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TWO-SCALE NUMERICAL SIMULATION OF SAND TRANSPORT PROBLEMS

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ABSTRACT. In this paper we consider the model built in [3] for short term dynamics of dunes in tidal area. We construct a Two-Scale Numerical Method based on the fact that the solution of the equation which has oscillations Two-Scale converges to the solution of a well-posed problem. This numerical method uses on Fourier series.

1. Introduction. This paper deals with numerical simulations of sand transport problems. Its goal is to build a Two-Scale Numerical Method to simulate dynamics of dunes in tidal area.

This paper enters a work program concerning the development of Two-Scale Numerical Methods to solve PDEs with oscillatory singular perturbations linked with physical phenomena. In Ailliot, Frénot and Monbet [2], such a method is used to manage the tide oscillation for long term drift forecast of objects in coastal ocean waters. Frénot, Mouton and Sonnendrücker [5] made simulations of the 1D Euler equation using a Two-Scale Numerical Method. In Frénot, Salvarani and Sonnendrücker [6], such a method is used to simulate a charged particle beam in a periodic focusing channel. Mouton [9, 10] developed a Two-Scale Semi Lagrangian Method for beam and plasma applications.

We consider the following model, valid for short-term dynamics of dunes, built and studied in [3]:

$$\begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon} \nabla \cdot (\mathcal{A}^\epsilon \nabla z^\epsilon) = \frac{1}{\epsilon} \nabla \cdot \mathcal{C}^\epsilon, \\ z^\epsilon|_{t=0} = z_0, \end{cases} \quad (1.1)$$

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where $z^\epsilon = z^\epsilon(t, x)$ is the dimensionless seabed altitude. For a given T , $t \in (0, T)$ stands for the dimensionless time and $x \in \mathbb{T}^2$, \mathbb{T}^2 being the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, stands for the dimensionless position and \mathcal{A}^ϵ , \mathcal{C}^ϵ are given by

$$\mathcal{A}^\epsilon(t, x) = \tilde{\mathcal{A}}^\epsilon(t, x) + \epsilon \tilde{\mathcal{A}}_1^\epsilon(t, x), \quad (1.2)$$

and

$$\mathcal{C}^\epsilon(t, x) = \tilde{\mathcal{C}}^\epsilon(t, x) + \epsilon \tilde{\mathcal{C}}_1^\epsilon(t, x), \quad (1.3)$$

where, for three positive constants a , b and c ,

$$\tilde{\mathcal{A}}^\epsilon(t, x) = \tilde{\mathcal{A}}(t, \frac{t}{\epsilon}, x) = a g_a(|\mathcal{U}(t, \frac{t}{\epsilon}, x)|), \quad (1.4)$$

$$\tilde{\mathcal{C}}^\epsilon(t, x) = \tilde{\mathcal{C}}(t, \frac{t}{\epsilon}, x) = c g_c(|\mathcal{U}(t, \frac{t}{\epsilon}, x)|) \frac{\mathcal{U}(t, \frac{t}{\epsilon}, x)}{|\mathcal{U}(t, \frac{t}{\epsilon}, x)|}, \quad (1.5)$$

and

$$\tilde{\mathcal{A}}_1^\epsilon(t, x) = \tilde{\mathcal{A}}_1(t, \frac{t}{\epsilon}, x), \quad \tilde{\mathcal{C}}_1^\epsilon(t, x) = \tilde{\mathcal{C}}_1(t, \frac{t}{\epsilon}, x), \quad (1.6)$$

with

$$\tilde{\mathcal{A}}_1(t, \theta, x) = -ab\mathcal{M}(t, \theta, x) g_a(|\mathcal{U}(t, \theta, x)|) \text{ and } \tilde{\mathcal{C}}_1(t, \theta, x) = -cb\mathcal{M}(t, \theta, x) g_c(|\mathcal{U}(t, \theta, x)|) \frac{\mathcal{U}(t, \theta, x)}{|\mathcal{U}(t, \theta, x)|}. \quad (1.7)$$

\mathcal{U} and \mathcal{M} are the dimensionless water velocity and height.

The small parameter ϵ involved in the model is the ratio between the main tide period $\frac{1}{\omega} = 13$ hours and an observation time which is about three months i.e. $\epsilon = \frac{1}{t\omega} = \frac{1}{200}$.

The following hypotheses on g_a , g_c , \mathcal{U} and \mathcal{M} given in (1.8) and (1.9) are technical assumptions and are needed to prove Theorem 1.1. Functions g_a and g_c are regular functions on \mathbb{R}^+ and satisfy

$$\left\{ \begin{array}{l} g_a \geq g_c \geq 0, \quad g_c(0) = g'_c(0) = 0, \\ \exists d \geq 0, \sup_{u \in \mathbb{R}^+} |g_a(u)| + \sup_{u \in \mathbb{R}^+} |g'_a(u)| \leq d, \\ \sup_{u \in \mathbb{R}^+} |g_c(u)| + \sup_{u \in \mathbb{R}^+} |g'_c(u)| \leq d, \\ \exists U_{thr} \geq 0, \exists G_{thr} > 0, \text{ such that } u \geq U_{thr} \implies g_a(u) \geq G_{thr}. \end{array} \right. \quad (1.8)$$

Functions \mathcal{U} and \mathcal{M} are regular and satisfy:

$$\left\{ \begin{array}{l} \theta \mapsto (\mathcal{U}, \mathcal{M}) \text{ is periodic of period } 1, \\ |\mathcal{U}|, \left| \frac{\partial \mathcal{U}}{\partial t} \right|, \left| \frac{\partial \mathcal{U}}{\partial \theta} \right|, |\nabla \mathcal{U}|, \\ |\mathcal{M}|, \left| \frac{\partial \mathcal{M}}{\partial t} \right|, \left| \frac{\partial \mathcal{M}}{\partial \theta} \right|, |\nabla \mathcal{M}| \text{ are bounded by } d, \\ \forall (t, \theta, x) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^2, |\mathcal{U}(t, \theta, x)| \leq U_{thr} \implies \\ \left(\frac{\partial \mathcal{U}}{\partial t}(t, \theta, x) = 0, \quad \nabla \mathcal{U}(t, \theta, x) = 0, \right. \\ \left. \frac{\partial \mathcal{M}}{\partial t}(t, \theta, x) = 0, \quad \text{and } \nabla \mathcal{M}(t, \theta, x) = 0 \right), \\ \exists \theta_\alpha < \theta_\omega \in [0, 1] \text{ such that } \forall \theta \in [\theta_\alpha, \theta_\omega] \implies |\mathcal{U}(t, \theta, x)| \geq U_{thr}. \end{array} \right. \quad (1.9)$$

To develop the Two-Scale Numerical Method, we use that in [3] we proved that under assumptions (1.8) and (1.9) the solution z^ϵ of (1.1) exists, is unique and moreover asymptotically behaves, as $\epsilon \rightarrow 0$, the way given by the following theorem.

Theorem 1.1. *Under assumptions (1.8) and (1.9), for any T , not depending on ϵ , the sequence (z^ϵ) of solutions to (1.1), with coefficients given by (1.2) coupled with (1.4) and (1.3), (1.5) and (1.6), Two-Scale converges to the profile $Z \in L^\infty([0, T], L^\infty_{\#}(\mathbb{R}, L^2(\mathbb{T}^2)))$ solution to*

$$\frac{\partial Z}{\partial \theta} - \nabla \cdot (\tilde{\mathcal{A}} \nabla Z) = \nabla \cdot \tilde{\mathcal{C}}, \quad (1.10)$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{C}}$ are given by

$$\tilde{\mathcal{A}}(t, \theta, x) = a g_a(|\mathcal{U}(t, \theta, x)|) \text{ and } \tilde{\mathcal{C}}(t, \theta, x) = c g_c(|\mathcal{U}(t, \theta, x)|) \frac{\mathcal{U}(t, \theta, x)}{|\mathcal{U}(t, \theta, x)|}. \quad (1.11)$$

Futhermore, if the supplementary assumption

$$U_{thr} = 0, \quad (1.12)$$

is done, we have

$$\tilde{\mathcal{A}}(t, \theta, x) \geq \tilde{G}_{thr} \text{ for any } t, \theta, x \in [0, T] \times \mathbb{R} \times \mathbb{T}^2, \quad (1.13)$$

and, defining $Z^\epsilon = Z^\epsilon(t, x) = Z(t, \frac{t}{\epsilon}, x)$, the following estimate holds for $z^\epsilon - Z^\epsilon$

$$\left\| \frac{z^\epsilon - Z^\epsilon}{\epsilon} \right\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \leq \alpha, \quad (1.14)$$

where α is a constant not depending on ϵ .

