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# Dimensional contraction in Wasserstein distance for diffusion semigroups on a Riemannian manifold

Ivan Gentil\*

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## Abstract

We prove a refined contraction inequality for diffusion semigroups with respect to the Wasserstein distance on a Riemannian manifold taking account of the dimension. The result generalizes in a Riemannian context, the dimensional contraction established in [BGG13] for the Euclidean heat equation. The theorem is proved by using a dimensional coercive estimate for the Hodge-de Rham semigroup on 1-forms.

**Key words:** Diffusion equations, Wasserstein distance, Hodge-de Rham operator, Curvature-dimension bounds.

**Mathematics Subject Classification (2000):** 58J65, 58J35, 53B21

## 1 Introduction

The von Renesse-Sturm Theorem (c.f. [vRS05]) insures that the contraction of the heat equation on a Riemannian manifold with respect to the Wasserstein distance is equivalent to a lower bound of the Ricci curvature. This result is one of the first theorem relating the Wasserstein distance and the Ricci curvature, actually there are many extensions of this result including on a metric measurable space. One of the problem is to take account of the dimension into such contraction.

Let us explain the history of the result and its extensions. For simplicity we focus on the heat equation on a Riemannian manifold. All of these results have been proved by the authors in a context of a general diffusion semigroup on a manifold or more general spaces. Let  $\Delta$  be the Laplace-Beltrami operator on a smooth Riemannian manifold  $(M, g)$  and let  $P_t f$  be the solution of the heat equation  $\partial_t u = \Delta u$  with  $f$  as the initial condition.

The von Renesse-Sturm Theorem states that: there exists  $R \in \mathbb{R}$ , such that

$$W_2^2(P_t f dx, P_t g dx) \leq e^{-2Rt} W_2^2(f dx, g dx), \quad \forall t \geq 0,$$

for any  $f, g$  probability densities with respect to  $dx$  (the Riemannian measure) if and only if  $\text{Ricci}_g \geq R$  (uniformly in  $M$ ) where  $\text{Ricci}_g$  is the Ricci tensor of  $(M, g)$ . The Wasserstein distance is as usual denoted  $W_2$ .

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They are actually many proofs and extensions of this result, one can see for instance [Wan04, OW05, AGS08, Kuw10, Wan11, BGL13a, BGL13b, Sav13].

Recently, many extensions have been given taking account of the dimension of the manifold. For instance in [Kuw13, BGL13b] the authors prove that if  $M$  is of dimension  $n$  with a non-negative Ricci curvature, then

$$W_2^2(P_s f dx, P_t g dx) \leq W_2^2(f dx, g dx) + 2n(\sqrt{s} - \sqrt{t})^2, \quad \forall s, t \geq 0, \quad (1)$$

for densities  $f$  and  $g$ . Non-negative curvature condition has been removed in [Kuw13, EKS13, GM13]. The main extensions are the following:

- In [Kuw13], K. Kuwada proves that the Ricci curvature is bounded from below by  $R \in \mathbb{R}$  if and only if

$$W_2^2(P_t f dx, P_s g dx) \leq A(s, t, R)W_2^2(f dx, g dx) + B(s, t, n, R), \quad \forall s, t \geq 0,$$

for probability densities  $f, g$  with respect to  $dx$ , for appropriate functions  $A, B \geq 0$ .

- In [EKS13], M. Erbar, K. Kuwada and K.-T. Sturm prove that the Ricci curvature of the  $n$ -dimensional manifold  $M$  is bounded from below by a constant  $R \in \mathbb{R}$  if and only if

$$s_{\frac{R}{n}} \left( \frac{1}{2} W_2(P_t f dx, P_s g dx) \right)^2 \leq e^{-R(t+s)} s_{\frac{R}{n}} \left( \frac{1}{2} W_2(f dx, g dx) \right)^2 + \frac{n}{R} (1 - e^{-R(s+t)}) \frac{(\sqrt{t} - \sqrt{s})^2}{2(t+s)}, \quad (2)$$

for any  $s, t \geq 0$  and any probability densities  $f$  and  $g$ . Here  $s_r(x) = \sin(\sqrt{r}x)/\sqrt{r}$  if  $r > 0$ ,  $s_r(x) = \sinh(\sqrt{-r}x)/\sqrt{-r}$  if  $r < 0$  and  $s_0(x) = x$ , hence recovering (1) for  $R = 0$ .

- In [BGG13] we prove that the classical heat equation in the Euclidean space  $\mathbb{R}^n$  satisfies, for any  $f, g$  probability densities with respect to the Lebesgue measure  $\lambda$ ,

$$W_2^2(P_t f \lambda, P_t g \lambda) \leq W_2^2(f \lambda, g \lambda) - \frac{2}{n} \int_0^t (\text{Ent}_\lambda(P_u f) - \text{Ent}_\lambda(P_u g))^2 du, \quad \forall t \geq 0, \quad (3)$$

where  $\text{Ent}$  is the Entropy (it will be defined later).

- On the same way, again in [BGG13], we prove a more general result for a  $n$ -dimensional Riemannian manifold with a Ricci curvature bounded from below by  $R$  and for the Markov transportation distance  $T_2$  (a new distance on measures). We obtain for any  $f$  and  $g$ ,

$$T_2^2(P_t f dx, P_t g dx) \leq e^{-2Rt} T_2^2(f dx, g dx) - \frac{2}{n} \int_0^t e^{-2R(t-u)} (\text{Ent}_\mu(P_u g) - \text{Ent}_\mu(P_u f))^2 du. \quad t \geq 0.$$

The goal of this paper is to prove the previous inequality for the Wasserstein distance instead the Markov transportation distance, in other words to prove inequality (3) on a  $n$ -dimensional Riemannian manifold with a lower bound on the Ricci curvature.

