Arithmetic properties of Apéry-like numbers

Eric Delaygue

ARITHMETIC PROPERTIES OF APÉRY-LIKE NUMBERS

É. DELAYGUE

Abstract. We provide lower bounds for p-adic valuations of multisums of factorial ratios which satisfy an Apéry-like recurrence relation: these include Apéry, Domb, Franel numbers, the numbers of abelian squares over a finite alphabet, and constant terms of powers of certain Laurent polynomials. In particular, we prove Beukers' conjectures on the p-adic valuation of Apéry numbers. Furthermore, we give an effective criterion for a sequence of factorial ratios to satisfy the p-Lucas property for almost all primes p.

1. Introduction

1.1. **Classical results of Lucas and Kummer.** It is a well-known result of Lucas [20] that, for all nonnegative integers m, n and all primes p, we have

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p, \tag{1.1}$$

where $m = m_0 + m_1 p + \cdots + m_k p^k$ and $n = n_0 + n_1 p + \cdots + n_k p^k$ are the base p expansions of m and n.

In particular, a prime p divides the binomial $\binom{m}{n}$ if, and only if there is $0 \leq i \leq k$ such that $m_i < n_i$. Precisely, Kummer proved in [17] that, for all natural integers $m \geq n$, the p-adic valuation ([1]) of the binomial $\binom{m}{n}$ is the number of carries which occur when n is added to $m - n$ in base p. As a consequence, we have

$$\binom{m}{n} \in p^{\alpha}\mathbb{Z}, \quad \text{where} \quad \alpha = \# \left\{ 0 \leq i \leq k \, : \, \binom{m_i}{n_i} = 0 \right\}. \tag{1.2}$$

In this article, we show that many sequences of Apéry-like numbers satisfy congruences similar to (1.1), that is, for all nonnegative integers n and all primes p, we have

$$A(n) \equiv \prod_{i=0}^{k} A(n_i) \mod p,$$

2010 *Mathematics Subject Classification.* Primary 11B50; Secondary 11B65 05A10.

Key words and phrases. Apéry numbers, Constant terms of powers of Laurent polynomials, p-Lucas property, Congruences.

Research supported by the project Holonomix (PEPS CNRS INS2I 2012).

[1]The p-adic valuation of an integer m is the maximum integer β such that p^{β} divides m.

where $n = n_0 + n_1 p + \cdots + n_k p^k$ is the base p expansion of n. Furthermore, we prove that an analogue of (1.2) holds for those numbers, that is

$$A(n) \in p^\alpha \mathbb{Z}, \quad \text{where } \alpha = \#\big\{0 \le i \le k \,:\, A(n_i) \equiv 0 \mod p\big\},$$

which proves Beukers' conjectures on the p-adic valuation of Apéry numbers.

1.2. **Beukers' conjectures on Apéry numbers.** For all natural integers n, we set

$$A_1(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A_2(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

Those sequences were used in 1979 by Apéry in his proofs of the irrationality of $\zeta(3)$ and $\zeta(2)$ (see [2]). In the 1980's, several congruences satisfied by those sequences were demonstrated (see for example [4], [5], [10], [16], [22]). In particular, Gessel proved in [16] that A_1 satisfies the p-Lucas property for all prime numbers p, that is, for any prime p, all v in $\{0, \ldots, p-1\}$ and all natural integers n, we have

$$A_1(v + np) \equiv A_1(v) A_1(n) \mod p.$$

Thereby, if $n = n_0 + n_1 p + \cdots + n_N p^N$ is the base p expansion of n, then we obtain

$$A_1(n) \equiv A_1(n_0) \cdots A_1(n_N) \mod p. \tag{1.3}$$

In particular, p divides $A_1(n)$ if and only if there exists k in $\{0, \ldots, N\}$ such that p divides $A_1(n_k)$. Beukers stated in [3] two conjectures, when $p = 5$ or 11, which generalize this property ([2]). Before stating these conjectures, we observe that the set of all v in $\{0, \ldots, 4\}$ (respectively v in $\{0, \ldots, 10\}$) satisfying $A_1(v) \equiv 0 \mod 5$ (respectively $A_1(v) \equiv 0 \mod 11$) is $\{1, 3\}$ (respectively $\{5\}$).

Conjecture A (Beukers, [3]). *Let n be a natural integer whose base 5 expansion is $n = n_0 + n_1 5 + \cdots + n_N 5^N$. Let α be the number of k in $\{0, \ldots, N\}$ such that $n_k = 1$ or 3. Then 5^α divides $A_1(n)$.*

Conjecture B (Beukers, [3]). *Let n be a natural integer whose base 11 expansion is $n = n_0 + n_1 11 + \cdots + n_N 11^N$. Let α be the number of k in $\{0, \ldots, N\}$ such that $n_k = 5$. Then 11^α divides $A_1(n)$.*

Similarly, Sequence A_2 satisfies the p-Lucas property for all primes p. Furthermore, Beukers and Stienstra proved in [6] that, if $p \equiv 3 \mod 4$, then $A_2\left(\frac{p-1}{2}\right) \equiv 0 \mod p$, and Beukers stated in [3] the following conjecture.

Conjecture C. *Let p be a prime number satisfying $p \equiv 3 \mod 4$. Let n be a natural integer whose base p expansion is $n = n_0 + n_1 p + \cdots + n_N p^N$. Let α be the number of k in $\{0, \ldots, N\}$ such that $n_k = \frac{p-1}{2}$. Then p^α divides $A_2(n)$.*

[2] If p is 2, 3 or 7, then for all v in $\{0, \ldots, p-1\}$, $A_1(v)$ is coprime to p so that, according to (1.3), for all natural integers n, $A_1(n)$ is coprime to p.

Conjectures A-C have been extended to generalized Apéry numbers and any prime p by Deutsch and Sagan in [13, Conjecture 5.13] but this conjecture is false for at least one generalization of Apéry numbers. Indeed, a counterexample is given by

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^3,$$

since $A(1) = 9 \equiv 0 \mod 3$ but $A(4) = A(1+3) = 1152501$ is not divisible by 3^2.

The main aim of this article is to prove Theorem 1, stated in Section 1.4, which demonstrates and generalizes Conjectures A-C. First, we introduce some notations which we use throughout this article.

1.3. **Notations.** Let d be a positive integer. If $\mathbf{m} = (m_1, \ldots, m_d)$ and $\mathbf{n} = (n_1, \ldots, n_d)$ belong to $\mathbb{R}^d$ and if $\lambda \in \mathbb{R}$ and $k \in \{1, \ldots, d\}$, then we write:

- $\mathbf{m} + \mathbf{n} = (m_1 + n_1, \ldots, m_d + n_d)$;
- $\mathbf{m} \cdot \mathbf{n} = m_1 n_1 + \cdots + m_d n_d$;
- $\mathbf{m}\lambda = (m_1\lambda, \ldots, m_d\lambda)$;
- $|\mathbf{m}| = m_1 + \cdots + m_d$;
- $\mathbf{m}^{(k)} = m_k$;
- $\mathbf{m} \geq \mathbf{n}$ if, and only if, for all i in $\{1, \ldots, d\}$, we have $m_i \geq n_i$.

Furthermore, we set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{N}^d$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$ and we write $\mathbf{1}_k$ for the vector in $\mathbb{N}^d$, all of whose coordinates equal zero except the k-th which is 1. If p is a prime number and $\mathbf{n}$ is nonzero, then we say that $\mathbf{n} = \mathbf{n}_0 + \mathbf{n}_1 p + \cdots + \mathbf{n}_N p^N$ is the base p expansion of $\mathbf{n}$ if, for all i in $\{0, \ldots, N\}$, we have $\mathbf{n}_i \in \{0, \ldots, p-1\}^d$, and $\mathbf{n}_N \neq \mathbf{0}$.

For all primes p, we write $\mathbb{Z}_p$ for the ring of p-adic integers. If $A = \big(A(\mathbf{n})\big)_{\mathbf{n} \in \mathbb{N}^d}$ is a $\mathbb{Z}_p$-valued family, then we say that A satisfies the p-Lucas property if and only if, for all vectors $\mathbf{v}$ in $\{0, \ldots, p-1\}^d$ and $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$A(\mathbf{v} + \mathbf{n}p) \equiv A(\mathbf{v})A(\mathbf{n}) \mod p\mathbb{Z}_p. \tag{1.4}$$

We write f_A for the generating series of A defined by $f_A(\mathbf{z}) := \sum_{\mathbf{n} \in \mathbb{N}^d} A(\mathbf{n})\mathbf{z}^{\mathbf{n}}$, where, if $\mathbf{z} = (z_1, \ldots, z_d)$ is a vector of variables and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $\mathbf{z}^{\mathbf{n}}$ denotes $z_1^{n_1} \cdots z_d^{n_d}$.

In addition, we write $\mathcal{Z}_p(A)$ for the set of all vectors $\mathbf{v}$ in $\{0, \ldots, p-1\}^d$ such that $A(\mathbf{v}) \in p\mathbb{Z}_p$. For every nonzero vector $\mathbf{n}$ in $\mathbb{N}^d$ whose base p expansion is $\mathbf{n} = \mathbf{n}_0 + \mathbf{n}_1 p + \cdots + \mathbf{n}_N p^N$, we write $\alpha_p(A, \mathbf{n})$ for the number of k in $\{0, \ldots, N\}$ such that $\mathbf{n}_k \in \mathcal{Z}_p(A)$, and we set $\alpha_p(A, \mathbf{0}) = 0$. Thereby, to prove Conjectures A-C, it is enough to show that $A_i(n) \in p^{\alpha_p(A_i, n)}\mathbb{Z}$ with $i = 1, p = 5$ or 11 and $i = 2$, $p \equiv 3 \mod 4$.

