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Invariant measure of scalar first-order conservation laws with stochastic forcing

A. Debussche*and J. Vovelle†

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Abstract

Under an hypothesis of non-degeneracy of the flux, we study the long-time behaviour of periodic scalar first-order conservation laws with stochastic forcing in any space dimension. For sub-cubic fluxes, we show the existence of an invariant measure. Moreover for sub-quadratic fluxes we show uniqueness and ergodicity of the invariant measure. Also, since this invariant measure is supported by L^p for some p small, we are led to generalize to the stochastic case the theory of L^1 solutions developed by Chen and Perthame [CP03].

Keywords: Stochastic partial differential equations, conservation laws, kinetic formulation, invariant measure.

MSC: 60H15 (35L65 35R60)

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1 Introduction

In this work we investigate the long-time behaviour of periodic scalar first-order conservation laws with stochastic forcing. In space dimension one, there is a famous paper of E, Khanin, Mazel, Sinai [EKMS00] where the author prove existence and uniqueness, and also analyse the invariant measure for the periodic inviscid Burgers equation with stochastic forcing. The analysis is done by use of the Lax-Oleinik formula, which means that the random unknown u is derived from a potential ψ which satisfies a periodic Hamilton-Jacobi with stochastic forcing. This can be extended to higher dimension (Iturriaga and Khanin [IK03]) and has also been extended to the case of fractional noise by Saussereau and Stoica, [SS12]. Recently, Boritchev ([B13]) has been able to derive sharp estimates on the solutions of the stochastic Burgers equation in the viscous case. They hold independently of the viscosity (see Remark 2 on that point).

Note that in all these articles, the space domain is compact. A recent work by Bakhtin [Bak13] deals with a scalar first-order conservation law with Poisson random forcing set on the whole line. We also would like to mention, for stochastically forced Hamilton-Jacobi equations, the thorough analysis of the invariant measure for such problems by Dirr and Souganidis in [DS05].

Our purpose is to generalize [EKMS00] to higher dimension and, mainly, to relax the hypothesis of uniform convexity which is necessary when the Lax-Oleinik formula is used. The first step was to have a satisfactory framework for existence and uniqueness of solutions. This has been accomplished by several authors and this point is now rather well understood (see Kim [K03], Vallet Wittbold [VW09], Feng, Nualart, [FN08], Debussche, Vovelle [DV10, DV], Bauzet, Vallet, Wittbold [BVW12], Chen, Ding, Karlsen [CDK12], Lions, Perthame, Souganidis [LPS13]).

In this article we aim at proving existence and uniqueness of invariant measures for the type of stochastic conservation laws studied in [DV10]. There, a general result of existence and uniqueness result was obtained thanks to the kinetic formulation (see Lions, Perthame, Tadmor [LPT94], Perthame [P02]) suitably generalized to the stochastic case. This kinetic formulation seems well adapted for this purpose since it allows to keep track of the dissipation due to the shocks. Note that, as explained below, we need here a notion of solution in L^1, which we develop in this article in the framework of kinetic formulation.

Under an hypothesis of non-degeneracy of the flux, we show the existence of an invariant measure in any dimension of space. For technical reasons we need to assume that the flux does not grow faster than a cubic polynomial. Moreover, for sub-quadratic fluxes we show the uniqueness of the invariant measure, see our main result, Theorem 1. An essential tool for the proofs is the use of averaging lemma for kinetic equations. We use such an averaging lemma well adapted to stochastic equation issued from Bouchut and Desvillettes [BD99].

More precisely, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis and let $T > 0$. We study the invariant measure for the first-order scalar conservation law with stochastic forcing

$$du + \operatorname{div}(A(u))dt = \Phi dW(t), \quad x \in \mathbb{T}^N, t \in (0, T). \tag{1}$$

The equation is periodic in the space variable x: $x \in \mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The flux function A in (1) is supposed to be of class C^2: $A \in C^2(\mathbb{R}; \mathbb{R}^N)$ and its derivatives have at most polynomial growth. We assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is complete and that W is a cylindrical Wiener process: $W = \sum_{k\geq 1} \beta_k e_k$, where the β_k are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space H. The map $\Phi\colon H \to L^2(\mathbb{T}^N)$ is defined by $\Phi e_k = g_k$ where g_k is a regular function on $\mathbb{T}^N$. More precisely, we assume $g_k \in C(\mathbb{T}^N)$, with the bounds

$$\mathbf{G}^2(x) = \sum_{k\geq 1} |g_k(x)|^2 \leq D_0, \tag{2}$$

$$\sum_{k\geq 1} |g_k(x) - g_k(y)|^2 \leq D_1 |x - y|^2, \tag{3}$$

for all $x, y \in \mathbb{T}^N$. Note in particular that $\Phi\colon H \to L^2(\mathbb{T}^N)$ is Hilbert-Schmidt since $\|g_k\|_{L^2(\mathbb{T}^N)} \leq \|g_k\|_{C(\mathbb{T}^N)}$ and thus

$$\sum_{k\geq 1} \|g_k\|^2_{L^2(\mathbb{T}^N)} \leq D_0.$$

Existence and uniqueness of a solution $u^\omega(t)$ to (1) satisfying a given initial condition $u(0) = u_0 \in L^\infty(\mathbb{T}^N)$ has been proved in [DV10]. This result can easily be extended to initial data in $L^p(\mathbb{T}^N)$ for p larger than the degree of polynomial growth of A. However, as stated below, the invariant measure is supported by $L^r(\mathbb{T}^N)$ with r small and we do not know whether r can be taken sufficiently large so that the result of [DV10] cannot be used for such invariant measures. Thus, we need to develop a theory of existence and uniqueness for initial data in $L^r(\mathbb{T}^N)$ for small r. In fact, we generalize to the stochastic context the notion of solutions in $L^1(\mathbb{T}^N)$ developed in [CP03] . Note however that in [CP03] second-order, possibly degenerate equations are considered, so that, strictly speaking, what we generalize is the "fully degenerate" case contained in [CP03]. Anyway, what matter technically here is to prove that the kinetic measure decay sufficiently fast at infinity.

Let us now give some precisions on our framework. We assume that the noise is additive and that the functions g_k satisfy the cancellation condition

$$\forall k \geq 1, \quad \int_{\mathbb{T}^N} g_k(x)dx = 0. \tag{4}$$

The solution then satisfies $\bar{u}(t) = \int_{\mathbb{T}^N} u(t,x)dx = \int_{\mathbb{T}^N} u(0,x)dx$ almost surely. We will consider initial data with non random space average $\bar{u}$. Then this remains the case for all time $t \geq 0$. Changing u to $u - \bar{u}$, we see that it is no loss of generality to consider the case $\bar{u} = 0$. Thus in all the article, we only consider such initial data.

To introduce the non-degeneracy condition on which we will work on, let us denote $a := A'$ and set

$$\iota(\varepsilon) = \sup_{\alpha \in \mathbb{R}, \beta \in \mathbb{S}^{N-1}} |\{\xi \in \mathbb{R}; |\alpha + \beta \cdot a(\xi)| < \varepsilon\}|, \tag{5}$$

and

$$\eta(\varepsilon) := \int_0^\infty e^{-t} \iota(t\varepsilon) dt, \tag{6}$$

where $|B|$ denotes the Lebesgue measure of a set $B \subset \mathbb{R}$. It is known that if a satisfies the following non-degeneracy condition:

$$\lim_{\varepsilon \to 0} \iota(\varepsilon) = 0, \tag{7}$$

then all solutions of the deterministic equation, *i.e.* with $\Phi = 0$, converge to zero (*cf.* Debussche, Vovelle [DV09] for example, in that case the condition can be localized. See also [CP09] for the quasilinear parabolic case). The condition (7) is a condition of non-stationarity of $\xi \mapsto a(\xi)$.

In this paper, we use the approach developed in [BD99]. There, an averaging Lemma based on the Fourier transform in x is developed. The non-degeneracy assumption is strengthened into:

$$\iota(\varepsilon) \leq c_1 \varepsilon^b, \tag{8}$$

for some $c_1 > 0$ and $b > 0$. Note that $b \leq 1$, unless $a \equiv 0$, and that (8) is equivalent to

$$\eta(\varepsilon) \leq c_1 \varepsilon^b, \tag{9}$$

for possibly a different constant c_1. We will use the averaging lemma (under the form developed in [BD99], *cf.* Section 3 and Section 4) to prove the following result.

Theorem 1 (Invariant measure)**.** *Let $A \in C^2(\mathbb{R}; \mathbb{R}^N)$ satisfy the non-degeneracy condition (8) where ι is defined by (5). Assume conditions (2)-(3) on the noise, the cancellation condition (4) and that A is at most cubic in the following sense:*

$$|a'(\xi)| \leq c_1(|\xi| + 1), \ \xi \in \mathbb{R}. \tag{10}$$

Then there exists an invariant measure *for (1) in $L^1(\mathbb{T}^N)$. Moreover, it is supported by $L^r(\mathbb{T}^N)$ for $r < 2 + \frac{b}{2}$ if $N = 1$ and $r < \frac{N}{N-1}$ if $N \geq 2$. If the condition (10) is strengthened into the hypothesis that A is sub-quadratic in the following sense:*

$$|a'(\xi)| \leq c_2, \quad \xi \in \mathbb{R}, \tag{11}$$

then the invariant measure is unique.

Thanks to the kinetic formulation, it is easy to see that a stationary solution develops shocks (see Remark 8). Thus the noise does not have a regularizing effect here (see Flandoli [F11] for a review of such effects in SPDEs). Note that the presence of shocks was already observed in [EKMS00] and their structure was carefully studied.