The main result of this short paper can be stated as follow, let  $d\mu = e^{-\Psi} dx$  be a probability measure with  $\psi$  a smooth function on  $M$  (for the sequel the manifold will be compact). We denote by  $(P_t)_{t \geq 0}$  the Markov semigroup associated to the generator  $L = \Delta - \nabla \Psi \cdot \nabla$ . (It is reversible in  $L^2(\mu)$ .) Then under the curvature-dimension condition

$$\text{Ricci}_g + \text{Hess}(\Psi) \geq R - \frac{1}{m-n} \nabla \Psi \otimes \nabla \Psi, \quad (4)$$

for some  $R \in \mathbb{R}$  and  $m \geq n$ , then for any probability densities  $f, g$  and any  $t \geq 0$ ,

$$W_2^2(P_t f \mu, P_t g \mu) \leq e^{-2Rt} W_2^2(f \mu, g \mu) - \frac{2}{m} \int_0^t e^{-2R(t-u)} [\text{Ent}_\mu(P_u g) - \text{Ent}_\mu(P_u f)]^2 du.$$

The main advantage of such inequality with respect to (2) is to obtain a contraction inequality with the same time  $t$  instead two different times  $s$  and  $t$ . Moreover, the additional term is given with a minus sign which shows the improvement given by the dimension.

The strategy used to prove such inequality is the Benamou-Brenier dynamical formulation of Wasserstein distance associated to a coercive bounds on the Hodge-de Rham semigroup.

The paper is organized as follow. First in Section 2, we recalled the Riemannian setting and the Wasserstein distance through the Benamou-Brenier dynamical formulation. We need to introduce the Hodge-de Rham operator on forms and its associated semigroup. In Section 3, we are improving the Bochner-Lichnerowicz-Weitzenböck identity for 1-forms to get coercive inequality for the Hodge-de Rham semigroup. Finally in Section 4 the main result is proved.

For simplicity reasons, the result will be stated and proved in the context of a compact Riemannian manifold. The generalization of this result in a metric space, including the equivalence with respect to the condition (4), is actually a project with F. Bolley, N. Gigli and A. Guillin. The method will be radically different since we will use the definition of the heat equation as gradient flow of the Entropy with respect to the Wasserstein distance.

## 2 Framework and main result

### 2.1 Geometrics tools

**Conventions and notations:** Let  $(M, g)$  be a  $n$ -dimensional complete, connected, compact and differentiable Riemannian manifold without boundary. We assume for simplicity that the manifold is  $\mathcal{C}^\infty$ . For each  $x \in M$  we denote by  $T_x M$  the tangent space to  $M$  and by  $TM$  the whole tangent bundle of  $M$ . Moreover  $g_x$  is a symmetric definite positive quadratic form on  $T_x M$ , in a local basis of  $T_x M$   $(e_i)_{1 \leq i \leq n}$ ,  $g_x = (g_{ij})$ . (For simplicity, the  $x$  dependance of the metric  $g$  is omitted.) For every  $x, y \in M$ ,  $d(x, y)$  denotes the usual (Riemannian) distance and  $dx$  its measure. In the sequel we will use the Einstein's convention of summation over repeated indices:  $x_i y^i = \sum_{i=1}^n x_i y^i$ ,  $x^i g_{ij} y^j = \sum_{i=1}^n x^i g_{ij} y^j$ .

The Riemannian scalar product between two tensors  $X$  and  $Y$  is denoted  $X \cdot Y$ , its associated norm is noted  $|X|$  (depending on  $g$ ). For instance, locally in a basis  $(e_i)_{1 \leq i \leq n}$  for two smooth functions  $f, h : M \mapsto \mathbb{R}$ ,  $\nabla f \cdot \nabla h = \partial_i f g^{ij} \partial_j h$  where  $(g^{ij}) = (g_{ij})^{-1}$ . As usual the covariant derivative in the direction  $e_i$  of a tensor  $X$  (vector field or form) is noted  $\nabla_i X$ . The geometric musicology will be used in force, for instance  $\nabla^i f = g^{ij} \nabla_j f = g^{ij} \partial_j f$  or  $\nabla^i X = g^{ij} \nabla_j X$ . If  $\omega$  is a 1-form then  $\omega^*$  is its dual representation as a vector field with components  $\omega^i = g^{ij} \omega_j$ . The Ricci tensor of  $(M, g)$  is noted  $\text{Ricci}_g$ .

The Laplace-Beltrami operator  $\Delta$  is acting on smooth functions  $f$ ,

$$\Delta f = \nabla \cdot \nabla f,$$

where  $\nabla \cdot$  is the divergence operator acting on vector fields,  $\nabla \cdot X = \nabla_i X^i$  for every vector field  $X$ . (We use the analyst's convention with respect to the sign.) A smooth function (or form) is

a  $C^\infty$ -function (or form). The divergence operator  $\nabla \cdot$  satisfies for any smooth vector fields  $X$  and any smooth functions  $f$ ,

$$\int_M \nabla \cdot X f dx = - \int_M \delta df \cdot X dx.$$

where  $\delta$  is the divergence operator on 1-form

$$\delta \omega = \nabla_i \omega^i = g^{ij} \nabla_i \omega_j = g^{ij} (\partial_i \omega_j - \Gamma_{ij}^p \omega_p),$$

where  $\Gamma_{ij}^p$  are Christoffel symbols. Operators  $\delta$  and  $\nabla \cdot$  are related by the formula  $\delta \omega = \nabla \cdot \omega^*$ , for any 1-forms  $\omega$ .