Given tuples of vectors in $\mathbb{N}^d$, $e = (\mathbf{e}_1, \ldots, \mathbf{e}_u)$ and $f = (\mathbf{f}_1, \ldots, \mathbf{f}_v)$, we write $|e| = \sum_{i=1}^{u} \mathbf{e}_i$ and, for all vectors $\mathbf{n}$ in $\mathbb{N}^d$ and all natural integers m, we set

$$\mathcal{Q}_{e,f}(\mathbf{n}) := \frac{\prod_{i=1}^{u}(\mathbf{e}_i \cdot \mathbf{n})!}{\prod_{i=1}^{v}(\mathbf{f}_i \cdot \mathbf{n})!} \quad \text{and} \quad \mathfrak{S}_{e,f}(m) := \sum_{\mathbf{n} \in \mathbb{N}^d, |\mathbf{n}| = m} \mathcal{Q}_{e,f}(\mathbf{n}).$$

Let $\mathcal{S} := \{1 \leq i \leq u \,:\, \mathbf{e}_i \geq \mathbf{1}\}$. For every positive integer r, we say that e is *r-admissible* if

$$\#\mathcal{S} + \min_{1\leq k\leq d} \#\{1 \leq i \leq u \,:\, i \notin \mathcal{S},\, \mathbf{e}_i \geq d\mathbf{1}_k\} \geq r.$$

For all primes p, we write $\mathfrak{F}_p^d$ for the set of all functions $g : \mathbb{N}^d \to \mathbb{Z}_p$ such that, for all natural integers K, there exists a sequence $(P_{K,k})_{k\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to g on $\{0,\dots,K\}^d$. For all tuples e and f of vectors in $\mathbb{N}^d$, all $g \in \mathfrak{F}_p^d$ and all natural integers m, we set

$$\mathfrak{S}_{e,f}^g(m) := \sum_{\mathbf{n}\in\mathbb{N}^d, |\mathbf{n}|=m} \mathcal{Q}_{e,f}(\mathbf{n})g(\mathbf{n}).$$

Finally, we set $\theta := z\frac{d}{dz}$ and we say that a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z,\theta]$ is of *type I* if there is a natural integer q such that:

- $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^qP_q(\theta)$ with $P_k(X) \in \mathbb{Z}_p[X]$ for $0 \leq k \leq q$;
- $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$;
- for all k in $\{2,\dots,q\}$, we have $P_k(X) \in \prod_{i=1}^{k-1}(X+i)^2\mathbb{Z}_p[X]$.

We say that a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z,\theta]$ is of *type II* if

- $\mathcal{L} = P_0(\theta) + zP_1(\theta) + z^2P_2(\theta)$ with $P_k(X) \in \mathbb{Z}_p[X]$ for $0 \leq k \leq 2$;
- $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$;
- $P_2(X) \in (X+1)\mathbb{Z}_p[X]$.

1.4. **Main results.** The main result of this article is the following.

Theorem 1. *Let e and $f = (\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$ be two disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$, for all i in $\{1,\dots,v\}$, k_i is in $\{1,\dots,d\}$, and e is 2-admissible. Let p be a fixed prime. Assume that $f_{\mathfrak{S}_{e,f}}$ is annihilated by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(\mathfrak{S}_{e,f})$.*

Then, for all natural integers n and all functions g in $\mathfrak{F}_p^d$, we have

$$\mathfrak{S}_{e,f}(n) \in p^{\alpha_p(\mathfrak{S}_{e,f},n)}\mathbb{Z} \quad \text{and} \quad \mathfrak{S}_{e,f}^g \in p^{\alpha_p(\mathfrak{S}_{e,f},n)-1}\mathbb{Z}_p.$$

In Section 1.6, we show that Theorem 1 applies to many classical sequences. In particular, Theorem 1 implies Conjectures A-C. Indeed, we have $A_1 = \mathfrak{S}_{e_1,f_1}$ and $A_2 = \mathfrak{S}_{e_2,f_2}$ with $d = 2$,

$$e_1 = \big((2,1),(2,1)\big) \quad \text{and} \quad f_1 = \big((1,0),(1,0),(1,0),(1,0),(0,1),(0,1)\big),$$

and

$$e_2 = \big((2,1),(1,1)\big) \quad \text{and} \quad f_2 = \big((1,0),(1,0),(1,0),(0,1),(0,1)\big).$$

Furthermore, it is well known that f_{A_1}, respectively f_{A_2}, is annihilated by the differential operator $\mathcal{L}_1$, respectively $\mathcal{L}_2$, defined by

$$\mathcal{L}_1 = \theta^3 - z(34\theta^3 + 51\theta^2 + 27\theta + 5) + z^2(\theta+1)^3$$

and

$$\mathcal{L}_2 = \theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta+1)^2.$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are of type I for all primes p, the conditions of Theorem 1 are satisfied by A_1 and A_2, and Conjectures A-C hold. In addition, for all primes p and all natural integers n and α, we obtain that

$$\sum_{k=0}^{n} k^\alpha \binom{n}{k}^2 \binom{n+k}{k}^2 \in p^{\alpha_p(A_1,n)-1}\mathbb{Z} \quad \text{and} \quad \sum_{k=0}^{n} k^\alpha \binom{n}{k}^2 \binom{n+k}{k} \in p^{\alpha_p(A_2,n)-1}\mathbb{Z}.$$

We provide a similar result which applies to the constant terms of powers of certain Laurent polynomials. Consider a Laurent polynomial

$$\Lambda(\mathbf{x}) = \sum_{i=1}^{k} \alpha_i \mathbf{x}^{\mathbf{a}_i} \in \mathbb{Z}_p[x_1^{\pm}, \ldots, x_d^{\pm}],$$

where $\mathbf{a}_i \in \mathbb{Z}^d$ and $\alpha_i \neq 0$ for i in $\{1, \ldots, k\}$. Recall that the Newton polyhedron of Λ is the convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ in $\mathbb{R}^d$. Hence we have the following result.

Theorem 2. *Let p be a fixed prime. Let $\Lambda(\mathbf{x}) \in \mathbb{Z}_p[x_1^{\pm}, \ldots, x_d^{\pm}]$ be a Laurent polynomial, and consider the sequence of the constant terms of powers of Λ defined, for all natural integers n, by*

$$A(n) := \left[\Lambda(\mathbf{x})^n\right]_{\mathbf{0}}.$$

Assume that the Newton polyhedron of Λ contains the origin as its only interior integral point, and that f_A is annihilated by a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z, \theta]$ such that at least one of the following conditions holds:

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p - 1 \in \mathscr{Z}_p(A)$.*

Then, for all natural integers n, we have

$$A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}_p.$$

For example, Theorem 2 applies to Apéry numbers A_1 thanks to the following formula of Lairez [18]:

$$A_1(n) = \left[\left(\frac{(1+z)(yz+z+1)(1+x)(xy+x+y)}{xyz}\right)^n\right]_{(0,0,0)}.$$

By a result of Samol and van Straten [23], if $\Lambda(\mathbf{x}) \in \mathbb{Z}_p[x_1^{\pm}, \ldots, x_d^{\pm}]$ contains the origin as its only interior integral point, then $\left([\Lambda(\mathbf{x})^n]_{\mathbf{0}}\right)_{n\geq 0}$ satisfies the p-Lucas property, which is essential for the proof of Theorem 2. Likewise, the proof of Theorem 1 rests on the fact that $\mathfrak{S}_{e,f}$ satisfies the p-Lucas property when $|e| = |f|$, e is 2-admissible and $f = (\mathbf{1}_{k_1}, \ldots, \mathbf{1}_{k_v})$.

Since those results deal with multisums of factorial ratios, it seems natural to study similar arithmetic properties for simpler numbers such as families of factorial ratios. To that purpose, we prove Theorem 3 below which gives an effective criterion for $\mathcal{Q}_{e,f}$ to satisfy the p-Lucas property for almost all primes p ([3]). Furthermore, Theorem 3 shows that if $A := \mathcal{Q}_{e,f}$ satisfies the p-Lucas property for almost all primes p, then, for all natural integers n and all primes p, we have $A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}$.

To state this result, we introduce some additional notations. For all tuples e and f of vectors in $\mathbb{N}^d$, we write $\Delta_{e,f}$ for Landau's function defined, for all $\mathbf{x}$ in $\mathbb{R}^d$, by

$$\Delta_{e,f}(\mathbf{x}) := \sum_{i=1}^{u} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor - \sum_{i=1}^{v} \lfloor \mathbf{f}_i \cdot \mathbf{x} \rfloor \in \mathbb{Z},$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Therefore, according to Landau's criterion [19] and a precision of the author [11], we have the following dichotomy.

- If, for all $\mathbf{x}$ in $[0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) \geq 0$, then $\mathcal{Q}_{e,f}$ is a family of integers;
- if there exists $\mathbf{x}$ in $[0,1]^d$ such that $\Delta_{e,f}(\mathbf{x}) \leq -1$, then there are only finitely many primes p such that $\mathcal{Q}_{e,f}$ is a family of p-adic integers.

In the rest of the article, we write $\mathcal{D}_{e,f}$ for the semi-algebraic set of all $\mathbf{x}$ in $[0,1)^d$ such that there exists a component $\mathbf{d}$ of e or f satisfying $\mathbf{d} \cdot \mathbf{x} \geq 1$. Observe that $\Delta_{e,f}$ vanishes on the nonempty set $[0,1)^d \setminus \mathcal{D}_{e,f}$.

Theorem 3. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers. Then we have the following dichotomy.*

(1) *If $|e| = |f|$ and if, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geq 1$, then for all primes p, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property;*
(2) *if $|e| \neq |f|$ or if there exists $\mathbf{x}$ in $\mathcal{D}_{e,f}$ such that $\Delta_{e,f}(x) = 0$, then there are only finitely many primes p such that $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property.*

Furthermore, if $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p, then, for all $\mathbf{n}$ in $\mathbb{N}^d$ and every prime p, we have

$$\mathcal{Q}_{e,f}(\mathbf{n}) \in p^{\alpha_p(\mathcal{Q}_{e,f},\mathbf{n})}\mathbb{Z}.$$

Remark. Theorem 3 implies that $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p if and only if all Taylor coefficients at the origin of the associated mirror maps $z_{e,f,k}$, $1 \leq k \leq d$, are integers (see Theorems 1 and 3 in [11]). Indeed, if $\Delta_{e,f}$ is nonnegative on $[0,1]^d$ and if $|e| \neq |f|$, then there exists k in $\{1,\ldots,d\}$ such that $|e|^{(k)} > |f|^{(k)}$.

Coster proved in [9] similar results to Theorems 1-3 for the coefficients of certain algebraic power series. Namely, given a prime $p \geq 3$, integers $a_1, \ldots, a_{p-1}$, and a sequence A such that

$$f_A(z) = (1 + a_1 z + \cdots + a_{p-1} z^{p-1})^{\frac{1}{1-p}},$$

[3]Throughout this article, we say that an assertion $\mathcal{A}_p$ is true for almost all primes p if there exists a constant $C \in \mathbb{N}$ such that $\mathcal{A}_p$ holds for all primes $p \geq C$.

Coster proved that, for all natural integers n, we have

$$v_p\big(A(n)\big) \geq \left\lfloor \frac{\alpha_p(A,n)+1}{2} \right\rfloor.$$

1.5. **Auxiliary results.** The proof of Theorem 1 rests on three results which may be useful to study other sequences.

