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| Bachar Alhajjar. The Filtration Induced by a Locally Nilpotent Derivation. 2013. hal-00872823

**HAL Id: hal-00872823**

**<https://hal.science/hal-00872823v1>**

Preprint submitted on 14 Oct 2013

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# THE FILTRATION INDUCED BY A LOCALLY NILPOTENT DERIVATION

BACHAR ALHAJJAR

ABSTRACT. We investigate the filtration corresponding to the degree function induced by a non-zero locally nilpotent derivations  $\partial$  and its associated graded algebra. As an application we provide an efficient method to recover the Makar-Limanov invariants, isomorphism classes and automorphism groups of classically known algebras. We also present a new class of examples which can be fully described with this method.

## Introduction

Let  $k$  be a field of characteristic zero, and let  $B$  be a commutative  $k$ -domain. A  $k$ -derivation  $\partial \in \text{Der}_k(B)$  is said to be *locally nilpotent* if for every  $a \in B$ , there is an integer  $n \geq 0$  such that  $\partial^n(a) = 0$ . An important invariant of  $k$ -domains admitting non-trivial locally nilpotent derivations is the so called *Makar-Limanov invariant* which was defined by Makar-Limanov as the intersection  $\text{ML}(B) \subset B$  of kernels of all locally nilpotent derivations of  $B$  ([2]). This invariant was initially introduced as a tool to distinguish certain  $k$ -domains from polynomial rings but it has many other applications for the study of  $k$ -algebras and their automorphism groups ([3]). One of the main difficulties in applications is to compute this invariant without a prior knowledge of all locally nilpotent derivations of a given  $k$ -domain.

In [4] S. Kaliman and L. Makar-Limanov developed general techniques to determine the ML-invariant for a class of finitely generated  $k$ -domains  $B$ . The idea is to reduce the problem to the study of homogeneous locally nilpotent derivations on graded algebras  $\text{Gr}(B)$  associated to  $B$ . For this, one considers appropriate filtrations  $\mathcal{F} = \{\mathcal{F}_i\}_{i \in \mathbb{R}}$  on  $B$  generated by so called real-valued weight degree functions in such a way that every non-zero locally nilpotent derivation on  $B$  induces a non-zero homogeneous locally nilpotent derivation on the associated graded algebra  $\text{Gr}_{\mathcal{F}}(B)$ .

In particular, every  $k$ -domain  $B$  admitting a non-zero locally nilpotent derivation  $\partial$  comes equipped with a natural filtration by the  $k$ -sub-vector-spaces  $\mathcal{F}_i = \ker(\partial^{i+1})$ ,  $i \geq 0$ , that we call the  $\partial$ -filtration.

In this article we show that this filtration is convenient for the computation of the ML-invariant, and we give general methods to describe the sub-spaces  $\ker(\partial^{i+1})$  and their associated graded algebra.

Knowing this filtration gives a very precise understanding of the structure of *semi-rigid*  $k$ -domains, that is,  $k$ -domains for which every locally nilpotent derivation gives rise to the same filtration. For such rings the study of the  $\partial$ -filtration is a very efficient tool to determine isomorphism types and automorphism groups. We

illustrate how the computation of ML-invariant of classically known semi-rigid  $k$ -domains can be simplified using these types of filtration. We also present a new class of semi-rigid  $k$ -domains which can be studied with this method.

## 1. Preliminaries

In this section we briefly recall basic facts on filtered algebra and their relation with derivation in a form appropriate to our needs.

In the sequel, unless otherwise specified  $B$  will denote a commutative domain over a field  $k$  of characteristic zero. The set  $\mathbb{Z}_{\geq 0}$  of non-negative integers will be denoted by  $\mathbb{N}$ .

### 1.1. Filtration and associated graded algebra.

**Definition 1.1.** An  $\mathbb{N}$ -filtration of  $B$  is a collection  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  of  $k$ -sub-vector-spaces of  $B$  with the following properties:

- 1-  $\mathcal{F}_i \subset \mathcal{F}_{i+1}$  for all  $i \in \mathbb{N}$ .
- 2-  $B = \cup_{i \in \mathbb{N}} \mathcal{F}_i$ .
- 3-  $\mathcal{F}_i \cdot \mathcal{F}_j \subset \mathcal{F}_{i+j}$  for all  $i, j \in \mathbb{N}$ .

The filtration is called *proper* if the following additional property holds:

- 4- If  $a \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$  and  $b \in \mathcal{F}_j \setminus \mathcal{F}_{j-1}$ , then  $ab \in \mathcal{F}_{i+j} \setminus \mathcal{F}_{i+j-1}$ .

There is a one-to-one correspondence between proper  $\mathbb{N}$ -filtrations and so called  $\mathbb{N}$ -degree functions:

**Definition 1.2.** An  $\mathbb{N}$ -degree function on  $B$  is a map  $\deg : B \longrightarrow \mathbb{N} \cup \{-\infty\}$  such that, for all  $a, b \in B$ , the following conditions are satisfied:

- (1)  $\deg(a) = -\infty \Leftrightarrow a = 0$ .
- (2)  $\deg(ab) = \deg(a) + \deg(b)$ .
- (3)  $\deg(a + b) \leq \max\{\deg(a), \deg(b)\}$ .

If the equality in (2) replaced by the inequality  $\deg(ab) \leq \deg(a) + \deg(b)$ , we say that  $\deg$  is an  $\mathbb{N}$ -semi-degree function.

Indeed, for an  $\mathbb{N}$ -degree function on  $B$ , the sub-sets  $\mathcal{F}_i = \{b \in B \mid \deg(b) \leq i\}$  are  $k$ -subvector spaces of  $B$  that give rise to a proper  $\mathbb{N}$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ . Conversely, every proper  $\mathbb{N}$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ , yields an  $\mathbb{N}$ -degree function  $\omega : B \longrightarrow \mathbb{N} \cup \{-\infty\}$  defined by  $\omega(0) = -\infty$  and  $\omega(b) = i$  if  $b \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ .

**Definition 1.3.** Given a  $k$ -domain  $B = \cup_{i \in \mathbb{N}} \mathcal{F}_i$  equipped with a proper  $\mathbb{N}$ -filtration, the associated graded algebra  $\text{Gr}(B)$  is the  $k$ -vector space

$$\text{Gr}(B) = \oplus_{i \in \mathbb{N}} \mathcal{F}_i / \mathcal{F}_{i-1}$$

equipped with the unique multiplicative structure for which the product of the elements  $a + \mathcal{F}_{i-1} \in \mathcal{F}_i / \mathcal{F}_{i-1}$  and  $b + \mathcal{F}_{j-1} \in \mathcal{F}_j / \mathcal{F}_{j-1}$ , where  $a \in \mathcal{F}_i$  and  $b \in \mathcal{F}_j$ , is the element

$$(a + \mathcal{F}_{i-1})(b + \mathcal{F}_{j-1}) := ab + \mathcal{F}_{i+j-1} \in \mathcal{F}_{i+j} / \mathcal{F}_{i+j-1}.$$

Property 4 (proper) in Definition 1.1 ensures that  $\text{Gr}(B)$  is a commutative  $k$ -domain when  $B$  is an integral domain. Since for each  $a \in B$  the set  $\{n \in \mathbb{N} \mid a \in \mathcal{F}_n\}$  has a minimum, there exists  $i$  such that  $a \in \mathcal{F}_i$  and  $a \notin \mathcal{F}_{i-1}$ . So we can define a  $k$ -linear map  $\text{gr} : B \longrightarrow \text{Gr}(B)$  by sending  $a$  to its class in  $\mathcal{F}_i / \mathcal{F}_{i-1}$ , i.e  $a \mapsto a + \mathcal{F}_{i-1}$ , and  $\text{gr}(0) = 0$ . We will frequently denote  $\text{gr}(a)$  simply by  $\bar{a}$ . Observe that  $\text{gr}(a) = 0$  if and only if  $a = 0$ .

Denote by  $\deg$  the  $\mathbb{N}$ -degree function  $\deg : B \longrightarrow \mathbb{N} \cup \{-\infty\}$  corresponding to the proper  $\mathbb{N}$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ . We have the following properties.

**Lemma 1.4.** *Given  $a, b \in B$  the following holds:*

- P1)  $\overline{ab} = \overline{a}\overline{b}$ , i.e.  $\text{gr}$  is a multiplicative map.
- P2) If  $\deg(a) > \deg(b)$ , then  $\overline{a+b} = \overline{a}$ .
- P3) If  $\deg(a) = \deg(b) = \deg(a+b)$ , then  $\overline{a+b} = \overline{a} + \overline{b}$ .
- P4) If  $\deg(a) = \deg(b) > \deg(a+b)$ , then  $\overline{a+b} = 0$ , in particular  $\text{gr}$  is not an additive map in general.

*Proof.* Let us assume that  $\deg(a) = i$  and  $\deg(b) = j$ . By definition,  $\deg(ab) = i+j$  means that  $ab \in \mathcal{F}_{i+j}$  and  $ab \notin \mathcal{F}_{i+j-1}$ , so  $\overline{ab} = ab + \mathcal{F}_{i+j-1} := (a + \mathcal{F}_{i-1})(b + \mathcal{F}_{j-1}) = \overline{a}\overline{b}$ . Which gives P1. For P2 we observe that since  $\deg(a+b) = \deg(a)$ , we have  $\overline{a+b} = (a+b) + \mathcal{F}_{i-1} = (a + \mathcal{F}_{i-1}) + (b + \mathcal{F}_{i-1})$ , and since  $\mathcal{F}_{j-1} \subset \mathcal{F}_j \subseteq \mathcal{F}_{i-1}$  as  $i > j$ , we get  $b + \mathcal{F}_{i-1} = 0$ . P3) is immediate, by definition. Finally, assume by contradiction that  $\overline{a+b} \neq 0$ , then  $\overline{a+b} = (a + \mathcal{F}_{i-1}) + (b + \mathcal{F}_{i-1}) = ((a+b) + \mathcal{F}_{i-1}) \neq 0$ , which means that  $a+b \notin \mathcal{F}_{i-1}$  and  $\deg(a+b) = i$ , which is absurd. So P4 follows.  $\square$

## 1.2. Derivations.

By a  $k$ -derivation of  $B$ , we mean a  $k$ -linear map  $D : B \longrightarrow B$  which satisfies the Leibniz rule: For all  $a, b \in B$ ;  $D(ab) = aD(b) + bD(a)$ . The set of all  $k$ -derivations of  $B$  is denoted by  $\text{Der}_k(B)$ .

