u=k_0+1}^{k}u^{\alpha-1}(1-\frac{u}{N})^\alpha(l-k+u)^{\alpha-1}(1-\frac{l-k+u}{N})^\alpha + S'_{1,\alpha}.$$

with

$$|S'_{1,\alpha}| = O\left(\sum_{u=k_0+1}^{k}|\rho_N(u+1)-\rho_N(u)|\right)$$

and

$$\rho_N(u) = u^{\alpha-1}(1-\frac{u}{N})^\alpha(l-k+u)^{\alpha-1}(1-\frac{l-k+u}{N})^\alpha.$$

With the main value theorem we can write

$$|S'_{1,\alpha}| = O\left(\sum_{u=k_0+1}^{k} |\rho'_N(c)|\right) \quad u < c < u+1.$$

Hence

$$|S'_{1,\alpha}| = O\left(\sum_{j=0}^{4} \Sigma_{1,\alpha}^{(j)}\right) = o(N^{2\alpha-1}).$$

Finally we obtain

$$\sum_{u=k_0+1}^{k} \overline{\delta^{(\alpha)}_{u,\theta_0}} \delta^{(\alpha)}_{l-k+u,\theta_0}$$
$$= \frac{|K_{\alpha,\theta_0,c_1}|^2}{\Gamma^2(\alpha)} \cos\left((l-k)\theta_0\right) \sum_{u=k_0+1}^{k} u^{\alpha-1}(1-\frac{u}{N})^{\alpha}(l-k+u)^{\alpha-1}(1-\frac{l-k+u}{N})^{\alpha} + o(N^{2\alpha-1}).$$

As for the equation (19) we get $\sum_{u=0}^{k_0} \tilde{\delta}^{(\alpha)}_u \tilde{\delta}^{(\alpha)}_{l-k+u}$. Consequently we can conclude

$$S_{1,\alpha} = |K_{\alpha,\theta_0,c_1}|^2 \cos\left((l-k)\theta_0\right) \sum_{u=0}^{k} \overline{\tilde{\delta}^{(\alpha)}_u} \tilde{\delta}^{(\alpha)}_{l-k+u} + o(N^{2\alpha-1}). \tag{20}$$

As previously we can split the sum $S_{2,\alpha}$ into $\sum_{u=0}^{k-k_1-1} \delta^{(\alpha)}_{N-k+u,\theta_0} \overline{\delta^{(\alpha)}_{N-l+u,\theta_0}}$ and $\sum_{u=k-k_1}^{k} \delta^{(\alpha)}_{N-k+u,\theta_0} \overline{\delta^{(\alpha)}_{N-l+u,\theta_0}}$.

Using Lemma 5 we obtain the bound

$$\begin{aligned}\Big|\sum_{u=k-k_1}^{k} \delta^{(\alpha)}_{N-k+u,\theta_0} \overline{\delta^{(\alpha)}_{N-l+u,\theta_0}}\Big| \leq & \sum_{u=k-k_1}^{k} |\beta^{(\alpha)}_{N-k+u,\theta_0,c_1} \delta^{(\alpha)}_{N-l+u,\theta_0}| \\ & + \sum_{u=k-k_1}^{k} |\delta^{(\alpha)}_{N-l+u,\theta_0}| \frac{1}{N} \sum_{v=0}^{N-k+u} |\beta^{(\alpha)}_{N-k+u-v,\theta_0,c_1}||F_{N,\alpha}(\frac{v}{N})| \\ & + o(N^{2\alpha-1}).\end{aligned}$$

Assume now $k_1 = o(N)$. We have

$$\begin{aligned}\sum_{u=k-k_1}^{k} |\beta^{(\alpha)}_{N-k+u,\theta_0,c_1} \delta^{(\alpha)}_{N-l+u,\theta_0}| &\leq O\left((N-l+k)^{\alpha-1}(\frac{l-k}{N})^{\alpha} \sum_{u=k-k_1}^{k} (N-k+u)^{\alpha-1}\right) \\ &\leq O\left(N^{2\alpha-1}\left(1-(1-\frac{k_1}{N})^{\alpha}\right)\right) = o(N^{2\alpha-1})\end{aligned}$$

and

$$\sum_{u=k-k_1}^{k} |\delta^{(\alpha)}_{N-l+u,\theta_0}| \frac{1}{N} \sum_{v=0}^{N-k+u} |\beta^{(\alpha)}_{N-k+u-v,\theta_0,c_1}||F_{N,\alpha}(\frac{v}{N})|$$
$$= O\left((N-l+k)^{\alpha-1}(\frac{l-k}{N})^{\alpha} k_1 N^{\alpha-1} \int_0^1 \ln(1-t+\frac{\alpha+1}{N})dt\right)$$
$$= o(N^{2\alpha-1})$$