**The heat semigroup:** Let  $\Psi : M \mapsto \mathbb{R}$  be a fixed  $C^\infty$ -function. Let  $\mu(dx) = e^{-\Psi} dx$  and since  $M$  is compact we can assume that  $\mu$  is a probability measure. Let  $Lf = \Delta f - \nabla \psi \cdot \nabla f$ , for any smooth functions  $f$ . Since the manifold is compact, the operator  $L$  defined a unique Markov semigroup  $(P_t)_{t \geq 0}$  on  $L^2(\mu)$  ( $(M, \mu, \Gamma)$  is then a compact Markov triple as defined in [BGL13a, Chap. 3]). This semigroup is symmetric in  $L^2(\mu)$  and for any  $f$ ,  $P_t f$  is solution of the equation  $\partial_t u = Lu$  with  $f$  as the initial condition. If  $f$  is a probability density (with respect to  $\mu$ ) then for all  $t \geq 0$ ,  $P_t f \mu$  remains a probability measure. Finally, for any smooth functions  $f$  and  $g$  on  $M$ ,

$$\int_M f Lg d\mu = \int_M g Lf \mu = - \int_M \nabla f \cdot \nabla g d\mu = - \int_M \Gamma(f, g) d\mu,$$

where  $\Gamma$  is Carré du champ operator defined on functions,  $\Gamma(f, g) = \nabla f \cdot \nabla g$ . Let  $\nabla_\Psi \cdot X = \nabla \cdot X - \nabla \Psi \cdot X$ , where  $X$  is a vector field, the generator  $L$  takes the form

$$L = \nabla_\Psi \cdot \nabla.$$

As before we note by  $\delta_\Psi$  the divergence operator acting on forms:

$$\delta_\Psi \omega = \nabla_\Psi \cdot \omega^*. \quad (5)$$

The following integration by parts is valid for any smooth 1-forms  $\omega$  and functions  $f$ ,

$$\int_M \delta_\Psi \omega f d\mu = - \int_M \omega \cdot df d\mu.$$

**The Hodge-de Rham semigroup:** Connected to the heat semigroup (associated to the generator  $L$ ) one can define the Hodge-de Rham semigroup. As explained in this context in [Bak87] the Hodge-de Rham (modified) operator is acting on smooth 1-form,

$$\vec{L} = -(d^{(0)} \delta_\Psi^{(1)} + \delta_\Psi^{(2)} d^{(1)}), \quad (6)$$

where  $d^{(i)}$  is the differential operator acting on  $i$ -forms and  $\delta_\Psi^{(i)}$  is its adjoint operator in  $L^2(\mu)$  (in the sequel, we omit the exponent  $(i)$ ). For computations, we use the Hodge-de Rham operator toward the Weitzenböck formula, that is for any  $1 \leq i \leq n$ ,

$$(\vec{L}\omega)_i = \nabla^k \nabla_k \omega_i - (\nabla_{\nabla \Psi} \omega)_i - Ricci(L)(\omega^*, e_i) = \nabla^k \nabla_k \omega_i - \nabla^k \Psi \nabla_k \omega_i - Ricci(L)(\omega^*, e_i), \quad (7)$$

where  $Ricci(L) = Ricci_g + Hess(\Psi)$  is the so-called Bakry-Émery tensor (see for instance [Bak87, Prop. 1.5]). (Again, we use the analyst's convention with respect to the sign.)

Again, since  $M$  is compact, the operator  $\vec{L}$  involves a semigroup  $(R_t)_{t \geq 0}$  on 1-forms. It is also symmetric in  $L^2(d\mu)$ , for any smooth 1-forms  $\omega$  and  $\eta$ ,

$$\int_M \omega \cdot \vec{L}\eta \, d\mu = \int_M \eta \cdot \vec{L}\omega \, d\mu.$$

Then for any smooth 1-forms  $\omega$ ,  $R_t\omega$  is the solution of  $\partial u = \vec{L}u$  where  $u : [0, \infty) \times M \mapsto TM^*$  ( $TM^*$  is the cotangent bundle) with  $\omega$  as the initial condition. The details of the construction of the Hodge-de Rham semigroup can be found in [Bak87] (see also the references therein).

The Hodge-de Rham semigroup is related to the heat equation by the following commutation property: for any smooth 1-forms  $\omega$  and  $t \geq 0$ ,

$$P_t \delta_\Psi \omega = \delta_\Psi R_t \omega. \quad (8)$$

The easiest way to prove this fundamental identity is to use the definition (6) since  $\delta_\Psi^{(1)} \delta_\Psi^{(2)} = 0$ .

## 2.2 The Wasserstein distance

Let  $\mathcal{P}_2(M)$  be the set of probability measures in  $M$  with a finite second moment  $\int_M d(x_0, x)^2 d\mu < \infty$  for some  $x_0 \in M$ . The Wasserstein distance between two probability measures  $\nu_1, \nu_2 \in \mathcal{P}_2(M)$ , is defined by

$$W_2(\nu_1, \nu_2) = \inf \left( \int_{M \times M} d^2(x, y) d\pi(x, y) \right)^{1/2},$$

where the infimum runs over all probability measures  $\pi$  in  $M \times M$  with marginals  $\mu$  and  $\nu$ . We refer to the monumental work [Vil09] for a reference presentation of this distance, its interplay with the optimal transportation problem and many other issues.

The Wasserstein distance has a dynamical formulation: for any probabilities measure  $\nu_1, \nu_2 \in \mathcal{P}_2(M)$ ,

$$W_2^2(\nu_1, \nu_2) = \inf \int_0^1 \int_M |\eta_s|^2 d\mu_s ds,$$

where the infimum is running over all path of probabilities  $(\mu_s)_{s \in [0,1]}$  and  $\eta_s \in TM^*$  satisfying in the distributional sense

$$\begin{cases} \partial_s \mu_s + \delta(\mu_s \eta_s) = 0 \\ \mu_0 = \nu, \mu_1 = \nu_2. \end{cases}$$

The Euclidean case has been proved by Benamou-Brenier in [BB00] and the Riemannian case by F. Otto and M. Westdickenberg an the other hand by L. De Pascale, M. S. Gelli and L. Granieri in [OW05, DPGG06].