In the statement of Theorem 1 and in the sequel, we use the letter C or c to denote a constant. Its value may change from one line to another. Sometimes, we precise its dependence on some parameters.

Remark 2. *The existence of an invariant measure is closely related to existence of uniform in time estimates for the solution to* (1) *(we do not specify in which norm in this discussion). Note that such uniform estimates, with respect to time and with respect to the viscosity parameter also since second-order equations are considered, have been given by Boritchev, [B13], for the generalized Burgers equation with a flux A satisfying an hypothesis of strict convexity and an hypothesis of (strict) sub-quadratic growth of A'. Compare to* (10) *here.*

To prove Theorem 1, we first describe in Section 2 the concept of L^1 solution for (1) used in this article. The well posedness in L^1 is proved in Appendix A. Then, in Section 3, we prove some bounds uniform in time on the solution which give the existence of the invariant measure. These bounds are used again in Section 4, together with a quite classical argument of smallness of the noise, to obtain the uniqueness of the invariant measure. The difficulty in the proof of uniqueness is that, contrary to the convex case treated in the references above, we do not have any uniformity with respect to the initial data on the proximity between the deterministic solution and a solution with small noise. A similar thing appears for the 2D stochastic Navier-Stokes equations with large viscosity (see Mattingly [M99]) but in this case the a priori estimates are easier to obtain and the contraction of the trajectories is much stronger.

2 Kinetic solution

Let us give the notion of solution to (1) we need use in this article. It has been introduced in [DV10]. We slightly modify the definition to adapt the L^1 setting.

Definition 3 (Kinetic measure)**.** *We say that a map m from Ω to the set of non-negative Radon measures over $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ is a kinetic measure if*

1. *m is measurable, in the sense that for each $\phi \in C_c(\mathbb{T}^N \times [0,T] \times \mathbb{R})$, $\langle m, \phi \rangle \colon \Omega \to \mathbb{R}$ is,*
2. *m vanishes for large ξ in the sense that*
$$\lim_{n \to +\infty} \mathbb{E} m(A_n) = 0, \tag{12}$$
where $A_n = \mathbb{T}^N \times [0,T] \times \{\xi \in \mathbb{R}, n \leq |\xi| \leq n+1\}$,
3. *for all $\phi \in C_c(\mathbb{T}^N \times \mathbb{R})$, the process*
$$t \mapsto \int_{\mathbb{T}^N \times [0,t] \times \mathbb{R}} \phi(x,\xi) dm(x,s,\xi)$$
is predictable.

Definition 4 (Solution)**.** *Let $u_0 \in L^1(\mathbb{T}^N)$. A measurable function $u\colon \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ is said to be a solution to (1) with initial datum u_0 if $(u(t))$ is predictable, if*

$$\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{L^1(\mathbb{T}^N)}\right) < +\infty, \tag{13}$$

and if there exists a kinetic measure m such that $f := \mathbf{1}_{u>\xi}$ satisfies: for all $\varphi \in C^1_c(\mathbb{T}^N \times [0,T) \times \mathbb{R})$,

$$\begin{aligned}\int_0^T \langle f(t), \partial_t\varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a(\xi)\cdot\nabla\varphi(t)\rangle dt \\ = -\sum_{k\geq 1}\int_0^T \int_{\mathbb{T}^N} g_k(x)\varphi(x,t,u(x,t))dxd\beta_k(t) \\ -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N} \partial_\xi\varphi(x,t,u(x,t))\mathbf{G}^2(x)dxdt + m(\partial_\xi\varphi),\end{aligned} \tag{14}$$

where $f_0(x,\xi) = \mathbf{1}_{u_0(x)>\xi}$.

We have used the brackets $\langle\cdot,\cdot\rangle$ to denote the duality between $C^\infty_c(\mathbb{T}^N\times\mathbb{R})$ and the space of distributions over $\mathbb{T}^N\times\mathbb{R}$. In what follows, we will denote similarly the integral

$$\langle F, G\rangle = \int_{\mathbb{T}^N}\int_{\mathbb{R}} F(x,\xi)G(x,\xi)dxd\xi, \quad F\in L^p(\mathbb{T}^N\times\mathbb{R}), G\in L^q(\mathbb{T}^N\times\mathbb{R}),$$

where $1\leq p\leq +\infty$ and q is the conjugate exponent of p. In (14) also, we have used the shorthand $m(\phi)$ for

$$m(\phi) = \int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} \phi(x,t,\xi)dm(x,t,\xi), \quad \phi\in C_b(\mathbb{T}^N\times[0,T]\times\mathbb{R}).$$

Equation (15) is naturally associated to the original problem (1). Indeed, if u is a solution to (1), we introduce a new variable ξ and consider the equation satisfied by $f(t,x,\xi) = \mathbf{1}_{u(t,x)>\xi}$. For a smooth function $\phi\in C^\infty(\mathbb{R})$ with compact support, we set $\theta(\xi) = \displaystyle\int_{-\infty}^{\xi}\phi(\zeta)d\zeta$. We have the obvious identity

$$(\mathbf{1}_{u>\xi},\phi) = \int_{\mathbb{R}} \mathbf{1}_{u>\xi}\theta'(\xi)d\xi = \int_{-\infty}^u \theta'(\xi)d\xi = \theta(u)$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R})$, with respect to the variable ξ. We use the same notations for the duality between continuous functions and measures. By Itô Formula, we deduce

$$\begin{aligned}&d(\mathbf{1}_{u>\xi},\phi) = d\theta(u)\\ &= \theta'(u)(-a(u)\cdot\nabla u dt + \Phi(u)dW) + \frac{1}{2}\theta''(u)\mathbf{G}^2(u)dt.\end{aligned}$$

Let us rewrite the three terms of the left hand side as follows:

$$\begin{aligned}
\theta'(u)a(u)\cdot\nabla u &= \operatorname{div}\left(\int_{-\infty}^{u} a(\xi)\theta'(\xi)d\xi\right) = \operatorname{div}(a\mathbf{1}_{u>\xi},\phi)\\
&= (a\cdot\nabla\mathbf{1}_{u>\xi},\phi),\\
\theta''(u)\mathbf{G}^2(u) &= (\mathbf{G}^2(\xi)\delta_{u=\xi},\phi'(\xi)) = -(\partial_\xi(\mathbf{G}^2(\xi)\delta_{u=\xi}),\phi(\xi)),\\
\theta'(u)\Phi(u)dW &= (\delta_{u=\xi}\Phi(\xi)dW,\phi(\xi)).
\end{aligned}$$

We deduce

$$d(\mathbf{1}_{u>\xi},\phi) = -(a\cdot\nabla\mathbf{1}_{u>\xi},\phi)dt - \tfrac{1}{2}(\partial_\xi(\mathbf{G}^2\delta_{u=\xi}),\phi)dt + (\delta_{u=\xi},\phi\Phi dW).$$

We then multiply by a smooth function ψ of x, t with compact support in $\mathbb{T}^d\times[0,T)$, integrate in x, t and perform intergration by parts to obtain (14) with $\varphi(x,t,\xi)=\psi(x,t)\phi(\xi)$. Recall that such functions are dense in $C^1_c(\mathbb{T}^N\times[0,T)\times\mathbb{R})$.
Note however that the kinetic measure is missing in the above equation. The reason is that in fact the computation is not rigorous: the solution of (1) are not smooth.
To understand why the measure is present, recall that solutions of (1) are not expected to be unique unless an extra condition is imposed. This is the entropy condition and entropy solutions are obtained as limit of solutions of the equation with an extra viscous term.
Thus, we consider the viscous stochastic equation, with $\eta>0$:

$$\begin{cases} du^\eta + \operatorname{div}(A(u^\eta))dt - \eta\Delta u^\eta dt &= \Phi(u^\eta)dW(t), \quad t>0, x\in\mathbb{T}^N,\\ u^\eta(x,0) &= u_0(x), \quad x\in\mathbb{T}^N. \end{cases}$$

We proceed as above and by Itô Formula, we have, for $\theta(\xi)=\int_{-\infty}^{\xi}\phi(\zeta)d\zeta$,

$$\begin{aligned}
&d(\mathbf{1}_{u^\eta>\xi},\phi) = d\theta(u^\eta)\\
&= \theta'(u^\eta)(-a(u^\eta)\cdot\nabla u^\eta dt + \eta\Delta u^\eta dt + \Phi(u^\eta)dW) + \frac{1}{2}\theta''(u^\eta)\mathbf{G}^2_\eta dt.
\end{aligned}$$

We rewrite the second term as

$$\begin{aligned}
\theta'(u^\eta)\eta\Delta u^\eta &= \eta\Delta\theta(u^\eta)dt - \eta|\nabla u^\eta|^2\theta''(u^\eta)\\
&= \eta\Delta(\mathbf{1}_{u^\eta>\xi},\theta')dt + (\partial_\xi(\eta|\nabla u^\eta|^2\delta_{u^\eta=\xi}),\theta'),
\end{aligned}$$

and obtain viscous kinetic formulation

$$\begin{aligned}
d\mathbf{1}_{u^\eta>\xi} + a\cdot\nabla\mathbf{1}_{u^\eta>\xi}dt &= \delta_{u^\eta=\xi}\Phi(\xi)dW - \partial_\xi(\frac{1}{2}\mathbf{G}^2_\eta\delta_{u^\eta=\xi})dt\\
&\quad +\eta\Delta\mathbf{1}_{u^\eta>\xi}dt + \partial_\xi m^\eta dt.
\end{aligned}$$

with the viscous kinetic measure $m^\eta=\eta|\nabla u^\eta|^2\delta_{u^\eta=\xi}$. The sequence $(m^\eta)_{\eta>0}$ does not converge to zero. At the limit $\eta\to 0$, a measure m remains.
Note that since u^η is smooth (see [Hof13]), the above computation is rigorous.
Also, it is easy to derive the energy inequality:

$$\mathbb{E}\left(\|u^\eta(t)\|^p_{L^p(\mathbb{T}^N)}\right) + \eta\mathbb{E}\int_0^T\int_{\mathbb{T}^N}|u^\eta(t,x)|^{p-2}|\nabla u^\eta(t)|^2dxdt \le C(p,u_0,T).$$