Because of assumptions (1.8) and (1.9),

$$\tilde{\mathcal{A}}, \tilde{\mathcal{C}}, \tilde{\mathcal{A}}_1, \tilde{\mathcal{C}}_1, \tilde{\mathcal{A}}^\epsilon, \tilde{\mathcal{A}}_1^\epsilon, \tilde{\mathcal{C}}^\epsilon, \text{ and } \tilde{\mathcal{C}}_1^\epsilon \text{ are regular and bounded.} \quad (1.15)$$

2. Two-Scale Numerical Method Building. In this section, we develop the Two-Scale Numerical Method in order to approach the solution z^ϵ of (1.1). The idea is to get a good approximation of $z^\epsilon(t, x)$ seeing Theorem 1.1 content as $z^\epsilon(t, x) \sim Z(t, \frac{t}{\epsilon}, x)$.

The strategy is to consider a Fourier expansion of Z solution to (1.10). In this equation, t is only a parameter.

The Fourier expansion of Z is given as follows:

$$Z(t, \theta, x) = \sum_{l, m, n} Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}, \quad (2.1)$$

where $Z_{l, m, n}(t)$, $l = 0, 1, 2, \dots$, $m = 0, 1, 2, \dots$, $n = 0, 1, 2, \dots$, are the unknown complex coefficients of the Fourier expansion of Z . Using (2.1), the Fourier expansion of $\frac{\partial Z}{\partial \theta}$ is given by

$$\frac{\partial Z}{\partial \theta}(t, \theta, x) = \sum_{l, m, n} 2i\pi l Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}. \quad (2.2)$$

To obtain the system satisfied by the Fourier expansion (2.1) of Z , it is necessary to compute the Fourier expansions of $\nabla \cdot (\tilde{\mathcal{A}} \nabla Z)$ and $\nabla \cdot \tilde{\mathcal{C}}$. As $\nabla \cdot (\tilde{\mathcal{A}} \nabla Z) = \nabla \tilde{\mathcal{A}} \cdot \nabla Z + \tilde{\mathcal{A}} \cdot \Delta Z$, let

$$\sum_{l, m, n} \tilde{\mathcal{A}}_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}, \quad (2.3)$$

and

$$\sum_{l, m, n} \tilde{\mathcal{A}}_{l, m, n}^{grad}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}, \quad (2.4)$$

be respectively the Fourier expansions of $\tilde{\mathcal{A}}$ and $\nabla \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}_{l, m, n}^{grad}(t) = 2i\pi \tilde{\mathcal{A}}_{l, m, n} \binom{m}{n}$ and then the Fourier expansions of ∇Z and ΔZ are respectively given by

$$\sum_{l, m, n} 2i\pi \binom{m}{n} Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}, \quad (2.5)$$

and

$$- \sum_{l, m, n} 4\pi^2 (m^2 + n^2) Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}. \quad (2.6)$$

In the same way the Fourier expansion of $\nabla \cdot \tilde{\mathcal{C}}$ is given by

$$\sum_{l, m, n} \tilde{\mathcal{C}}_{l, m, n} e^{2i\pi(l\theta + mx_1 + nx_2)}. \quad (2.7)$$

Using (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7), equation (1.10) becomes

$$\begin{aligned} & \sum_{l, m, n} 2i\pi l Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)} \\ & - \left(\sum_{l, m, n} \tilde{\mathcal{A}}_{l, m, n}^{grad}(t) e^{2i\pi(l\theta + mx_1 + nx_2)} \right) \cdot \left(\sum_{l, m, n} 2i\pi \binom{m}{n} Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)} \right) \\ & + \left(\sum_{l, m, n} \tilde{\mathcal{A}}_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)} \right) \left(\sum_{l, m, n} 4\pi^2 (m^2 + n^2) Z_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)} \right) = \\ & \sum_{l, m, n} \tilde{\mathcal{C}}_{l, m, n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}, \end{aligned} \quad (2.8)$$

which gives after identification, the following algebraic system for $(Z_{l,m,n})$:

$$\begin{aligned} & 2i\pi l Z_{l,m,n}(t) - \sum_{i,j,k} 2i\pi \tilde{\mathcal{A}}_{i,j,k}^{grad}(t) \cdot \binom{m-j}{n-k} Z_{l-i,m-j,n-k}(t) \\ & + 4\pi^2 \sum_{i,j,k} \tilde{\mathcal{A}}_{i,j,k}(t) ((m-j)^2 + (n-k)^2) Z_{l-i,m-j,n-k}(t) = \tilde{\mathcal{C}}_{l,m,n}(t). \end{aligned} \quad (2.9)$$

In formula (2.1), the integers m, n and l vary from $-\infty$ to $+\infty$. But in practice, we will consider the truncated Fourier series of order $P \in \mathbb{N}$ defined by

$$Z_P(t, \theta, x) = \sum_{0 \leq l \leq P, 0 \leq m \leq P, 0 \leq n \leq P} Z_{l,m,n}(t) e^{2i\pi(l\theta + mx_1 + nx_2)}. \quad (2.10)$$

Using (2.10), formula (2.9) becomes:

$$\begin{aligned} & 2i\pi l Z_{l,m,n}(t) - \sum_{0 \leq i \leq P, 1 \leq j \leq P, 0 \leq k \leq P} 2i\pi \tilde{\mathcal{A}}_{i,j,k}^{grad}(t) \cdot \binom{m-j}{n-k} Z_{l-i,m-j,n-k}(t) \\ & + 4\pi^2 \sum_{0 \leq i \leq P, 0 \leq j \leq P, 0 \leq k \leq P} \tilde{\mathcal{A}}_{i,j,k}(t) ((m-j)^2 + (n-k)^2) Z_{l-i,m-j,n-k}(t) = \tilde{\mathcal{C}}_{l,m,n}(t). \end{aligned} \quad (2.11)$$

3. Convergence result.

Proof. of Theorem 1.1. For self-containedness, we recall the proof of Theorem 1.1. Firstly, we obtain an estimate leading to that z^ϵ is bounded in $L^\infty((0, T); L^2(\mathbb{T}^2))$. Secondly, defining test function $\psi^\epsilon(t, x) = \psi(t, \frac{t}{\epsilon}, x)$ for any $\psi(t, \theta, x)$, regular with a compact support over $[0, T) \times \mathbb{T}^2$ and 1-periodic in θ , multiplying (1.1) by ψ^ϵ and integrating over $[0, T) \times \mathbb{T}^2$ gives

$$\int_{\mathbb{T}^2} \int_0^T \frac{\partial z^\epsilon}{\partial t} \psi^\epsilon dt dx - \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \nabla \cdot (\mathcal{A}^\epsilon \nabla z^\epsilon) \psi^\epsilon dt dx = \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \nabla \cdot \mathcal{C}^\epsilon \psi^\epsilon dt dx. \quad (3.1)$$

Then integrating by parts in the first integral over $[0, T)$ and using the Green formula in \mathbb{T}^2 in the second integral we have

$$\begin{aligned} & - \int_{\mathbb{T}^2} z_0(x) \psi(0, 0, x) dx - \int_{\mathbb{T}^2} \int_0^T \frac{\partial \psi^\epsilon}{\partial t} z^\epsilon dt dx \\ & + \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \mathcal{A}^\epsilon \nabla z^\epsilon \nabla \psi^\epsilon dt dx = \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \nabla \cdot \mathcal{C}^\epsilon \psi^\epsilon dt dx. \end{aligned} \quad (3.2)$$

Again using the Green formula in the third integral we obtain

$$\begin{aligned} & - \int_{\mathbb{T}^2} z_0(x) \psi(0, 0, x) dx - \int_{\mathbb{T}^2} \int_0^T \frac{\partial \psi^\epsilon}{\partial t} z^\epsilon dt dx \\ & - \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T z^\epsilon \nabla \cdot (\mathcal{A}^\epsilon \nabla \psi^\epsilon) dt dx = \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \nabla \cdot \mathcal{C}^\epsilon \psi^\epsilon dt dx. \end{aligned} \quad (3.3)$$