Let  $\omega_s = \eta_s \rho_s$  where  $\frac{d\mu_s}{d\mu} = \rho_s$ , in [OW05] the authors state that for any  $f$  and  $g$ , smooth and positive probability densities (with respect to  $\mu$ ),  $f\mu, g\mu \in \mathcal{P}_2(M)$  since  $M$  is compact,

$$W_2^2(f\mu, g\mu) = \inf \int_0^1 \int_M \frac{|\omega_s|^2}{\rho_s} d\mu ds, \quad (9)$$

where the infimum is running over smooth couples  $(\rho_s, \omega_s)_{s \in [0,1]}$  where for any  $s \in [0, 1]$ ,  $\rho_s$  is a positive probability density (with respect to  $\mu$ ) and  $\omega_s$  is a 1-form, satisfying

$$\begin{cases} \partial_s \rho_s + \delta_\Psi \omega_s = 0 \\ \rho_0 = f, \rho_1 = g. \end{cases}$$

This dynamical formulation of optimal transportation has been manly used to get contraction result or Evolutional Variational Inequality (see e.g. [DNS09, DNS12, BGG13]). These papers are the starting point of this article.

## 2.3 Main result

We can now state the main result of this short paper.

**Theorem 2.1 (Dimensional contraction in Wasserstein distance)** *Let  $(M, g)$  be a  $C^\infty$ , connected, complete and compact Riemannian manifold of dimension  $n$ . We assume that there exists  $R \in \mathbb{R}$  and  $m \geq n$  such that uniformly in  $M$ ,*

$$\text{Ricci}(L) \geq R - \frac{1}{m-n} \nabla \Psi \otimes \nabla \Psi. \quad (10)$$

Then for any probability densities  $f, g$  with respect to  $\mu$  such that  $f\mu, g\mu \in \mathcal{P}_2(M)$  and any  $t \geq 0$ ,

$$W_2^2(P_t f\mu, P_t g\mu) \leq e^{-2Rt} W_2^2(f\mu, g\mu) - \frac{2}{m} \int_0^t e^{-2R(t-u)} [\text{Ent}_\mu(P_u g) - \text{Ent}_\mu(P_u f)]^2 du, \quad (11)$$

where  $\text{Ent}_\mu(h) = \int_M h \log h d\mu$  for every probability density  $h$  (with respect to  $\mu$ ).

**Remark 2.2** *When  $m \rightarrow \infty$  we recover the von Renesse-Sturm result: the exponential contraction of the heat equation with respect to the Wasserstein distance. When  $\psi = 0$ , one can choose  $m = n$ , and then the Laplace-Beltrami operator satisfies the condition (10) under a lower bound on the Ricci curvature.*

*As it will be explained in Remark 3.7, condition (10) is equivalent to the so-called Bakry-Émery curvature-dimension condition  $CD(R, m)$ .*

## 3 Coercive inequality for the Hodge-de Rham semi-group

The next proposition is a refined Bochner-Lichnerowicz-Weitzenböck formula for 1-forms.

**Proposition 3.1 (Refined Bochner-Lichnerowicz-Weitzenböck formula)** *For any smooth 1-forms  $\alpha, \eta$  and any  $a \in \mathbb{R}$ ,*

$$L \frac{|\eta|^2}{2} - \eta \cdot \vec{L} \eta + 2a\alpha \cdot d|\eta|^2 + 4a^2 |\alpha|^2 |\eta|^2 = |\nabla \eta + 2a\alpha \otimes \eta|^2 + \text{Ricci}(L)(\eta^*, \eta^*), \quad (12)$$

where  $|\nabla \eta + 2a\alpha \otimes \eta|$  has to be understood as the norm of the tensor  $\nabla_i \eta_j + 2a\alpha_i \eta_j$ .

When  $\Psi = 0$  let  $\vec{\Delta}$  be the usual Hodge-de Rham operator. The Weitzenböck (7) implies

$$\vec{L} \omega = \vec{\Delta} \omega - \nabla_{\nabla \Psi} \omega - \text{Hess}(\psi)(\omega^*, \cdot). \quad (13)$$

According to Lemma 3.2, equation (12) is equivalent to the equation

$$\Delta \frac{|\eta|^2}{2} - \eta \cdot \vec{\Delta} \eta + 2a\alpha \cdot d|\eta|^2 + 4a^2 |\alpha|^2 |\eta|^2 = |\nabla \eta + 2a\alpha \otimes \eta|^2 + \text{Ricci}_g(\eta^*, \eta^*),$$

which will be proved in Proposition 3.3. When  $a = 0$ , equation (12) is classic and is the starting point of our proof.

Next Lemma is classic in Riemannian geometry (see for instance [Lic58, p. 3]).

**Lemma 3.2** For any 1-forms  $\eta$  and any  $1 \leq k \leq n$ ,  $\left(d\frac{|\eta|^2}{2}\right)_k = \eta^i \nabla_k \eta_i$ .