This can rewritten as a bound on the moments of the measures m^η and $\nu^\eta = \delta_{u^\eta=\xi}$:

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p d\nu^\eta_{x,t}(\xi)dx + \mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}|\xi|^{p-2}dm^\eta(x,t,\xi) \le C(p,u_0,T).$$

Using other similar estimates, we have for a subsequence $(\eta_n)\downarrow 0$

1. $\nu^{\eta_n}\to\nu$ in the sense of Young measures indexed by $\Omega\times\mathbb{T}^N\times[0,T]$ and $f^{\eta_n} = \mathbf{1}_{u^{\eta_n}>\xi} \rightharpoonup f$ in $L^\infty(\Omega\times\mathbb{T}^N\times(0,T)\times\mathbb{R}) - weak - *$. Moreover $\nu_{x,t} = -\partial_\xi f$

2. $m^{\eta_n}\rightharpoonup m$ in $L^2(\Omega;\mathcal{M}_b)$-weak star, where $\mathcal{M}_b$ denote the space of bounded Borel measures over $\mathbb{T}^N\times[0,T]\times\mathbb{R}$

Then, since the equation is linear in f, m, ν, it is easy to pass to the limit and prove that (14) holds. We say (*cf.* [DV10]) that f is a generalized kinetic solution.
We only have $f\in[0,1]$ and $\nu_{x,t}$ is a probability measure. To get a kinetic solution, we need $f\in\{0,1\}$; then, necessarily, $f=\mathbf{1}_{u>\xi}$ and $\nu_{x,t}=\delta_{u=\xi}$. To prove this fact, one first prove uniqueness of generalized solutions using a doubling of variables and see that this implies the result. The difficulty at that point is to get time regularity of f.
This proof is given in [DV10] (see also [DV] where minor errors of [DV10] have been corrected) in an $L^p(\mathbb{T}^d)$ framework. It is also shown that u is an entropy solution.

As a last comment on the notion of kinetic solution, let us remark that it is sometimes problematic to work with the unknown $\mathbf{1}_{u>\xi}$. Indeed, it is not integrable with respect to ξ. However, switching from $\mathbf{1}_{u>\xi}$ to $\chi_u(\xi) := \mathbf{1}_{u>\xi>0} - \mathbf{1}_{0>\xi>u} = \mathbf{1}_{u>\xi} - \mathbf{1}_{0>\xi}$, it turns out that Equation (14) is the weak form of the following equation

$$(\partial_t + a(\xi)\cdot\nabla)f = \delta_{u=\xi}\dot{W} + \partial_\xi(m - \frac{1}{2}\mathbf{G}^2\delta_{u=\xi}), \quad f=\chi_u, \tag{15}$$

which has now a distribution f as unknown. Note in particular that $f\in L^1(\mathbb{T}^N\times[0,T]\times\mathbb{R})$ if $u\in L^1(\mathbb{T}^N\times[0,T])$,. Actually $u\mapsto f=\chi_u$ is then an isometry. This simple fact is at the basis of the resolution of (1) in L^1. In Appendix A we prove the following result which generalizes [DV10] (*cf.* Theorem 11, Corollary 12, and Theorem 19 in [DV10] in particular for a precise definition of the "parabolic approximation").

Theorem 5 (Resolution of (1) in L^1)**.** *Let $u_0\in L^1(\mathbb{T}^N)$. There exists a unique measurable $u\colon\mathbb{T}^N\times[0,T]\times\Omega\to\mathbb{R}$ solution to (1) with initial datum u_0 in the sense of Definition 4. Besides, u has almost surely continuous trajectories in $L^1(\mathbb{T}^N)$ and u is the a.s. limit in $L^1(\mathbb{T}^N\times(0,T))$ of the parabolic approximation to (1). Moreover, given u_0^1 and $u_0^2\in L^1(\mathbb{T}^N)$, the following holds:*

$$\|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^N)} \le \|u_0^1-u_0^2\|_{L^1(\mathbb{T}^N)},\ a.s. \tag{16}$$

We need this generalization because we are not able to prove that the invariant measure we construct has its support in $L^p(\mathbb{T}^N)$ for p sufficiently large to apply the result in [DV10]. Note that it follows from the proof that if $u_0 \in L^r(\mathbb{T}^N)$ for some $r > 1$ then the solution lives in $L^r(\mathbb{T}^N)$ and an estimate similar to (13) but in $L^r(\mathbb{T}^N)$ holds.

By Theorem 5, we can define the transition semigroup in $L^1(\mathbb{T}^N)$:

$$P_t\phi(u_0) = \mathbb{E}(\phi(u(t))),\ \phi \in \mathcal{B}_b(L^1(\mathbb{T}^N)).$$

It follows easily from the property of L^1-contraction (16) that (P_t) is Feller.

Our aim is to construct an invariant measure for (P_t) under assumption (10). Recall that this is a probability measure mu on $L^1(\mathbb{T}^d)$ satisfying:

$$P_t^*\mu = \mu,\ t \geq 0.$$

We prove the first part of Theorem 1 (existence of an invariant measure) in the next section thanks to Krylov-Bogoliubov theorem. We conclude this section with some remarks.

Since the solution of (1) has continuous trajectories, it is easy to prove that it satisfies the following weak formulation:

$$\begin{aligned} -\langle f(t), \varphi\rangle dt + \langle f_0, \varphi\rangle + \int_0^t \langle f(s), a(\xi)\cdot\nabla\varphi\rangle ds \\ = -\sum_{k\geq 1}\int_0^t \int_{\mathbb{T}^N} g_k(x)\varphi(x, u(x,t))dxd\beta_k(s) \\ -\frac{1}{2}\int_0^t\int_{\mathbb{T}^N} \partial_\xi\varphi(x, u(x,s))\mathbf{G}^2(x)dxdt + m(\partial_\xi\varphi), \end{aligned} \tag{17}$$

for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$.

In fact, we will even use stronger formulation by using the following remark.

Remark 6 (Regularity of the parabolic approximation)**.** *Let the parabolic approximation to* (1) *be given by:*

$$du^\eta + \operatorname{div}(A_\eta(u))dt - \eta\Delta u^\eta = \Phi_\eta dW(t), \quad x \in \mathbb{T}^N, t \in (0,T), \tag{18}$$

where A_η and Φ_η are smooth. By Hofmanová [Hof13], if $u^\eta(0)$ is in $W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)$, then u^η belongs to $L^q(\Omega; C([0,T]; W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)))$ for any $q \geq 1$. Also by Theorem 5, if one approximates the solution of (1) *by the solution of* (18) *with $u^\eta(0) \to u_0$ in $L^1(\mathbb{T}^N)$, then $u^\eta \to u$ in $L^\infty(0,T; L^1(\Omega \times \mathbb{T}^N))$.*

Remark 7 (Multiplicative noise)**.** *A natural extension of our study is to consider a noise depending on the solution of the form*

$$\Phi(u)dW = \sum_{k\geq 1}\Phi(u)e_k d\beta_k = \sum_{k\geq 1} g_k(u)d\beta_k.$$

With such noise, the spatial average satisfies:

$$\int_{\mathbb{T}^N} u(x,t)dx = \int_{\mathbb{T}^N} u_0(x)dx + \sum_{k\geq 1}\int_0^t \int_{\mathbb{T}^N} g_k(u(x,s)dxd\beta_k(s).$$

It seems difficult to provide any bound on this average for a general noise. A first possibility is to consider a noise satisfying

$$\int_{\mathbb{T}^N} g_k(u(x))dx = 0,\ k \geq 1.$$

Then the mass is preserved exactly. This would hold for $g_k(u) = \frac{\partial}{\partial x}h_k(u)$ *or* $g_k(u) = h_k(u) - \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} h_k(u)dx$. *But such noises are not suited to the framework of kinetic solutions and are not covered by our theory. The first one is considered in [LPS13] but the longtime behaviour is not analysed there. The second noise is non local and does not seem to be justified physically.*

It would be also possible to consider noises of the form $u(1-u)dW$. *Then, the solution is easily proved to live in* $[0,1]$ *and the existence of an invariant measure is easy. The longtime behaviour in this case is expected to be completely different (see Ikeda, Watanabe [IK81] for the case of an SDE and Bergé, Saussereau [BS05] for the case of a parabolic SPDE).*

Remark 8 (Shocks do not disappear)**.** *Once we have proved Theorem 1, it is not difficult to construct a stationary solution of (1) with the invariant law given by this result. It is of course a kinetic solution. Assume that* $N = 1$, *then the invariant measure is supported by* $L^p(\mathbb{T}^N)$ *for some* $p > 2$. *It is easy to see that a preliminary step of truncation allows to take the test function* $\varphi(\xi) = \xi$ *in (17) and we obtain:*

$$\mathbb{E}\left(\|u(t_0+1)\|^2_{L^2(\mathbb{T}^N)}\right) + \mathbb{E}m\left(\mathbb{T}^N \times \mathbb{R} \times [t_0, t_0+1]\right) = \mathbb{E}\left(\|u(t)\|^2_{L^2(\mathbb{T}^N)}\right) + D_0.$$