But

$$\frac{\partial \psi^\epsilon}{\partial t} = \left(\frac{\partial \psi}{\partial t} \right)^\epsilon + \frac{1}{\epsilon} \left(\frac{\partial \psi}{\partial \theta} \right)^\epsilon, \quad (3.4)$$

where

$$\left(\frac{\partial \psi}{\partial t} \right)^\epsilon(t, x) = \frac{\partial \psi}{\partial t}(t, \frac{t}{\epsilon}, x) \quad \text{and} \quad \left(\frac{\partial \psi}{\partial \theta} \right)^\epsilon(t, x) = \frac{\partial \psi}{\partial \theta}(t, \frac{t}{\epsilon}, x), \quad (3.5)$$

then we have

$$\begin{aligned} & \int_{\mathbb{T}^2} \int_0^T z^\epsilon \left(\left(\frac{\partial \psi}{\partial t} \right)^\epsilon + \frac{1}{\epsilon} \left(\frac{\partial \psi}{\partial \theta} \right)^\epsilon + \frac{1}{\epsilon} \nabla \cdot (\mathcal{A}^\epsilon \nabla \psi^\epsilon) \right) dx dt \\ & + \frac{1}{\epsilon} \int_{\mathbb{T}^2} \int_0^T \nabla \cdot \mathcal{C}^\epsilon \psi^\epsilon dt dx = - \int_{\mathbb{T}^2} z_0(x) \psi(0, 0, x) dx. \end{aligned} \quad (3.6)$$

Using the Two-Scale convergence due to Nguetseng [11] and Allaire [1] (see also Frénot Raviart and Sonnendrücker [7]), since z^ϵ is bounded in $L^\infty([0, T), L^2(\mathbb{T}^2))$, there exists a profile $Z(t, \theta, x)$, periodic of period 1 with respect to θ , such that for all $\psi(t, \theta, x)$, regular with a compact support with respect to (t, x) and 1-periodic with respect to θ , we have

$$\int_{\mathbb{T}^2} \int_0^T z^\epsilon \psi^\epsilon dt dx \longrightarrow \int_{\mathbb{T}^2} \int_0^T \int_0^1 Z \psi d\theta dt dx, \quad \text{as } \epsilon \text{ tends to zero,} \quad (3.7)$$

for a subsequence extracted from (z^ϵ) .

Multiplying (3.6) by ϵ , passing to the limit as $\epsilon \rightarrow 0$ and using (3.7) we have

$$\int_{\mathbb{T}^2} \int_0^T \int_0^1 Z \frac{\partial \psi}{\partial \theta} d\theta dt dx + \lim_{\epsilon \rightarrow 0} \int_{\mathbb{T}^2} \int_0^T z^\epsilon \nabla \cdot (\mathcal{A}^\epsilon \nabla \psi^\epsilon) dt dx = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{T}^2} \int_0^T C^\epsilon \cdot \nabla \psi^\epsilon dt dx, \quad (3.8)$$

for an extracted subsequence. As \mathcal{A}^ϵ and C^ϵ are bounded and ψ^ϵ is a regular function, $\mathcal{A}^\epsilon \nabla \psi^\epsilon$ and $\nabla \psi^\epsilon$ can be considered as test functions. Using (3.7) we have

$$\int_{\mathbb{T}^2} \int_0^T z^\epsilon \nabla \cdot (\mathcal{A}^\epsilon \nabla \psi^\epsilon) dt dx \rightarrow \int_{\mathbb{T}^2} \int_0^T \int_0^1 Z \nabla \cdot (\tilde{\mathcal{A}} \nabla \psi) d\theta dt dx, \quad (3.9)$$

and

$$\int_{\mathbb{T}^2} \int_0^T C^\epsilon \cdot \nabla \psi^\epsilon dt dx \text{ Two-Scale converges to } \int_{\mathbb{T}^2} \int_0^T \int_0^1 \tilde{C} \cdot \nabla \psi d\theta dt dx. \quad (3.10)$$

Passing to the limit as $\epsilon \rightarrow 0$ we obtain from (3.8) a weak formulation of the equation (1.10) satisfied by Z .

Using (1.2) and (1.3) equation (1.1) becomes

$$\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon} \nabla \cdot (\tilde{\mathcal{A}}^\epsilon \nabla z^\epsilon) = \frac{1}{\epsilon} \nabla \cdot \tilde{C}^\epsilon + \nabla \cdot (\tilde{\mathcal{A}}_1^\epsilon \nabla z^\epsilon) + \nabla \cdot \tilde{C}_1^\epsilon. \quad (3.11)$$

For Z^ϵ , we have

$$\frac{\partial Z^\epsilon}{\partial t} = \left(\frac{\partial Z}{\partial t} \right)^\epsilon + \frac{1}{\epsilon} \left(\frac{\partial Z}{\partial \theta} \right)^\epsilon, \quad (3.12)$$

where

$$\left(\frac{\partial Z}{\partial t} \right)^\epsilon (t, x) = \frac{\partial Z}{\partial t} \left(t, \frac{t}{\epsilon}, x \right) \text{ and } \left(\frac{\partial Z}{\partial \theta} \right)^\epsilon (t, x) = \frac{\partial Z}{\partial \theta} \left(t, \frac{t}{\epsilon}, x \right). \quad (3.13)$$

Using (1.10), Z^ϵ is solution to

$$\frac{\partial Z^\epsilon}{\partial t} - \frac{1}{\epsilon} \nabla \cdot (\tilde{\mathcal{A}}^\epsilon \nabla Z^\epsilon) = \frac{1}{\epsilon} \nabla \cdot \tilde{C}^\epsilon + \left(\frac{\partial Z}{\partial t} \right)^\epsilon. \quad (3.14)$$

Formulas (3.11) and (3.14) give

$$\frac{\partial (z^\epsilon - Z^\epsilon)}{\partial t} - \frac{1}{\epsilon} \nabla \cdot (\tilde{\mathcal{A}}^\epsilon \nabla (z^\epsilon - Z^\epsilon)) = \nabla \cdot \tilde{C}_1^\epsilon + \left(\frac{\partial Z}{\partial t} \right)^\epsilon + \nabla \cdot (\tilde{\mathcal{A}}_1^\epsilon \nabla z^\epsilon). \quad (3.15)$$

Multiplying equation (3.15) by $\frac{1}{\epsilon}$ and using the fact that $z^\epsilon = z^\epsilon - Z^\epsilon + Z^\epsilon$ in the right hand side of equation (3.15), $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is solution to:

$$\frac{\partial \left(\frac{z^\epsilon - Z^\epsilon}{\epsilon} \right)}{\partial t} - \frac{1}{\epsilon} \nabla \cdot \left((\tilde{\mathcal{A}}^\epsilon + \epsilon \tilde{\mathcal{A}}_1^\epsilon) \nabla \left(\frac{z^\epsilon - Z^\epsilon}{\epsilon} \right) \right) = \frac{1}{\epsilon} \left(\nabla \cdot \tilde{C}_1^\epsilon + \left(\frac{\partial Z}{\partial t} \right)^\epsilon + \nabla \cdot (\tilde{\mathcal{A}}_1^\epsilon \nabla z^\epsilon) \right). \quad (3.16)$$

Our aim here is to prove that $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is bounded by a constant α not depending on ϵ . For this let us use that $\tilde{\mathcal{A}}^\epsilon$, $\tilde{\mathcal{A}}_1^\epsilon$, \tilde{C}^ϵ and \tilde{C}_1^ϵ are regular and bounded coefficients (see (1.15)) and that $\tilde{\mathcal{A}}^\epsilon \geq G_{thr}$ (see (1.13)). Hence, $\nabla \cdot \tilde{C}_1^\epsilon$ is bounded, $\nabla \cdot (\tilde{\mathcal{A}}_1^\epsilon \nabla z^\epsilon)$ is also bounded. Since Z^ϵ is solution to (3.14), $\frac{\partial Z}{\partial t}$ satisfies the following equation

$$\frac{\partial \left(\frac{\partial Z}{\partial t} \right)}{\partial \theta} - \nabla \cdot \left(\tilde{\mathcal{A}} \nabla \frac{\partial Z}{\partial t} \right) = \frac{\partial \nabla \cdot \tilde{C}}{\partial t} + \nabla \cdot \left(\frac{\partial \tilde{\mathcal{A}}}{\partial t} \nabla Z \right). \quad (3.17)$$

Equation (3.17) is linear with regular and bounded coefficients. Using a result of Ladyzenskaja, Solonnikov and Ural'ceva [8], $\frac{\partial Z}{\partial t}$ is regular and bounded and so the coefficients of equations (3.16) are regular and bounded. Then, using the same arguments as in the proof of Theorem 1.1 in [3] we obtain that $\left(\frac{z^\epsilon - Z^\epsilon}{\epsilon} \right)$ is bounded.