The *kernel* of a derivation  $D$  is the subalgebra  $\ker D = \{b \in B; D(b) = 0\}$  of  $B$ .

The *plinth ideal* of  $D$  is the ideal  $\text{pl}(D) = \ker D \cap D(B)$  of  $\ker D$ , where  $D(B)$  denotes the image of  $B$ .

An element  $s \in B$  such that  $D(s) \in \ker(D) \setminus \{0\}$  is called a *local slice* for  $D$ .

**Definition 1.5.** Given a  $k$ -algebra  $B = \cup_{i \in \mathbb{N}} \mathcal{F}_i$  equipped with a proper  $\mathbb{N}$ -filtration, a  $k$ -derivation  $D$  of  $B$  is said to respect the filtration if there exists an integer  $d$  such that  $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$  for all  $i \in \mathbb{N}$ .

If so, we define a  $k$ -linear map  $\overline{D} : \text{Gr}(B) \longrightarrow \text{Gr}(B)$  as follows: If  $D = 0$ , then  $\overline{D} = 0$  the zero map. Otherwise, if  $D \neq 0$  then we let  $d$  be the least integer such that  $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$  for all  $i \in \mathbb{N}$  and we define

$$\overline{D} : \mathcal{F}_i / \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_{i+d} / \mathcal{F}_{i+d-1}$$

by the rule  $\overline{D}(a + \mathcal{F}_{i-1}) = D(a) + \mathcal{F}_{i+d-1}$ . One checks that  $\overline{D}$  satisfies the Leibniz rule, therefore it is a  $k$ -derivation of the graded algebra  $\text{Gr}(B)$ . Moreover it is homogeneous of degree  $d$ , i.e  $\overline{D}$  sends homogeneous elements of degree  $i$  to zero or to homogeneous elements of degree  $i+d$ .

Observe that  $\overline{D} = 0$  if and only if  $D = 0$ . In addition,  $\text{gr}(\ker D) \subset \ker \overline{D}$ .

## 2. LND-Filtrations and Associated Graded Algebras

In this section we introduce the  $\partial$ -filtration associated with a locally nilpotent derivation  $\partial$ . We explain how to compute this filtration and its associated graded algebra in certain situation.

**Definition 2.1.** A  $k$ -derivation  $\partial \in \text{Der}_k(B)$  is said to be *locally nilpotent* if for every  $a \in B$ , there exists  $n \in \mathbb{N}$  (depending of  $a$ ) such that  $\partial^n(a) = 0$ . The set of all locally nilpotent derivations of  $B$  is denoted by  $\text{LND}(B)$ .

In particular, every locally nilpotent derivation  $\partial$  of  $B$  gives rise to a proper  $\mathbb{N}$ -filtration of  $B$  by the sub-spaces  $\mathcal{F}_i = \ker \partial^i$ ,  $i \in \mathbb{N}$ , that we call the  $\partial$ -filtration. It is straightforward to check (see Prop. 1.9 in [1]) that the  $\partial$ -filtration corresponds to the  $\mathbb{N}$ -degree function  $\deg_\partial : B \longrightarrow \mathbb{N} \cup \{-\infty\}$  defined by

$$\deg_\partial(a) := \min\{i \in \mathbb{N} \mid \partial^{i+1}(a) = 0\}, \text{ and } \deg_\partial(0) := -\infty.$$

Note that by definition  $\mathcal{F}_0 = \ker \partial$  and that  $\mathcal{F}_1 \setminus \mathcal{F}_0$  consists of all local slices for  $\partial$ . Let  $\text{Gr}_\partial(B) = \bigoplus_{i \in \mathbb{N}} \mathcal{F}_i / \mathcal{F}_{i-1}$  denote the associated graded algebra relative to the  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ . Let  $\text{gr}_\partial : B \longrightarrow \text{Gr}_\partial(B)$ ;  $a \xrightarrow{\text{gr}_\partial} \bar{a}$  be the natural map between  $B$  and  $\text{Gr}_\partial(B)$  defined in 1.3, where  $\bar{a}$  denote  $\text{gr}_\partial(a)$ .

The next Proposition, which is due to Daigle (Theorem 2.11 in [1]), implies in particular that if  $B$  is of finite transcendence degree over  $k$ , then every non-zero  $D \in \text{LND}(B)$  respects the  $\partial$ -filtration and therefore induces a non-zero homogeneous locally nilpotent derivation  $\bar{D}$  of  $\text{Gr}_\partial(B)$ .

**Proposition 2.2. (Daigle)** *Suppose that  $B$  is a commutative domain, of finite transcendence degree over  $k$ . Then for every pair  $D \in \text{Der}_k(B)$  and  $\partial \in \text{LND}(B)$ ,  $D$  respects the  $\partial$ -filtration. Consequently,  $\bar{D}$  is a well defined homogeneous derivation of the integral domain  $\text{Gr}_\partial(B)$  relative to this filtration, and it is locally nilpotent if  $D$  is locally nilpotent.*

### 2.1. Computing the $\partial$ -filtration.

Here, given a finitely generated  $k$ -domain  $B$ , we describe a general method which enables the computation of the  $\partial$ -filtration for a locally nilpotent derivation  $\partial$  with finitely generated kernel. First we consider a more general situation where the plinth ideal  $\text{pl}(\partial)$  is finitely generated as an ideal in  $\ker \partial$  then we deal with the case where  $\ker \partial$  is itself finitely generated as a  $k$ -algebra.

Let  $B = k[X_1, \dots, X_n]/I = k[x_1, \dots, x_n]$  be a finitely generated  $k$ -domain, and let  $\partial \in \text{LND}(B)$  be such that  $\text{pl}(\partial)$  is generated by precisely  $m$  elements  $f_1, \dots, f_m$  as an ideal in  $\ker \partial$ . Denote by  $\mathcal{F} = \{\mathcal{F}_i\}_{i \in \mathbb{N}}$  the  $\partial$ -filtration, then:

By definition  $\mathcal{F}_0 = \ker \partial$ . Furthermore, given elements  $s_i \in \mathcal{F}_1$  such that  $\partial(s_i) = f_i$  for every  $i \in \{1, \dots, m\}$ , it is straightforward to check that

$$\mathcal{F}_1 = \mathcal{F}_0 s_1 + \dots + \mathcal{F}_0 s_m + \mathcal{F}_0.$$

Letting  $\deg_\partial(x_i) = d_i$ , we denote by  $H_j$  the  $\mathcal{F}_0$ -sub-module in  $B$  generated by elements of degree  $j$  relative to  $\deg_\partial$  of the form  $s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n}$ , i.e.,

$$H_j := \sum_{\sum_j u_j + \sum_i d_i v_i = j} \mathcal{F}_0 (s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n})$$

where  $u_j, v_i \in \mathbb{N}$  for all  $i$  and  $j$ . The integer  $\sum_j u_j + \sum_i d_i v_i$  is nothing but  $\deg_\partial(s_1^{u_1} s_2^{u_2} \dots s_m^{u_m} x_1^{v_1} x_2^{v_2} \dots x_n^{v_n})$ . Then we define a new  $\mathbb{N}$ -filtration  $\mathcal{G} = \{\mathcal{G}_i\}_{i \in \mathbb{N}}$  of  $B$  by setting

$$\mathcal{G}_i = \sum_{j \leq i} H_j.$$

By construction  $\mathcal{G}_i \subseteq \mathcal{F}_i$  for all  $i \in \mathbb{N}$ , with equality for  $i = 0, 1$ . The following result provides a characterization of when these two filtrations coincide:

**Lemma 2.3.** *The filtrations  $\mathcal{F}$  and  $\mathcal{G}$  are equal if and only if  $\mathcal{G}$  is proper.*

*Proof.* One direction is clear since  $\mathcal{F}$  is proper. Conversely, suppose that  $\mathcal{G}$  is proper with the corresponding  $\mathbb{N}$ -degree function  $\omega$  on  $B$  (see §1.1). Given  $f \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ ,  $i > 1$ , for every local slice  $s \in \mathcal{F}_1 \setminus \mathcal{F}_0$ , there exist  $f_0 \neq 0, a_i \neq 0, a_{i-1}, \dots, a_0 \in \mathcal{F}_0$  such that  $f_0 f = a_i s^i + a_{i-1} s^{i-1} + \dots + a_0$  (see the proof of Lemma 4 in [7]). Since  $\omega(g) = 0$  (resp.  $\omega(g) = 1$ ) for every  $g \in \mathcal{F}_0$  (resp.  $g \in \mathcal{F}_1 \setminus \mathcal{F}_0$ ), we obtain

$$\omega(f) = \omega(f_0 f) = \omega(a_i s^i + a_{i-1} s^{i-1} + \dots + a_0) = \max\{\omega(a_i s^i)\} = i,$$

and so  $f \in \mathcal{G}_i$ .  $\square$

Next, we determine the  $\partial$ -filtration, for a locally nilpotent derivation  $\partial$  with finitely generated kernel, by giving an effective criterion to decide when the  $\mathbb{N}$ -filtration  $\mathcal{G}$  defined above is proper.

Hereafter, we assume that  $0 \in \text{Spec}(B)$  and that  $\ker(\partial)$  is generated by elements  $z_i(x_1, \dots, x_n) \in B$  such that  $z_i(0, \dots, 0) = 0, i \in \{1, \dots, r\}$ , which is always possible since  $k \subset \ker \partial$ . Since  $\ker(\partial)$  is finitely generated  $k$ -algebra, the plinth ideal  $\text{pl}(\partial)$  is finitely generated. So there exist  $s_1(x_1, \dots, x_n), \dots, s_m(x_1, \dots, x_n) \in \mathcal{F}_1$  such that  $\mathcal{F}_1 = \mathcal{F}_0 s_1 + \dots + \mathcal{F}_0 s_m + \mathcal{F}_0$ . We can also assume that  $s_i(0, \dots, 0) = 0$ .