Proposition 1. *Let p be a fixed prime and A a $\mathbb{Z}_p$-valued sequence satisfying the p-Lucas property with $A(0)$ in $\mathbb{Z}_p^\times$. Let $\mathfrak{A}$ be the $\mathbb{Z}_p$-module spanned by A. Assume that*

(*a*) *there exists a set $\mathfrak{B}$ of $\mathbb{Z}_p$-valued sequences with $\mathfrak{A} \subset \mathfrak{B}$ such that, for all B in $\mathfrak{B}$, all v in $\{0,\dots,p-1\}$ and all positive integers n, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\geq 0}$, B_k in $\mathfrak{B}$, such that*

$$B(v+np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k);$$

(*b*) *$f_A(z)$ is annihilated by a differential operator $\mathcal{L}$ in $\mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*
- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(A)$.*

Then, for all B in $\mathfrak{B}$ and all natural integers n, we have

$$A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}_p \quad \text{and} \quad B(n) \in p^{\alpha_p(A,n)-1}\mathbb{Z}_p.$$

In Proposition 1 and throughout this article, if $(A(n))_{n\geq 0}$ is a sequence taking its values in $\mathbb{Z}$ or $\mathbb{Z}_p$, then, for all negative integers n, we set $A(n) := 0$. Therefore, to prove Theorem 1, it suffices to show that $\mathfrak{S}_{e,f}$ satisfies the p-Lucas property and Condition (a) of Proposition 1 with $\mathfrak{B} = \{\mathfrak{S}_{e,f}^g : g \in \mathfrak{F}_p^d\}$. To that purpose, we shall prove the following results.

Proposition 2. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$ and, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, $\Delta_{e,f}(\mathbf{x}) \geq 1$. Assume that e is 1-admissible. Then, $\mathfrak{S}_{e,f}$ is integer-valued and satisfies the p-Lucas property for all primes p.*

Proposition 3. *Let p be a fixed prime. We write Γ_p for the p-adic Gamma function. Then, there exists a function g in $\mathfrak{F}_p^2$ such that, for all natural integers n and m, we have*

$$\frac{\Gamma_p\big((m+n)p\big)}{\Gamma_p(mp)\Gamma_p(np)} = 1 + g(m,n)p.$$

1.6. **Application of Theorem 1.** By applying Theorem 1, we obtain similar results to Conjectures A-C for numbers satisfying Apéry-like recurrence relations which we list below. Characters in brackets in the last column of the following table form the sequence number

in the Online Encyclopedia of Integer Sequences [25].

Sequence	$\mathcal{Q}_{e,f}(n_1,n_2)$	$\mathcal{L}$	Reference
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$	$\frac{(2n_1+n_2)!^2}{n_1!^4 n_2!^2}$	[1, (γ)]	Apéry numbers (A005259)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$	$\frac{(2n_1+n_2)!(n_1+n_2)!}{n_1!^3 n_2!^2}$	[26, **D**]	Apéry numbers (A005258)
$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$	$\frac{(n_1+n_2)!^2}{n_1!^2 n_2!^2}$	type I	Central binomial coefficients (A000984)
$\sum_{k=0}^{n}\binom{n}{k}^3$	$\frac{(n_1+n_2)!^3}{n_1!^3 n_2!^3}$	[26, **A**]	Franel numbers (A000172)
$\sum_{k=0}^{n}\binom{n}{k}^4$	$\frac{(n_1+n_2)!^4}{n_1!^4 n_2!^4}$	[14],[15]	(A005260)
$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!(2n_1)!(2n_2)!}{n_1!^3 n_2!^3}$	[1, (d)]	(A081085)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$	$\frac{(n_1+n_2)!^2(2n_1)!}{n_1!^4 n_2!^2}$	[26, **C**]	Number of abelian squares of length $2n$ over an alphabet with 3 letters (A002893)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!^2(2n_1)!(2n_2)!}{n_1!^4 n_2!^4}$	[1, (α)]	Domb numbers (A002895)
$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2$	$\frac{(2n_1)!^2(2n_2)!^2}{n_1!^4 n_2!^4}$	[1, (β)]	(A036917)

All differential operators listed in the above table are of type I for all primes p, except the one associated with $A_5(n):=\sum_{k=0}^{n}\binom{n}{k}^4$ which reads

$$\mathcal{L}_5=\theta^3-z2(2\theta+1)(3\theta^2+3\theta+1)-z^2 4(\theta+1)(4\theta+5)(4\theta+3).$$

Hence $\mathcal{L}_5$ is of type II for all primes p. By a result of Calkin [8, Proposition 3], for all primes p, we have $A_5(p-1)\equiv 0 \mod p$, *i. e.* $p-1$ is in $\mathcal{Z}_p(A_5)$. Thus we can apply Theorem 1 to A_5.

Observe that the generating function of the central binomial coefficients is annihilated by the differential operator $\mathcal{L}=\theta-z(4\theta+2)$ which is of type I for all primes p.

According to the recurrence relation found by Almkvist and Zudilin (see Case (d) in [1]), $A_6(n):=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$ is also Sequence **E** in Zagier's list [26], that is

$$A_6(n)=\sum_{k=0}^{\lfloor n/2\rfloor}4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2.$$

Furthermore, according to [24], Domb numbers $A_8(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$ are also the numbers of abelian squares of length $2n$ over an alphabet with 4 letters.

Now we consider the numbers $C_i(n)$ of abelian squares of length $2n$ over an alphabet with i letters which, for all positive integers $i \geq 2$, satisfy (see [24])

$$C_i(n) = \sum_{\substack{k_1+\cdots+k_i=n \\ k_1,\ldots,k_i\in\mathbb{N}}} \left(\frac{n!}{k_1!\cdots k_i!}\right)^2.$$

According to [7], $C_i(n)$ is also the $2n$-th moment of the distance to the origin after i steps traveled by a walk in the plane with unit steps in random directions.

To apply Theorem 1 to C_i, it suffices to show that f_{C_i} is annihilated by a differential operator of type I for all primes p. Indeed, by Proposition 1 and Theorem 2 in [7], for all $j \geq 2$, $C_j(n)$ satisfies the recurrence relation of order $\lceil j/2 \rceil$ with polynomial coefficients of degree $j-1$:

$$n^{j-1}C_j(n) + \sum_{i\geq 1}\left(n^{j-1}\sum_{\alpha_1,\ldots,\alpha_i}\prod_{k=1}^{i}(-\alpha_k)(j+1-\alpha_k)\left(\frac{n-k}{n-k+1}\right)^{\alpha_k-1}\right)C_j(n-i) = 0, \tag{1.5}$$

where the sum is over all sequences of positive integers $\alpha_1, \ldots, \alpha_i$ satisfying $\alpha_k \leq j$ and $\alpha_{k+1} \leq \alpha_k - 2$. We consider $i \geq 2$ and i positive integers $\alpha_1, \ldots, \alpha_i \leq j$ satisfying $\alpha_{k+1} \leq \alpha_k - 2$. We have

$$n^{j-1}\prod_{k=1}^{i}\left(\frac{n-k}{n-k+1}\right)^{\alpha_k-1} = \frac{n^{j-1}}{n^{\alpha_1-1}}\left(\prod_{k=1}^{i-1}(n-k)^{\alpha_k-\alpha_{k+1}}\right)(n-i)^{\alpha_i-1},$$

with $j-\alpha_1 \geq 0$, $\alpha_k - \alpha_{k+1} \geq 2$ and $\alpha_i - 1 \geq 0$. Then, $f_{C_j}(z)$ is annihilated by a differential operator $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^q P_q(\theta)$ with $P_0(\theta) = \theta^{j-1}$ and, for all $i \geq 2$,

$$P_i(\theta) \in \prod_{k=1}^{i-1}(\theta+i-k)^2\mathbb{Z}[\theta] \subset \prod_{k=1}^{i-1}(\theta+k)^2\mathbb{Z}[\theta],$$

so that $\mathcal{L}$ is of type I for all primes p, as expected.

1.7. **Structure of the article.** In Section 2, we use several results of [11] to prove Theorem 3. Section 3 is devoted to the proofs of Theorem 2 and Proposition 1. In particular, we prove Lemma 1 which points out the role played by differential operators in our proofs. In Section 4, we prove Theorem 1 by applying Proposition 1 to $\mathfrak{S}_{e,f}$. It is the most technical part of this article.

2. Proof of Theorem 3

First, we prove that if $|e| = |f|$, then, for all primes p, all $\mathbf{a}$ in $\{0, \dots, p-1\}^d$ and all $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} \in \frac{\prod_{i=1}^{u}\prod_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}}{j}\right)}(1+p\mathbb{Z}_p). \tag{2.1}$$

Indeed, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} = \frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)}\cdot\frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})}.$$

Since $|e| = |f|$, we can apply [11, Lemma 7] ([4]) with $\mathbf{c} = \mathbf{0}$, $\mathbf{m} = \mathbf{n}$ and $s = 0$ which yields

$$\frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})} \in 1 + p\mathbb{Z}_p.$$

Furthermore, we have

$$\begin{aligned}
\frac{\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)} &= \frac{1}{\mathcal{Q}_{e,f}(\mathbf{a})}\frac{\prod_{i=1}^{u}\prod_{j=1}^{\mathbf{e}_i\cdot\mathbf{a}}(j+\mathbf{e}_i\cdot\mathbf{n}p)}{\prod_{i=1}^{v}\prod_{j=1}^{\mathbf{f}_i\cdot\mathbf{a}}(j+\mathbf{f}_i\cdot\mathbf{n}p)} \\
&= \frac{\prod_{i=1}^{u}\prod_{j=1}^{\mathbf{e}_i\cdot\mathbf{a}}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}p}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\mathbf{f}_i\cdot\mathbf{a}}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}p}{j}\right)} \\
&\in \frac{\prod_{i=1}^{u}\prod_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{n}}{j}\right)}{\prod_{i=1}^{v}\prod_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{a}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{n}}{j}\right)}(1+p\mathbb{Z}_p),
\end{aligned}$$

because, if p does not divide j, then $1 + (\mathbf{e}_i\cdot\mathbf{n}p)/j$ belongs to $1 + p\mathbb{Z}_p$. This finishes the proof of (2.1).

Now we prove Assertion (1) in Theorem 3. Let p be a fixed prime number. It is well known that, for all natural integers n, we have

$$v_p(n!) = \sum_{\ell=1}^{\infty}\left\lfloor \frac{n}{p^\ell}\right\rfloor.$$

Thus, for all vectors $\mathbf{n}$ in $\mathbb{N}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big) = \sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\frac{\mathbf{n}}{p^\ell}\right).$$

[4]The proof of this lemma uses a lemma of Lang which contains an error. Fortunately, Lemma 7 remains true. Details of this correction are presented in [12, Section 2.4].