To determine the value of the constant α , we proceed in the same way as in the proof of Theorem 3.16 of [3]. Since the coefficients $(\tilde{\mathcal{A}}^\epsilon, \tilde{\mathcal{A}}_1^\epsilon, \tilde{C}^\epsilon$ and $\tilde{C}_1^\epsilon, \nabla \cdot \tilde{C}_1^\epsilon, \nabla \cdot (\tilde{\mathcal{A}}_1^\epsilon \nabla z^\epsilon)$, and $\frac{\partial Z}{\partial t})$ are bounded by constants, let β denotes the maximum between all these constants. Then we use the same argument as in the proof of Theorems 1.1 and 3.16 and we get:

$$\left\| \frac{z^\epsilon - Z^\epsilon}{\epsilon} \right\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \leq \|z_0(\cdot) - Z(0, 0, \cdot)\|_2 \sqrt{\frac{\beta + \beta^3}{\sqrt{\tilde{G}_{thr}}}} + 2\beta T. \quad (3.18)$$

□

Theorem 3.1. *Let ϵ be a positive real, z^ϵ be the solution to (1.1), Z_P be the truncated Fourier series (defined by (2.10)) of Z solution to (1.10) and Z_P^ϵ defined by $Z_P^\epsilon(t, x) = Z_P(t, \frac{t}{\epsilon}, x)$. Then, under assumptions (1.8), (1.9) and (1.12), $z^\epsilon - Z_P^\epsilon$ satisfies the following estimate:*

$$\|z^\epsilon - Z_P^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \leq \epsilon \|z_0(\cdot) - Z(0, 0, \cdot)\|_2 \sqrt{\frac{\beta + \beta^3}{\sqrt{\tilde{G}_{thr}}} + 2\beta T + f(P)}, \quad (3.19)$$

where f is a non-negative function of P not depending on ϵ and satisfying $\lim_{P \rightarrow +\infty} f(P) = 0$.

Proof. We can write:

$$\begin{aligned} \|z^\epsilon - Z_P^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} &= \|z^\epsilon - Z^\epsilon + Z^\epsilon - Z_P^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \\ &\leq \|z^\epsilon - Z^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} + \|Z^\epsilon - Z_P^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))}. \end{aligned} \quad (3.20)$$

Using (3.18), the first term in the right hand side of (3.20) is bounded by

$$\|z^\epsilon - Z^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \leq \epsilon \|z_0(\cdot) - Z(0, 0, \cdot)\|_2 \sqrt{\frac{\beta + \beta^3}{\sqrt{\tilde{G}_{thr}}} + 2\beta T}. \quad (3.21)$$

For the second term of (3.20), using classical results of Fourier series theory, since $Z - Z_P$ is nothing but the rest of the Fourier series of order P of Z and since Z is regular (because it is the solution of (1.10) which has regular coefficients), the non-negative function f satisfying $\lim_{P \rightarrow +\infty} f(P) = 0$ such that

$$\|Z - Z_P\|_{L^\infty([0, T], L^\infty(\mathbb{R}, L^2(\mathbb{T}^2)))} \leq f(P), \quad (3.22)$$

exists. From this last inequality,

$$\|Z^\epsilon - Z_P^\epsilon\|_{L^\infty([0, T], L^2(\mathbb{T}^2))} \leq f(P), \quad (3.23)$$

follows and coupling this with (3.21) and (3.20) gives inequality (3.19). \square

4. Numerical illustration of Theorem 3.1.

4.1. Reference solution. Having Fourier coefficients of Z on hand, we will do the same for function $z^\epsilon(t, x)$ solution to (1.1) in order to compare it to the profile Z for a given ϵ , in a fixed time. The Fourier expansion of z^ϵ is given by

$$z^\epsilon(t, x_1, x_2) = \sum_{m, n} z_{m, n}(t) e^{2\pi i(m x_1 + n x_2)}, \quad (4.1)$$

where $m = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$, then the Fourier expansion of $\frac{\partial z^\epsilon}{\partial t}$ is

$$\frac{\partial z^\epsilon}{\partial t} = \sum_{m, n} \dot{z}_{m, n}(t) e^{2\pi i(m x_1 + n x_2)}. \quad (4.2)$$

Using the same idea as in the Fourier expansion of Z , we obtain the following infinite system of Ordinary Differential Equations

$$\begin{aligned} \frac{\partial z_{m, n}}{\partial t}(t) - \frac{1}{\epsilon} \sum_{i, j} 2i\pi \mathcal{A}_{i, j}^{grad}(t) \cdot \begin{pmatrix} m - i \\ n - j \end{pmatrix} z_{m-i, n-j}(t) \\ + \frac{1}{\epsilon} 4\pi^2 \sum_{i, j} \mathcal{A}_{i, j}(t) ((m - i)^2 + (n - j)^2) z_{m-i, n-j}(t) = \frac{1}{\epsilon} \mathcal{C}_{m, n}(t), \end{aligned} \quad (4.3)$$

where $\mathcal{A}_{i, j}^{grad}(t)$, $\mathcal{A}_{i, j}(t)$ and $\mathcal{C}_{m, n}(t)$ are respectively the Fourier coefficients of $\nabla \mathcal{A}^\epsilon$, \mathcal{A}^ϵ and $\nabla \cdot \mathcal{C}^\epsilon$.

In the same way, the truncated Fourier series of order $P \in \mathbb{N}$ of z^ϵ is given by

$$z_P^\epsilon(t, x_1, x_2) = \sum_{m, n=0}^P z_{m, n}(t) e^{2\pi i(m x_1 + n x_2)}, \quad (4.4)$$

which gives from (4.3) the following system Ordinary Differential Equations

$$\begin{aligned} \frac{\partial z_{m, n}}{\partial t}(t) - \frac{1}{\epsilon} \sum_{i, j=0}^P 2i\pi \mathcal{A}_{i, j}^{grad}(t) \cdot \begin{pmatrix} m - i \\ n - j \end{pmatrix} z_{m-i, n-j}(t) \\ + \frac{1}{\epsilon} 4\pi^2 \sum_{i, j=0}^P \mathcal{A}_{i, j}(t) ((m - i)^2 + (n - j)^2) z_{m-i, n-j}(t) = \frac{1}{\epsilon} \mathcal{C}_{m, n}(t). \end{aligned} \quad (4.5)$$

In (4.5), we will use an initial condition $z_{m, n}(0, x)$. To solve (4.5) we use, for the discretization in time, a Runge-Kutta method (ode45).

4.2. Comparison Two-Scale Numerical Solution and reference solution. In this paragraph, we consider the truncated solution $z_P^\epsilon(t, x_1, x_2)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$. The objective here is to compare for a fixed ϵ and a given time, the quantity $|z_P^\epsilon(t, x_1, x_2) - Z_P(t, \frac{t}{\epsilon}, x_1, x_2)|$ when the water velocity \mathcal{U} is given.

4.2.1. *Comparisons of $z_P^\epsilon(t, x)$ and $Z_P(t, \frac{t}{\epsilon}, x)$ with \mathcal{U} given by (4.6).* For the numerical simulations, concerning z^ϵ , we take $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ and $z_0(x_1, x_2) = Z(0, 0, x_1, x_2)$. In what concerns the water velocity field, we consider the function

$$\mathcal{U}(t, \theta, x_1, x_2) = \sin \pi x_1 \sin 2\pi \theta \mathbf{e}_1, \quad (4.6)$$

where \mathbf{e}_1 and \mathbf{e}_2 are respectively the first and the second vector of the canonical basis of \mathbb{R}^2 and x_1, x_2 are the first and the second components of x .

In Figure 1, we can see the space distribution of the first component of the velocity \mathcal{U} for a given time $t = 1$ and for various values of θ : 0.3, 0.55, and 0.7. In Figure 2, we see, for a fixed point $x = (x_1, x_2)$, how the water velocity $\tilde{\mathcal{U}}(\theta)$

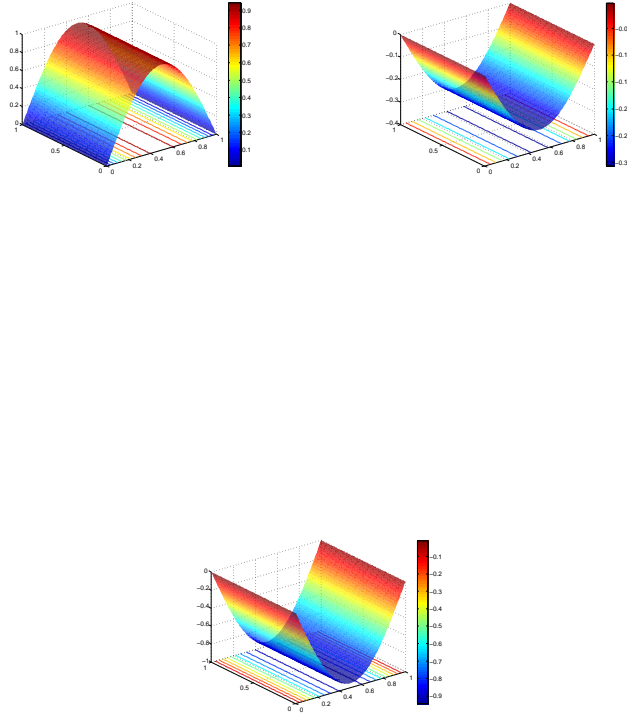


FIGURE 1. Space distribution of the first component of $\mathcal{U}(1, 0.3, (x_1, x_2))$, $\mathcal{U}(1, 0.55, (x_1, x_2))$ and $\mathcal{U}(1, 0.7, (x_1, x_2))$ when \mathcal{U} is given by (4.6).

evolves with respect to θ . In Figure 3, the θ -evolution of $\tilde{\mathcal{A}}(\theta)$ is also given in various points $(x_1, x_2) \in \mathbb{R}^2$.