Letting  $J \subset k^{[r+n+m]} = k[Z_1, \dots, Z_r][X_1, \dots, X_n][S_1, \dots, S_m]$  be the ideal generated by  $I$  and the elements  $Z_1 = z_1(X_1, \dots, X_n), \dots, Z_r = z_r(x_1, \dots, x_n), S_1 = s_1(X_1, \dots, X_n), \dots, S_m = s_m(x_1, \dots, x_n)$ , then we have

$$B = k[Z_1, \dots, Z_r][X_1, \dots, X_n][S_1, \dots, S_m]/J.$$

Note that by construction  $(0, \dots, 0) \in \text{Spec}(k^{[r+n+m]}/J)$ .

We define an  $\mathbb{N}$ -degree function  $\omega$  on  $k^{[r+n+m]}$  by declaring that  $\omega(Z_i) = 0 = \deg_{\partial}(z_i)$  for all  $i \in \{1, \dots, r\}$ ,  $\omega(S_i) = 1 = \deg_{\partial}(s_i)$  for all  $i \in \{1, \dots, m\}$ , and  $\omega(X_i) = \deg_{\partial}(x_i) = d_i$  for all  $i \in \{1, \dots, n\}$ . The corresponding proper  $\mathbb{N}$ -filtration  $\mathcal{Q}_i := \{P \in k^{[n]} \mid \omega(P) \leq i\}$ ,  $i \in \mathbb{N}$ , on  $k^{[r+n+m]}$  has the form  $\mathcal{Q}_i = \bigoplus_{j \leq i} \mathcal{H}_j$  where

$$\mathcal{H}_j := \bigoplus_{\sum_j u_j + \sum_i d_i v_i = j} k[Z_1, \dots, Z_r] S_1^{u_1} \dots S_m^{u_m} X_1^{v_1} \dots X_n^{v_n}.$$

By construction  $\pi(\mathcal{Q}_i) = \mathcal{G}_i$  where  $\pi : k^{[r+n+m]} \longrightarrow B$  denotes the natural projection. Indeed, since

$$\pi(\mathcal{Q}_i) = \sum_{j \leq i} \pi(\mathcal{H}_j)$$

and  $\pi(\mathcal{H}_j) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} k[z_1, \dots, z_r] s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n}$ , we get

$$\pi(\mathcal{H}_j) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} (\ker \partial) s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n}$$

which means precisely that  $\pi(\mathcal{Q}_i) = \mathcal{G}_i$ .

Let  $\hat{J} \subset k^{[r+n+m]}$  be the homogeneous ideal generated by the highest homogeneous components relative to  $\omega$  of all elements in  $J$ . Then we have the following result, which is inspired by the technique developed by S. Kaliman and L. Makar-Limanov:

**Proposition 2.4.** *The  $\mathbb{N}$ -filtration  $\mathcal{G}$  is proper if and only if  $\hat{J}$  is prime.*

*Proof.* It is enough to show that  $\mathcal{G} = \{\pi(\mathcal{Q}_i)\}_{i \in \mathbb{N}}$  coincides with the filtration corresponding to the  $\mathbb{N}$ -semi-degree function  $\omega_B$  on  $B$  defined by  $\omega_B(p) := \min_{P \in \pi^{-1}(p)} \{\omega(P)\}$ . Indeed, if so, the result will follow from Lemma 3.2 in [4] which asserts in particular

that  $\omega_B$  is an  $\mathbb{N}$ -degree function on  $B$  if and only if  $\hat{J}$  is prime. Let  $\{\mathcal{G}'_i\}_{i \in \mathbb{N}}$  be the filtration corresponding to  $\omega_B$ . Given  $f \in \mathcal{G}'_i$  there exists  $F \in \mathcal{Q}_i$  such that  $\pi(F) = f$ , which means that  $\mathcal{G}'_i \subset \pi(\mathcal{Q}_i)$ . Conversely, it is clear that  $\omega_B(z_i) = \omega(Z_i) = 0$  for all  $i \in \{1, \dots, r\}$ . Furthermore  $\omega_B(s_i) = \omega(S_i) = 1$  for all  $i \in \{1, \dots, m\}$ , for otherwise  $s_i \in \ker \partial$  which is absurd. Finally, if  $d_i \neq 0$  and  $\omega_B(x_i) < \omega(X_i) = d_i \neq 0$ , then  $x_i \in \pi(\mathcal{Q}_{d_i-1}) \subset \ker \partial^{d_i-1}$  which implies that  $\deg_{\partial}(x_i) < d_i$ , a contradiction. So  $\omega_B(x_i) = d_i$ . Thus  $\omega_B(f) \leq i$  for every  $f \in \pi(\mathcal{Q}_i)$  which means that  $\pi(\mathcal{Q}_i) \subset \mathcal{G}'_i$ .  $\square$

The next Proposition, which is a reinterpretation of Prop. 4.1 in [4], describes in particular the associated graded algebra  $Gr_{\partial}(B)$  of the filtered algebra  $(B, \mathcal{F})$  in the case where the  $\mathbb{N}$ -filtration  $\mathcal{G}$  is proper:

**Proposition 2.5.** *If the  $\mathbb{N}$ -filtration  $\mathcal{G}$  is proper then  $Gr_{\partial}(B) \simeq k^{[r+n+m]}/\hat{J}$ .*

*Proof.* By virtue of (Prop. 4.1 in [4]) the graded algebra associated to the filtered algebra  $(B, \mathcal{G})$  is isomorphic to  $k^{[r+n+m]}/\hat{J}$ . So the assertion follows from Lemma 2.3.  $\square$

### 3. Semi-Rigid and Rigid $k$ -Domains

In [6] D. Finston and S. Maubach considered rings  $B$  whose sets of locally nilpotent derivations are “one-dimensional” in the sense that  $\text{LND}(B) = \ker(\partial) \cdot \partial$  for some non-zero  $\partial \in \text{LND}(B)$ . They called them almost-rigid rings. Hereafter, we consider the following definition which seems more natural in our context (see Prop. 3.2 below for a comparison between the two notions).

**Definition 3.1.** A commutative domain  $B$  over a field  $k$  of characteristic zero is called *semi-rigid* if all non-zero locally nilpotent derivations of  $B$  induce the same proper  $\mathbb{N}$ -filtration (equivalently, the same  $\mathbb{N}$ -degree function).

Recall that the *Makar-Limanov invariant* of a commutative  $k$ -domain  $B$  over a field  $k$  of characteristic zero is defined by

$$\text{ML}(B) := \bigcap_{D \in \text{LND}(B)} \ker(D).$$

In particular,  $B$  is semi-rigid if and only if  $\text{ML}(B) = \ker(\partial)$  for any non-zero  $\partial \in \text{LND}(B)$ . Indeed, given  $D, E \in \text{LND}(B) \setminus \{0\}$  such that  $A := \ker(D) = \ker(E)$ , there exist non-zero elements  $a, b \in A$  such that  $aD = bE$  (see [1] Principle 12) which implies that the  $D$ -filtration is equal to the  $E$ -filtration. So if  $\text{ML}(B) = \ker(\partial)$  for any non-zero  $\partial \in \text{LND}(B)$  then  $B$  is semi-rigid. The other implication is clear by definition.

Recall that  $D \in \text{Der}_k(B)$  is *irreducible* if and only if  $D(B)$  is contained in no proper principal ideal of  $B$ , and that  $B$  is said to satisfy the ascending chain condition (ACC) on principal ideals if and only if every infinite chain  $(b_1) \subset (b_2) \subset (b_3) \subset \dots$  of principal ideals of  $B$  eventually stabilizes.  $B$  is said to be a *highest common factor ring*, or HCF-ring, if and only if the intersection of any two principal ideals of  $B$  is again principal.

**Proposition 3.2.** *Let  $B$  be a semi-rigid  $k$ -domain satisfying the ACC on principal ideals. If  $\text{ML}(B)$  is an HCF-ring, then there exists a unique irreducible  $\partial \in \text{LND}(B)$  up to multiplication by unit.*

*Proof.* Existence: since  $B$  satisfies the ACC on principal ideals, then for every non-zero  $T \in \text{LND}(B)$ , there exists an irreducible  $T_0 \in \text{LND}(B)$  and  $c \in \ker(T)$  such that  $T = cT_0$ . (see [1], Prop. 2.2 and Principle 7).

Uniqueness: the following argument is similar to that in [1] Prop. 2.2.b, but with a little difference, that is, in [1] it is assumed that  $B$  itself is an HCF-ring while here we only require that  $\text{ML}(B)$  is an HCF-ring.

Let  $D, E \in \text{LND}(B)$  be irreducible derivations, and denote  $A = \text{ML}(B)$ . By hypothesis  $\ker(D) = \ker(E) = A$ , so there exist non-zero  $a, b \in A$  such that  $aD = bE$  (see [1] Principle 12). Here we can assume that  $a, b$  are not units otherwise we are done. Set  $T = aD = bE$ . Since  $A$  is an HCF-ring, there exists  $c \in A$  such that  $aA \cap bA = cA$ . Therefore,  $T(B) \subset cB$ , and there exists  $T_0 \in \text{LND}(B)$  such that  $T = cT_0$ . Write  $c = as = bt$  for  $s, t \in B$ . Then  $cT_0 = asT_0 = aD$  implies  $D = sT_0$ , and likewise  $E = tT_0$ . By irreducibility,  $s$  and  $t$  are units of  $B$ , and we are done.  $\square$

In particular, for a ring  $B$  as in 3.2, every  $D \in \text{LND}(B)$  has the form  $D = f\partial$  for some irreducible  $\partial \in \text{LND}(B)$  and  $f \in \ker(\partial)$ , and so  $B$  is almost rigid in the sense introduced by Finston and Maubach.