Thus, by stationarity,

$$\mathbb{E}m\left(\mathbb{T}^N \times \mathbb{R} \times [t_0, t_0+1]\right) = D_0.$$

As expected, the lhs does not depend on t_0. *More interestingly, it is not* 0. *This shows that shocks are present in the stationary solutions and noise has no regularizing effect. If* $N \geq 2$, *one can use the test functions of the proof of Proposition 16 to see that again shocks are present for a stationary solution.*

Remark 9. *We could also have constructed a random dynamical system associated to (1) and consider invariant measure in the co-cycle sense. This is the approach in [EKMS00]. It is not difficult to see that, with the results obtained in the following section, we can prove that for almost all* $\omega \in \Omega$ *there exists a unique solution defined on the whole real line and that there exists a random attractor which is reduced to a single random point.*

3 Uniform bound and tightness for the stochastically forced equation

3.1 Decomposition of u

Let $\alpha, \gamma, \delta > 0$ be some positive parameters. On $L^2(\mathbb{T}^N)$, we consider the following regularization of the operator $-a(\xi) \cdot \nabla$:

$$A_\gamma := -a(\xi) \cdot \nabla - B_\gamma, \quad A_{\gamma,\delta} = A_\gamma - \delta \mathrm{Id}, \quad B_\gamma := \gamma(-\Delta)^\alpha.$$

Let also $S_{A_\gamma}(t)$ and $S_{A_{\gamma,\delta}}(t)$ be the associated semi-groups on $L^2(\mathbb{T}^N)$:

$$S_{A_\gamma}(t)v(x) = (e^{-tB_\gamma}v)(x - a(\xi)t), \quad S_{A_{\gamma,\delta}}(t)v(x) = e^{-\delta t}(e^{-tB_\gamma}v)(x - a(\xi)t).$$

Note that the semi-group e^{-tB_γ} has the following regularizing properties, which we will use later.

Lemma 10. *For $\gamma > 0$, $\alpha \in (0, 1]$, let $B_\gamma = \gamma(-\Delta)^\alpha$ on $\mathbb{T}^N$ and let e^{-tB_γ} denote the associated semi-group. Then there exists a constant c_3 depending on N, n, m, α, β such that*

$$\|(-\Delta)^{\beta/2} e^{-tB_\gamma}\|_{L^m \to L^n} \leq \frac{c_3}{(\gamma t)^{\frac{N}{2\alpha}\left(\frac{1}{m} - \frac{1}{n}\right) + \frac{\beta}{2\alpha}}},$$

for all $1 \leq m \leq n \leq +\infty$ and $\beta \geq 0$.

Proof: it is sufficient to consider the case $\gamma = 1$. Note that $e^{-tB_1}\varphi = K_t * \varphi$ for all $\varphi \in \mathcal{C}^2(\mathbb{T}^N)$, where

$$K_t(x) = \sum_{n \in \mathbb{Z}^d} e^{-t|n|^{2\alpha} + 2\pi i n \cdot x}.$$

Let $\mathcal{F}$ denote the Fourier transform

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^N} u(x) e^{-2\pi i x \cdot \xi} dx,$$

for $u \in \mathcal{S}(\mathbb{R}^N)$. Let H_t denote the inverse Fourier transform of $\xi \mapsto e^{-t|\xi|^{2\alpha}}$ (this is the "heat" kernel associated to $(-\Delta)^\alpha$ on $\mathbb{R}^N$) and let H_t^{per} denote the periodic function generated by H_t:

$$H_t^{\mathrm{per}} = \sum_{l \in \mathbb{Z}^N} H_t(x + l), \quad x \in \mathbb{R}^N.$$

For a fixed $x \in \mathbb{R}^N$, we apply the Poisson formula

$$\frac{1}{(2\pi)^N} \sum_{k \in \mathbb{Z}^N} \mathcal{F}\theta(k) = \sum_{l \in \mathbb{Z}^N} \theta(2\pi l), \quad \theta \in \mathcal{S}'(\mathbb{R}^N)$$

to the function $\theta(z) = H_t(x + z/2\pi)$. We obtain $H_t^{\mathrm{per}} = K_t$. In particular, we have

$$\|K_t\|_{L^r(\mathbb{T}^N)} \leq \|H_t\|_{L^r(\mathbb{R}^N)}, \quad 1 \leq r \leq +\infty.$$

By the Young inequality, we also have

$$\|e^{-tB_1}\|_{L^m(\mathbb{T}^N)\to L^n(\mathbb{T}^N)} \leq \|K_t\|_{L^r(\mathbb{T}^N)}, \quad \frac{1}{m}+\frac{1}{r}=1+\frac{1}{n},$$

and, therefore,

$$\|e^{-tB_1}\|_{L^m(\mathbb{T}^N)\to L^n(\mathbb{T}^N)} \leq \|H_t\|_{L^r(\mathbb{R}^N)}.$$

By homogeneity, $\|H_t\|_{L^r(\mathbb{R}^N)} = \frac{c}{t^{\frac{N}{2\alpha}\left(\frac{1}{m}-\frac{1}{n}\right)}}$. The case $\beta \neq 0$ is similar. ■

Now, we formally decompose the solutions using the following rewriting of (15):

$$(\partial_t - A_{\gamma,\delta})f = (B_\gamma + \delta \mathrm{Id}) + p + \partial_\xi q, \quad f = \chi_u. \tag{19}$$

with $p = \delta_{u=\xi}\dot{W}$, $q = m - \frac{1}{2}\mathbf{G}^2\delta_{u=\xi}$.

More precisely, by a preliminary step of regularization, we may use the test function $S^*_{A_{\gamma,\delta}}(T-t)\varphi$ with $\varphi \in C(\mathbb{T}^N)$ in (14). Then, by the commutation identity

$$S_{A_{\gamma,\delta}}(t)\partial_\xi g = \partial_\xi\left(S_{A_{\gamma,\delta}}(t)g\right) + tXS_{A_{\gamma,\delta}}(t)g, \quad X = a'(\xi)\cdot\nabla,$$

we obtain the following decomposition of the solution:

$$u = u^0 + u^\flat + P + Q, \tag{20}$$

where

$$u^0(t) = \int_{\mathbb{R}} S_{A_{\gamma,\delta}}(t)f(0,\xi)d\xi, \tag{21}$$

$$u^\flat(t) = \int_{\mathbb{R}}\int_0^t S_{A_{\gamma,\delta}}(s)(B_\gamma f + \delta f)(t-s,\xi)dsd\xi, \tag{22}$$

$$\langle P(t),\varphi\rangle = \sum_{k\geq 1}\int_{\mathbb{T}^N}\int_0^t \left(S^*_{A_{\gamma,\delta}}(t-s)\varphi\right)(x,u(s,x))g_k(x)d\beta_k(s)dx, \tag{23}$$

and

$$\langle Q(t),\varphi\rangle = \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} (t-s)a'(\xi)\cdot\nabla S^*_{A_{\gamma,\delta}}(t-s)\varphi dq(x,s,\xi), \tag{24}$$

where $\varphi \in C(\mathbb{T}^N)$ and $\langle M,\varphi\rangle$ denote the duality product between the space of finite Borel measures on $\mathbb{T}^N$ and $C(\mathbb{T}^N)$.

We estimate each term separately. In fact, we estimate the parabolic approximation of u. Since it is as smooth as we need (*cf.* Remark 6), the computations below are easily justified. In particular, for the parabolic approximation, the kinetic measure is given by

$$m^\eta = \eta|\nabla u^\eta|^2\delta_{u^\eta=\xi}$$

which is smooth in t and x. To lighten the computation below, we will however omit to write the dependence on η, except in section 3.7 where we derive the final estimate on the true solution u.

3.2 Estimate of u^0

We use the Fourier transform with respect to $x \in \mathbb{T}^d$, *i.e.*

$$\hat{v}(n) = \int_{\mathbb{T}^N} v(x) e^{-2\pi i n \cdot x}, \quad n \in \mathbb{Z}^N, \quad v \in L^1(\mathbb{T}^N).$$

After Fourier transform, $S_{A_{\gamma,\delta}}(t)$ is multiplication by $e^{-(ia(\xi)\cdot n + \gamma|n|^{2\alpha}+\delta)t}$, hence, for all $n \in \mathbb{Z}^N$, $n \neq 0$,

$$\hat{u}^0(n,t) = \int_{\mathbb{R}} e^{-(ia(\xi)\cdot n + \gamma|n|^{2\alpha}+\delta)t} \widehat{f}_0(n,\xi) d\xi.$$

Let us set, for $\varphi : \mathbb{Z}^N \times \mathbb{R} \to \mathbb{C}$ satisfying $\varphi(n,\cdot) \in L^2(\mathbb{R})$ for all $n \in \mathbb{Z}^N$:

$$\mathcal{G}\varphi(n,z) = \int_{\mathbb{R}} e^{-ia(\xi)\cdot z} \varphi(n,\xi) d\xi. \tag{25}$$