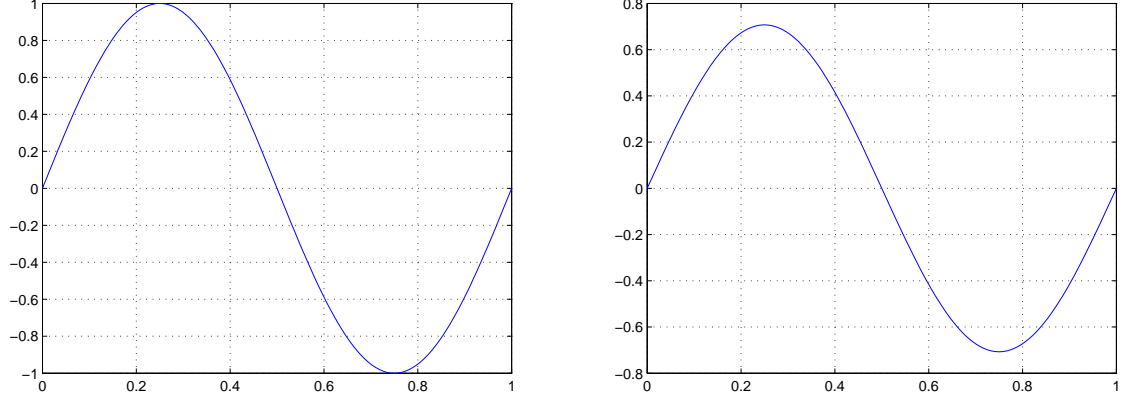


FIGURE 2. θ -evolution of $\tilde{\mathcal{U}}(\theta, (1/2, 0))$ and $\tilde{\mathcal{U}}(\theta, (1/4, 0))$ when \mathcal{U} is given by (4.6)

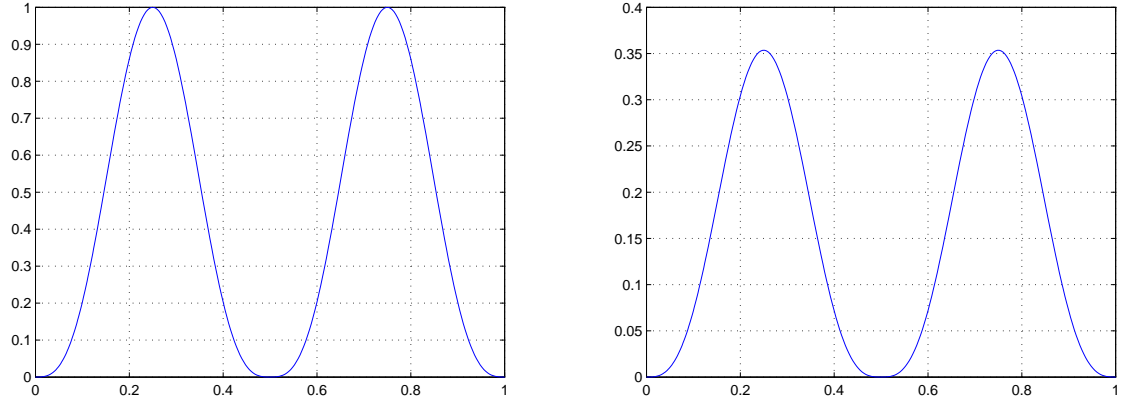


FIGURE 3. θ -evolution of $\tilde{\mathcal{A}}(\theta, (1/2, 0))$ and $\tilde{\mathcal{A}}(\theta, (1/4, 0))$ when \mathcal{U} is given by (4.6)

In this paragraph, we present numerical simulations in order to validate the Two-Scale convergence presented in Theorem 1.1. For a given ϵ , we compare $Z_P(t, \frac{t}{\epsilon}, x)$, where Z_P is the Fourier expansion of order P of the solution to (1.10) and $z_P^\epsilon(t, x)$ the Fourier expansion of order P of the solution to the reference problem. The simulations presented are given for $P = 4$. The calculation of $z_P^\epsilon(t, x)$ implies knowledge of $z_0(x)$. For an initial condition $z_0(x)$ well prepared and equal to $Z(0, 0, x)$, we obtain the results of Figure 4 and we remark that the results obtained are the same for $z_P^\epsilon(t, x)$ and $Z_P(t, \frac{t}{\epsilon}, x)$.

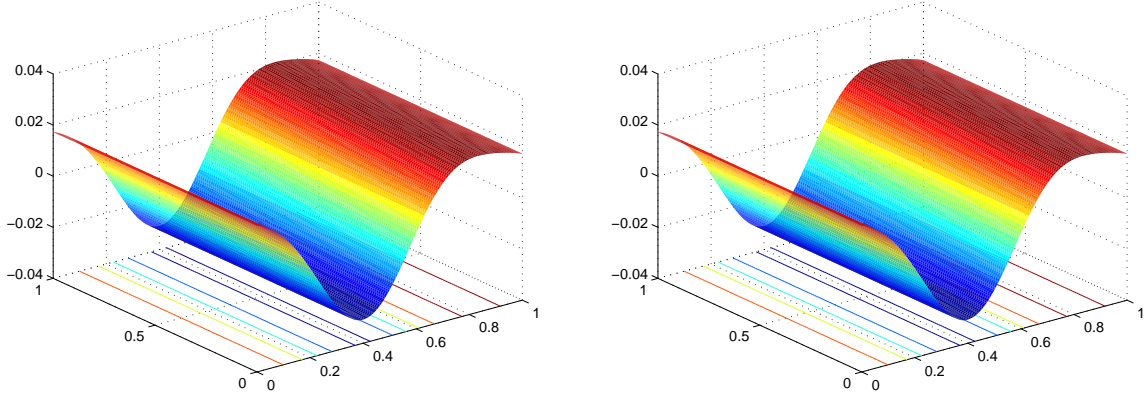


FIGURE 4. Comparison of $z_P^\epsilon(t, \cdot)$ and $Z_P(t, \frac{t}{\epsilon}, \cdot)$, $P = 4$, at time $t = 1$, $\epsilon = 0.001$, when \mathcal{U} is given by (4.6) and when $z_0(\cdot) = Z(0, 0, \cdot)$. On the left $z_P^\epsilon(t, \cdot)$, on the right $Z_P(t, \frac{t}{\epsilon}, \cdot)$.

In practice, the solution Z_P , $P \in \mathbb{N}$ evolves according to P . For the simulations, we made the value of the integer P vary and we saw that this variation is very low from $P \geq 6$. To better show that $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$ is close to the reference solution $z_P^\epsilon(t, x_1, x_2)$, we plot and compare $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$ and $z_P^\epsilon(t, x_1, 0)$, at different times t . In these comparisons the initial condition $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ is different from $Z(0, 0, x_1, x_2)$. The results are shown in Figure 5 and Figure 6. We see in these figures that the solution $z_P^\epsilon(t, x)$ get closer and closer to $Z_P(t, \frac{t}{\epsilon}, x)$ with time of order ϵ .

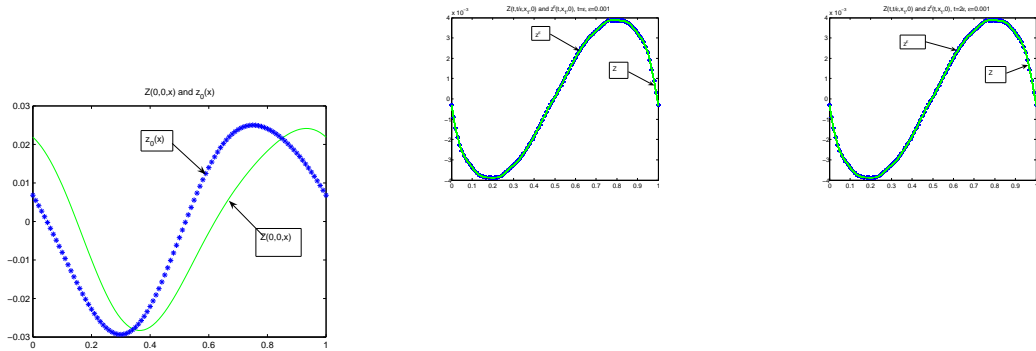


FIGURE 5. Comparison of $z_P^\epsilon(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$, $P = 4$. On the left $t = 0$, in the middle $t = \epsilon$ and $t = 2\epsilon$ on the right, $\epsilon = 0.001$.