Lastly we obtain, still with an Abel summation

$$\sum_{u=0}^{k-k_1-1} \delta^{(\alpha)}_{N-k+u,\theta_0} \overline{\delta^{(\alpha)}_{N-l+u,\theta_0}}$$
$$= \frac{|K_{\alpha,\theta_0,c_1}|^2}{\Gamma^2(\alpha)} \cos\left((l-k)\theta_0\right) \sum_{u=0}^{k-k_1-1} (N-k+u)^{\alpha-1}(\frac{k-u}{N})^{\alpha}(N-l+u)^{\alpha-1}(\frac{l-u}{N})^{\alpha}$$
$$+ o(N^{2\alpha-1}).$$

Merging this last equality with (20) we obtain

$$S_{2,\alpha} = |K_{\alpha,\theta_0,c_1}|^2 \cos\left((l-k)\theta_0\right) \sum_{u=0}^{k} \tilde{\delta}^{(\alpha)}_{N-k+u} \overline{\tilde{\delta}^{(\alpha)}_{N-l+u}} + o\left(N^{2\alpha-1}\right). \tag{21}$$

The equations (20) and (21) and Lemma 6 provide Theorem 6 for the case $\frac{1}{2} > \alpha > 0$. The uniformity is a direct consequence of Theorem 4 and Lemmas 4 and 5.

6 Proof of Corollary 4 and 5

Lemma 7 *For $\theta_0 \in]0,\pi[$ and $\alpha \in]0,\frac{1}{2}[$ we have*

$$\|T_N\left(2(\cos\theta - \cos\theta_0)c_1\right) - T_N\left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha}c_1\right)\| \leq K(\frac{1}{2}-\alpha)N$$

where K is a constant no depending from N.

Proof : By the main value Theorem we have

$$|2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha} - 2(\cos\theta-\cos\theta_0)| \leq 4(1-2\alpha)2^{c_\alpha(\theta)}(\cos\theta-\cos\theta_0)^{c_\alpha(\theta)}|$$

with $0 < c_\alpha(\theta) < 1-2\alpha$. Hence the function $\psi_\alpha \mapsto \theta \mapsto 2^{c_\alpha(\theta)}(\cos\theta-\cos\theta_0)^{c_\alpha(\theta)}c_1(\theta)$ is in $L^1(\mathbb{T})$. For all integer k, $0 \leq k \leq N$ we consider the integral
$I_k = \int_0^{2\pi} \left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha} - 2(\cos\theta-\cos\theta_0)\right) c_1(\theta)e^{-ik\theta}d\theta$.
Assume $\frac{1}{2}-\alpha \to 0$ and put ϵ, $0 < \epsilon < 1-2\alpha$, for α sufficiently closed from $\frac{1}{2}$. Put $I_k = I_{k,1} + I_{k,2} + I_{k,3}$ with

$$I_{k,1} = \int_0^{\theta_0-\epsilon} \left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha} - 2(\cos\theta-\cos\theta_0)\right) c_1(\theta)e^{-ik\theta}d\theta,$$
$$I_{k,2} = \int_{\theta_0-\epsilon}^{\theta_0+\epsilon} \left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha} - 2(\cos\theta-\cos\theta_0)\right) c_1(\theta)e^{-ik\theta}d\theta,$$
$$I_{k,3} = \int_{\theta_0+\epsilon}^{2\pi} \left(2^{2\alpha}(\cos\theta-\cos\theta_0)^{2\alpha} - 2(\cos\theta-\cos\theta_0)\right) c_1(\theta)e^{-ik\theta}d\theta.$$

It is easy to see that $|I_{k,1}|$ and $|I_{k,3}|$ are bounded by $M(1-2\alpha)$ with M is a positive real no depending from k or N. Easily $|I_{k,2}| \leq 1-2\alpha\|\psi_\alpha\|_1$. Hence $|I_k| \leq M_2(1-2\alpha)$ where M_2 is a positive real no depending from k or N.
In the other hand it is well known that for a $N \times N$ matrix A we have

$$\|A\| \leq \left(\sum_{i=1}^{N} \sum_{j=1}^{N} A_{i,j}^2 \right)^{\frac{1}{2}}.$$