The operator $\mathcal{G}$ is well-defined by [BD99]. Let us also define

$$\omega_n = \gamma|n|^{2\alpha-1} + \delta|n|^{-1}. \tag{26}$$

Then, for $T \geq 0$, $n \neq 0$,

$$\begin{aligned}\int_0^T |\hat{u}^0(n,t)|^2 dt &= \int_0^T e^{-2(\gamma|n|^\alpha+\delta)t} \left|\mathcal{G}\widehat{f}_0(n,nt)\right|^2 dt \\ &\leq \frac{1}{|n|} \int_{\mathbb{R}^+} e^{-2\omega_n s} \left|\mathcal{G}\widehat{f}_0\left(n, \frac{n}{|n|}s\right)\right|^2 ds,\end{aligned}$$

thanks to the change of variable $s = |n|t$. We now use Lemma 2.4 in [BD99], which gives an estimate of the oscillatory integral (25); we also use the obvious inequality $e^{-a} \leq 1/(1+a^2)$ to deduce:

$$\begin{aligned}\int_0^T |\hat{u}^0(n,t)|^2 dt &\leq \frac{1}{|n|} \int_{\mathbb{R}^+} \frac{1}{1+4\omega_n^2 s^2} \left|\mathcal{G}\widehat{f}_0\left(n, \frac{n}{|n|}s\right)\right|^2 ds \\ &\leq \frac{1}{|n|} \frac{2\pi}{2\omega_n} \eta(\omega_n) \left\|\widehat{f}_0(n,\cdot)\right\|^2_{L^2_\xi}.\end{aligned}$$

Recall that η was defined in (6) and satisfies the decay condition (9). Consequently, we have the estimate

$$\int_0^T |\hat{u}^0(n,t)|^2 dt \leq c \frac{1}{|n|} \omega_n^{b-1} \left\|\widehat{f}_0(n,\cdot)\right\|^2_{L^2_\xi}.$$

Clearly,

$$|n|\omega_n^{1-b} \geq \gamma^{1-b}|n|^{2\alpha(1-b)+b},$$

for $n \neq 0$. Summing over $n \in \mathbb{Z}^N$ and noticing that

$$\begin{aligned}\hat{u}^0(0,t) = \int_{\mathbb{T}^N} u^0(t,x) dx &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} e^{-\delta t} f(0,x,\xi) d\xi dx \\ &= \int_{\mathbb{T}^d} e^{-\delta t} u_0(x) dx = 0,\end{aligned}$$

we obtain:

$$\int_0^T \|u^0(t)\|^2_{H^{\alpha+\left(\frac{1}{2}-\alpha\right)_\flat}} dt \leq c\gamma^{b-1} \|f_0\|^2_{L^2_{x,\xi}} .$$

Since $\|f_0\|^2_{L^2_{x,\xi}} = \|f_0\|_{L^1_{x,\xi}} = \|u_0\|_{L^1_x}$, this gives

$$\int_0^T \|u^0(t)\|^2_{H^{\alpha+\left(\frac{1}{2}-\alpha\right)_\flat}} dt \leq c\gamma^{b-1} \|u_0\|_{L^1_x} . \tag{27}$$

3.3 Estimate of $u^\flat$

Recall that

$$u^\flat(t) = \int_{\mathbb{R}} \int_0^t S_{A_{\gamma,\delta}}(s)(B_\gamma f + \delta f)(t-s,\xi) ds d\xi.$$

We use the same argument as above. We first notice that

$$\widehat{u}^\flat(0,t) = \int_{\mathbb{T}^N} u^\flat(x,t) dx = 0$$

for all $t \geq 0$, almost surely. This is easily seen using that

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} f(x,t,\xi) d\xi \, dx = 0,$$

which follows from (4) and (17) with test functions approximating $\varphi(x,\xi) = 1$. Then, we write the Fourier transform of $u^\flat$ for $n \neq 0$:

$$\begin{aligned}
\hat{u}^\flat(n,t) &= \int_0^t \int_{\mathbb{R}} (\gamma|n|^{2\alpha} + \delta) e^{-s(-ia(\xi)\cdot n + \gamma|n|^{2\alpha} + \delta)} \widehat{f}(n,\xi,t-s) d\xi ds \\
&= \int_0^t (\gamma|n|^{2\alpha} + \delta) e^{-s(\gamma|n|^{2\alpha}+\delta)} \mathcal{G}\widehat{f}(n,ns,t-s) ds,
\end{aligned}$$

with, as in (25),

$$\mathcal{G}\widehat{f}(n,z,s) = \int_{\mathbb{R}} e^{-ia(\xi)\cdot z} \widehat{f}(n,\xi,s) d\xi.$$

By Jensen's inequality, we have

$$|\hat{u}^\flat(n,t)|^2 \leq \int_0^t (\gamma|n|^{2\alpha} + \delta) e^{-s(\gamma|n|^{2\alpha}+\delta)} \left|\mathcal{G}\widehat{f}(n,ns,t-s)\right|^2 ds.$$

Then, by simple manipulation and Lemma 2.4 in [BD99], we deduce the follo-

wing sequence of inequality for $T \geq 0$ (recall that ω_n is defined by (26)):

$$\int_0^T |\hat{u}^\flat(n,t)|^2 dt \leq \int_0^T \int_0^t (\gamma|n|^{2\alpha}+\delta)e^{-s(\gamma|n|^{2\alpha}+\delta)} \left|\mathcal{G}\widehat{f}(n,ns,t-s)\right|^2 dsdt$$

$$\leq \int_0^T \int_0^T (\gamma|n|^{2\alpha}+\delta)e^{-s(\gamma|n|^{2\alpha}+\delta)} \left|\mathcal{G}\widehat{f}(n,ns,t)\right|^2 dsdt$$

$$\leq \int_0^T \int_{\mathbb{R}^+} \omega_n e^{-\omega_n s} \left|\mathcal{G}\widehat{f}\left(n,\frac{n}{|n|}s,t\right)\right|^2 dsdt$$

$$\leq \int_0^T \int_{\mathbb{R}^+} \omega_n \frac{1}{1+\omega_n^2 s^2} \left|\mathcal{G}\widehat{f}\left(n,\frac{n}{|n|}s,t\right)\right|^2 dsdt$$

$$\leq c\omega_n^b \int_0^T \left|\widehat{f}(n,t)\right|^2_{L^2_\xi} dt.$$

We conclude by summing over $n \in \mathbb{Z}^N$, $n \neq 0$:

$$\int_0^T \|u^\flat(t)\|^2_{H^{\left(\frac{1}{2}-\alpha\right)^b}} dt \leq c\gamma^b \int_0^T \|f(t)\|^2_{L^2_{x,\xi}}.$$

Since $\|f(t)\|^2_{L^2_{x,\xi}} = \|u(t)\|_{L^1_x}$, we obtain

$$\int_0^T \|u^\flat(t)\|^2_{H^{\left(\frac{1}{2}-\alpha\right)^b}} dt \leq c\gamma^b \int_0^T \|u(t)\|_{L^1_x} dt. \tag{28}$$

3.4 Estimate of P

Recall that P is a random measure on $\mathbb{T}^N$ given by:

$$\langle P(t),\varphi\rangle = \sum_{k\geq 1} \int_{\mathbb{T}^N} \int_0^t \left(S^*_{A_{\gamma,\delta}}(t-s)\varphi\right)(x,u(s,x))g_k(x)d\beta_k(s)dx,\ \varphi \in C(\mathbb{T}^N).$$

Note that, for each $k \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{T}^N} \int_0^t \left(S^*_{A_{\gamma,\delta}}(t-s)\varphi\right)(x,u(s,x))g_k(x)d\beta_k(s)dx \\
&\quad = \int_{\mathbb{T}^N} \int_0^t e^{-\delta(t-s)} \left(e^{-B_\gamma(t-s)}\varphi\right)\big(x+a(u(s,x))(t-s)\big)g_k(x)d\beta_k(s)dx \\
&\quad = \int_{\mathbb{T}^N} \int_0^t e^{-\delta(t-s)}\varphi(x)e^{-B_\gamma(t-s)}\big(g_k(\cdot - a(u(\cdot,s))(t-s))\big)(x)d\beta_k(s)dx.
\end{aligned}$$

By (2), the mapping $x \mapsto g_k(x-a(u(x,s))(t-s))$ is a bounded function, therefore by Lemma 10 we can write for $\sigma \geq 0$:

$$\|e^{B_\gamma(t-s)}\left(g_k(\cdot - a(u(\cdot,s))(t-s))\right)\|_{H^\sigma(\mathbb{T}^N)} \leq c\left(\gamma(t-s)\right)^{-\frac{\sigma}{2\alpha}} \|g_k\|_{L^2(\mathbb{T}^N)}.$$

We deduce that, for $\sigma \in [0, \alpha)$, this defines a function in $H^\sigma(\mathbb{T}^N)$ and

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t e^{-\delta(t-s)} e^{-B_\gamma(t-s)} \left(g_k(\cdot - a(u(\cdot,s))(t-s))\right) d\beta_k(s)\right\|^2_{H^\sigma(\mathbb{T}^N)}\right) \\
&\leq c \int_0^t e^{-2\delta(t-s)} \left(\gamma(t-s)\right)^{-\frac{\sigma}{\alpha}} ds \|g_k\|^2_{L^2(\mathbb{T}^N)} \\
&\leq c\gamma^{-\frac{\sigma}{\alpha}} \delta^{\frac{\sigma}{\alpha}-1} \int_{\mathbb{R}^+} e^{2s} s^{-\frac{\sigma}{\alpha}} ds \|g_k\|^2_{L^2(\mathbb{T}^N)}
\end{aligned} \tag{29}$$

Since the sum over k of the right hand side of (29) is finite, we may then conclude:

$$\begin{aligned}
&\mathbb{E}\left(\left\|\sum_{k\geq 1}\int_0^t e^{B_\gamma(t-s)} \left(g_k(\cdot - a(u(\cdot,s))(t-s))\right) d\beta_k(s)\right\|^2_{H^\sigma(\mathbb{T}^N)}\right) \\
&\leq c\gamma^{-\frac{\sigma}{\alpha}} \delta^{\frac{\sigma}{\alpha}-1} \left(\int_{\mathbb{R}^+} e^{-2s} s^{-\frac{\sigma}{\alpha}} ds\right) D_0.
\end{aligned}$$

This shows that in fact $P(t)$ is more regular than a measure. It is in the space $L^2(\Omega; H^\sigma(\mathbb{T}^N))$ for $\sigma < \alpha$ and

$$\mathbb{E}\left(\|P(t)\|^2_{H^\sigma(\mathbb{T}^N)}\right) \leq c\gamma^{-\frac{\sigma}{\alpha}} \delta^{\frac{\sigma}{\alpha}-1} \left(\int_{\mathbb{R}^+} e^{-2s} s^{-\frac{\sigma}{\alpha}} ds\right) D_0. \tag{30}$$

3.5 Bound on the kinetic measure

Before we estimate Q, we give an estimate on the kinetic measure.