Recall that a ring  $A$  is called *rigid* if the zero derivation is the only locally nilpotent derivation of  $A$ . Equivalently,  $A$  is rigid if and only if  $\text{ML}(A) = A$ . It is well known that the only non rigid  $k$ -domains of transcendence degree one are polynomial rings in one variable over algebraic extensions of  $k$  ([1] Corollary 1.24 and Corollary 1.29). In particular, we have the following elementary criterion for rigidity that we will use frequently in the sequel:

**Lemma 3.3.** *A domain  $B$  of transcendence degree one over a field  $k$  of characteristic zero is rigid if one of the following properties hold:*

- (1)  *$B$  is not factorial.*
- (2)  *$\text{Spec}(B)$  has a singular point.*

### 3.1. Elementary examples of semi-rigid $k$ -domains.

The next Proposition, which is due to Makar-Limanov ([7] Lemma 21, also [8] Theorem 3.1), presents some of the simplest examples of semi-rigid  $k$ -domains.

**Proposition 3.4. (Makar-Limanov)** *Let  $A$  be a rigid domain of finite transcendence degree over a field  $k$  of characteristic zero. Then the polynomial ring  $A[x]$  is semi-rigid.*

*Proof.* For the convenience of the reader, we provide an argument formulated in the LND-filtration language. Let  $\partial$  be the locally nilpotent derivation of  $A[x]$  defined by  $\partial(a) = 0$  for every  $a \in A$  and  $\partial(x) = 1$ . Then the  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by  $\mathcal{F}_i = Ax^i \oplus \mathcal{F}_{i-1}$  where  $\mathcal{F}_0 = \ker(\partial) = A$ . So the associated graded algebra is  $\text{Gr}(A[x]) = \bigoplus_{i \in \mathbb{N}} A\bar{x}^i$ , where  $\bar{x} := \text{gr}(x)$ . By Proposition 2.2, every non-zero  $D \in \text{LND}(A[x])$  respects the  $\partial$ -filtration and induces a non-zero homogeneous locally nilpotent derivation  $\bar{D}$  of  $\text{Gr}(A[x])$  of a certain degree  $d$ .

Let  $f \in \ker(D)$  and assume that  $f \notin A$ . Then  $\bar{x}$  divides  $\bar{f}$ . Since  $\bar{f} \in \ker(\bar{D})$  and  $\ker(\bar{D})$  is factorially closed ([1] Principle 1), we have  $\bar{x} \in \ker(\bar{D})$ . Note that  $\bar{D}$  sends homogeneous elements of degree  $i$  to zero or to homogeneous elements of degree  $i + d$ , therefore  $\bar{D}$  has the form  $\bar{D} = a\bar{x}^d E$ , where  $a \in A$ ,  $d \in \mathbb{N}$ , and  $E \in \text{LND}(A)$  see [1] Principle 7. But since  $A$  is rigid,  $E = 0$ . Thus  $\bar{D} = 0$ , a contradiction. This means  $f \in A$  which implies that  $\ker(D) \subset A$ . Finally, since



$\text{tr.deg}_k(A) = \text{tr.deg}_k(\ker(D))$  and  $A$  is algebraically closed in  $A[x]$ , we get the equality  $\ker(D) = A$ . Hence  $\text{ML}(A[x]) = A$ .  $\square$

#### 4. Computing the ML-Invariant using LND-Filtration

Here we illustrate the use of the  $\partial$ -filtration in the computation of ML-invariants, for classes of already well-studied examples.

##### 4.1. Classical examples of semi-rigid $k$ -domains.

We first consider in 4.1.1, certain surfaces in  $\mathbb{A}^{[3]}$  defined by equations  $X^n Z - P(Y)$  where  $n > 1$  and  $\deg_Y P(Y) > 1$ , which were first discussed by Makar-Limanov in [7], where he computed their ML-invariants. Later on Poloni [10] used similar methods to compute ML-invariants for a larger class. In the second example 4.1.2, we consider certain threefolds whose invariants were computed by S. Kaliman and L. Makar-Limanov [5] in the context of the linearization problem for  $\mathbb{C}^*$ -action on  $\mathbb{C}^3$ .

In these examples, the use of LND-filtrations is more natural and less tedious than other existing approaches.

##### 4.1.1. Danielewski hypersurfaces.

Let

$$B_{n,P} = k[X, Y, Z] / \langle X^n Z - P(X, Y) \rangle$$

where

$$P(X, Y) = Y^m + f_{m-1}(X)Y^{m-1} + \dots + f_0(X),$$

$f_i(X) \in k[X]$ ,  $n \geq 2$ , and  $m \geq 2$ .

Let  $x, y, z$  be the images of  $X, Y, Z$  in  $B_{n,P}$ . Define  $\partial$  by  $\partial(x) = 0$ ,  $\partial(y) = x^n$ ,  $\partial(z) = \frac{\partial P}{\partial y}$  where

$$\frac{\partial P}{\partial y} = my^{m-1} + (m-1)f_{m-1}(x)y^{m-2} + \dots + f_1(x)$$

We see that  $\partial \in \text{LND}(B_{n,P})$  with  $\ker(\partial) = k[x]$ , and  $y$  is a local slice for  $\partial$ . Moreover, we have  $\deg_\partial(x) = 0$ ,  $\deg_\partial(y) = 1$ ,  $\deg_\partial(z) = m$ . The plinth ideal is  $\text{pl}(\partial) = \langle x^n \rangle$ . Up to a change of variable of the form  $Y \mapsto Y - c$  where  $c \in k$ , we can always assume that  $0 \in \text{Spec}(B)$ . A consequence of Lemma 2.3, Proposition 2.4, and Proposition 2.5 (see the Proof of Prop. 5.1 for more details) is that

1- The  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by:

$$\mathcal{F}_{mi+j} = k[x]y^jz^i + \mathcal{F}_{mi+j-1}$$

where  $i \in \mathbb{N}$  and  $j \in \{0, \dots, m-1\}$ .

2- The associated graded algebra  $\text{Gr}(B_{n,P}) = \bigoplus_{i \in \mathbb{N}} \overline{B}_i$ , where  $\overline{B}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ , is generated by  $\overline{x} = gr_\partial(x)$ ,  $\overline{y} = gr_\partial(y)$ ,  $\overline{z} = gr_\partial(z)$  as an algebra over  $k$  with relation  $\overline{x}^n \overline{z} = \overline{y}^m$ , i.e.  $\text{Gr}(B_{n,P}) = k[\overline{X}, \overline{Y}, \overline{Z}] / \langle \overline{X}^n \overline{Z} - \overline{Y}^m \rangle$ . And we have :

$$\overline{B}_{mi+j} = k[\overline{x}]\overline{y}^j\overline{z}^i$$

where  $i \in \mathbb{N}$  and  $j \in \{0, \dots, m-1\}$ .

**Proposition 4.1.** *With the notation above we have:*

- (1)  $\text{ML}(B_{n,P}) = k[x]$ . Consequently  $B_{n,P}$  is semi-rigid.
- (2) Every  $D \in \text{LND}(B_{n,P})$  has the form  $D = f(x)\partial$ .

*Proof.* (1) By Proposition 2.2 a non-zero  $D \in \text{LND}(B)$  induces a non-zero  $\overline{D} \in \text{LND}(\text{Gr}(B))$ . Let  $f \in \ker(D) \setminus k$ , then  $\overline{f} \in \ker(\overline{D}) \setminus k$ . There exists  $i \in \mathbb{N}$  such that  $\overline{f} \in \overline{B}_i$ .

Assume that  $\overline{f} \notin k[\overline{x}] = \overline{B}_0$ , then one of the elements  $\overline{y}, \overline{z}$  must divide  $\overline{f}$ . Which leads to a contradiction as follows:

If  $\overline{y}$  divides  $\overline{f}$ , then  $\overline{y} \in \ker(\overline{D})$  because  $\ker(\overline{D})$  is factorially closed in  $\text{Gr}(B)$  ([1] Principle 1). For the same reason  $\overline{x}, \overline{z} \in \ker(\overline{D})$  because  $\overline{x}^n \overline{z} = \overline{y}^m$ . So  $\overline{D} = 0$ , a contradiction.

If  $\overline{z}$  divides  $\overline{f}$ , then  $\overline{D}(\overline{z}) = 0$ . So  $\overline{D}$  extends to a locally nilpotent derivation  $\tilde{D}$  of the ring  $\tilde{B} = k(\overline{z})[\overline{x}, \overline{y}] / \langle \overline{x}^n \overline{z} - \overline{y}^m \rangle$ . Since  $0 \in \text{Spec}(B)$  is a singular point when  $n \geq 2$  and  $m \geq 2$ ,  $\tilde{B}$  is rigid (Lemma 3.3). Therefore,  $\tilde{D} = 0$  which means  $\overline{D} = 0$ , a contradiction.

So the only possibility is that  $\overline{f} \in k[\overline{x}]$ . This means that  $\deg_{\partial}(f) = 0$ , and hence that  $f \in k[x]$ . So  $\ker(D) \subset k[x]$ , and finally  $k[x] = \ker(D)$  because  $\text{tr.deg}_k(\ker(D)) = 1$  and  $k[x]$  is algebraically closed in  $B$ . So we get  $\text{ML}(B) = k[x]$ .

(2) is an immediate consequence of Proposition 3.2.  $\square$

#### 4.1.2. Koras-Russell hypersurfaces of the second type.

Here we consider hypersurfaces associated with  $k$ -algebras of the form:

$$B_{n,e,l,Q} = k[X, Y, Z, T] / \langle Y(X^n + Z^e)^l - Q(X, Z, T) \rangle$$

where

$$Q(X, Z, T) = T^m + f_1(X, Z)T^{m-1} + \dots + f_m(X, Z)$$

$f_i(X, Z) \in k[X, Z]$ ,  $n > 1$ ,  $e > 1$ ,  $l > 1$ , and  $m > 1$ . We may assume without loss of generality that  $Q(0, 0, 0) = 0$ . A particular case of this family corresponds to the so called Koras-Russell hypersurfaces of the second type considered by S. Kaliman and L. Makar-Limanov ([5]) where they computed their ML-invariants. Here we explain how to apply the LND-filtration method to compute this invariant for all algebras  $B_{n,e,l,Q}$ .