Let fix $\mathbf{n}$ in $\mathbb{N}^d$ and $\mathbf{a}$ in $\{0,\dots,p-1\}^d$. Let $\{\cdot\}$ denote the fractional part function. For any vector of real numbers $\mathbf{x}=(x_1,\dots,x_d)$, we set $\{\mathbf{x}\}:=(\{x_1\},\dots,\{x_d\})$. Since $|e|=|f|$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)\big)=\sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\left\{\frac{\mathbf{a}+\mathbf{n}p}{p^\ell}\right\}\right)\geq\Delta_{e,f}\left(\frac{\mathbf{a}}{p}\right),$$

because $\Delta_{e,f}$ is nonnegative on $[0,1]^d$. On the one hand, if $\mathbf{a}/p$ is in $\mathcal{D}_{e,f}$, then both $\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)$ and $\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})$ are congruent to 0 modulo p. On the other hand, if $\mathbf{a}/p$ is not in $\mathcal{D}_{e,f}$, then, for all $\mathbf{d}$ in e or f, we have $\lfloor\mathbf{d}\cdot\mathbf{a}/p\rfloor=0$ so that (2.1) yields

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p)\equiv\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})\mod p\mathbb{Z}_p,$$

as expected. This proves Assertion (1) in Theorem 3.

Now we prove Assertion (2) in Theorem 3. If $|e|\neq|f|$ then, since $\Delta_{e,f}$ is nonnegative on $[0,1]^d$, there exists k in $\{1,\dots,d\}$ such that $|e|^{(k)}-|f|^{(k)}=\Delta_{e,f}(\mathbf{1}_k)\geq 1$. Thereby, for almost all primes p, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{1}_k+\mathbf{1}_kp)\big)=\sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\frac{\mathbf{1}_k+\mathbf{1}_kp}{p^\ell}\right)\geq\Delta_{e,f}\left(\frac{\mathbf{1}_k}{p}+\mathbf{1}_k\right)\geq 1,$$

but $v_p\big(\mathcal{Q}_{e,f}(\mathbf{1}_k)\big)=0$ so that $\mathcal{Q}_{e,f}$ does not satisfy the p-Lucas property.

Throughout the rest of this proof, we assume that $|e|=|f|$. According to Section 7.3.2 in [11], there exist k in $\{1,\dots,d\}$ and a rational fraction $R(X)$ in $\mathbb{Q}(X)$, $R(X)\neq 1$, such that, for all large enough prime numbers p, we can choose $\mathbf{a}_p$ in $\{0,\dots,p-1\}^d$ satisfying $\mathcal{Q}_{e,f}(\mathbf{a}_p)\in\mathbb{Z}_p^\times$, and such that, for all natural integers n, we have (see [11, (7.10)])

$$\mathcal{Q}_{e,f}(\mathbf{a}_p+\mathbf{1}_knp)\in R(n)\mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_kn)(1+p\mathbb{Z}_p).$$

We fix a natural integer n satisfying $R(n)\neq 1$. For almost all primes p, the numbers $R(n)$, $\mathcal{Q}_{e,f}(\mathbf{1}_kn)$ and $\mathcal{Q}_{e,f}(\mathbf{a}_p)$ are invertible in $\mathbb{Z}_p$, and $R(n)\not\equiv 1 \mod p\mathbb{Z}_p$. Thus, we obtain

$$\mathcal{Q}_{e,f}(\mathbf{a}_p+\mathbf{1}_knp)\not\equiv\mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_kn)\mod p\mathbb{Z}_p,$$

which finishes the proof of Assertion (2) in Theorem 3.

Now we assume that $|e|=|f|$ and that, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x})\geq 1$. Hence, for every prime p, we have

$$\mathcal{Z}_p(\mathcal{Q}_{e,f})=\big\{\mathbf{v}\in\{0,\dots,p-1\}^d\,:\,\mathbf{v}/p\in\mathcal{D}_{e,f}\big\}.$$

Furthermore, if $\mathbf{v}/p$ belongs to $\mathcal{D}_{e,f}$, then, for all positive integers N and all vectors $\mathbf{a}_0,\dots,\mathbf{a}_{N-1}$ in $\{0,\dots,p-1\}^d$, we have

$$\frac{\mathbf{v}}{p}\leq\left\{\frac{\mathbf{a}_0+\mathbf{a}_1p+\cdots+\mathbf{a}_{N-1}p^{N-1}+\mathbf{v}p^N}{p^{N+1}}\right\}\in\mathcal{D}_{e,f},$$

so that, for every $\mathbf{n}$ in $\mathbb{N}^d$, $\mathbf{n}=\sum_{k=0}^{\infty}\mathbf{n}_kp^k$ with $\mathbf{n}_k\in\{0,\dots,p-1\}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big)=\sum_{\ell=1}^{\infty}\Delta_{e,f}\left(\left\{\frac{\sum_{k=0}^{\ell-1}\mathbf{n}_kp^k}{p^\ell}\right\}\right)\geq\alpha_p(\mathcal{Q}_{e,f},\mathbf{n}),$$

and Theorem 3 is proved.

3. Proofs of Theorem 2 and Proposition 1

3.1. **Induction *via* Apéry-like recurrence relations.** In this section, we fix a prime p. If A is a $\mathbb{Z}_p$-valued sequence, then, for all natural integers r, we write $\mathcal{U}_A(r)$ for the assertion "For all $n, i \in \mathbb{N}$, $i \leq r$, if $\alpha_p(A,n) \geq i$, then $A(n) \in p^i\mathbb{Z}_p$". As a first step, we shall prove the following result.

Lemma 1. *Let A be a $\mathbb{Z}_p$-valued sequence satisfying the p-Lucas property with $A(0)$ in $\mathbb{Z}_p^\times$. Assume that f_A is annihilated by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(A)$.*

Let r be a natural integer such that $\mathcal{U}_A(r)$ holds. Then, for all n_0 in $\mathcal{Z}_p(A)$ and all natural integers m satisfying $\alpha_p(A,m) \geq r$, we have

$$A(n_0 + mp) \in p^{r+1}\mathbb{Z}_p.$$

Proof. Since A satisfies the p-Lucas property, we can assume that r is nonzero. The series $f_A(z)$ is annihilated by a differential operator $\mathcal{L} = P_0(\theta) + zP_1(\theta) + \cdots + z^qP_q(\theta)$ with $P_k(X)$ in $\mathbb{Z}_p[X]$ and $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$. Thus, for every natural integer n, we have

$$\sum_{k=0}^{q} P_k(n-k)A(n-k) = 0. \tag{3.1}$$

We fix a natural integer m satisfying $\alpha_p(A,m) \geq r$. In particular, since r is nonzero and $A(0)$ is invertible in $\mathbb{Z}_p$, we have $m \geq 1$. Furthermore, for all v in $\{0,\ldots,p-1\}$, we also have $\alpha_p(A, v+mp) \geq r$. According to $\mathcal{U}_A(r)$, we obtain that, for all v in $\{0,\ldots,p-1\}$, $A(v+mp)$ belongs to $p^r\mathbb{Z}_p$ so that $A(v+mp) =: \beta(v,m)p^r$, with $\beta(v,m) \in \mathbb{Z}_p$.

By (3.1), for all v in $\{q,\ldots,p-1\}$, we have

$$\begin{aligned} 0 = \sum_{k=0}^{q} P_k(v-k+mp)A(v-k+mp) &= p^r \sum_{k=0}^{q} P_k(v-k+mp)\beta(v-k,m) \\ &\equiv p^r \sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p, \end{aligned}$$

because, for all polynomials P in $\mathbb{Z}_p[X]$ and all integers a and c, we have $P(a+cp) \equiv P(a) \mod p\mathbb{Z}_p$. Thus, for all v in $\{q,\ldots,p-1\}$, we obtain

$$\sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.2}$$

We claim that if v is in $\{1,\ldots,q-1\}$, then, for all k in $\{v+1,\ldots,q\}$, we have

$$P_k(v+mp-k)A(v+mp-k) \in p^{r+1}\mathbb{Z}_p. \tag{3.3}$$

Indeed, on the one hand, if $\mathcal{L}$ is of type II, then we have $q = 2$ and $P_2(X)$ belongs to $(X+1)\mathbb{Z}_p[X]$ which yields

$$P_2(-1+mp)A(-1+mp) \in pA\big(p-1+(m-1)p\big)\mathbb{Z}_p.$$

Since 0 is not in $\mathcal{Z}_p(A)$, we have $\alpha_p(A, m-1) \geq r-1$ which, together with $p-1 \in \mathcal{Z}_p(A)$, leads to

$$\alpha_p\big(A, p-1+(m-1)p\big) \geq r.$$

According to $\mathcal{U}_A(r)$, we obtain that $pA\big(p-1+(m-1)p\big)$ is in $p^{r+1}\mathbb{Z}_p$, as expected. On the other hand, if $\mathcal{L}$ is of type I, then for all v in $\{1, \ldots, q-1\}$ and all k in $\{v+1, \ldots, q\}$, we have

$$v_p\big(P_k(v+mp-k)\big) \geq v_p\left(\prod_{i=1}^{k-1}(v+mp-k+i)^2\right).$$

Writing $k-v = a+bp$ with a in $\{0, \ldots, p-1\}$ and b in $\mathbb{N}$, we obtain $k-1 \geq a+bp$ so that

$$v_p\left(\prod_{i=1}^{k-1}(mp+i-a-bp)\right) \geq \begin{cases} b & \text{if } a = 0; \\ b+1 & \text{if } a \geq 1. \end{cases}.$$

Thus, it is enough to prove that

$$A(v+mp-k) \in \begin{cases} p^{r+1-2b}\mathbb{Z}_p & \text{if } a = 0; \\ p^{r-1-2b}\mathbb{Z}_p & \text{if } a \geq 1. \end{cases}. \tag{3.4}$$

We have $v+mp-k = -a+(m-b)p$. If $-a+(m-b)p$ is negative, then $A(v+mp-k) = 0$ and (3.4) holds. If $m-b$ is nonnegative, then we have $\alpha_p(A, m-b) \geq r-b$. Thus, we have either $a = 0$ and $\alpha_p(A, v+mp-k) \geq r-b$, or $a, m-b \geq 1$ and

$$\alpha_p(A, v+mp-k) = \alpha_p\big(A, p-a+(m-b-1)p\big) \geq r-b-1.$$

Hence Assertion $\mathcal{U}_A(r)$ yields

$$A(v+mp-k) \in \begin{cases} p^{r-b}\mathbb{Z}_p & \text{if } a = 0; \\ p^{r-1-b}\mathbb{Z}_p & \text{if } a \geq 1. \end{cases}.$$

If $a = 0$, then $b \geq 1$ so that (3.4) holds and (3.3) is proved.