Lemma 11. *Let $u\colon \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ be the solution to (1) with initial datum u_0. Then the measure $q := m - \frac{1}{2}\mathbf{G}^2\delta_{u=\xi}$ satisfies*

$$\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} \theta(\xi) d|q|(x,t,\xi) \leq D_0 \mathbb{E}\|\theta(u)\|_{L^1(\mathbb{T}^N\times[0,T])} + \mathbb{E}\int_{\mathbb{T}^N} \Theta(u_0(x))dx \tag{31}$$

for all non-negative $\theta \in C_c(\mathbb{R})$, where $\Theta(s) = \int_0^s \int_0^\sigma \theta(r) dr d\sigma$.

Proof: Let $\theta \in C_c(\mathbb{R})$ be non-negative. Set

$$\Theta(s) = \int_0^s \int_0^\sigma \theta(r) dr d\sigma.$$

We test the kinetic formulation (14) with $\Theta'(\xi)$ and sum over $x \in \mathbb{T}^N$, $\xi \in \mathbb{R}$:

$$\frac{d}{dt}\int_{\mathbb{T}^N} \mathbb{E}\Theta(u)dx = \frac{1}{2}\mathbb{E}\int_{\mathbb{T}^N} \mathbf{G}^2\theta(u)dx - \mathbb{E}\int_{\mathbb{T}^N\times\mathbb{R}} \theta(\xi)dm(x,t,\xi).$$

Since $\Theta(0) = 0$ and $\Theta(u) \geq 0$, we deduce that

$$\begin{aligned}
\mathbb{E}|\langle |q|, \theta\rangle| &\leq \frac{1}{2}\mathbb{E}\int_{\mathbb{T}^N\times[0,T]} \mathbf{G}^2\theta(u)dx + \mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} \theta(\xi)dm(x,t,\xi), \\
&\leq \mathbb{E}\int_{\mathbb{T}^N\times[0,T]} \mathbf{G}^2\theta(u)dx + \mathbb{E}\int_{\mathbb{T}^N} \Theta(u_0(x))dx \\
&\leq D_0\mathbb{E}\|\theta(u)\|_{L^1(\mathbb{T}^N\times[0,T])} + \mathbb{E}\int_{\mathbb{T}^N} \Theta(u_0(x))dx. \qquad \blacksquare
\end{aligned}$$

3.6 Estimate of Q

Recall that

$$\langle Q(t), \varphi\rangle = \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} (t-s)a'(\xi)\cdot\nabla S^*_{A_{\gamma,\delta}}(t-s)\varphi dq(x,s,\xi).$$

For $\varepsilon > 0$, we define the measure:

$$\langle Q_\varepsilon(t), \varphi\rangle = \int_{\mathbb{T}^N\times[0,(t-\varepsilon)\wedge 0]\times\mathbb{R}} (t-s)a'(\xi)\cdot\nabla S^*_{A_{\gamma,\delta}}(t-s)\varphi dq(x,s,\xi),$$

where $\varphi \in C(\mathbb{T}^N)$. We choose φ depending also on $\omega \in \Omega$ and $t \in [0,T]$. We then have, for $\lambda \in (0,2]$,

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q_\varepsilon(t), \varphi(t)\rangle dt \\
&\quad = \mathbb{E}\int_\varepsilon^T \int_{\mathbb{T}^N\times[0,t-\varepsilon]\times\mathbb{R}} (t-s)a'(\xi)\cdot\nabla(-\Delta)^{\frac{\lambda}{2}} S^*_{A_{\gamma,\delta}}(t-s)\varphi(t) dq(x,s,\xi)dt.
\end{aligned}$$

By Lemma 10, for $p \geq 1$ and p' its conjugate exponent, we have

$$\begin{aligned}
&\|(-\Delta)^{\frac{\lambda}{2}}\nabla S^*_{A_{\gamma,\delta}}(t-s)\varphi(t)\|_{L^\infty(\mathbb{T}^N\times\mathbb{R})} \\
&\qquad = e^{-\delta(t-s)}\|(-\Delta)^{\frac{\lambda}{2}}\nabla e^{B_\gamma(t-s)}\varphi(t)\|_{L^\infty(\mathbb{T}^N)} \\
&\qquad \leq c(\gamma(t-s))^{-2+\mu_{N,\alpha,\lambda,p}} e^{-\delta(t-s)}\|\varphi(t)\|_{L^{p'}(\mathbb{T}^N)},
\end{aligned}$$

with

$$\mu_{N,\alpha,\lambda,p} = 2 - \frac{N+\lambda+1}{2\alpha} + \frac{N}{2p\alpha}. \tag{32}$$

Assume $\mu_{N,\alpha,\lambda,p} > 0$. By the Fubini Theorem, we have then

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q_\varepsilon(t), \varphi(t)\rangle dt \\
&\qquad \leq \frac{\kappa}{\gamma}\|\varphi\|_{L^\infty(\Omega\times(0,T);L^{p'}(\mathbb{T}^N))}\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} |a'(\xi)|d|q|(x,s,\xi),
\end{aligned}$$

where

$$\kappa = \sup_{s\in[0,T],\ T>0}\int_{s+\varepsilon}^T (\gamma(t-s))^{-1+\mu}e^{-\delta(t-s)}dt \leq \gamma^{-1+\mu}\delta^{-\mu}\int_{\mathbb{R}^+}\sigma^{\mu-1}e^{-\sigma}d\sigma,$$

with $\mu = \mu_{N,\alpha,\lambda,p}$. It follows that for all $\varepsilon > 0$, $Q_\varepsilon \in L^1(\Omega\times(0,T); W^{\lambda,p}(\mathbb{T}^N))$ with

$$\begin{aligned}
&\mathbb{E}\|Q_\varepsilon\|_{L^1((0,T);W^{\lambda,p}(\mathbb{T}^N))} \\
&\qquad \leq C_{N,\alpha,\lambda,p}\gamma^{-2}\left(\frac{\gamma}{\delta}\right)^{\mu_{N,\alpha,\lambda,p}}\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} |a'(\xi)|d|q|(x,s,\xi),
\end{aligned}$$

for some constant $C_{N,\alpha,\lambda,p}$. We let $\varepsilon \to 0$ and use Lemma 11 to obtain eventually the following estimate on Q:

$$\begin{aligned}
&\mathbb{E}\|Q\|_{L^1((0,T);W^{\lambda,p}(\mathbb{T}^N))} \\
&\leq C_{N,\alpha,\lambda,p}\gamma^{-2}\left(\frac{\gamma}{\delta}\right)^{\mu_{N,\alpha,\lambda,p}}\left(D_0\mathbb{E}\|a'(u)\|_{L^1(\mathbb{T}^N\times(0,T))} + \mathbb{E}\int_{\mathbb{T}^N}\Theta(u_0)dx\right),
\end{aligned} \tag{33}$$

where $\Theta(s) = \int_0^s\int_0^\sigma |a'(r)|drd\sigma$, provided $\mu_{N,\alpha,\lambda,p} > 0$.

3.7 Conclusion and proof of the existence part in Theorem 1

Under the growth hypothesis (10) (sub-linearity of a'), the bound (33) gives

$$\begin{aligned}&\mathbb{E}\|Q\|_{L^1((0,T);W^{\lambda,p}(\mathbb{T}^N))}\\&\leq C_{N,\alpha,\lambda,p,D_0}\gamma^{-2}\left(\frac{\gamma}{\delta}\right)^{\mu_{N,\alpha,\lambda,p}}\left(1+\mathbb{E}\|u\|_{L^1(\mathbb{T}^N\times(0,T))}+\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^N)}\right).\end{aligned}$$

We choose γ, $\delta>0$ such that:

$$C_{N,\alpha,\lambda,p,D_0}\gamma^{-2}\left(\frac{\gamma}{\delta}\right)^{\mu_{N,\alpha,\lambda,p}}\leq\frac{1}{4}$$

and obtain

$$\begin{aligned}&\mathbb{E}\|Q\|_{L^1(0,T;W^{\lambda,p}(\mathbb{T}^N))}\\&\qquad\leq\frac{1}{4}\mathbb{E}\|u\|_{L^1(0,T;L^p(\mathbb{T}^1))}+C(N,\alpha,\lambda,p,\gamma,\delta,D_0)(\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^N)}+1).\end{aligned}\tag{34}$$

Consider now the estimates (27), (28), (30) and assume that $H^{\left(\frac{1}{2}-\alpha\right)b}$ and H^σ are both imbedded in a given Sobolev space $W^{s,q}(\mathbb{T}^N)$ with $s>0$ (the exponent s, q will be determined later). Then (27), (28), (30) give

$$\begin{aligned}&\mathbb{E}\big(\|u^0+u^\flat+P\|^2_{L^2(0,T;W^{s,q}(\mathbb{T}^N))}\big)\\&\qquad\leq C(\delta,\gamma,D_0)\left(1+\mathbb{E}\|u\|_{L^1(\mathbb{T}^N\times(0,T))}+\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^1)}+T\right).\end{aligned}$$