This last result achieves the proof. □

Lemma 8 *Let $\delta > 0$ a fixed real. For $\alpha < \frac{1}{2}$ such that $\frac{1}{2} - \alpha$ sufficiently near of zero we have for all integer k, $0 \leq k \leq N$*

$$\|T_N^{-1}\left(|\chi-\chi_0||\chi-\bar{\chi}_0|c_1\right)(\chi^k) - T_N^{-1}\left(|\chi-\chi_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1\right)(\chi^k) \leq o(N^{-\delta}).$$

Proof : Let us denote by $T_{1/2,N}$ the matrix $T_N^{-1}\left(|\chi-\chi_0||\chi-\bar{\chi}_0|c_1\right)$ and by $T_{\alpha,N}$ the matrix $T_N^{-1}\left(|\chi-\chi_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1\right)$. Obviously

$$T_{1/2,N} = T_{\alpha,N}\left(Id + T_{\alpha,N}^{-1}\left(T_{1/2,N} - T_{\alpha,N}\right)\right).$$

Corollary 2 and Lemma 6 imply the existence of a positve real C such that for all integers k, l, $0 \leq k, l \leq N$, we have

$$\left(T_{\alpha,N}\right)_{k,l} \leq N^C. \tag{22}$$

Since

$$\|T_{\alpha,N}\left(T_{1/2,N} - T_{\alpha,N}\right)\| \leq \|T_{\alpha,N}^{-1}\|\|T_{1/2,N} - T_{\alpha,N}\|$$

the previous Lemma implies

$$\|T_{\alpha,N}^{-1}\|\|T_{1/2,N} - T_{\alpha,N}\| \leq C'(\frac{1}{2}-\alpha)N^{C+2}. \tag{23}$$

Put $\alpha = \frac{1}{2} - o(N^{-(C+3)})$. From (23) the matrix $\left(Id + T_{\alpha,N}^{-1}(T_{1/2,N} - T_{\alpha,N})\right)^{-1}$ is defined and $T_{1/2,N}^{-1} = \left(Id + T_{\alpha,N}^{-1}(T_{1/2,N} - T_{\alpha,N})\right)^{-1} T_{\alpha,N}^{-1}$. Then we can write, for all integer k, $0 \leq k \leq N$

$$\begin{aligned}\|T_{1/2,N}^{-1}(\chi^k) - T_{\alpha,N}^{-1}(\chi^k)\| &\leq \left\| \left(\left(Id + T_{\alpha,N}^{-1}(T_{1/2,N} - T_{\alpha,N})\right)^{-1} - Id \right) T_{\alpha,N}^{-1}(\chi^k) \right\| \\ &\leq \frac{\|T_{\alpha,N}^{-1}(T_{1/2,N} - T_{\alpha,N})\|\|T_{\alpha,N}^{-1}(\chi^k)\|}{1 - \|T_{\alpha,N}^{-1}(T_{1/2,N} - T_{\alpha,N})\|}.\end{aligned}$$

As for the equation (22) we have obviously a constant J no depending from k or N such that $\|T_{\alpha,N}^{-1}(\chi^k)| \leq O(N^J)$. If $\alpha = \frac{1}{2} + o(N^{-(C+J+3+\delta)})$ we have

$$\|T_N^{-1}\left(|\chi-\chi_0||\chi-\bar{\chi}_0|c_1\right)(\chi^k) - T_N^{-1}\left(|\chi-\chi_0|^{2\alpha}|\chi-\bar{\chi}_0|^{2\alpha}c_1\right)(\chi^k) \leq o(N^{-\delta})$$

□

7 Appendix

7.1 Estimation of a trigonometric sum

Lemma 9 *Let M_0, M_1 two integers with $0 < M_0 < M_1$, $\chi \neq 1$ and f a function in $\mathcal{C}^1(]M_0, M_1[)$ such that for all $t \in]M_0, M_1[$ $f(t) = O(t^\beta)$ and $f'(t) = O(t^{\beta-1})$. Then*

$$\Big| \sum_{u=M_0}^{M_1} f(u)\chi^u \Big| = \begin{cases} O(M_1^\beta) & \text{if} \quad \beta > 0 \\ O(M_0^\beta) & \text{if} \quad \beta < 0. \end{cases}$$