By (3.3), for all natural integers v satisfying $1 \leq v \leq \min(q-1, p-1)$, we have

$$\begin{aligned}
0 &= \sum_{k=0}^{q} P_k(v-k+mp)A(v-k+mp) \\
&\equiv \sum_{k=0}^{v} P_k(v-k+mp)A(v-k+mp) \mod p^{r+1}\mathbb{Z}_p \\
&\equiv p^r \sum_{k=0}^{v} P_k(v-k+mp)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p \\
&\equiv p^r \sum_{k=0}^{v} P_k(v-k)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p.
\end{aligned}$$

Thus, for all natural integers v satisfying $1 \leq v \leq \min(q-1, p-1)$, we have

$$\sum_{k=0}^{v} P_k(v-k)\beta(v-k,m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.5}$$

Both sequences $(\beta(v,m))_{0\leq v\leq p-1}$ and $(A(v))_{0\leq v\leq p-1}$ satisfy Equations (3.2) and (3.5). Furthermore, for all v in $\{1,\ldots,p-1\}$, $P_0(v)$ and $A(0)$ are invertible in $\mathbb{Z}_p$. Hence there exists $\gamma(m)$ in $\{0,\ldots,p-1\}$ such that, for all v in $\{0,\ldots,p-1\}$, we have $\beta(v,m) \equiv A(v)\gamma(m) \mod p\mathbb{Z}_p$ so that

$$A(v+mp) \equiv A(v)\gamma(m)p^r \mod p^{r+1}\mathbb{Z}_p.$$

Since n_0 is in $\mathcal{Z}_p(A)$, we obtain that $A(n_0+mp)$ belongs to $p^{r+1}\mathbb{Z}_p$ and Lemma 1 is proved. □

3.2. **Proof of Theorem 2.** Let p be a fixed prime number. For every positive integer n, we set $\ell(n) := \lfloor \log_p(n) \rfloor + 1$ the length of the expansion of n to the base p, and $\ell(0) := 1$. For all natural integers $n_1,\ldots,n_r$, we set

$$n_1 * \cdots * n_r := n_1 + n_2 p^{\ell(n_1)} + \cdots + n_r p^{\ell(n_1)+\cdots+\ell(n_{r-1})},$$

so that the expansion of $n_1 * \cdots * n_r$ to the base p is the concatenation of the respective expansions of $n_1,\ldots,n_r$. Then, by a result of Mellit and Vlasenko [21, Lemma 1], there exists a $\mathbb{Z}_p$-valued sequence $(c_n)_{n\geq 0}$ such that, for all positive integers n, we have

$$A(n) = \sum_{\substack{n_1*\cdots*n_r=n \\ 1\leq r\leq \ell(n),\, n_r>0}} c_{n_1}\cdots c_{n_r} \quad \text{and} \quad c_n \equiv 0 \mod p^{\ell(n)-1}\mathbb{Z}_p. \tag{3.6}$$

For every natural integer r, we write $\mathcal{U}(r)$ for the assertion: "For all $n, i \in \mathbb{N}$, $i \leq r$, if $\alpha_p(A,n) \geq i$, then $A(n), c_n \in p^i\mathbb{Z}_p$". To prove Theorem 2, it suffices to show that, for all natural integers r, Assertion $\mathcal{U}(r)$ holds.

First we prove $\mathcal{U}(1)$. By Theorem 1 in [21], A satisfies the p-Lucas property. In addition, if v is in $\mathcal{Z}_p(A)$, then v is nonzero because $A(0) = 1$, and by (3.6) we have $c_v = A(v) \in p\mathbb{Z}_p$.

Now, if a natural integer n satisfies $\ell(n) = 2$ and $\alpha_p(A, n) \geq 1$, then Equation (3.6) yields $A(n) \equiv c_n \mod p\mathbb{Z}_p$, so that c_n is in $p\mathbb{Z}_p$. Hence, by induction on $\ell(n)$, we obtain that, for all natural integers n satisfying $\alpha_p(A, n) \geq 1$, c_n belongs to $p\mathbb{Z}_p$, so that $\mathcal{U}(1)$ holds.

Let r be a positive integer such that $\mathcal{U}(r)$ holds. We shall prove that $\mathcal{U}(r+1)$ is true. For all positive integers M, we write $\mathcal{U}_M(r+1)$ for the assertion:

"For all $n, i \in \mathbb{N}$, $n \leq M$, $i \leq r+1$, if $\alpha_p(A, n) \geq i$, then $A(n), c_n \in p^i\mathbb{Z}_p$".

Hence $\mathcal{U}_M(r+1)$ is true if $\ell(M) \leq r$. Let M be a positive integer such that $\mathcal{U}_M(r+1)$ holds. We shall prove $\mathcal{U}_{M+1}(r+1)$. By Assertions $\mathcal{U}(r)$ and $\mathcal{U}_M(r+1)$, it suffices to prove that if $\alpha_p(A, M+1)$ is greater than r, then $A(M+1)$ and c_{M+1} belong to $p^{r+1}\mathbb{Z}_p$. In the rest of the proof, we assume that $\alpha_p(A, M+1)$ is greater than r.

If u and $n_1, \ldots, n_u$ are natural integers satisfying $2 \leq u \leq \ell(M+1)$ and $n_1 * \cdots * n_u = M+1$ with $n_u > 0$, then, for all i in $\{1, \ldots, u\}$, we have $n_i \leq M$ and

$$\alpha_p(A, n_1) + \cdots + \alpha_p(A, n_u) = \alpha_p(A, M+1) \geq r+1.$$

Then there exist $1 \leq a_1 < \cdots < a_k \leq u$ and $1 \leq i_1, \ldots, i_k \leq r+1$ such that $\alpha_p(A, n_{a_j}) \geq i_j$ and $i_1 + \cdots + i_k \geq r+1$. Thereby, Assertion $\mathcal{U}_M(r+1)$ yields $c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p$, so that

$$\sum_{\substack{n_1 * \cdots * n_u = M+1 \\ 2 \leq u \leq \ell(M+1),\, n_u > 0}} c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p.$$

By (3.6), we obtain

$$A(M+1) \equiv c_{M+1} \mod p^{r+1}\mathbb{Z}_p \quad \text{and} \quad c_{M+1} \equiv 0 \mod p^{\ell(M+1)-1}\mathbb{Z}_p.$$

Hence it suffices to consider the case $\ell(M+1) = r+1$. In particular, we have $M+1 = v+mp$ where v is in $\mathcal{Z}_p(A)$ and m is a natural integer satisfying $\alpha_p(A, m) = r$. Since $\mathcal{U}(r)$ holds, Lemma 1 yields $A(M+1) \in p^{r+1}\mathbb{Z}_p$. Thus we also have $c_{M+1} \in p^{r+1}\mathbb{Z}_p$ and Assertion $\mathcal{U}_{M+1}(r+1)$ holds. This finishes the proof of $\mathcal{U}(r+1)$ so that of Theorem 2. □

3.3. **Proof of Proposition 1.** Let p be a prime and A a $\mathbb{Z}_p$-valued sequence satisfying hypothesis of Proposition 1. For every natural integer n, we write $\alpha(n)$, respectively $\mathcal{Z}$, as a shorthand for $\alpha_p(A, n)$, respectively for $\mathcal{Z}_p(A)$. For every natural integer r, we define Assertions

$$\mathcal{U}(r): \text{ "For all } n, i \in \mathbb{N},\ i \leq r, \text{ if } \alpha(n) \geq i, \text{ then } A(n) \in p^i\mathbb{Z}_p.",$$

and

$$\mathcal{V}(r): \text{ "For all } n, i \in \mathbb{N},\ i \leq r, \text{ and all } B \in \mathfrak{B}, \text{ if } \alpha(n) \geq i, \text{ then } B(n) \in p^{i-1}\mathbb{Z}_p".$$

To prove Proposition 1, we have to show that, for all natural integers r, Assertions $\mathcal{U}(r)$ and $\mathcal{V}(r)$ are true. We shall prove those assertions by induction on r.

Observe that Assertions $\mathcal{U}(0)$, $\mathcal{V}(0)$ and $\mathcal{V}(1)$ are trivial. Furthermore, since A satisfies the p-Lucas property, Assertion $\mathcal{U}(1)$ holds. Let r_0 be a fixed positive integer, $r_0 \geq 2$, such that Assertions $\mathcal{U}(r_0 - 1)$ and $\mathcal{V}(r_0 - 1)$ are true. First, we prove Assertion $\mathcal{V}(r_0)$.

Let B in $\mathfrak{B}$ and m in $\mathbb{N}$ be such that $\alpha(m) \geq r_0$. We write $m = v + np$ with v in $\{0, \ldots, p-1\}$. Since $r_0 \geq 2$ and 0 does not belong to $\mathcal{Z}$, we have $n \geq 1$ and, by Assertion (a) in Proposition 1, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\geq 0}$, with B_k in $\mathfrak{B}$, such that

$$B(v+np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k). \tag{3.7}$$

In addition, we have $\alpha(n) \geq r_0 - 1$ and, since 0 is not in $\mathcal{Z}$, we have $\alpha(n-1) \geq r_0 - 2$. By induction, for all natural integers k satisfying $k \leq n$, we have $\alpha(n-k) \geq r_0 - 1 - k$. Thus, by (3.7) in combination with $\mathcal{U}(r_0-1)$ and $\mathcal{V}(r_0-1)$, we obtain

$$A'(n) \in p^{r_0-1}\mathbb{Z} \quad \text{and} \quad p^{k+1}B_k(n-k) \in p^{k+1+r_0-2-k}\mathbb{Z}_p \subset p^{r_0-1}\mathbb{Z}_p,$$

so that $B(v+np)$ belongs to $p^{r_0-1}\mathbb{Z}_p$ and $\mathcal{V}(r_0)$ is true.

Now we prove Assertion $\mathcal{U}(r_0)$. We write $\mathcal{U}_N(r_0)$ for the assertion:

"For all $n, i \in \mathbb{N}$, $n \leq N$, $i \leq r_0$, if $\alpha(n) \geq i$, then $A(n) \in p^i\mathbb{Z}_p$".

We shall prove $\mathcal{U}_N(r_0)$ by induction on N. Assertion $\mathcal{U}_1(r_0)$ holds. Let N be a positive integer such that $\mathcal{U}_N(r_0)$ is true. Let $n := n_0 + mp \leq N+1$ with n_0 in $\{0, \ldots, p-1\}$ and m in $\mathbb{N}$. We can assume that $\alpha(n) \geq r_0$.