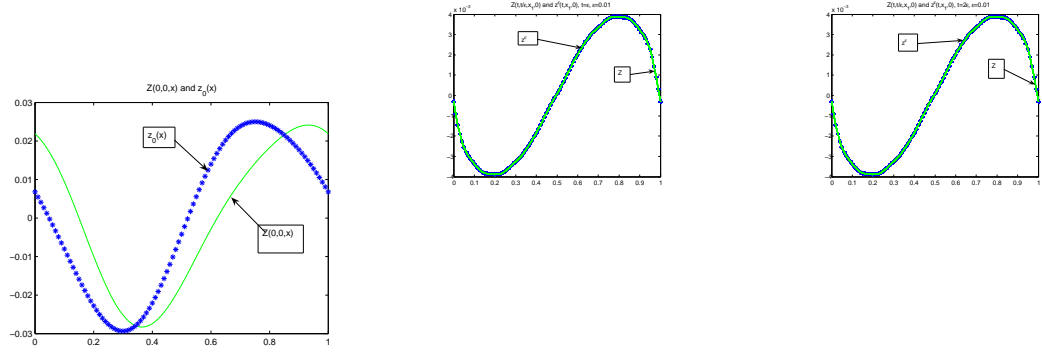


FIGURE 6. Comparison of $z_P^\epsilon(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$. On the left $t = 0$, in the middle $t = \epsilon$ and $t = 2\epsilon$ on the right, $\epsilon = 0.01$.

So we can see from these figures that the solution Z of the Two-Scale limit problem is such that $Z(t, \frac{t}{\epsilon}, \cdot, \cdot)$ is close to the solution $z^\epsilon(t, \cdot, \cdot)$ of the reference problem. In the presently considered case where the initial condition for z^ϵ is not $Z(0, 0, \cdot, \cdot)$, we saw in Figure 5 and Figure 6 that z_P^ϵ tends to reach a steady state. This steady state is an oscillatory one in the sense that for large t , $z_P^\epsilon(t, \cdot, \cdot)$ behaves like $Z_P(t, \frac{t}{\epsilon}, \cdot, \cdot)$. This is illustrated by Figure 7 where $z_P^\epsilon(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$ are given for various value of t in a period of length ϵ .

More precisely, in this figure we see that within a period of time of length ϵ , $z_P^\epsilon(t, \cdot, \cdot)$ and $Z_P(t, \frac{t}{\epsilon}, \cdot, \cdot)$ do not glue together completely. Nevertheless, despite this phenomenon which is linked with the fact that the Two-Scale approximation of $z^\epsilon(t, \cdot, \cdot)$ by $Z(t, \frac{t}{\epsilon}, \cdot, \cdot)$ is only of order 1 in ϵ , the two solutions re-glug well together at the end of the period.

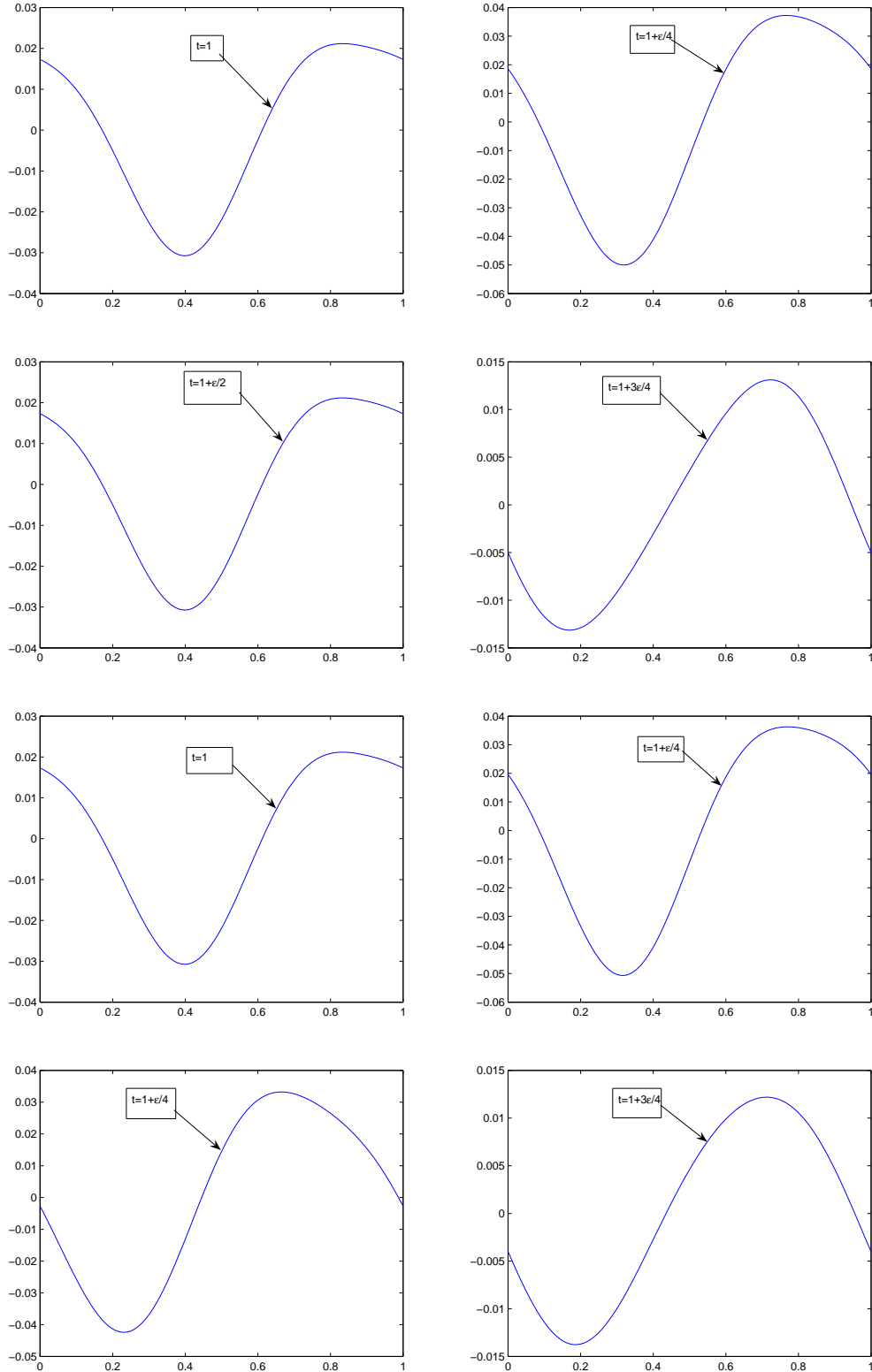


FIGURE 7. Evolution of $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$ in the top and $z_P^\epsilon(t, x_1, 0)$ in the bottom, $t = 1 + \frac{n\epsilon}{4}$, $n = 0, 1, 2, 3$.

4.2.2. *Comparisons of $z^\epsilon(t, x)$ and $Z(t, \frac{t}{\epsilon}, x)$ with \mathcal{U} is given by (4.7).* In this subsection, we do the same as in the precedent one, but when the velocity fields \mathcal{U} given by (4.7). The results are all identical to the precedent one i.e. the Two-Scale limit $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$ is very close to the solution $z_P^\epsilon(t, x_1, x_2)$ to the reference problem when $P = 4$. The initial condition $z_0(x_1, x_2) \neq Z(0, 0, x_1, x_2)$ and is the same as in the subsection 4.2.1. The results are given for $\epsilon = 0.1$ and $\epsilon = 0.005$ and for various time t . We notice that z^ϵ comes very close to $Z(t, \frac{t}{\epsilon}, x_1, x_2)$ when ϵ is very small. We begin by giving the space distribution of \mathcal{U} at various time and the θ -evolution of \mathcal{U} and $\tilde{\mathcal{A}}$. The second velocity fields is given by