Proof : With an Abel summation we obtain, if $\sigma_u = 1 + \cdots + \chi^u$,

$$\sum_{u=M_0}^{M_1} f(u)\chi^u = \sum_{u=M_0}^{M_1-1} \left(f(u+1) - f(u)\right)\sigma_u + f(M_1)\sigma_{M_1} + f(M_0)\sigma_{M_0-1}$$

and

$$\begin{aligned} \sum_{u=M_0}^{M_1-1} \left(f(u+1) - f(u)\right)\sigma_u &= \left(f(M_0) + f(M_1)\right)\left(\frac{1}{1-\chi}\right) - \sum_{u=M_0}^{M_1-1} \left(f(u+1) - f(u)\right)\frac{\chi^{u+1}}{1-\chi} \\ &= \sum_{u=M_0}^{M_1-1} f'(c_u)\frac{\chi^{u+1}}{1-\chi} + \left(f(M_0) + f(M_1)\right)\left(\frac{1}{1-\chi}\right) \end{aligned}$$

with $c_u \in]u, u+1[$. We have

$$\Big| \sum_{u=M_0}^{M_1-1} f'(c_u)\frac{\chi^u}{1-\chi} \Big| \leq O\left(\sum_{u=M_0}^{M_1-1} u^{\beta-1} \right)$$

hence

$$\Big| \sum_{u=M_0}^{M_1} f(u)\chi^u \Big| = \begin{cases} O(M_1^\beta) & \text{if} \quad \beta > 0 \\ O(M_0^\beta) & \text{if} \quad \beta < 0. \end{cases}$$

□

References

[1] P. Beaumont and R. Ramachandran. Robust estimation of GARMA model parameters with an application to cointegration among interest rates of industrialized country. *Computatinal economics*, 17:179–201, 2001.

[2] J. Beran. *Statistics for long memory process*. Chapmann and Hall, 1994.

[3] A. Böttcher and B. Silbermann. Toeplitz matrices and determinants with Fisher-Hartwig symbols. *J. Funct. Anal.*, 63:178–214, 1985.

[4] A. Böttcher and B. Silbermann. Toeplitz operators and determinants generated by symbols with one Fisher-Hartwig singularity. *Math. Nachr.*, 127:95–124, 1986.

[5] P. J. Brockwell and R. A. Davis. *Times series: theory and methods*. Springer Verlag, 1986.

[6] Q.C. Cheng, H. L. Gray, and W. A. Wayne. A k-factor GARMA long-memory model. *Journal of time series analysis*, 19(4):485–504, 1998.

[7] R. Dahlhaus. Efficient parameter estimation for self-similar processes. *Ann. Statist.*, 17:1749–1766, 1989.

[8] P. Doukhan, G. Oppenheim, and M. S. Taqqu. *Theory and applications of long-range dependence*, volume 54. Birkhäuser, Boston, 2003.

[9] U. Grenander and G. Szegö. *Toeplitz forms and their applications*. Chelsea, New York, 2nd ed. edition, 1984.

[10] A.P. Kirman and G. Teyssiere. *Long memory in economic*. Mathematical Review, 2007.

[11] H.J. Landau. Maximum entropy and the moment problem. *Bulletin (New Series) of the american mathematical society*, 16(1):47–77, 1987.

[12] Y. Lu and C. M. Hurvich. On the complexity of the preconditioned conjugate gradient algorithm for solving Toeplitz systems with a Fisher-Hartwig singularity. *SIAM J. Matrix Anal. Appl.*, 27:638–653, 2005.

[13] P. Rambour and A.Seghier. Inversion des matrices de Toeplitz dont le symbole admet un zéro d'ordre rationnel positif,valeur propre minimale. *Annales de la Faculté des Sciences de Toulouse*, XXI, n° 1:173–2011, 2012.

[14] P. Rambour and A. Seghier. Formulas for the inverses of Toeplitz matrices with polynomially singular symbols. *Integr. equ. oper. theory*, 50:83–114, 2004.

[15] P. Rambour and A. Seghier. Inverse asymptotique des matrices de Toeplitz de symbole $(1-\cos\theta)^\alpha f_1$, $\frac{-1}{2} < \alpha \leq \frac{1}{2}$, et noyaux intégraux. *Bull. des Sci. Math.*, 134:155–188, 2008.

[16] B. Simon. *Orthogonal polynomials on the unit circle, Part 1: classical theory*, volume 54. American Mathematical Society, 2005.

[17] B. Simon. *Orthogonal polynomials on the unit circle, Part 2: spectral theory*, volume 54. American Mathematical Society, 2005.

[18] G. Szegö. *Orthogonal polynomials*. American Mathematical Society, colloquium publication, Providence, Rhodes Island, 3nd edition, 1967.

[19] A. Zygmund. *Trigonometric series*, volume 1. Cambridge University Press,, 1968.