**Proposition 3.3** For any smooth 1-forms  $\alpha, \eta$  and any  $a \in \mathbb{R}$ ,

$$\Delta \frac{|\eta|^2}{2} - \eta \cdot \overrightarrow{\Delta} \eta + 2a\alpha \cdot d|\eta|^2 + 4a^2|\alpha|^2|\eta|^2 = |\nabla \eta + 2a\alpha \otimes \eta|^2 + \text{Ricci}_g(\eta^*, \eta^*). \quad (14)$$

**Proof**

$\triangleleft$  The Bochner-Lichnerowicz-Weitzenböck formula (c.f. [Lic58, P. 3]) insures that for any smooth 1-forms  $\omega$ ,

$$\Delta \frac{|\omega|^2}{2} - \omega \cdot \overrightarrow{\Delta} \omega = |\nabla \omega|^2 + \text{Ricci}_g(\omega^*, \omega^*), \quad (15)$$

recall that  $|\nabla \omega|^2 = \nabla_i \omega_j g^{il} g^{jk} \nabla_l \omega_k$ , is the square of the norm of the 2-tensor  $\nabla_i \omega_j$ .

The idea is to change of variables into this formula. We would like to prove this result at some  $x_0 \in M$  which is supposed to be fixed. Let

$$\omega = (ag + 1)\eta + af\alpha,$$

where  $\eta$  and  $\alpha$  are 1-forms and  $f, g$  are actually smooth functions satisfying  $f(x_0) = g(x_0) = 0$ .

First we have

$$|\omega|^2 = (ag + 1)^2 |\eta|^2 + a^2 f^2 |\alpha|^2 + 2a(ag + 1)f\alpha \cdot \eta.$$

Since, for any smooth functions  $F$  and  $G$  we have  $\Delta(FG) = 2\Gamma(F, G) + F\Delta G + G\Delta F$ , where  $\Gamma$  is the carré du champ operator, a straight forward computation gives at the point  $x_0$ ,

$$\begin{aligned} \Delta |\omega|^2 &= 4a\Gamma(g, |\eta|^2) + 2a|\eta|^2 \Delta g + 2a^2 |\eta|^2 |\nabla g|^2 + \Delta |\eta|^2 + 2a^2 |\alpha|^2 |\nabla f|^2 \\ &\quad + 4a\Gamma(f, \eta \cdot \alpha) + 4a^2 (\eta \cdot \alpha) \Gamma(f, g) + 2a(\eta \cdot \alpha) \Delta f. \end{aligned}$$

Since  $\omega(x_0) = \eta(x_0)$ , we have in  $x_0$ ,

$$2\omega \cdot \overrightarrow{\Delta} \omega = 2a\eta \overrightarrow{\Delta} (g\eta) + 2\eta \overrightarrow{\Delta} \eta + 2a\eta \overrightarrow{\Delta} (f\alpha).$$

Thanks to Lemma 3.5, at the point  $x_0$  (recall that  $f(x_0) = g(x_0) = 0$ ),

$$2\omega \cdot \overrightarrow{\Delta} \omega = 2a|\eta|^2 \Delta g + 4a\eta \cdot \nabla_{\nabla g} \eta + 2\eta \overrightarrow{\Delta} \eta + 2a\eta \cdot \alpha \Delta f + 4a\eta \cdot \nabla_{\nabla f} \alpha.$$

Moreover, at the point  $x_0$ ,

$$\begin{aligned} |\nabla \omega|^2 &= |\nabla_i \omega_j|^2 = a^2 |\eta|^2 \Gamma(g) + |\nabla_i \eta_j|^2 + a^2 \Gamma(f) |\alpha|^2 + 2a\eta^j \nabla^i g \nabla_i \eta_j \\ &\quad + 2a \nabla^i f \alpha^j \nabla_i \eta_j + 2a^2 \eta_j \nabla_i g \nabla^i f \alpha^j, \end{aligned}$$

and  $\text{Ricci}_g(\omega^*, \omega^*) = \text{Ricci}_g(\eta^*, \eta^*)$ . At the point  $x_0$ , the identity (15) applied to  $\omega$ , becomes

$$\begin{aligned} \Delta \frac{|\eta|^2}{2} - \eta \overrightarrow{\Delta} \eta + 2a\Gamma(g, |\eta|^2) + a^2 |\eta|^2 \Gamma(g) + a^2 |\alpha|^2 \Gamma(f) + 2a\Gamma(f, \eta \cdot \alpha) \\ + 2a^2 (\eta \cdot \alpha) \Gamma(f, g) - 2a\eta \cdot \nabla_{\nabla g} \eta - 2a\eta \cdot \nabla_{\nabla f} \alpha &= a^2 |\eta|^2 \Gamma(g) + |\nabla_i \eta_j|^2 \\ + a^2 \Gamma(f) |\alpha|^2 + 2a\eta^j \nabla^i g \nabla_i \eta_j + 2a \nabla^i f \alpha^j \nabla_i \eta_j + 2a^2 \eta_j \nabla_i g \nabla^i f \alpha^j &+ \text{Ricci}_g(\eta^*, \eta^*) \end{aligned}$$



Let now assume that  $f$  and  $g$  satisfied moreover  $df(x_0) = \eta(x_0)$  and  $dg(x_0) = \alpha(x_0)$  (which is allays possible), we obtain

$$\begin{aligned} \Delta \frac{|\eta|^2}{2} - \eta \overrightarrow{\Delta} \eta + 2a \alpha \cdot \nabla |\eta|^2 + 4a^2 |\eta|^2 |\alpha|^2 \\ = 4a^2 |\eta|^2 |\alpha|^2 - 2a \eta \cdot \nabla (\eta \cdot \alpha) + 2a \eta \cdot \nabla_\alpha \eta - 2a \eta \cdot \nabla_\eta \alpha + |\nabla_i \eta_j|^2 \\ + 2a \eta^j \alpha^i \nabla_i \eta_j + 2a \eta^i \alpha^j \nabla_i \eta_j + Ricci_g(\eta^*, \eta^*). \end{aligned}$$

Then Lemma 3.4 implies that

$$\begin{aligned} 4a^2 |\eta|^2 |\alpha|^2 - 2a \eta \cdot d(\eta \cdot \alpha) + 2a \eta \cdot \nabla_\alpha \eta - 2a \eta \cdot \nabla_\eta \alpha + |\nabla_i \eta_j|^2 + 2a \eta^j \alpha^i \nabla_i \eta_j + 2a \eta^i \alpha^j \nabla_i \eta_j \\ = 4a^2 |\eta|^2 |\alpha|^2 + 4a \eta^j \alpha^i \nabla_i \eta_j + \nabla_i \eta_j \nabla^i \eta^j = |\nabla \eta + 2a \alpha \otimes \eta|^2, \end{aligned}$$

which implies the identity (14).  $\triangleright$

The next lemma is classic in the Riemannian context and we skip the proof.