$$\mathcal{U}(t, \theta, x_1, x_2) = \mathcal{U}(t, \theta, x) = \left\{ \begin{array}{l} 0 \text{ in } [0, \theta_1], \\ \frac{\theta - \theta_1}{\theta_2 - \theta_1} U_{thr} \mathbf{e}_2 \text{ in } [\theta_1, \theta_2], \\ U_{thr} \mathbf{e}_2 + \phi\left(\frac{\theta - \theta_2}{\theta_3 - \theta_2}\right) \psi(t, x) \text{ in } [\theta_2, \theta_3], \\ \frac{\theta - \theta_3}{\theta_4 - \theta_3} U_{thr} \mathbf{e}_2 \text{ in } [\theta_3, \theta_4], \\ 0 \text{ in } [\theta_4, \theta_5], \\ \frac{\theta - \theta_5}{\theta_6 - \theta_5} U_{thr} \mathbf{e}_2 \text{ in } [\theta_5, \theta_6], \\ -U_{thr} \mathbf{e}_2 - \phi\left(\frac{\theta - \theta_6}{\theta_7 - \theta_6}\right) \psi(t, x) \text{ in } [\theta_6, \theta_7], \\ -\frac{\theta - \theta_7}{\theta_8 - \theta_7} U_{thr} \mathbf{e}_2 \text{ in } [\theta_7, \theta_8], \\ 0 \text{ in } [\theta_8, 1], \end{array} \right. \quad (4.7)$$

where $U_{thr} > 0$, ϕ is a regular positive function satisfying $\phi(s) = s(1 - s)$ and $\psi(t, x_1) = (1 + \sin \frac{\pi}{30} t)(U_{thr} \mathbf{e}_2 + \frac{1}{10}(1 + \sin 2\pi x_1) \mathbf{e}_1)$, $\theta_i = \frac{i+1}{10}$, $i = 1, \dots, 8$.

The θ -evolution of \mathcal{U} , given by (4.7), is given in Figure 9 for various position in $[0, 1]^2$.

Function $g_a(\mathbf{u}) = g_c(\mathbf{u}) = |\mathbf{u}|^3$, $a = c = 1$ and $\mathcal{M}(t, \theta, x) = 0$ which yields a θ -evolution of $\tilde{\mathcal{A}}(\theta)$ which is drawn for various positions in Figure 10.

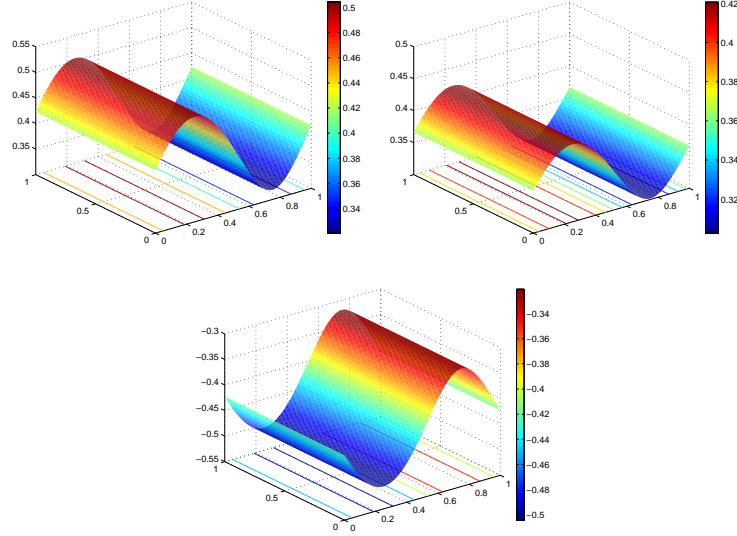


FIGURE 8. Space distribution of the first component of $\mathcal{U}(1, 0.25, (x_1, x_2))$, $\mathcal{U}(1, 0.275, (x_1, x_2))$ and $\mathcal{U}(1, 0.75, (x_1, x_2))$ when \mathcal{U} is given by (4.7).

Using this, we compute $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$ and $z_P^\epsilon(t, x)$ for $P = 4$. To compute $z_P^\epsilon(t, x)$ we take $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ which is not $Z(0, 0, x_1, x_2)$. First we study the errors $Z_P(t, \frac{t}{\epsilon}, x_1, x_2) - z_P^\epsilon(t, x)$ at $t = 1$. This quantity decreases when ϵ decreases as illustrated in the following tabular.

value of ϵ	norm L^1	norm L^2	norm L^∞
0.01	0.012212	0.00048013	0.003376
0.03	0.019082	0.0005753	0.0017347
0.05	0.030769	0.01348	0.0069818
0.07	0.045123	0.029055	0.009
0.09	0.17067	0.10562	0.038790
0.1	0.3053	0.10562	0.04878

Table: Errors norm $Z_P(t, \frac{t}{\epsilon}, x_1, x_2) - z_P^\epsilon(t, x_1, x_2)$, $\bar{P} = (4, 4)$, $P = (4, 4, 4)$, $t = 1$.

The results given in this table show that, at time $t = 1$, $z^\epsilon(t, x)$ is closer to $Z(t, \frac{t}{\epsilon}, x)$ when ϵ is very small. These results validate the results obtained in Theorem 1.1.

In Figures 11 and 12, we present simulations at times $t = 0.75$ and $t = 0.775$. We see that $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$ is close to $z_P^\epsilon(t, x_1, x_2)$. The numerical results shown in these figures are made with $\epsilon = 0.1$.

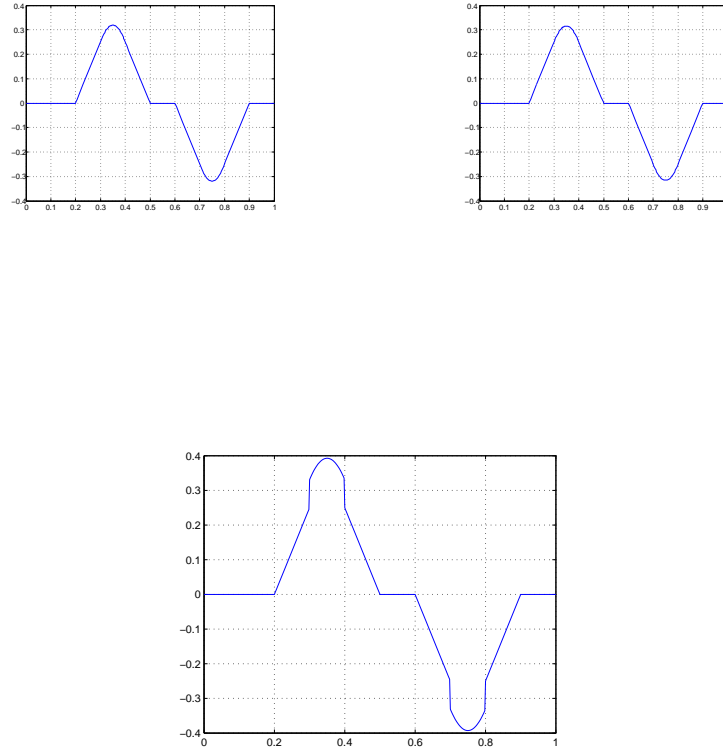


FIGURE 9. θ -evolution of $\mathcal{U}(1, \theta, (1, 0))$, $\mathcal{U}(1, \theta, (4, 0))$ and $\mathcal{U}(1, \theta, (1/3, 1/3))$ when \mathcal{U} is given by (4.7).

In Figure 13 and 14, we do the same but for $\epsilon = 0.005$. The numerical results show that $z_P^\epsilon(t, x)$ is also very close to $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$.

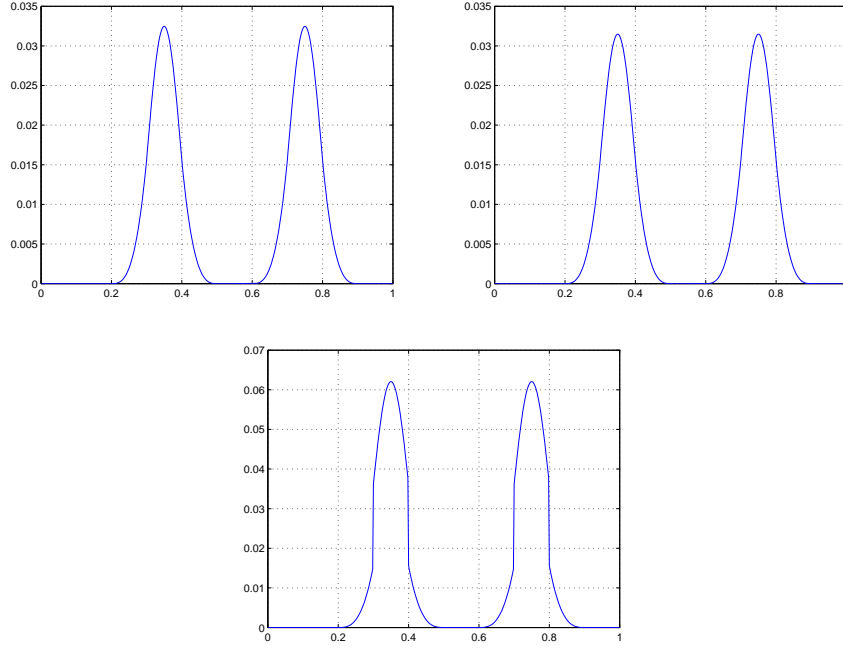


FIGURE 10. θ -evolution of $\tilde{\mathcal{A}}(1, \theta, (1, 0))$, $\tilde{\mathcal{A}}(1, \theta, (4, 0))$ and $\tilde{\mathcal{A}}(1, \theta, (1/3, 1/3))$ when \mathcal{U} is given by (4.7).