Let  $x, y, z, t$  be the images of  $X, Y, Z, T$  in  $B_{n,e,l,Q}$ . Define  $\partial$  by

$$\partial = \frac{\partial Q}{\partial t} \partial_y + (X^n + Z^e)^l \partial_t$$

We see that  $\partial \in \text{LND}(B_{n,e,l,Q})$  with  $\ker(\partial) = k[x, z]$ , and  $t$  is a local slice for  $\partial$ . Moreover, we have  $\deg_{\partial}(x) = 0$ ,  $\deg_{\partial}(y) = m$ ,  $\deg_{\partial}(z) = 0$ , and  $\deg_{\partial}(t) = 1$ . The plinth ideal is  $\text{pl}(\partial) = \langle (X^n + Z^e)^l \rangle$ . By Lemma 2.3, Prop. 2.4, and Prop. 2.5 we get the following.

1- The  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by:

$$\mathcal{F}_{mi+j} = k[x, z]t^j y^i + \mathcal{F}_{mi+j-1}$$

where  $i \in \mathbb{N}$ , and  $j \in \{0, \dots, m-1\}$ .

2- The associated graded algebra  $\text{Gr}(B_{n,e,l,Q}) = \bigoplus_{i \in \mathbb{N}} \overline{B}_i$ , where  $\overline{B}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ , is generated by  $\overline{x} = gr_{\partial}(x)$ ,  $\overline{y} = gr_{\partial}(y)$ ,  $\overline{z} = gr_{\partial}(z)$ ,  $\overline{t} = gr_{\partial}(t)$  as an algebra over  $k$  with the relation  $\overline{y}(\overline{x}^n + \overline{z}^e)^l = \overline{t}^m$ , i.e.  $\text{Gr}(B_{n,e,l,Q}) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{T}] / \langle \overline{Y}(\overline{X}^n + \overline{Z}^e)^l - \overline{T}^m \rangle$ . And we have :

$$\overline{B}_{mi+j} = k[\overline{x}, \overline{z}] \overline{t}^j \overline{y}^i$$

where  $i \in \mathbb{N}$ , and  $j \in \{0, \dots, m-1\}$ .

**Proposition 4.2.** *With the notation above we have:*

- (1)  $\text{ML}(B_{n,e,l,Q}) = k[x, z]$ . *Consequently  $B$  is semi-rigid.*
- (2) *Every  $D \in \text{LND}(B_{n,e,l,Q})$  has the form  $D = f(x, z)\partial$ .*

*Proof.* (1) Given a non-zero  $D \in \text{LND}(B)$ . By Proposition 2.2  $D$  induces a non-zero  $\overline{D} \in \text{LND}(\text{Gr}(B))$ . Suppose that  $f \in \ker(D) \setminus k$ , then  $\overline{f} \in \ker(\overline{D}) \setminus k$ . So there exists  $i \in \mathbb{N}$  such that  $\overline{f} \in \overline{B}_i$ . Assume that  $\overline{f} \notin k[\overline{x}, \overline{z}] = \overline{B}_0$ , then one of the elements  $\overline{t}, \overline{y}$  must divides  $\overline{f}$  (see §4.1.2 above). Which leads to a contradiction as follows:

If  $\overline{t}$  divides  $\overline{f}$ , then  $\overline{t} \in \ker(\overline{D})$  as  $\ker(\overline{D})$  is factorially closed, and for the same reason  $\overline{y}, (\overline{x}^n + \overline{z}^e)^l \in \ker(\overline{D})$  due to the relation  $\overline{y}(\overline{x}^n + \overline{z}^e)^l = \overline{t}^m$ . So  $\overline{x}^n + \overline{z}^e \in \ker(\overline{D})$  which implies that  $\overline{x}, \overline{z} \in \ker(\overline{D})$  ([1] Lemma 9.3). This means  $\overline{D} = 0$ , a contradiction.

Finally, if  $\overline{y}$  divides  $\overline{f}$ , then  $\overline{D}(\overline{y}) = 0$ . Choose  $H \in \ker(\overline{D})$  which is homogeneous and algebraically independent of  $\overline{y}$ , which is possible, since  $\text{tr.deg}_k \ker(\overline{D}) = 2$  and  $\ker(\overline{D})$  is generated by homogeneous elements. Then by §4.1.2,  $H$  has the form  $H = h(\overline{x}, \overline{z}) \cdot \overline{y}^l$ . By algebraic dependence, we may assume  $H = h(\overline{x}, \overline{z})$ , which is non-constant, and that  $h(0, 0) = 0$ . So  $\overline{D}$  extends to a locally nilpotent derivation  $\tilde{D}$  of the ring  $\tilde{B} = k(\overline{y}, H)[\overline{x}, \overline{z}, \overline{t}] / \langle h(\overline{x}, \overline{z}) - H, \overline{y}(\overline{x}^n + \overline{z}^e)^l - \overline{t}^m \rangle$ . But  $\tilde{B}$  is of transcendence degree one over the field  $k(\overline{y}, H)$  whose spectrum has a singular point at 0. This means that  $\tilde{B}$  is rigid (Lemma 3.3). Thus  $\tilde{D} = 0$ , which implies  $\overline{D} = 0$ , a contradiction.

So the only possibility is that  $\overline{f} \in k[\overline{x}, \overline{z}]$ , and this means  $\deg_\partial(f) = 0$ , thus  $f \in k[x, z]$  and  $\ker(D) \subset k[x, z]$ . Finally,  $k[x, z] = \ker(D)$  because  $\text{tr.deg}_k(\ker(D)) = 2$ . So we get  $\text{ML}(B) = k[x, z]$

(2) follows again from Proposition 3.2. □

## 5. A new class of semi-rigid rings

In this section, we use the LND-filtration method to establish the semi-rigidity of new families of two dimensional domains of the form

$$R = k[X, Y, Z] / \langle X^n Y - P(X, Q(X, Y) - X^e Z) \rangle$$

for suitable integers  $n, e \geq 2$  and polynomials  $P(X, T), Q(X, T) \in k[X, T]$ . They share with Danielewski hypersurfaces discussed in 4.1.1 above, the property to come naturally equipped with an irreducible locally nilpotent derivation induced by a locally nilpotent derivation of  $k[X, Y, Z]$ . But in contrast with the Danielewski hypersurfaces case, the corresponding derivation on  $k[X, Y, Z]$  are no longer triangular, in fact not even triangulable by virtue of characterization due to Daigle [9].

We will begin with a very elementary example illustrating the steps needed to determine the LND-filtration and its associated graded algebra, and then we proceed to the general case.

### 5.1. A toy example.

We let

$$R = k[X, Y, Z] / \langle X^2 Y - (Y^2 - XZ)^2 \rangle$$

and we let  $x, y, z$  be the images of  $X, Y, Z$  in  $R$ . A direct computation reveals that the derivation

$$2XS\partial_Y + (4YS - X^2)\partial_Z$$

of  $k[X, Y, Z]$  where  $S := Y^2 - XZ$  is locally nilpotent and annihilates the polynomial  $X^2Y - (Y^2 - XZ)^2$ . It induces a locally nilpotent derivation  $\partial$  of  $R$  for which we have  $\partial(x) = 0$ ,  $\partial^3(y) = 0$ ,  $\partial^5(z) = 0$ . Furthermore, the element  $s = y^2 - xz$  is a local slice for  $\partial$  with  $\partial(s) = x^3$ . The kernel of  $\partial$  is  $k[x]$  and the plinth ideal is the principal ideal generated by  $x^3$ . We have  $\deg_\partial(x) = 0$ ,  $\deg_\partial(y) = 2$ ,  $\deg_\partial(z) = 4$ ,  $\deg_\partial(s) = 1$ .

**Proposition 5.1.** *With the notation above, we have:*

(1) *The  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by :*

$$\mathcal{F}_{4i+2j+l} = k[x]s^l y^j z^i + \mathcal{F}_{4i+2j+l-1}$$

where  $i \in \mathbb{N}$ ,  $j \in \{0, 1\}$ ,  $l \in \{0, 1\}$ .

(2) *The associated graded algebra  $\text{Gr}_\partial(R) = \bigoplus_{i \in \mathbb{N}} \overline{R}_i$ , where  $\overline{R}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ , is generated by  $\overline{x} = \text{gr}_\partial(x)$ ,  $\overline{y} = \text{gr}_\partial(y)$ ,  $\overline{z} = \text{gr}_\partial(z)$ ,  $\overline{s} = \text{gr}_\partial(s)$  as an algebra over  $k$  with relations  $\overline{x}^2 \overline{z} = \overline{s}^2$  and  $\overline{x} \overline{z} = \overline{y}^2$ , i.e.  $\text{Gr}_\partial(R) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{S}] / \langle \overline{X}^2 \overline{Z} - \overline{S}^2, \overline{X} \overline{Z} - \overline{Y}^2 \rangle$ . Furthermore:*

$$\overline{R}_{4i+2j+l} = k[\overline{x}] \overline{s}^l \overline{y}^j \overline{z}^i$$

where  $i \in \mathbb{N}$ ,  $j \in \{0, 1\}$ ,  $l \in \{0, 1, 2, 3\}$ .

*Proof.* 1) First, the  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by  $\mathcal{F}_r = \sum_{h \leq r} H_h$  where  $H_h := \sum_{u+2v+4w=h} k[x](s^u y^v z^w)$  and  $u, v, w, h \in \mathbb{N}$ . To show this, let  $J$  be the ideal in  $k^{[4]} = k[X, Y, Z, S]$  defined by  $J = (X^2Y - (Y^2 - XZ)^2, Y^2 - XZ - S)$ . Define an  $\mathbb{N}$ -degree function  $\omega$  on  $k^{[4]}$  by declaring that  $\omega(X) = 0$ ,  $\omega(S) = 1$ ,  $\omega(Y) = 2$ , and  $\omega(Z) = 4$ . By Lemma 2.4, the  $\mathbb{N}$ -filtration  $\{\mathcal{G}_r\}_{r \in \mathbb{N}}$  where  $\mathcal{G}_r = \sum_{h \leq r} H_h$  is proper if and only if  $\hat{J}$  is prime. Which is the case since  $\hat{J} = \langle X^2Y - S^2, Y^2 - XZ \rangle$  is prime. Thus by Lemma 2.3 we get the desired description.