**Lemma 3.4** *For any 1-forms  $\eta$  and  $\alpha$ ,  $d(\eta \cdot \alpha) = \nabla \eta \cdot \alpha + \nabla \alpha \cdot \eta$ .*

The next result can be seen as a kind of diffusion property as defined in [BGL13a]. The result is probably classic but I didn't find it in the literature.

**Lemma 3.5** *For any smooth functions  $f$  and 1-form  $\omega$ ,*

$$\overrightarrow{\Delta}(f\omega) = f \overrightarrow{\Delta}(\omega) + \omega \Delta f + 2 \nabla_{\nabla f} \omega. \quad (16)$$

*In other words, for any  $i$ ,  $(\overrightarrow{\Delta}(f\omega))_i = f(\overrightarrow{\Delta}(\omega))_i + \omega_i \Delta f + 2 \nabla^j f \nabla_j \omega_i$ .*

**Proof**

$\triangleleft$  The Weitzenböck formula (7) insures that

$$(\overrightarrow{\Delta}(f\omega))_i = \nabla^k \nabla_k (f\omega)_i - Ricci_g(f\omega^*, e_i) = \nabla^k \nabla_k (f\omega)_i - f Ricci_g(\omega^*, e_i).$$

Now, using in force the formula  $\nabla_k (f\omega)_i = \omega_i \partial_k f + f \nabla_k \omega_i$ , we get

$$\begin{aligned} \nabla^k \nabla_k (f\omega)_i &= g^{kl} \nabla_l \nabla_k (f\omega)_i \\ &= g^{kl} (\partial_l \nabla_k (f\omega)_i - \Gamma_{li}^p \nabla_k (f\omega)_p - \Gamma_{lk}^p \nabla_p (f\omega)_i) \\ &= g^{kl} (\partial_l \partial_k f - \Gamma_{lk}^p \partial_p f) \omega_i + f g^{kl} (\partial_l (\nabla_k \omega)_i - \Gamma_{lk}^p (\nabla_p \omega)_i - \Gamma_{li}^p (\nabla_k \omega)_p) \\ &\quad + g^{kl} (\partial_l \omega_i \partial_k f - \partial_k f \Gamma_{li}^p \omega_p + \partial_l f (\nabla_k \omega)_i) \\ &= g^{kl} (\nabla_k \nabla_l f) \omega_i + f g^{kl} \nabla_l \nabla_k \omega_i + 2 g^{kl} \partial_k f \nabla_l \omega_i \\ &= \omega_i \Delta f + f \nabla^j \nabla_j \omega_i + 2 \nabla^j f \nabla_j \omega_i = \omega_i \Delta f + f \nabla^j \nabla_j \omega_i + 2 \nabla_{\nabla f} \omega_i, \end{aligned}$$

which implies (16).  $\triangleright$

Finally we can state our main estimation:

**Corollary 3.6** *Assume that there exists  $R \in \mathbb{R}$  and  $m \geq n$  such that uniformly in  $M$ ,*

$$Ricci(L) \geq R - \frac{1}{m-n} \nabla \Psi \otimes \nabla \Psi. \quad (17)$$

*Then for any smooth 1-forms  $\eta$ ,  $\alpha$  and  $a \in \mathbb{R}$ , (with  $\delta_\Psi$  defined in (5)),*

$$L \frac{|\eta|^2}{2} - \eta \cdot \overrightarrow{L} \eta + 2a \alpha \cdot d|\eta|^2 + 4a^2 |\alpha|^2 |\eta|^2 \geq \frac{1}{m} (\delta_\Psi \eta + 2a \alpha \cdot \eta)^2 + R |\eta|^2. \quad (18)$$

**Proof**

◁ From (12) we only have to verify

$$|\nabla\eta + 2a\alpha \otimes \eta|^2 + \text{Ricci}(L)(\eta^*, \eta^*) \geq \frac{1}{m}(\delta_\Psi\eta + 2a\alpha \cdot \eta)^2 + R|\eta|^2.$$

Let  $x \in M$  and let assume that  $(e_i)_{1 \leq i \leq n}$  is an orthonormal basis of  $T_x M$ . Then the left hand side can be written

$$\begin{aligned} |\nabla\eta + 2a\alpha \otimes \eta|^2 + \text{Ricci}(L)(\eta^*, \eta^*) &= \sum_{i,j=1}^n (\nabla_i \eta_j + 2a\alpha_i \eta_j)^2 + \text{Ricci}(L)(\eta^*, \eta^*) \\ &\geq \sum_{i=1}^n (\nabla_i \eta_i + 2a\alpha_i \eta_i)^2 + \text{Ricci}(L)(\eta^*, \eta^*) \geq \frac{1}{n} \left( \sum_{i=1}^n \nabla_i \eta_i + 2a\alpha \cdot \eta \right)^2 + \text{Ricci}(L)(\eta^*, \eta^*), \end{aligned}$$

from the Cauchy-Schwartz inequality. So it remains to prove that for any  $t \in \mathbb{R}$

$$\frac{1}{n}(t + 2a\alpha \cdot \eta)^2 + \text{Ricci}(L)(\eta^*, \eta^*) \geq \frac{1}{m}(t - d\psi \cdot \eta + 2a\alpha \cdot \eta)^2 + R|\eta|^2,$$

where  $t = \delta\eta = \sum_{i=1}^n \nabla_i \eta_i$  (recall that  $(e_i)$  is an orthonormal basis). The formula is valid since  $m \geq n$  and from (17) the discriminant of this two order polynomial function in variable  $t$  is non-positive (it doesn't depend on the parameter  $a$ ). ▷