If n_0 is in $\mathcal{Z}$, then we have $\alpha(m) \geq r_0 - 1$ and, by Lemma 1, we obtain that $A(n)$ belongs to $p^{r_0}\mathbb{Z}_p$ as expected. If n_0 is not in $\mathcal{Z}$, then we have $\alpha(m) \geq r_0$. By Assertion (a) in Proposition 1, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\geq 0}$ with B_k in $\mathfrak{B}$ such that

$$A(n) = A'(m) + \sum_{k=0}^{\infty} p^{k+1} B_k(m-k).$$

We have $m \leq N$, $\alpha(m) \geq r_0$ and $\alpha(m-k) \geq r_0 - k$, hence, by Assertions $\mathcal{U}_N(r_0)$ and $\mathcal{V}(r_0)$, we obtain that $A(n)$ belongs to $p^{r_0}\mathbb{Z}_p$. This finishes the induction on N and proves $\mathcal{U}(r_0)$. Therefore, by induction on r_0, Proposition 1 is proved. □

4. Proof of Theorem 1

To prove Theorem 1, we shall apply Proposition 1 to $\mathfrak{S}_{e,f}$. As a first step, we prove that this sequence satisfies the p-Lucas property.

Proof of Proposition 2. For all $\mathbf{x}$ in $[0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) = \Delta_{e,f}(\{\mathbf{x}\}) \geq 0$ so that, by Landau's criterion, $\mathcal{Q}_{e,f}$ is integer-valued. Let p be a fixed prime, v in $\{0, \ldots, p-1\}$ and n a natural integer. We have

$$\mathfrak{S}_{e,f}(v+np) = \sum_{\substack{k_1+\cdots+k_d = v+np \\ k_i \in \mathbb{N}}} \mathcal{Q}_{e,f}(k_1, \ldots, k_d).$$

Write $k_i = a_i + m_i p$ with a_i in $\{0, \ldots, p-1\}$ and m_i in $\mathbb{N}$. If $a_1 + \cdots + a_d \neq v$, then we have $a_1 + \cdots + a_d \geq p$ and there exists i in $\{1, \ldots, d\}$ such that $a_i \geq p/d$. Since e is 1-admissible, $(a_1, \ldots, a_d)/p$ belongs to $\mathcal{D}_{e,f}$ so that $\Delta_{e,f}\big((a_1, \ldots, a_p)/p\big) \geq 1$ and $\mathcal{Q}_{e,f}(k_1, \ldots, k_d)$ is in

$p\mathbb{Z}_p$. In addition, by Theorem 3, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p. Hence we obtain

$$\begin{aligned}\mathfrak{S}_{e,f}(v+np) &\equiv \sum_{\substack{a_1+\cdots+a_d=v\\0\le a_i\le p-1}} \sum_{\substack{m_1+\cdots+m_d=n\\m_i\in\mathbb{N}}} \mathcal{Q}_{e,f}(a_1+m_1p,\dots,a_d+m_dp) \mod p\mathbb{Z}_p\\ &\equiv \sum_{\substack{a_1+\cdots+a_d=v\\0\le a_i\le p-1}} \sum_{\substack{m_1+\cdots+m_d=n\\m_i\in\mathbb{N}}} \mathcal{Q}_{e,f}(a_1,\dots,a_d)\mathcal{Q}_{e,f}(m_1,\dots,m_d) \mod p\mathbb{Z}_p\\ &\equiv \mathfrak{S}_{e,f}(v)\mathfrak{S}_{e,f}(n) \mod p\mathbb{Z}_p.\end{aligned}$$

This finishes the proof of Proposition 2. □

If e is 2-admissible then e is also 1-admissible. Furthermore, if $f=(\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$, then, for all $\mathbf{x}$ in $\mathcal{D}_{e,f}$, we have

$$\Delta_{e,f}(\mathbf{x})=\sum_{i=1}^{u}\lfloor \mathbf{e}_i\cdot\mathbf{x}\rfloor \ge 1.$$

Hence, if e and f satisfy the conditions of Theorem 1, then Proposition 2 implies that, for all primes p, $\mathfrak{S}_{e,f}$ has the p-Lucas property and $\mathfrak{S}_{e,f}(0)=1$ is invertible in $\mathbb{Z}_p$. Thereby, to prove Theorem 1, it remains to prove that $\mathfrak{S}_{e,f}$ satisfies Condition (a) in Proposition 1 with

$$\mathfrak{B}=\left\{\mathfrak{S}_{e,f}^{g}\,:\,g\in\mathfrak{F}_p^d\right\}.$$

First we prove that some special functions belong to $\mathfrak{F}_p^1$.

4.1. **Special functions in $\mathfrak{F}_p^1$.** For all primes p, we write $|\cdot|_p$ for the ultrametric norm on $\mathbb{Q}_p$ (the field of p-adic numbers) defined by $|a|_p:=p^{-v_p(a)}$. Note that $(\mathbb{Z}_p,|\cdot|_p)$ is a compact space. Furthermore, if $(c_n)_{n\ge0}$ is a $\mathbb{Z}_p$-valued sequence, then $\sum_{n=0}^{\infty}c_n$ is convergent in $(\mathbb{Z}_p,|\cdot|_p)$ if and only if $|c_n|_p$ tends to 0 as n tends to infinity. In addition, if $\sum_{n=0}^{\infty}c_n$ converges, then $(c_n)_{n\in\mathbb{N}}$ is a summable family in $(\mathbb{Z}_p,|\cdot|_p)$.

In the rest of the article, for all primes p and all positive integers k, we set $\Psi_{p,k,0}(0)=1$, $\Psi_{p,k,i}(0)=0$ for $i\ge1$ and, for all natural integers i and m, $m\ge1$, we set

$$\Psi_{p,k,i}(m):=(-1)^i\sigma_{m,i}\left(\frac{1}{k},\frac{1}{k+p},\dots,\frac{1}{k+(m-1)p}\right),$$

where $\sigma_{m,i}$ is the i-th elementary symmetric polynomial of m variables. Let us remind to the reader that, for all natural integers m and i satisfying $i>m\ge1$, we have $\sigma_{m,i}=0$.

The aim of this section is to prove that, for all primes p, all k in $\{1,\dots,p-1\}$ and all natural integers i, we have

$$i!\Psi_{p,k,i}\in\mathfrak{F}_p^1. \tag{4.1}$$

Proof of (4.1). Throughout this proof, we fix a prime number p and an integer k in $\{1,\dots,p-1\}$. Furthermore, for all nonnegative integers i, we use Ψ_i as a shorthand for $\Psi_{p,k,i}$ and $\mathbb{N}_{\ge i}$ as a shorthand for the set of integers larger than or equal to i. We shall

prove (4.1) by induction on i. To that end, for all natural integers i, we write $\mathcal{A}_i$ for the following assertion:

> "There exists a sequence $(T_{i,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $i!\Psi_i$ on $\mathbb{N}$".

First, observe that, for all natural integers m, we have $\Psi_0(m) = 1$, so that Assertion $\mathcal{A}_0$ is true. Let i be a fixed positive integer such that assertions $\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}$ are true. According to the Newton-Girard formulas, for all integers $m \geq i$, we have

$$i(-1)^i \sigma_{m,i}(X_1, \ldots, X_m) = -\sum_{t=1}^{i} (-1)^{i-t} \sigma_{m,i-t}(X_1, \ldots, X_m) \Lambda_t(X_1, \ldots, X_m),$$

where $\Lambda_t(X_1, \ldots, X_m) := X_1^t + \cdots + X_m^t$. Thereby, for all integers $m \geq i$, we have

$$i\Psi_i(m) = -\sum_{t=1}^{i} \Psi_{i-t}(m) \Lambda_t\left(\frac{1}{k}, \ldots, \frac{1}{k+(m-1)p}\right). \tag{4.2}$$

For all natural integers j and t, we have

$$\frac{1}{(k+jp)^t} = \frac{1}{k^t} \frac{1}{(1+\frac{j}{k}p)^t} = \frac{1}{k^t} + \sum_{s=1}^{\infty} \frac{(-1)^s}{k^t} \binom{t-1+s}{s} \left(\frac{j}{k}\right)^s p^s, \tag{4.3}$$

where the right hand side of (4.3) is a convergent series in $(\mathbb{Z}_p, |\cdot|_p)$ because k is invertible in $\mathbb{Z}_p$. Therefore, we obtain that

$$\begin{aligned} \Lambda_t\left(\frac{1}{k}, \ldots, \frac{1}{k+(m-1)p}\right) &= \frac{m}{k^t} + \sum_{j=0}^{m-1} \sum_{s=1}^{\infty} \frac{(-1)^s}{k^t} \binom{t-1+s}{s} \left(\frac{j}{k}\right)^s p^s \\ &= \frac{m}{k^t} + \sum_{s=1}^{\infty} \frac{(-1)^s}{k^{t+s}} \binom{t-1+s}{s} p^s \left(\sum_{j=0}^{m-1} j^s\right). \end{aligned} \tag{4.4}$$

According to Faulhaber's formula, for all positive integers s, we have

$$p^s \sum_{j=0}^{m-1} j^s = \sum_{c=1}^{s+1} (-1)^{s+1-c} \binom{s+1}{c} p^s \frac{B_{s+1-c}}{s+1} (m-1)^c,$$

where B_k is the k-th first Bernoulli number. For all positive integers s and t, we set $R_{0,t}(X) := X/k^t$ and

$$R_{s,t}(X) := \frac{1}{k^{t+s}} \binom{t-1+s}{s} \sum_{c=1}^{s+1} (-1)^{1-c} \binom{s+1}{c} p^s \frac{B_{s+1-c}}{s+1} (X-1)^c,$$

so that

$$\Lambda_t\left(\frac{1}{k}, \ldots, \frac{1}{k+(m-1)p}\right) = \sum_{s=0}^{\infty} R_{s,t}(m).$$

In the rest of this article, for all polynomials $P(X) = \sum_{n=0}^{N} a_n X^n$ in $\mathbb{Z}_p[X]$, we set

$$\|P\|_p := \max\big\{|a_n|_p \,:\, 0 \leq n \leq N\big\}.$$

We claim that, for all natural integers s and t, $t \geq 1$, we have

$$R_{s,t}(X) \in \mathbb{Z}_p[X], \quad \|R_{s,t}\|_p \underset{s\to\infty}{\longrightarrow} 0 \quad \text{and} \quad R_{s,t}(0) = 0. \tag{4.5}$$

Indeed, on the one hand, if $p = 2$ and $s = 1$, then we have

$$R_{1,t}(X) = \frac{-t}{k^{t+1}}\big(X - 1 + (X-1)^2\big) \in X\mathbb{Z}_2[X].$$