Second, let  $l \in \{0, 1\}$  and  $j \in \{0, 1, 2, 3\}$  be such that  $l := r \bmod 2$ ,  $j := r - l \bmod 4$ , and  $i := \frac{r-2j-l}{4}$ . Then we get the following unique expression  $r = 4i + 2j + l$ . Since  $\mathcal{F}_r = \sum_{u+2v+4w=r} k[x](s^u y^v z^w) + \mathcal{F}_{r-1}$ , we conclude in particular that  $\mathcal{F}_r \supseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$ . For the other inclusion, the relation  $x^2y = s^2$  allows to write  $s^u y^v z^w = x^e s^l y^{v_0} z^w$  and from the relation  $y^2 = s + xz$  we get  $x^e s^l y^{v_0} z^w = x^e s^l y^j (s + xz)^n z^w$ . Since the monomial with the highest degree relative to  $\deg_\partial$  in  $(s + xz)^n$  is  $x^n z^n$ , we deduce that  $x^e s^l y^j (s + xz)^n z^w = x^{e+n} s^l y^j z^{w+n} + \sum M_\beta$  where  $M_\beta$  is monomial in  $x, y, s, z$  of degree less than  $r$ . Since the expression  $r = 4i + 2j + l$  is unique, we get  $w + n = i$ . So  $s^u y^v z^w = x^{e+n} s^l y^j z^i + f$  where  $f \in \mathcal{F}_{r-1}$ . Thus  $k[x](s^u y^v z^w) \subseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$  and finally  $\mathcal{F}_r = k[x]s^l y^j z^i + \mathcal{F}_{r-1}$ .

2) By part (1), an element  $f$  of degree  $r$  can be written as  $f = g(x)s^l y^j z^i + f_0$  where  $f_0 \in \mathcal{F}_{r-1}$ ,  $l = r \bmod 2$ ,  $j = r - l \bmod 4$ ,  $i = \frac{r-2j-l}{4}$ , and  $i \in \mathbb{N}$ ,  $j \in \{0, 1\}$ ,  $l \in \{0, 1\}$ . So by Lemma 1.4, P2, P1, P3 respectively we get

$$\overline{f} = \overline{g(x)s^l y^j z^i + f_0} = \overline{g(x)s^l y^j z^i} = \overline{g(x)} \overline{s}^l \overline{y}^j \overline{z}^i = g(\overline{x}) \overline{s}^l \overline{y}^j \overline{z}^i$$

and therefore  $\overline{B}_{4i+2j+l} = k[\overline{x}] \overline{s}^l \overline{y}^j \overline{z}^i$ .

Finally, by Proposition 2.5,  $\text{Gr}_\partial(B) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{S}] / \langle \overline{X}^2 \overline{Z} - \overline{S}^2, \overline{X} \overline{Z} - \overline{Y}^2 \rangle$ .  $\square$

## 5.2. A more general family.

We now consider more generally rings  $R$  of the form

$$k[X, Y, Z] / \langle X^n Y - P(X, Q(X, Y) - X^e Z) \rangle$$

where

$$P(X, S) = S^d + f_{d-1}(X)S^{d-1} + \cdots + f_1(X)S + f_0(X)$$

$$Q(X, Y) = Y^m + g_{m-1}(X)Y^{m-1} + \cdots + g_1(X)Y + g_0(X)$$

$n \geq 2$ ,  $d \geq 2$ ,  $m \geq 1$ , and  $e \geq 1$ . Up to a change of variable of the form  $Y \mapsto Y - c$  where  $c \in k$ , we may assume that  $0 \in \text{Spec}(R)$ .

Let  $x, y, z$  be the images of  $X, Y, Z$  in  $R$ . Define  $\partial$  by:  $\partial(x) = 0$ ,  $\partial(s) = x^{n+e}$  where  $s := Q(x, y) - x^e z$ . Considering the relation  $x^n y = P(x, Q(x, y) - x^e z)$ , a simple computation leads to  $\partial(y) = x^e \frac{\partial P}{\partial s}$ ,  $\partial(z) = \frac{\partial Q}{\partial y} \frac{\partial P}{\partial s} - x^n$ , i.e.

$$\partial := x^e \frac{\partial P}{\partial s} \partial_y + \left( \frac{\partial Q}{\partial y} \frac{\partial P}{\partial s} - x^n \right) \partial_z$$

where  $\frac{\partial P}{\partial s} = ds^{d-1} + (d-1)f_{d-1}(x)s^{d-2} + \cdots + f_1(x)$ , and  $\frac{\partial Q}{\partial y} = my^{m-1} + (m-1)g_{m-1}(x)y^{m-2} + \cdots + g_1(x)$ . Since  $\partial(x^n y - P(x, Q(x, y) - x^e z)) = 0$  and  $\partial^{d+1}(y) = 0$ ,  $\partial^{md+1}(z) = 0$ ,  $\partial$  is a well-defined locally nilpotent derivation of  $R$ . The kernel of  $\partial$  is equal to  $k[x]$  and the element  $s$  is a local slice for  $\partial$  by construction. One checks further that the plinth ideal is equal to  $\text{pl}(\partial) = \langle x^{n+e} \rangle$ . A direct computation shows that  $\deg_\partial(x) = 0$ ,  $\deg_\partial(y) = d$ ,  $\deg_\partial(z) = md$  and  $\deg_\partial(s) = 1$ . Furthermore:

1- The  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by :

$$\mathcal{F}_{mdi+dj+l} = k[x]s^l y^j z^i + \mathcal{F}_{mdi+dj+l-1}$$

where  $i \in \mathbb{N}$ ,  $j \in \{0, \dots, m-1\}$ ,  $l \in \{0, \dots, d-1\}$ .

2- The associated graded algebra  $\text{Gr}(R) = \bigoplus_{i \in \mathbb{N}} \overline{R}_i$ , where  $\overline{R}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ , is generated by  $\overline{x} = gr_\partial(x)$ ,  $\overline{y} = gr_\partial(y)$ ,  $\overline{z} = gr_\partial(z)$ ,  $\overline{s} = gr_\partial(s)$  as an algebra over  $k$  with relations  $\overline{x}^n \overline{z} = \overline{s}^d$  and  $\overline{x}^e \overline{z} = \overline{y}^m$ , i.e.  $\text{Gr}(R) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{S}] / \langle \overline{X}^n \overline{Z} - \overline{S}^d, \overline{X}^e \overline{Z} - \overline{Y}^m \rangle$ .

And we have :

$$\overline{R}_{mdi+dj+l} = k[\overline{x}] \overline{s}^l \overline{y}^j \overline{z}^i$$

where  $i \in \mathbb{N}$ ,  $j \in \{0, \dots, m-1\}$ ,  $l \in \{0, \dots, d-1\}$ .

**Theorem 5.2.** *With the above notation the following hold:*

- (1)  $\text{ML}(R) = k[x]$ . Consequently  $R$  is semi-rigid.
- (2) Every  $D \in \text{LND}(R)$  has form  $D = f(x)\partial$ , i.e.  $R$  is almost rigid.

*Proof.* (1) Given a non-zero  $D \in \text{LND}(R)$ . By Proposition 2.2,  $D$  respects the  $\partial$ -filtration and induces a non-zero locally nilpotent derivation  $\overline{D}$  of  $\text{Gr}(R)$ . Suppose that  $f \in \ker(D) \setminus k$ , then  $\overline{f} \in \ker(\overline{D}) \setminus k$  is an homogenous element of  $\text{Gr}(R)$ . So there exists  $i \in \mathbb{N}$  such that  $\overline{f} \in \overline{R}_i$ .

Assume that  $\overline{f} \notin k[\overline{x}] = \overline{R}_0$ , then one of the elements  $\overline{s}, \overline{y}, \overline{z}$  must divides  $\overline{f}$  by 5.2.2. Which leads to a contradiction as follows :

If  $\overline{s}$  divides  $\overline{f}$ , then  $\overline{s} \in \ker(\overline{D})$  as  $\ker(\overline{D})$  is factorially closed, and for the same reason  $\overline{x}, \overline{y} \in \ker(\overline{D})$  due to the relation  $\overline{x}^n \overline{y} = \overline{s}^d$ . Then by the relation  $\overline{x}^e \overline{z} = \overline{y}^m$ , we must have  $\overline{z} \in \ker(\overline{D})$ , which means  $\overline{D} = 0$ , a contradiction. In the same way, we get a contradiction if  $\overline{y}$  divides  $\overline{f}$ .

Finally, if  $\overline{z}$  divides  $\overline{f}$ , then  $\overline{D}(\overline{z}) = 0$ . So  $\overline{D}$  induces in a natural way a locally nilpotent derivation  $\tilde{D}$  of the ring  $\tilde{R} = k(\overline{z})[\overline{x}, \overline{y}, \overline{s}] / \langle \overline{x}^n \overline{z} - \overline{s}^d, \overline{x}^e \overline{z} - \overline{y}^m \rangle$ . But since  $0 \in \text{Spec}(\tilde{R})$  is a singular point,  $\tilde{R}$  is rigid (Lemma 3.3). So  $\tilde{D} = 0$ , which implies  $\overline{D} = 0$ , a contradiction.

So the only possibility is that  $\overline{f} \in k[x]$ , and this means  $\deg_{\partial}(f) = 0$ , thus  $f \in k[x]$  and  $\ker(D) \subset k[x]$ . Finally,  $k[x] = \ker(D)$  because  $\text{tr.deg}_k(\ker(D)) = 1$  and  $k[x]$  is algebraically closed in  $B$ . So  $\text{ML}(R) = k[x]$ .

(2) follows again from Proposition 3.2.  $\square$

## 6. Further applications of the LND-filtration

Given a commutative domain  $B$  over an algebraically closed field  $k$  of characteristic zero, we denote  $\text{Aut}_k(B)$  the group of algebraic  $k$ -automorphisms of  $B$ . This group acts by conjugation on  $\text{LND}(B)$ . An immediate consequence is that  $\alpha(\text{ML}(B)) = \text{ML}(B)$  for every  $\alpha \in \text{Aut}_k(B)$  which yield in particular an induced action of  $\text{Aut}_k(B)$  on  $\text{ML}(B)$ . Let  $\partial_{\alpha} = \alpha^{-1}\partial\alpha$  be the conjugate of  $\partial$  by a given automorphism  $\alpha$  of  $B$ , it is straightforward to check that  $\alpha\{\ker(\partial_{\alpha})\} = \ker(\partial)$  and more generally that  $\deg_{\partial_{\alpha}}(b) = \deg_{\partial}(\alpha(b))$  for any  $b \in B$ . In other words,  $\alpha$  respects  $\deg_{\partial}$  and  $\deg_{\partial_{\alpha}}$  (i.e.  $\alpha$  sends an element of degree  $n$  relative to  $\deg_{\partial_{\alpha}}$ , to an element of the same degree  $n$  relative to  $\deg_{\partial}$ ).