**Remark 3.7 (Link with the curvature-dimension condition)** *The so-called Bakry-Émery curvature-dimension condition  $CD(R, m)$  for an operator  $L$  is satisfied when for every smooth function  $f$ ,*

$$\Gamma_2(f) \geq R\Gamma(f) + \frac{1}{m}(Lf)^2,$$

where

$$\Gamma_2(f) = \Gamma_2(f, f) = \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf)).$$

*The same procedure in the case of closed 1-forms has been stated in [Bak94] (see also [ABC<sup>+</sup>00, Chap. 5]) in the context of  $\Gamma_2$ -calculus. It is proved that if  $\eta = df$  and  $\alpha = dg$ , then under the curvature-dimension inequality  $CD(R, m)$ , and any  $a \in \mathbb{R}$ ,*

$$\Gamma_2(f) + 2a\Gamma(f, \Gamma(g)) + 4a^2\Gamma(f)\Gamma(g) \geq \frac{1}{n}(Lf + 2a\Gamma(f, g))^2 + \rho\Gamma(f), \quad (19)$$

*which is (18) for closed 1-form. Moreover, inequality (19) for every function  $f$  (with  $a = 0$ ) is equivalent to  $CD(R, m)$ . Since inequality (18) is a generalization of (19), it is also equivalent to (17) and then to  $CD(R, m)$ .*

We can now give the main estimation of our semigroups:

**Theorem 3.8 (Coercive estimation)** *Assume that there exists  $R \in \mathbb{R}$ ,  $m \geq n$  such that,*

$$\text{Ricci}(L) \geq R - \frac{1}{m-n}\nabla\Psi \otimes \nabla\Psi. \quad (20)$$

*Then for any smooth 1-forms  $\omega$  and smooth functions  $g > 0$ , for any  $t \geq 0$ ,*

$$\frac{|R_t\omega|^2}{P_tg} \leq e^{-2Rt} P_t \left( \frac{|\omega|^2}{g} \right) - \frac{2}{m} \int_0^t \frac{e^{-2Ru}}{P_tg} [P_t\delta_\Psi\omega - P_u(d(\log P_{t-u}g) \cdot R_{t-u}\omega)]^2 du. \quad (21)$$

**Proof**

◁ For regularity problem, one can assume that there exists  $\varepsilon > 0$  such that  $g = f + \varepsilon$  with  $f > 0$  and then  $\varepsilon \rightarrow 0$  in (21). Let  $\omega$  be a smooth 1-form and  $t \geq 0$ . For any  $s \in [0, t]$ , we define

$$\Lambda(s) = P_s \left( \frac{|R_{t-s}\omega|^2}{P_{t-s}g} \right).$$

For any  $s \in [0, 1]$ ,

$$\Lambda'(s) = P_s \left( L \left( \frac{|R_{t-s}\omega|^2}{P_{t-s}g} \right) - 2 \frac{R_{t-s}\omega \cdot \vec{L} R_{t-s}\omega}{P_{t-s}g} + LP_{t-s}g \frac{|R_{t-s}\omega|^2}{(P_{t-s}g)^2} \right).$$

Since for any smooth functions  $F, G$ ,  $L(FG) = 2\Gamma(F, G) + FLG + GLF$ , the identity becomes, with  $\eta = R_{t-s}\omega$  and  $G = P_{t-s}g$ ,

$$\begin{aligned} \Lambda'(s) &= P_s \left( -\frac{2}{G^2} \Gamma(|\eta|^2, G) - \frac{2}{G} \eta \cdot \vec{L} \eta + \frac{2}{G^3} |\eta|^2 \Gamma(G) + \frac{1}{G} L|\eta|^2 \right) \\ &= P_s \left[ \frac{2}{G} \left( L \frac{|\eta|^2}{2} - \eta \cdot \vec{L} \eta - \Gamma(|\eta|^2, \log G) + |\eta|^2 \Gamma(\log G) \right) \right]. \end{aligned}$$

Corollary 3.6 applied to  $\alpha = d \log G$  and  $a = -1/2$ , implies

$$\Lambda'(s) \geq \frac{2}{m} P_s \left( \frac{1}{G} (\delta_\Psi \eta - d(\log G) \cdot \eta)^2 \right) + 2RP_s \left( \frac{|\eta|^2}{G} \right).$$

Thanks to the Cauchy-Schwarz inequality,

$$P_s \left( \frac{1}{G} (\delta_\Psi \eta - d(\log G) \cdot \eta)^2 \right) \geq \frac{1}{P_s G} \left[ P_s (\delta_\Psi \eta - d(\log G) \cdot \eta) \right]^2.$$

Since  $P_s G = P_s P_{t-s} g = P_t g$  and from (8),  $P_s \delta_\Psi R_{t-s} \omega = P_t \delta_\Psi \omega$ , the inequality becomes

$$(\Lambda(s) e^{-2Rs})' \geq \frac{2}{m} \frac{e^{-2Rs}}{P_t g} \left[ P_t \delta_\Psi \omega - P_s (d(\log P_{t-s} g) \cdot R_{t-s} \omega) \right]^2.$$