On the other hand, if $p \geq 3$ or $s \geq 2$, then we have $p^s > s + 1$ so that $v_p(s+1) \leq s - 1$. Furthermore, according to the von Staudt-Clausen theorem, we have $v_p(B_{s+1-c}) \geq -1$. Thus, the coefficients of $R_{s,t}(X)$ belong to $\mathbb{Z}_p$. To be more precise, we have $v_p(s+1) \leq \log_p(s+1)$, so that $\|R_{s,t}\|_p \underset{s\to\infty}{\longrightarrow} 0$ as expected. In addition, we have

$$\begin{aligned} R_{s,t}(0) &= -\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{c=1}^{s+1}\binom{s+1}{c}B_{s+1-c} \\ &= -\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{d=0}^{s}\binom{s+1}{d}B_d = 0, \end{aligned}$$

where we used the well known relation satisfied by the Bernoulli numbers

$$\sum_{d=0}^{s}\binom{s+1}{d}B_d = 0, \quad (s \geq 1).$$

According to $\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}$, for all j in $\{0, \ldots, i-1\}$, there exists a sequence $(T_{j,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $j!\Psi_j$ on $\mathbb{N}$. According to (4.2) and (4.5), for all natural integers N, there exists S_N in $\mathbb{N}$ such that, for all $r \geq S_N$ and all $m \geq i$, we have

$$i!\Psi_i(m) \equiv -\sum_{t=1}^{i}\frac{(i-1)!}{(i-t)!}T_{i-t,r}(m)\sum_{s=0}^{r}R_{s,t}(m) \mod p^N\mathbb{Z}_p.$$

Thus, the sequence $(T_{i,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$, defined by

$$T_{i,r}(x) := -\sum_{t=1}^{i}\frac{(i-1)!}{(i-t)!}T_{i-t,r}(x)\sum_{s=0}^{r}R_{s,t}(x), \quad (x, r \in \mathbb{N}), \tag{4.6}$$

converges uniformly to $i!\Psi_i$ on $\mathbb{N}_{\geq i}$. To prove $\mathcal{A}_i$, it suffices to show that, for all m in $\{0, \ldots, i-1\}$, we have

$$T_{i,r}(m) \underset{r\to\infty}{\longrightarrow} 0. \tag{4.7}$$

Observe that Equations (4.6) and (4.5) lead to $T_{i,r}(0) = 0$. In particular, if $i = 1$, then (4.7) holds. Now we assume that $i \geq 2$. For all $m \geq 2$, we have

$$\begin{aligned}\sum_{j=0}^{m} \Psi_j(m) X^j &= \prod_{w=0}^{m-1} \left(1 - \frac{X}{k+wp}\right) \\ &= \left(1 - \frac{X}{k+(m-1)p}\right) \prod_{w=0}^{m-2} \left(1 - \frac{X}{k+wp}\right) \\ &= \left(1 - \frac{X}{k+(m-1)p}\right) \sum_{j=0}^{m-1} \Psi_j(m-1) X^j.\end{aligned}$$

Thereby, for all j in $\{1, \ldots, m-1\}$, we obtain that

$$\Psi_j(m) = \Psi_j(m-1) - \frac{\Psi_{j-1}(m-1)}{k+(m-1)p},$$

with

$$\frac{1}{k+(m-1)p} = \sum_{s=0}^{\infty} \frac{(-1)^s}{k^{s+1}} p^s (m-1)^s.$$

Thus, there exists a sequence $(U_r)_{r\geq 0}$ of polynomials with coefficients in $\mathbb{Z}_p$ such that, for all positive integers N, there exits a natural integer S_N such that, for all $r \geq S_N$ and all $m \geq i+1$, we have

$$T_{i,r}(m) \equiv T_{i,r}(m-1) - T_{i-1,r}(m-1) U_r(m-1) \mod p^N \mathbb{Z}_p. \tag{4.8}$$

But, if $V_1(X)$ and $V_2(X)$ are polynomials with coefficients in $\mathbb{Z}_p$ and if there exists a natural integer a such that, for all $m \geq a$, we have $V_1(m) \equiv V_2(m) \mod p^N \mathbb{Z}_p$, then, for all integers n, we have $V_1(n) \equiv V_2(n) \mod p^N \mathbb{Z}_p$. Indeed, let n be an integer, there exists a natural integer v such that $n + vp^N \geq a$. Thus, we obtain that

$$V_1(n) \equiv V_1(n + vp^N) \equiv V_2(n + vp^N) \equiv V_2(n) \mod p^N \mathbb{Z}_p.$$

In particular, Equation (4.8) also holds for all positive integers m.

Furthermore, according to $\mathcal{A}_{i-1}$, for all m in $\{0, \ldots, i-2\}$, $T_{i-1,r}(m)$ tends to zero as r tends to infinity. Thus, for all positive integers N, there exists a natural integer S_N such that, for all $r \geq S_N$ and all m in $\{1, \ldots, i-1\}$, we have

$$T_{i,r}(m) \equiv T_{i,r}(m-1) \mod p^N \mathbb{Z}_p.$$

Since $T_{i,r}(0) = 0$, we obtain that $T_{i,r}(m) \equiv 0 \mod p^N \mathbb{Z}_p$ for all m in $\{0, \ldots, i-1\}$, so that (4.7) holds. This finishes the induction on i and proves (4.1). □

4.2. **On the p-adic Gamma function.** For every prime p, we write Γ_p for the p-adic Gamma function, so that, for all natural integers n, we have

$$\Gamma_p(n) = (-1)^n \prod_{\substack{\lambda=1 \\ p \nmid \lambda}}^{n-1} \lambda.$$

The aim of this section is to prove Proposition 3.

Proof of Proposition 3. Let p be a fixed prime number. For all natural integers n and m, we have

$$\begin{aligned}\frac{\Gamma_p\big((m+n)p\big)}{\Gamma_p(mp)\Gamma_p(np)} &= \left(\prod_{\substack{\lambda=np\\ p\nmid\lambda}}^{(m+n)p}\lambda\right)\Big/\left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\lambda\right)\\ &= \left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}(np+\lambda)\right)\Big/\left(\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\lambda\right)\\ &= \prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\left(1+\frac{np}{\lambda}\right). \end{aligned} \tag{4.9}$$

Let $X, T_1, \ldots, T_m$ be $m+1$ variables. Then, we have

$$\prod_{j=1}^{m}(X-T_j) = X^m + \sum_{i=1}^{\infty}(-1)^i\sigma_{m,i}(T_1,\ldots,T_m)X^{m-i}.$$

Therefore, we obtain that

$$\begin{aligned}\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\left(1+\frac{np}{\lambda}\right) &= \prod_{k=1}^{p-1}\prod_{\omega=0}^{m-1}\left(1+\frac{np}{k+\omega p}\right)\\ &= \prod_{k=1}^{p-1}\left(1+\sum_{i=1}^{\infty}(-1)^i\sigma_{m,i}\left(\frac{-np}{k},\cdots,\frac{-np}{k+(m-1)p}\right)\right)\\ &= \prod_{k=1}^{p-1}\left(1+\sum_{i=1}^{\infty}(-1)^i n^i p^i \Psi_{p,k,i}(m)\right). \end{aligned} \tag{4.10}$$

Let k in $\{1,\ldots,p-1\}$ be fixed. By (4.1), for all positive integers i, there exists a sequence $(P_{i,\ell})_{\ell\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to $i!\Psi_{p,k,i}$. We fix a natural integer K. For all positive integers N, we set

$$f_N(x,y) := 1 + \sum_{i=1}^{K+1}(-1)^i x^i \frac{p^i}{i!}P_{i,N}(y).$$

If n and m belong to $\{0,\dots,K\}$, then we have

$$\begin{aligned} R_N &:= 1+\sum_{i=1}^{\infty}(-1)^i n^i p^i \Psi_{p,k,i}(m) - f_N(n,m) \\ &= \sum_{i=1}^{K+1}(-1)^i n^i \frac{p^i}{i!}\big(i!\Psi_{p,k,i}(m) - P_{i,N}(m)\big) \underset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

Furthermore, we have $f_N(x,y) \in 1+p\mathbb{Z}_p[x,y]$. Indeed, if $i = i_0 + i_1 p + \cdots + i_a p^a$ with i_j in $\{0,\dots,p-1\}$, then we set $\mathfrak{s}_p(i) := i_0 + \cdots + i_a$ so that, for all positive integers i, we have

$$i - v_p(i!) = i - \frac{i-\mathfrak{s}_p(i)}{p-1} = \frac{i(p-2)+\mathfrak{s}_p(i)}{p-1} > 0.$$

Hence, by (4.10), we obtain that there exists a function g in $\mathfrak{F}_p^2$ such that, for all natural integers n and m, we have

$$\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp}\Big(1+\frac{np}{\lambda}\Big) = 1+g(n,m)p,$$

which, together with (4.9), finishes the proof of Proposition 3. □

4.3. **Last step in the proof of Theorem 1.** Let $\mathfrak{A}$ be the $\mathbb{Z}_p$-module spanned by $\mathfrak{S}_{e,f}$. We set $\mathfrak{B} = \{\mathfrak{S}_{e,f}^g,\ g \in \mathfrak{F}_p^d\}$. We shall prove that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) in Proposition 1.

Obviously, $\mathfrak{B}$ is constituted of $\mathbb{Z}_p$-valued sequences and $\mathfrak{A}$ is a subset of $\mathfrak{B}$. For all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$ and $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p) = \frac{\prod_{i=1}^{u}(\mathbf{e}_i\cdot\mathbf{m}p)!\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^{v}(\mathbf{f}_i\cdot\mathbf{m}p)!\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)}.$$

For every natural integer n, we have

$$\frac{(np)!}{n!} = p^n(-1)^{np}\Gamma_p(np),$$

so that we have

$$\frac{\prod_{i=1}^{u}(\mathbf{e}_i\cdot\mathbf{m}p)!}{\prod_{i=1}^{v}(\mathbf{f}_i\cdot\mathbf{m}p)!} = p^{(|e|-|f|)\cdot\mathbf{m}}\mathcal{Q}_{e,f}(\mathbf{m})\frac{\prod_{i=1}^{u}(-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^{v}(-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)}.$$

Furthermore, we have

$$\begin{aligned} &\frac{\prod_{i=1}^{u}\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^{v}\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)} \\ &\qquad = \frac{\prod_{i=1}^{u}\prod_{k=1,p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^{v}\prod_{k=1,p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)}\cdot p^{\Delta_{e,f}(\mathbf{a}/p)}\frac{\prod_{i=1}^{u}\prod_{k=1}^{\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k)}{\prod_{i=1}^{v}\prod_{k=1}^{\lfloor\mathbf{f}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{f}_i\cdot\mathbf{m}+k)}. \end{aligned}$$

Since $|e| = |f|$, we have

$$\frac{\prod_{i=1}^{u}(-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^{v}(-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)} = \frac{\prod_{i=1}^{u}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^{v}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)}.$$