**Definition 6.1.** We say that an algebraic  $k$ -automorphism  $\alpha$  preserves the  $\partial$ -filtration for some  $\partial \in \text{LND}(B)$  if  $\deg_{\partial}(\alpha(b)) = \deg_{\partial}(b)$  for any  $b \in B$ .

**Lemma 6.2.** Let  $\partial \in \text{LND}(B)$  and  $\alpha \in \text{Aut}_k(B)$ . Then  $\partial$  and  $\partial_{\alpha}$  are equivalent, i.e. have the same kernel, if and only if  $\alpha$  preserve the  $\partial$ -filtration.

*Proof.* Suppose that  $\partial$  and  $\partial_{\alpha}$  are equivalents, then  $\deg_{\partial}(\alpha(b)) = \deg_{\partial_{\alpha}}(b)$  for every  $b \in B$ , and by hypothesis  $\ker(\partial) = \ker(\partial_{\alpha})$ , so  $\deg_{\partial} = \deg_{\partial_{\alpha}}$ . Then we obtain  $\deg_{\partial}(\alpha(b)) = \deg_{\partial}(b)$ . Thus  $\alpha$  preserves the  $\partial$ -filtration. Since the other direction is obvious we are done.  $\square$

The following Corollary shows a nice property of a semi-rigid ring. That is, every algebraic automorphism  $\alpha$  has to preserve the unique filtration induced by any locally nilpotent derivation  $\partial$ , i.e.  $\alpha$  sends an element of degree  $i$  relative to  $\partial$  to an element of the same degree relative to  $\partial$ . Which makes the computation of the group of automorphisms easier up to the automorphism group of  $\ker(\partial)$ .

**Corollary 6.3.** Let  $B$  be a semi-rigid  $k$ -domain, then every  $k$ -automorphism of  $B$  preserve the  $\partial$ -filtration for every  $\partial \in \text{LND}(B)$ .

*Proof.* A direct consequence of Definition 3.1 and Lemma 6.2.  $\square$

### 6.1. The group of algebraic $k$ -automorphisms of a semi-rigid $k$ -domain.

Suppose that  $B$  is a semi-rigid  $k$ -domain. Then, it has a unique proper filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  which is the  $\partial$ -filtration corresponding to any non-zero locally nilpotent derivation  $\partial$  of  $B$ . Since every algebraic  $k$ -automorphism of  $B$  preserves this filtration (Corollary 6.3), we obtain an exact sequence

$$0 \rightarrow \text{Aut}_k(B, \text{ML}(B)) \rightarrow \text{Aut}_k(B) \rightarrow \text{Aut}_k(\text{ML}(B))$$

where  $\text{Aut}_k(B, \text{ML}(B))$  is by definition the sub-group of  $\text{Aut}_k(B)$  consisting of elements whose induced action on  $\text{ML}(B)$  is trivial. Furthermore, every element of  $\text{Aut}_k(B, \text{ML}(B))$  induces for every  $i \geq 1$  an automorphism of  $\mathcal{F}_0$ -module of each  $\mathcal{F}_i$ . In this section we illustrate how to exploit these information to compute  $\text{Aut}_k(B)$  for certain semi-rigid  $k$ -domains  $B$ .

### 6.1.1. $\text{Aut}_k$ for example 4.1.1.

In [7] Makar-Limanov computed the  $k$ -automorphism group for surfaces in  $k^{[3]}$  defined by equation  $X^n Z - P(Y) = 0$  where  $n > 1$  and  $\deg_Y P(Y) > 1$ . Then Poloni, see [10], generalized Makar-Limanov's method to obtained similar results for the rings considered in 4.1.1 above. Here we briefly indicate how to recover these results using LND-filtrations. So let

$$B_{n,P} = k[X, Y, Z] / \langle X^n Z - P(X, Y) \rangle$$

where  $P(X, Y) = Y^m + f_{m-1}(X)Y^{m-1} + \dots + f_0(X)$ ,  $f_i(X) \in k[X]$ ,  $n \geq 2$ , and  $m \geq 2$ . Up to change of variable of the form  $Y$  by  $Y - \frac{f_{m-1}(X)}{m}$  we may assume without loss of generality that  $f_{m-1}(X) = 0$ .

**Proposition 6.4.** *Let  $B_{n,P}$  be as above. Then every algebraic  $k$ -automorphism  $\alpha$  of  $B$  has the form*

$$\alpha(x, y, z) = (\lambda x, \mu y + x^n a(x), \frac{\mu^m}{\lambda^n} z + \frac{P(\lambda x, \mu y + x^n a(x)) - \mu^m P(x, y)}{\lambda^n x^n})$$

where  $\lambda, \mu \in k^*$  satisfy  $f_{m-i}(\lambda x) \equiv \mu^i \cdot f_{m-i}(x) \pmod{x^n}$  for all  $i$ , and  $a(x) \in k[x]$ .

*Proof.* By Proposition 4.1, (1) and §4.1.1,  $\text{ML}(B) = k[x]$  and the  $\partial$ -filtration  $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$  is given by  $\mathcal{F}_{im+j} = k[x]y^{im+j} + k[x]y^{im+j-m}z + \dots + k[x]y^j z^i + \mathcal{F}_{im+j-1}$ , where  $\partial = x^n \cdot \partial_y + \frac{\partial P}{\partial y} \cdot \partial_z$ ,  $\deg_\partial(x) = 0$ ,  $\deg_\partial(y) = 1$ , and  $\deg_\partial(z) = m$ . In particular  $\mathcal{F}_0 = k[x]$ ,  $\mathcal{F}_1 = k[x]y + \mathcal{F}_0$ , and  $\mathcal{F}_m = k[x]y^m + k[x]z + \mathcal{F}_{m-1}$ .

Now by Corollary 6.3  $\alpha$  preserve  $\deg_\partial$ , so we must have  $\alpha(x) \in \mathcal{F}_0 = k[x]$ ,  $\alpha(y) \in \mathcal{F}_1 = k[x]y + k[x]$  and  $\alpha(z) \in \mathcal{F}_m = k[x]y^m + k[x]z + \mathcal{F}_{m-1}$ . Since  $\alpha$  is invertible we get  $\alpha(x) = \lambda x + c$ ,  $\alpha(y) = \mu y + b(x)$ , and  $\alpha(z) = \xi z + h(x, y)$  where  $\lambda, \mu, \xi \in k^*$ ,  $c \in k$ ,  $b \in k[x]$ ,  $h(x, y) \in k[x, y]$ , and  $\deg_y h(x, y) \leq m$ .

By Proposition 4.1 (2) every  $D \in \text{LND}(B)$  has the form  $D = f(x)\partial$ . In particular,  $\partial_\alpha = f(x)\partial$  for some  $f(x) \in k[x]$ . Since  $\alpha\partial_\alpha = \partial\alpha$  we have  $\partial(\alpha(y)) = \alpha(f(x)\partial(y)) = f(\alpha(x))\alpha(x^n)$  where  $(\partial(y) = x^n)$ . So we get  $\partial(\mu y + b(x)) = f(\alpha(x))(\lambda x + c)^n$ . Since  $\partial(\mu y + b(x)) = \mu x^n$ ,  $x$  divides  $(\lambda x + c)^n$  in  $k[x]$ , and this is possible only if  $c = 0$ , so we get  $\alpha(x) = \lambda x$ .

Applying  $\alpha$  to the relation  $x^n z = P(x, y)$  in  $B_{P,n}$ , we get  $\lambda^n x^n \alpha(z) = P(\lambda x, \mu y + b(x)) = \mu^m P(x, y) + m\mu^{m-1}y^{m-1}b(x) + H(x, y)$  where  $\deg_y H \leq m-2$ . Since  $x^n$  divides  $P = x^n z$  and  $\deg_y H \leq m-2$ ,  $x^n$  divides  $m\mu^{m-1}y^{m-1}b(x) + H(x, y)$  in  $k[x, y]$ . So  $x^n$  divides  $b(x)$ , i.e.  $\alpha(y) = \mu y + x^n a(x)$ .

In addition,  $x^n$  divides every coefficient of  $H$  as a polynomial in  $y$ , so  $x^n$  divides  $-\mu^m f_{m-i}(x) + \mu^{m-i} f_{m-i}(\lambda x)$  because coefficients of  $H(y)$  are of the form  $q(x, y)b(x) - \mu^m f_{m-i}(x) + \mu^{m-i} f_{m-i}(\lambda x)$  and  $b(x)$  is divisible by  $x^n$ . So  $x^n$  divides  $-\mu^i f_{m-i}(x) + f_{m-i}(\lambda x)$  for every  $i$ . And we are done.  $\square$

### 6.1.2. $\text{Aut}_k$ for example 5.2.

The same method as in 6.1.1 can be applied to compute automorphism groups of rings  $R$  defined as in Theorem 5.2. For simplicity we only deal with the case where  $Q(X, Y) = Y^m$ , the general case can be deduced in the same way at the cost of longer and more complicated computation. Again we make a substitution in  $S$  as in 6.1.1 to get relation of the form presented in the following result:

**Theorem 6.5.** *Let  $R$  denote the ring  $R = k[X, Y, Z] / \langle X^n Y - P(X, Y^m - X^e Z) \rangle = k[x, y, z]$  where  $P(X, S) = S^d + f_{d-2}(X)S^{d-2} + \dots + f_1(X)S + f_0(S)$ ,  $n \geq 2$ ,  $d \geq 2$ ,*

$m \geq 1$ , and  $e \in \mathbb{N}$ . Then, every algebraic  $k$ -automorphism of  $B$  has the form

$$\alpha(x, y, z) = (\lambda x, \frac{\mu^d}{\lambda^n} y + F, \frac{\mu^{dm}}{\lambda^{nm+e}} z + \frac{(\frac{\mu^d}{\lambda^n} y + F)^m - \frac{\mu^{dm}}{\lambda^{nm}} y^m + x^{n+e} a(x)}{\lambda x^e})$$

where:

$\lambda, \mu \in k^*$  verify both  $\frac{\mu^{dm}}{\lambda^{nm}} = \mu$  and  $f_{d-i}(\lambda x) \equiv \mu^i f_{d-i}(x) \pmod{x^{n+e}}$  for every  $i \in \{2, \dots, d\}$ .  $s = y^m - x^e z$ , and  $F = \frac{P(\lambda x, \mu s + x^{n+e} a(x)) - \mu^d P(x, s)}{\lambda^n x^n}$ ,  $a(x) \in k[x]$ .