The integration over  $s \in [0, t]$  of the previous inequality implies (21). ▷

## 4 Proof of Theorem 2.1

◁ **Regularity assumption:** Let  $f_\varepsilon = (P_\varepsilon f + \varepsilon)/(1 + \varepsilon)$  and  $g_\varepsilon = (P_\varepsilon g + \varepsilon)/(1 + \varepsilon)$ , for  $\varepsilon > 0$ . The probability measure  $f_\varepsilon \mu$  (resp.  $g_\varepsilon \mu$ ) converges weakly to  $f \mu$  (resp.  $g \mu$ ), when  $\varepsilon \rightarrow 0$  and since  $M$  is compact  $W_2^2(f_\varepsilon \mu, g_\varepsilon \mu)$  converges to  $W_2(f \mu, g \mu)$ . The same is also true for  $W_2^2(P_t f_\varepsilon \mu, P_t g_\varepsilon \mu)$ . Thus, one can assume that  $f$  and  $g$  are two smooth functions satisfying  $f, g \geq \varepsilon$  for some  $\varepsilon > 0$ .

**Contraction inequality:** Let  $f, g \geq \varepsilon$ , be two smooth functions and let  $(\rho_s, \omega_s)_{s \in [0, 1]}$  be a smooth couple satisfying

$$\begin{cases} \partial_s \rho_s + \delta_\Psi \omega_s = 0 \\ \rho_0 = f, \quad \rho_1 = g. \end{cases} \quad (22)$$

For any  $t \geq 0$ , from the commutation property (8), the couple  $(P_t \rho_s, R_t \omega_s)_{s \in [0, 1]}$  satisfies

$$\begin{cases} \partial_s P_t \rho_s + \delta_\Psi R_t \omega_s = 0 \\ P_t \rho_0 = P_t f, \quad P_t \rho_1 = P_t g. \end{cases} \quad (23)$$

On can apply Theorem 3.8 to get

$$\int_0^1 \int_M \frac{|R_t \omega_s|^2}{P_t \rho_s} d\mu ds \leq e^{-2Rt} \int_0^1 \int_M \frac{|\omega_s|^2}{\rho_s} d\mu ds - \frac{2}{m} \int_0^1 \int_0^t e^{-2Ru} \int_M \frac{[P_t \delta_\Psi \omega_s - P_u(d(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s)]^2}{P_t \rho_s} d\mu du ds. \quad (24)$$

Cauchy-Schwarz inequality implies

$$\begin{aligned} & \int_M \frac{[P_t \delta_\Psi \omega_s - P_u(d(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s)]^2}{P_t \rho_s} d\mu \\ & \geq \frac{1}{P_t \rho_s} \left[ \int_M (P_t \delta_\Psi \omega_s - P_u(d(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s)) d\mu \right]^2 = \left[ \int_M d(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s d\mu \right]^2, \end{aligned}$$

since  $\int_M P_t \delta_\Psi \omega_s d\mu = \int_M \delta_\Psi \omega_s d\mu = 0$ . Integrating over  $s \in [0, 1]$ , thanks again to the Cauchy-Schwartz inequality,

$$\begin{aligned} \int_0^1 \left[ \int_M \nabla(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s d\mu \right]^2 ds & \geq \left[ \int_0^1 \int_M d(\log P_{t-u} \rho_s) \cdot R_{t-u} \omega_s d\mu ds \right]^2 \\ & = [\text{Ent}_\mu(P_{t-u} f) - \text{Ent}_\mu(P_{t-u} g)]^2, \end{aligned}$$

since coming from (23) we have (since  $M$  is compact, the next integration by parts is valid)

$$\begin{aligned} \frac{d}{ds} \text{Ent}_\mu(P_{t-u} \rho_s) & = \frac{d}{ds} \int_M P_{t-u} \rho_s \log P_{t-u} \rho_s d\mu = \int_M \partial_s P_{t-u} \rho_s \log P_{t-u} \rho_s d\mu \\ & = - \int_M \delta_\Psi R_{t-u} \omega_s \log P_{t-u} \rho_s d\mu = \int_M R_{t-u} \omega_s \cdot d(\log P_{t-u} \rho_s) d\mu. \end{aligned}$$

The inequality (24) becomes

$$\int_0^1 \int_M \frac{|R_t \omega_s|^2}{P_t \rho_s} d\mu ds \leq e^{-2Rt} \int_0^1 \int_M \frac{|\omega_s|^2}{\rho_s} d\mu ds - \frac{2}{m} \int_0^t e^{-2Ru} [\text{Ent}_\mu(P_{t-u} f) - \text{Ent}_\mu(P_{t-u} g)]^2 du$$

Thanks to the Brenier-Benamou formulation (9) with respect  $P_t f \mu$  and  $P_t g \mu$ ,

$$W_2^2(P_t f \mu, P_t g \mu) \leq e^{-2Rt} \int_0^1 \int_M \frac{|\omega_s|^2}{\rho_s} d\mu ds - \frac{2}{m} \int_0^t e^{-2Ru} [\text{Ent}_\mu(P_{t-u} f) - \text{Ent}_\mu(P_{t-u} g)]^2 du.$$

Taking now the infimum over all couples  $(\rho_s, \omega_s)_{s \in [0,1]}$  satisfying (22),

$$W_2^2(P_t f \mu, P_t g \mu) \leq e^{-2Rt} W_2^2(f \mu, g \mu) - \frac{2}{m} \int_0^t e^{-2Ru} [\text{Ent}_\mu(P_{t-u} f) - \text{Ent}_\mu(P_{t-u} g)]^2 du,$$

which is the inequality desired changing  $t - u$  by  $u$ .  $\triangleright$

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