By the Cauchy-Schwarz inequality and the Young inequality, we obtain

$$\begin{aligned}&\mathbb{E}\big(\|u^0+u^\flat+P\|_{L^1(0,T;W^{s,q}(\mathbb{T}^N))}\big)\\&\qquad\leq C(\delta,\gamma,D_0)\left(1+\mathbb{E}\|u_0\|_{L^1(\mathbb{T}^N)}+T\right)+\frac{1}{4}\mathbb{E}\|u\|_{L^1(0,T;L^1(\mathbb{T}^N))}.\end{aligned}\tag{35}$$

Assuming also that $W^{\lambda,p}(\mathbb{T}^N)$ is embedded in $W^{s,q}(\mathbb{T}^N)$, we deduce from (20), (34), (35) the estimate

$$\begin{aligned}&\mathbb{E}\|u\|_{L^1(0,T;W^{s,q}(\mathbb{T}^N))}\\&\qquad\leq C(N,\alpha,\lambda,p,\gamma,\delta,D_0)(\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^1)}+1+T)+\frac{1}{2}\mathbb{E}\|u\|_{L^1(0,T;L^1(\mathbb{T}^N))},\end{aligned}$$

and thus

$$\mathbb{E}\|u\|_{L^1(0,T;W^{s,q}(\mathbb{T}^N))}\leq C(N,\alpha,\lambda,p,\gamma,\delta,D_0)(\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^N)}+1+T).\tag{36}$$

Recall that the estimates above are actually obtained for u^η, the solution of (18) which converges to the solution of (1) in $L^\infty(0,T;L^1(\Omega\times\mathbb{T}^1))$. Thanks to the closedness of balls of $L^1(\Omega\times(0,T);W^{\lambda,p}(\mathbb{T}^1))$ for the $L^\infty(0,T;L^1(\Omega\times\mathbb{T}^1))$ topology, these estimates actually hold for the true solution u. Then, by Krylov-Bogoliubov theorem and the compactness of the embedding of $W^{s,q}(\mathbb{T}^N)$ in $L^1(\mathbb{T}^N)$, existence of an invariance measure follows. Moreover, by the Sobolev embedding (37) below, this invariant measure is supported by $L^r(\mathbb{T}^N)$ for any r such that $\frac{s}{N}-\frac{1}{q}\geq-\frac{1}{r}$.

There remains to determine for which s, q we do have (36). Let us recall that we have the injection

$$W^{r,p}(\mathbb{T}^N) \hookrightarrow W^{s,q}(\mathbb{T}^N) \tag{37}$$

if $r \geq s$ and $\frac{r}{N} - \frac{1}{p} \geq \frac{s}{N} - \frac{1}{q}$, where $\frac{s}{N} - \frac{1}{q}$ can be considered as the index of regularity of the Sobolev space $W^{s,q}(\mathbb{T}^N)$. Then (27), (28) and (30) provide us with the indices

$$\mathrm{ind}_0 = \frac{\alpha + \left(\frac{1}{2} - \alpha\right) b}{N} - \frac{1}{2}, \ \mathrm{ind}_\flat = \frac{\left(\frac{1}{2} - \alpha\right) b}{N} - \frac{1}{2}, \ \mathrm{ind}_P = \frac{\alpha-}{N} - \frac{1}{2},$$

where $\alpha-$ denotes any positive number $\sigma < \alpha$. The condition $\mu_{N,\alpha,\lambda,p} > 0$ on the parameter $\mu_{N,\alpha,\lambda,p}$ (defined by (32)) reads

$$\mathrm{ind}_Q := \frac{\lambda}{N} - \frac{1}{p} < \mathrm{ind}_Q^+ := \frac{4\alpha}{N} - \frac{N+1}{N},$$

where ind_Q is associated to (33). Clearly, we have $\mathrm{ind}_0 > \mathrm{ind}_\flat$. We look for a sobolev space $W^{s,q}(\mathbb{T}^N)$ of index lower than the minimum of $\mathrm{ind}_\flat$, ind_P, ind_Q^+, *i.e.*

$$s < \frac{N}{q} + \min\left(\left(\frac{1}{2} - \alpha\right) b - \frac{N}{2}, \alpha - \frac{N}{2}, 4\alpha - (N+1)\right), \tag{38}$$

and satisfying also

$$0 < s, \tag{39}$$

$$s < \min\left(\left(\frac{1}{2} - \alpha\right) b, \alpha\right). \tag{40}$$

The constraint (40) is contained in (38) as soon as $q \geq 2$. The constraint (39), together with (38) gives (for the maximal admissible value $\alpha = \frac{1}{2}$) the condition $\frac{1}{N} + \frac{1}{q} > 1$, which is compatible with $q \geq 2$ only if $N = 1$. Therefore we treat separately the cases $N = 1$ and $N > 1$.

Case $N = 1$: We choose $q = 2$ and then optimize the right-hand side of (38). The maximum is $\frac{b}{2(b+4)}$, reached for $\alpha = \frac{b+3}{2(b+4)}$. Then we choose $p = q$ and $s = \lambda \in \left(0, \frac{b}{2(b+4)}\right)$. We obtain that the invariant measure is supported by $L^r(\mathbb{T}^1)$ for any $r < 2 + \frac{b}{2}$.

Case $N > 1$: take q satisfying $\frac{1}{N} + \frac{1}{q} = 1 + \frac{\eta}{N}$, $\eta > 0$ small. We then have to consider the maximum of

$$\min\left(\left(\frac{1}{2} - \alpha\right) b, \alpha, 4\alpha - 2 + \eta\right),$$

for $\alpha \in [0, \frac{1}{2}]$, which is $\frac{\eta b}{b+4}$, obtained for $\alpha = \frac{1}{2} - \frac{\eta}{b+4}$. We choose then $p = q$, $\lambda = s \in \left(0, \frac{\eta b}{b+4}\right)$. We obtain that the invariant measure is supported by $L^r(\mathbb{T}^N)$ for any $r < \frac{N}{N-1}$. ■

4 Uniqueness of the invariant measure, ergodicity.

We have proved in (36) that there exists $q > 1$ and $s > 0$ such that:

$$\mathbb{E}\|u\|_{L^1(0,T;W^{s,q}(\mathbb{T}^N))} \leq \kappa_0(\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^N)} + 1 + T) \tag{41}$$

where κ_0 depends on λ, p, D_0. Below, in Section 4.1, we prove that this estimate implies that any solution enters a ball of some fixed radius in finite time. Then in the case of subquadratic flux A, we show that if the driving noise is small for some time, then the solution becomes very small, *cf.* Section 4.2. This smallness depends on the size of the initial data. These two facts are then shown to imply the second part of Theorem 1 (Section 4.3).

4.1 Time to enter a ball in $L^1(\mathbb{T}^N)$

For further purposes, we need to consider two solutions u^1 and u^2 starting from u^1_0, u^2_0 deterministic and in $L^3(\mathbb{T}^N)$. It is easy to generalize (41) to two solutions on the interval $[t, t+T]$ for t, $T \geq 0$. This implies:

$$\begin{aligned}\mathbb{E}\Big(\int_t^{t+T} \|u^1(s)\|_{L^1(\mathbb{T}^N)} + \|u^2(s)\|_{L^1(\mathbb{T}^N)} ds \big| \mathcal{F}_t\Big) \\ \leq \kappa_1(\|u^1(t)\|^3_{L^3(\mathbb{T}^N)} + \|u^2(t)\|^3_{L^3(\mathbb{T}^N)} + 1 + T)\end{aligned} \tag{42}$$

Moreover, we easily prove for $i = 1, 2$, thanks to the Itô formula:

$$\begin{aligned}&\|u^i(t)\|^3_{L^3(\mathbb{T}^N)} \\ &\leq \|u^i_0\|^3_{L^3(\mathbb{T}^N)} + 3\int_0^t ((u^i)^2(s), \Phi dW(s))_{L^2(\mathbb{T}^N)} + 3D_0 \int_0^t \|u^i(s)\|_{L^1(\mathbb{T}^N)} ds.\end{aligned} \tag{43}$$

It follows that

$$\begin{aligned}&\mathbb{E}\Big(\int_t^{t+T} \|u^1(s)\|_{L^1(\mathbb{T}^N)} + \|u^2(s)\|_{L^1(\mathbb{T}^N)} ds \big| \mathcal{F}_t\Big) \\ &\leq \kappa_1 \Big[\|u^1_0\|^3_{L^3(\mathbb{T}^N)} + \|u^2_0\|^3_{L^3(\mathbb{T}^N)} + 3D_0 \int_0^t \|u^1(s)\|_{L^1(\mathbb{T}^N)} + \|u^2(s)\|_{L^1(\mathbb{T}^N)} ds \\ &\quad + 3\int_0^t ((u^1(s))^2 + (u^2(s))^2, \Phi dW(s))_{L^2(\mathbb{T}^N)} + 1 + T\Big]\end{aligned}$$

We now define recursively the sequences of deterministic times $(t_k)_{k\geq 0}$ and $(r_k)_{k\geq 0}$ by

$$t_0 = 0,$$

$$t_{k+1} = t_k + r_k,$$

where $(r_k)_{k\geq 0}$ will be chosen below. We also define the events:

$$A_k = \Big\{\inf_{s\in[t_\ell, t_{\ell+1}]} \big(\|u^1(s)\|_{L^1(\mathbb{T}^N)} + \|u^2(s)\|_{L^1(\mathbb{T}^N)}\big) \geq 2\kappa_1,\ \ell = 0, \ldots, k-1\Big\}.$$