*Proof.* A similar argument as in the proof of Proposition 6.4 leads to  $\alpha(x) = \lambda x$  and  $\alpha(s) = \mu s + x^n a(x)$  where  $\lambda, \mu \in k^*$  verify  $f_{d-i}(\lambda x) \equiv \mu^i f_{d-i}(x) \pmod{x^n}$  for all  $i$ . Now  $\alpha(x)$  and  $\alpha(s)$  determine

$$\alpha(y) = \frac{\mu^d}{\lambda^n} y + \frac{P(\lambda x, \mu s + x^n a(x)) - \mu^d P(x, s)}{\lambda^n x^n}.$$

Apply  $\alpha$  to  $x^e z = y^m - s$  to get  $\lambda^e x^e \alpha(z) = (\frac{\mu^h}{\lambda^n} y + F)^m - \mu s - x^n a(x)$  where  $F = \frac{P(\lambda x, \mu s + x^n a(x)) - \mu^h P(x, s)}{\lambda^n x^n} \in k[x, s]$ . So we have  $\lambda^e x^e \alpha(z) = [\frac{\mu^{hm}}{\lambda^{nm}} y^m - \frac{\mu^{hm}}{\lambda^{nm}} s] + (\frac{\mu^{hm}}{\lambda^{nm}} - \mu) s + m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \dots + F^m - x^n a(x)$ . Since  $\frac{\mu^{hm}}{\lambda^{nm}} y^m - \frac{\mu^{hm}}{\lambda^{nm}} s = \frac{\mu^{hm}}{\lambda^{nm}} x^e z$ , we see that  $x^e$  divides  $G := (\frac{\mu^{hm}}{\lambda^{nm}} - \mu) s + m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \dots + F^m - x^n a(x)$  in  $k[x, s, y] \subset B$  because  $F \in k[x, s]$ . Thus  $x^e$  divides every coefficients of  $G$  as a polynomial in  $y$  because  $\deg_s G \leq d-1$ . So  $x^e$  divides  $F$  and  $\frac{\mu^{hm}}{\lambda^{nm}} - \mu = 0$ . This means that  $x^{n+e}$  divides  $f_{d-i}(\lambda x) - \mu^i f_{d-i}(x)$  for all  $i \in \{2, \dots, d\}$  (see proof of Proposition 6.4), and  $\frac{\mu^{hm}}{\lambda^{nm}} = \mu$ . Finally, by the relation  $s = y^m - x^e z$ , we get  $\alpha(z) = \frac{\mu^{hm}}{\lambda^{nm+e}} z + \frac{m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \dots + F^m + x^n a(x)}{\lambda x^e}$ , and we are done.  $\square$

## 6.2. Isomorphisms.

We are going to use the previous facts about semi-rigid  $k$ -domains to give a necessary and sufficient condition for two hypersurfaces of the family 4.1.1 to be isomorphic.

Let  $\Psi : A \rightarrow B$  be an algebraic isomorphism, we refer to this by  $\Psi \in \text{Isom}_k(A, B)$ , between two finitely generated  $k$ -domains  $A = k[y_1, \dots, y_r]$ ,  $B = k[x_1, \dots, x_r]$  where  $y_1, \dots, y_r$  and  $x_1, \dots, x_r$  are minimal sets of generators. Since  $\Psi(A) = k[\Psi(y_1), \dots, \Psi(y_r)] = k[x_1, \dots, x_r]$ , there exists an automorphism  $\psi : B \rightarrow B$  such that for every  $i \in \{1, \dots, r\}$  there exists  $j \in \{1, \dots, r\}$  such that  $\psi(x_i) = \Psi(y_j)$ .

Given  $\partial \in \text{LND}(B)$  and  $\Psi \in \text{Isom}_k(A, B)$ . For any  $n \in \mathbb{N}$  we have  $(\Psi^{-1} \partial \Psi)^n = \Psi^{-1} \partial^n \Psi$ . So we see that  $\Psi^{-1} \partial \Psi \in \text{LND}(A)$ . An immediate result is  $\Psi(\text{ML}(A)) = \text{ML}(B)$  for any  $\Psi \in \text{Isom}_k(A, B)$ . Denote  $\partial_\Psi := \Psi^{-1} \partial \Psi$ , we have the following properties

- (1)  $\Psi\{\ker(\partial_\Psi)\} = \ker(\partial)$ .
- (2)  $\deg_{\partial_\Psi}(a) = \deg_\partial(\Psi(a))$  for all  $a \in A$ .

In [10], Poloni obtained similar results to the next Proposition.

**Proposition 6.6.** *Let  $B_1 = k[x_1, y_1, z_1]$  and  $B_2 = k[x_2, y_2, z_2]$  be as in 4.1.1 where*

$$P_1 = y_1^{m_1} + f_{m_1-2}(x_1) y_1^{m_1-2} + \dots + f_0(x_1)$$

$$P_2 = y_2^{m_2} + g_{m_2-2}(x_2) y_2^{m_2-2} + \dots + g_0(x_2)$$

*$f_i(x_1) \in k[x_1]$ ,  $g_i(x_2) \in k[x_2]$  and  $n_i > 1, m_i > 1$ . Then*



$B_1$  and  $B_2$  are isomorphic if and only if  $n = n_1 = n_2$ ,  $m = m_1 = m_2$ , and  $f_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x) \pmod{x^n}$  for every  $i \in \{2, \dots, m\}$ .

In addition, every isomorphism between  $B_1$  and  $B_2$  takes the form

$$\Psi(x_1, y_1, z_1) = (\lambda x_2, \mu y_2 + x_2^n a(x_2), \frac{\mu^m}{\lambda^n} z_2 + \frac{P_1(\lambda x_2, \mu y_2 + x_2^n a(x_2)) - \mu^m P_2(x_2, y_2)}{\lambda^n x_2^n}),$$

where  $a(x) \in k[x]$  and  $\lambda \in k^*$ ,  $\mu \in k^*$  satisfy  $f_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x) \pmod{x^n}$  for all  $i$ .

*Proof.* Let  $D \in \text{LND}(B_2)$ , and let  $\Psi \in \text{Isom}_k(B_1, B_2)$ . By property 2,  $\deg_{D_\Psi}(x_1) = \deg_D(\Psi(x_1))$ , but we saw before that  $\deg_E(x_1) = 0$  for all  $E \in \text{LND}(B_1)$ , so we get  $\deg_{D_\Psi}(x_1) = 0 = \deg_D(\Psi(x_1))$  and  $\Psi(x_1) \in \mathcal{F}_0 = k[x_2]$ . The same argument shows that  $\deg_{D_\Psi}(y_1) = 1 = \deg_D(\Psi(y_1))$ ,  $\Psi(y_1) \in \mathcal{F}_1 - \mathcal{F}_0$ , and  $\deg_{D_\Psi}(z_1) = m_1 = \deg_D(\Psi(z_1))$ . This implies that the only possibility for  $\psi$  defined as in 6.2 is  $\psi(x_2) = \Psi(x_1)$ ,  $\psi(y_2) = \Psi(y_1)$  and  $\psi(z_2) = \Psi(z_1)$ .

Now by Proposition 6.4

$$\psi(x_2, y_2, z_2) = (\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2), \frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}),$$

where  $a(x_2) \in k[x_2]$  and  $\lambda \in k^*$ ,  $\mu \in k^*$  such that  $g_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x) \pmod{x^{n_2}}$  for all  $i$ . So we get

$$\Psi(x_1, y_1, z_1) = (\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2), \frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}).$$

Since  $\deg_D(z_2) = m_2$  for any non-zero  $D \in \text{LND}(B_2)$ , and since  $\psi$  preserves  $\deg_D$ , we get  $m_2 = \deg_D(z_2) = \deg_D(\psi(z_2)) = \deg_D(\Psi(z_1)) = \deg_{D_\Psi}(z_1) = m_1$ , i.e.  $m_1 = m_2$ .

By applying  $\Psi$  to relation  $x_1^{n_1} z_1 = Q_1(x_1, y_1)$  in  $B_1$ , we obtain

$$\lambda^{n_1} x_2^{n_1} (\frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}) = Q_1(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)).$$

Applying the map  $gr_D$  to the last equation, we get

$$\lambda^{n_1} \frac{\mu^{m_2}}{\lambda^{n_2}} \bar{x}_2^{n_1} \bar{z}_2 = \mu^{m_2} \bar{y}_2^{m_2}.$$

On the other hand  $\bar{x}_2^{n_2} \bar{z}_2 = \bar{y}_2^{m_2}$  (apply  $gr_D$  to  $x_2^{n_2} z_2 = Q_2(x_2, y_2)$ ), the last two equations give  $\lambda^{n_1} \frac{\mu^{m_2}}{\lambda^{n_2}} \bar{x}_2^{n_1} = \mu^{m_2} \bar{x}_2^{n_2}$  which means that  $n_1 = n_2$ . We could have obtained that  $n_1 = n_2$  in this way: from  $\lambda^{n_1} x_2^{n_1} \Psi(z_1) = Q_1(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2))$  we conclude that  $\lambda^{n_1} x_2^{n_1} \partial(\Psi(z_1)) = \partial(y_2) H(x_2, y_2)$  where  $\deg_{y_2} H < m_2$  and  $x_2$  does not divide  $H$ . So  $x_2^{n_1}$  divides  $\partial(y_2) = x_2^{n_2}$  where  $\partial$  is defined as in 4.1.1, which mean that  $n_1 \leq n_2$ . Since  $B_1$  and  $B_2$  play symmetric roles the equality follows.  $\square$

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