Let $\alpha_1,\dots,\alpha_d$ be natural integers with $\alpha_{i_0}\geq 1$ for some i_0 in $\{1,\dots,d\}$. By Proposition 3, there exists a function h in $\mathfrak{F}_p^d$ such that, for all natural integers $m_1,\dots,m_d$, we have

$$\frac{\Gamma_p\big((\alpha_1m_1+\cdots+\alpha_dm_d)p\big)}{\Gamma_p\big((\alpha_1m_1+\cdots+(\alpha_{i_0}-1)m_{i_0}+\cdots+\alpha_dm_d)p\big)\Gamma_p(m_{i_0}p)} = 1+h(m_1,\dots,m_d)p.$$

Hence, there exists a function h' in $\mathfrak{F}_p^d$ such that, for all natural integers $m_1,\dots,m_d$, we have

$$\frac{\Gamma_p\big((\alpha_1m_1+\cdots+\alpha_dm_d)p\big)}{\Gamma_p(m_1p)^{\alpha_1}\cdots\Gamma_p(m_dp)^{\alpha_d}} = 1+h'(m_1,\dots,m_d)p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, there exists g' in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^{u}\Gamma_p(\mathbf{e}_i\cdot\mathbf{m}p)}{\prod_{i=1}^{v}\Gamma_p(\mathbf{f}_i\cdot\mathbf{m}p)} = 1+g'(\mathbf{m})p.$$

Furthermore, if k is an integer coprime with p, and $\mathbf{d}$ a vector in $\mathbb{N}^d$, then for every $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{1}{\mathbf{d}\cdot\mathbf{m}p+k} = \sum_{s=0}^{\infty}(-1)^s\frac{(\mathbf{d}\cdot\mathbf{m})^s}{k^{s+1}}p^s,$$

so that there is a function g'' in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{1}{\mathbf{d}\cdot\mathbf{m}p+k} = \frac{1}{k}+g''(\mathbf{m})p.$$

Hence, for all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, there exist a p-adic integer $\lambda_{\mathbf{a}}$ and a function $g_{\mathbf{a}}$ in $\mathfrak{F}_p^d$ such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^{u}\prod_{k=1,p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}}(\mathbf{e}_i\cdot\mathbf{m}p+k)}{\prod_{i=1}^{v}\prod_{k=1,p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}}(\mathbf{f}_i\cdot\mathbf{m}p+k)} = \lambda_{\mathbf{a}}+g_{\mathbf{a}}(\mathbf{m})p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, for all i in $\{1,\dots,v\}$, we have $\lfloor\mathbf{f}_i\cdot\mathbf{a}/p\rfloor = 0$. Thereby, for all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, there exists a function $h_{\mathbf{a}}$ in $\mathbb{Z}_p+p\mathfrak{F}_p^d$, such that, for all $\mathbf{m}$ in $\mathbb{N}^d$, we have

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p) = \mathcal{Q}_{e,f}(\mathbf{m})h_{\mathbf{a}}(\mathbf{m})p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k).$$

Furthermore, if $\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor\geq 1$ for some i, then $\Delta_{e,f}(\mathbf{a}/p)\geq 1$. Hence we obtain that

$$\mathbf{m}\mapsto p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor\mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k)\in\mathbb{Z}_p+p\mathfrak{F}_p^d.$$

Let g be a function in $\mathfrak{F}_p^d$. For all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$ and $\mathbf{m}$ in $\mathbb{N}^d$, we set

$$\tau_{\mathbf{a}}(\mathbf{m}) := g(\mathbf{a}+\mathbf{m}p)h_{\mathbf{a}}(\mathbf{m})p^{\Delta_{e,f}(\mathbf{a}/p)}\prod_{i=1}^{u}\prod_{k=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor}(\mathbf{e}_i\cdot\mathbf{m}+k),$$

so that $\tau_{\mathbf{a}} \in \mathbb{Z}_p + p\mathfrak{F}_p^d$. Therefore, for all v in $\{0,\dots,p-1\}$ and n in $\mathbb{N}$, we have

$$\begin{aligned}\mathfrak{S}_{e,f}^g(v+np) &= \sum_{\mathbf{0}\le\mathbf{a}\le\mathbf{1}(p-1)}\sum_{\substack{\mathbf{m}\ge\mathbf{0}\\ |\mathbf{a}+\mathbf{m}p|=v+np}} g(\mathbf{a}+\mathbf{m}p)\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p)\\ &= \sum_{\mathbf{0}\le\mathbf{a}\le\mathbf{1}(p-1)}\sum_{\substack{\mathbf{m}\ge\mathbf{0}\\ |\mathbf{a}+\mathbf{m}p|=v+np}} \mathcal{Q}_{e,f}(\mathbf{m})\tau_{\mathbf{a}}(\mathbf{m}).\end{aligned}$$

If $|\mathbf{a}+\mathbf{m}p| = v+np$, then we have $|\mathbf{a}| = v+jp$ with

$$0 \le j \le \min\left(n, \left\lfloor \frac{d(p-1)-v}{p}\right\rfloor\right) =: M.$$

Furthermore, we have $\lfloor|\mathbf{a}|/p\rfloor = j$ and there is k in $\{1,\dots,d\}$ such that $\mathbf{a}^{(k)} \ge (v+jp)/d$. Since e is 2-admissible and f is constituted by vectors $\mathbf{1}_k$, we obtain that

$$\Delta_{e,f}(\mathbf{a}/p) = \sum_{i=1}^{u}\left\lfloor \frac{\mathbf{e}_i\cdot\mathbf{a}}{p}\right\rfloor \ge 2j.$$

In particular, there is $\tau'_{\mathbf{a}}$ in $\mathfrak{F}_p^d$ such that $\tau_{\mathbf{a}} = p^{2j}\tau'_{\mathbf{a}}$. Hence, for all $\mathbf{a}$ in $\{0,\dots,p-1\}^d$, we have

$$\mathfrak{S}_{e,f}^g(v+np) = \sum_{\substack{\mathbf{0}\le\mathbf{a}\le\mathbf{1}(p-1)\\ |\mathbf{a}|=v}}\sum_{|\mathbf{m}|=n}\mathcal{Q}_{e,f}(\mathbf{m})\tau_{\mathbf{a}}(\mathbf{m}) + \sum_{j=1}^{M}p^{2j}\sum_{\substack{\mathbf{0}\le\mathbf{a}\le\mathbf{1}(p-1)\\ |\mathbf{a}|=v+jp}}\sum_{|\mathbf{m}|=n-j}\mathcal{Q}_{e,f}(\mathbf{m})\tau'_{\mathbf{a}}(\mathbf{m}).$$

Therefore, there exist A' in $\mathfrak{A}$ and a sequence $(B_k)_{k\ge0}$, with B_k in $\mathfrak{B}$, such that

$$\mathfrak{S}_{e,f}^g(v+np) = A'(n) + pB_0(n) + \sum_{k=1}^{\infty}p^{k+1}B_k(n-k).$$

This shows that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) in Proposition 1, so that Theorem 1 is proved. □

References

[1] G. Almkvist, W. Zudilin, *Differential equations, mirror maps and zeta values*. Mirror symmetry. V,481–515, AMS/IP Stud. Adv. Math., **38**, Amer. Math. Soc., Providence, RI, 2006.

[2] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[3] F. Beukers, *Congruence properties of coefficients of solutions of Picard-Fuchs equations*, Groupe de travail d'analyse ultrametrique, tome **14** (1986-1987), Exposé 11, 1–6.

[4] F. Beukers, *Some congruences for the Apéry numbers*, J. Number Theory, **21** (1985), 141–155.

[5] F. Beukers, *Another congruence for the Apéry numbers*, J. Number Theory, **25** (1987), 201–210.

[6] F. Beukers, J. Stienstra, *On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces*, Math. Ann., **271** (1985), 269–304.
[7] J. M. Borwein, D. Nuyens, A. Straub, J. Wan, *Some arithmetic properties of short random walk integrals*, Ramanujan J. **26** (2011), no. 1, 109–132.
[8] N. J. Calkin, *Factors sums of powers of binomial coefficients*, Acta Arith. **86** (1998), 17–26.
[9] M. J. Coster, *Congruence properties of coefficients of certain algebraic power series*, Compositio Math. **68** (1988), no. 1, 41–57.
[10] S. Chowla, J. Cowles, M. Cowles, *Congruences properties of Apéry numbers*, J. Number Theory, **12** (1980), 188–190.
[11] E. Delaygue, *A criterion for the integrality of the Taylor coefficients of mirror maps in several variables*, Adv. Math. **234** (2013), pp. 414–452.
[12] E. Delaygue, T. Rivoal, J. Roques, *On Dwork's p-adic formal congruences theorem and hypergeometric mirror maps*, preprint (2013), arXiv:1309.5902v1 [math.NT].
[13] ,E. Deutsch, B. E. Sagan, *Congruences for Catalan and Motzkin numbers and related sequences*, J. Number Theory **117** (2006), no. 1, 191–215.
[14] J. Franel, *On a question of Laisant*, L'intermédiaire des mathématiciens **1** (3) (1894), 45–47.
[15] J. Franel, *On a question of J. Franel*, L'intermédiaire des mathématiciens **2** (1895), 33–35.
[16] I. Gessel, *Some congruences for Apéry numbers*, J. Number Theory, **14** (1982), 362–368.
[17] E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math., **44** (1852), 93–146.
[18] P. Lairez, *Personal communication*, 2013.
[19] E. Landau, *Sur les conditions de divisibilité d'un produit de factorielles par un autre*, collected works, **I**, page 116. Thales-Verlag (1985).
[20] E. Lucas, *Sur les congruences des nombres eulériens et les coefficients différentiels des fonctions trigonométriques suivant un module premier*, Bulletin de la S.M.F., tome **6** (1878), 49–54.
[21] A. Mellit, M. Vlasenko, *Dwork's congruences for the constant terms of powers of a Laurent polynomial*, preprint (2013), available at arXiv:1306.5811v1 [math.NT].
[22] Y. Mimura, *Congruence properties of Apéry numbers*, J. Number Theory, **16** (1983), 138–146.
[23] K. Samol, D. van Straten, *Dwork congruences and reflexive polytopes*, preprint (2009), arXiv:0911.0797v1 [math.AG], 14 pp.
[24] L. B. Richmond, J. Shallit, *Counting abelian squares*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 72, 9 pp.
[25] *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org, 2013.
[26] D. Zagier, *Integral solutions of Apéry-like reccurence equations*, Groups and symmetries, 349–366, CRM Proc. Lecture Notes, **47**, Amer. Math. Soc., Providence, RI, 2009.

E. Delaygue, Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 Novembre 1918, 69622 Villeurbanne cedex, France. Email: delaygue@math.univ-lyon1.fr