We remark that for $\epsilon = 0.1$ and $\epsilon = 0.005$, the solution $z_P^\epsilon(t, x)$ is very close to $Z_P(t, \frac{t}{\epsilon}, x)$. But the approximation $z_P^\epsilon(t, x) \sim Z_P(t, \frac{t}{\epsilon}, x)$ is very good when ϵ is very small.

To show that z_P^ϵ is very close to Z_P , we construct the same figures as previously but in dimension 2 with $\epsilon = 0.005$ i.e. we construct $z_P^\epsilon(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$ for $\epsilon = 0.005$ at time $t = 0.775$. This is given in Figure 15.

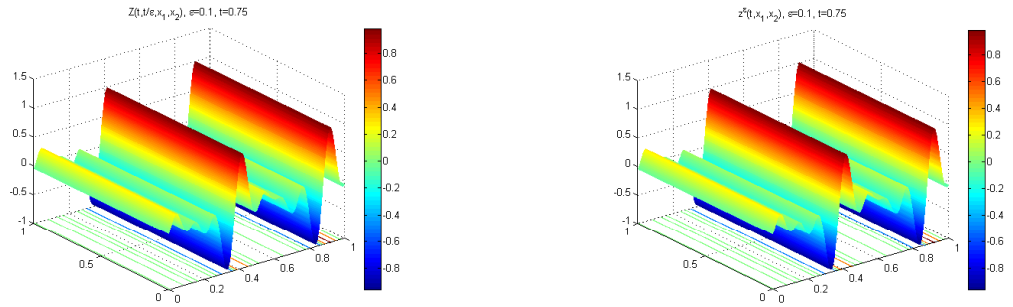
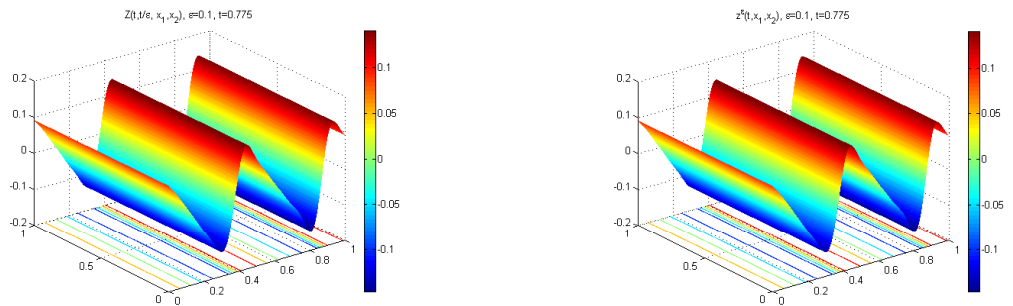


FIGURE 11. Comparison of $z_P^\epsilon(t, x_1, x_2)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$, $P = 4$; $t = 0.75$, $\epsilon = 0.1$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the left $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$, on the right $z_P^\epsilon(t, x_1, x_2)$.



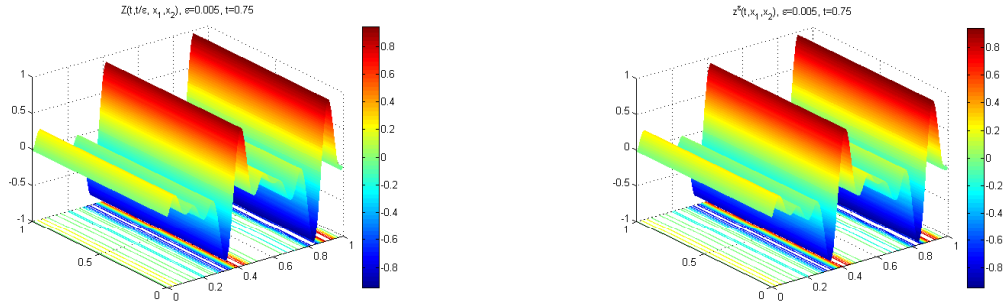
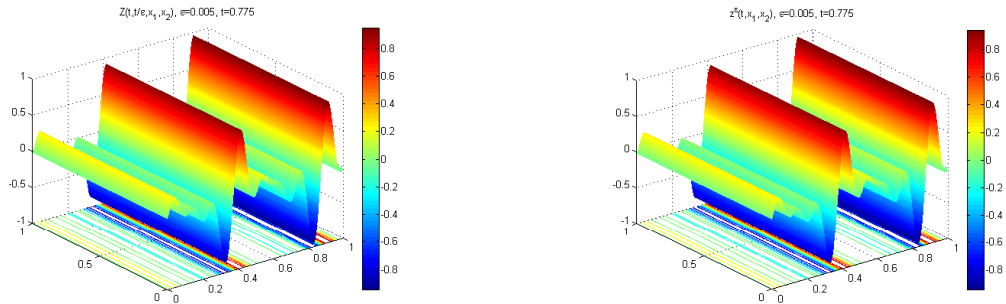


FIGURE 13. Comparison of $z_P^\epsilon(t, x_1, x_2)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$, $P = 4$; $t = 0.75$, $\epsilon = 0.005$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the left $Z_P(t, \frac{t}{\epsilon}, x_1, x_2)$, on the right $z_P^\epsilon(t, x_1, x_2)$.



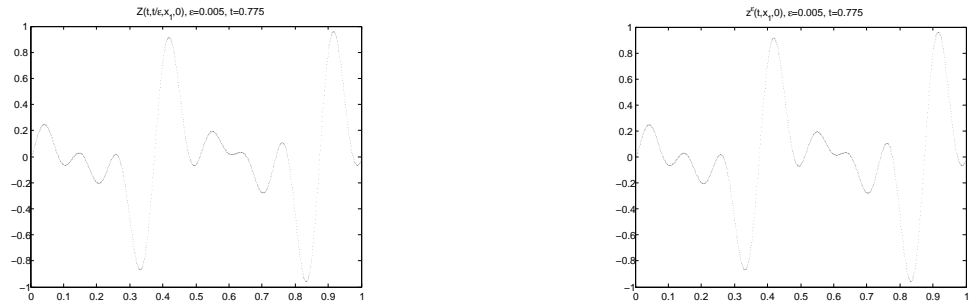
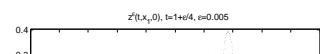
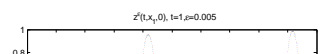
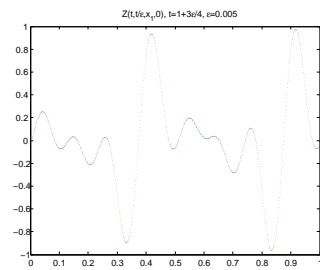
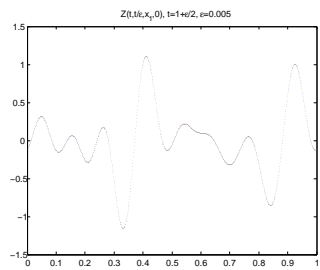
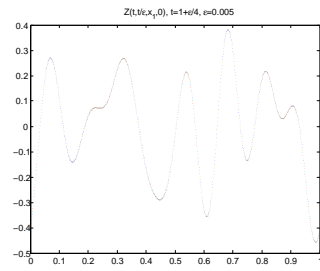
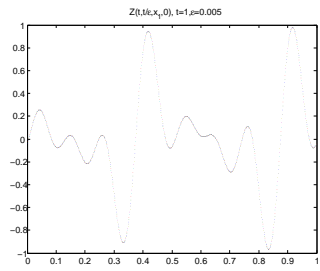


FIGURE 15. Comparison of $z_P^\epsilon(t, x_1, 0)$ and $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$, $t = 0.775$, $\epsilon = 0.005$.
On the left $Z_P(t, \frac{t}{\epsilon}, x_1, 0)$, on the right $z_P^\epsilon(t, x_1, 0)$.

The results in Figure 16 show that Z_P and z_P^ϵ have the same behavior in the same period and Z_P is very close to z_P^ϵ . We also notice that, despite the small shifts that occur during a period, the two solutions glue together.



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