Then, for all $k \geq 0$,

$$
\begin{aligned}
&\mathbb{P}\left(\inf_{s\in[t_k,t_{k+1}]}\left(\|u^1(s)\|_{L^1(\mathbb{T}^N)}+\|u^2(s)\|_{L^1(\mathbb{T}^N)}\right)\geq 2\kappa_1\middle|\mathcal{F}_{t_k}\right)\\
\leq&\mathbb{P}\left(\frac{1}{r_k}\int_{t_k}^{t_k+r_k}\|u^1(s)\|_{L^1(\mathbb{T}^N)}+\|u^2(s)\|_{L^1(\mathbb{T}^N)}ds\geq 2\kappa_1\middle|\mathcal{F}_{t_k}\right)\\
\leq&\frac{1}{2r_k}\Bigg[\|u_0^1\|^3_{L^3(\mathbb{T}^N)}+\|u_0^2\|^3_{L^3(\mathbb{T}^N)}\\
&+3D_0\int_0^{t_k}\|u^1(s)\|_{L^1(\mathbb{T}^N)}+\|u^2(s)\|_{L^1(\mathbb{T}^N)}ds+1\Bigg]\\
&+\frac{1}{2}+\frac{3}{2r_k}\int_0^{t_k}((u^1(s))^2+(u^2(s))^2,\Phi dW(s))_{L^2(\mathbb{T}^N)}.
\end{aligned}
\tag{44}
$$

We will choose r_k so that the following inequality is satisfied for all $k \geq 0$:

$$
\frac{1}{2r_k}(\|u_0^1\|^3_{L^3(\mathbb{T}^N)}+\|u_0^2\|^3_{L^3(\mathbb{T}^N)}+1)\leq\frac{1}{8}.
$$

Let us then multiply (44) by $\mathbf{1}_{A_k}$ and take the expectation to obtain

$$
\begin{aligned}
\mathbb{P}\left(A_{k+1}\right)\leq\quad&\frac{5}{8}\mathbb{P}\left(A_k\right)+\frac{3D_0}{2r_k}\mathbb{E}\left(\int_0^{t_k}\|u^1(s)\|_{L^1(\mathbb{T}^N)}+\|u^2(s)\|_{L^1(\mathbb{T}^N)}ds\mathbf{1}_{A_k}\right)\\
&+\frac{3}{2r_k}\mathbb{E}\left(\int_0^{t_k}((u^1(s))^2+(u^2(s))^2,\Phi dW(s))_{L^2(\mathbb{T}^N)}\mathbf{1}_{A_k}\right).
\end{aligned}
$$

By the Cauchy-Schwarz inequality, Itô isometry and the bound (2) on the covariance of the noise, we have:

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^t((u^1(s))^2+(u^2(s))^2,\Phi dW(s))_{L^2(\mathbb{T}^N)}\mathbf{1}_{A_k}\Big)\\
&\qquad\leq(2D_0)^{1/2}\left(\mathbb{E}\int_0^t\|u^1(s)\|^4_{L^2(\mathbb{T}^N)}+\|u^2(s)\|^4_{L^2(\mathbb{T}^N)}ds\right)^{1/2}\mathbb{P}(A_k)^{1/2}.
\end{aligned}
$$

By Itô formula applied to $\|u^i(t)\|^2_{L^2(\mathbb{T}^N}$ and $\|u^i(t)\|^4_{L^2(\mathbb{T}^N}$, $i=1,2$, we obtain the bound:

$$
\mathbb{E}\left(\|u^i(t)\|^2_{L^2(\mathbb{T}^N)}\right)\leq\mathbb{E}\left(\|u_0^i\|^2_{L^2(\mathbb{T}^N)}\right)+D_0t
$$

and

$$
\mathbb{E}\left(\|u^i(t)\|^4_{L^2(\mathbb{T}^N)}\right)\leq\mathbb{E}\left(\|u_0^i\|^4_{L^2(\mathbb{T}^N)}+6D_0\int_0^t\|u^i(s)\|^2_{L^2(\mathbb{T}^N)}ds\right).
$$

Thus, integrating in time, we obtain a bound

$$
\mathbb{E}\int_0^{t_k}\|u^1(s)\|^4_{L^2(\mathbb{T}^N)}+\|u^2(s)\|^4_{L^2(\mathbb{T}^N)}ds\leq 2[\|u_0^1\|^4_{L^2(\mathbb{T}^N)}+\|u_0^2\|^4_{L^2(\mathbb{T}^N)}]t_k+24D_0^2t_k^3.
$$

Similarly, we have the estimate

$$
\mathbb{E}\left|\int_0^{t_k}\|u^1(s)\|_{L^1(\mathbb{T}^N)}+\|u^2(s)\|_{L^1(\mathbb{T}^N)}ds\right|^2\leq t_k^2[\|u_0^1\|^2_{L^2(\mathbb{T}^N)}+\|u_0^2\|^2_{L^2(\mathbb{T}^N)}]+D_0t_k^3.
$$

It follows that

$$\mathbb{P}\left(A_{k+1}\right) \leq \frac{3}{4}\mathbb{P}\left(A_k\right) + \frac{C}{r_k^2}\left([\|u_0^1\|_{L^2(\mathbb{T}^N)}^4 + \|u_0^2\|_{L^2(\mathbb{T}^N)}^4]t_k + t_k^4 + 1\right),$$

where the constant C can be written explicitely in term of D_0. We choose r_k so that:

$$\frac{C}{r_k^2}\left([\|u_0^1\|_{L^2(\mathbb{T}^N)}^4 + \|u_0^2\|_{L^2(\mathbb{T}^N)}^4]t_k + t_k^4 + 1\right) \leq \left(\frac{3}{4}\right)^k,$$

and obtain: $\mathbb{P}\left(A_{k+1}\right) \leq \frac{3}{4}\mathbb{P}\left(A_k\right) + \left(\frac{3}{4}\right)^k$, which gives therefore

$$\mathbb{P}\left(A_k\right) \leq k\left(\frac{3}{4}\right)^{k-1}.$$

By Borel-Cantelli Lemma, we deduce that

$$k_0 = \inf\{k \geq 0 \mid \inf_{s\in[t_\ell, t_{\ell+1}]} \|u^1(s)\|_{L^1(\mathbb{T}^N)} + \|u^2(s)\|_{L^1(\mathbb{T}^N)} \leq 2\kappa_1\}$$

is almost surely finite. We then define the stopping time

$$\tau^{u_0^1,u_0^2} = \inf\{t \geq 0 \mid \|u^1(t)\|_{L^1(\mathbb{T}^N)} + \|u^2(t)\|_{L^1(\mathbb{T}^N)} \leq 2\kappa_1\}.$$

Clearly $\tau^{u_0^1,u_0^2} \leq t_{k_0+1}$ so that $\tau^{u_0^1,u_0^2} < \infty$ almost surely. It follows that for $T > 0$ the following stopping times are also almost surely finite:

$$\tau_\ell = \inf\{t \geq \tau_{\ell-1} + T \mid \|u^1(t)\|_{L^1(\mathbb{T}^N)} + \|u^2(t)\|_{L^1(\mathbb{T}^N)} \leq 2\kappa_1\},\ \tau_0 = 0.$$

4.2 The solution is small if the noise is small

Proposition 12. *Assume that a satisfies (11). Then, for any $\varepsilon > 0$, there exists $T > 0$ and $\eta > 0$ such that:*

$$\frac{1}{T}\int_0^T \|u(s)\|_{L^1(\mathbb{T}^N)} ds \leq \frac{\varepsilon}{2}$$

if

$$\|u(0)\|_{L^1(\mathbb{T}^N)} \leq 2\kappa_1 \text{ and } \sup_{t\in[0,T]} \|W\|_{W^{1,\infty}(\mathbb{T}^N)} \leq \eta.$$

Proof: We first take $\tilde{u}_0 \in L^2(\mathbb{T}^N)$ such that

$$\|u_0 - \tilde{u}_0\|_{L^1(\mathbb{T}^N)} \leq \frac{\varepsilon}{8},\ \|\tilde{u}_0\|_{L^2(\mathbb{T}^N)} \leq C\kappa_1\varepsilon^{-N/2}.$$

It is easy to see that this can be achieved by taking $\tilde{u}_0 = \rho_\varepsilon * u_0$ for some regularizing kernel $(\rho_\varepsilon)_{\varepsilon>0}$. By (16), we know that

$$\|u(t) - \tilde{u}(t)\|_{L^1(\mathbb{T}^N)} \leq \frac{\varepsilon}{8},\ t \geq 0,$$

where $\tilde{u}$ is the solution starting from $\tilde{u}_0$. We set $v = \tilde{u} - W$ so that v is a kinetic solution to

$$\partial_t v + \mathrm{div} A(v + W) = 0$$

and $g = \mathbf{1}_{v(x,t)>\xi} - \mathbf{1}_{0>\xi}$ satisfies:

$$\partial_t g + a(\xi)\cdot\nabla g = \partial_\xi \mathrm{n} + (a(\xi) - a(\xi+W))\cdot\nabla g - a(\xi+W)\cdot\nabla W \delta_{v=\xi}. \quad (45)$$

The kinetic measure n is easily related to the kinetic measure in the equation satisfied by $f = \mathbf{1}_{u>\xi} - \mathbf{1}_{0>\xi}$. Also since both u and W have a zero spatial average, so does v.

We use again B_γ and $A_{\gamma,\delta}$ introduced in section 3.1. We now take $\alpha = \frac{1}{2}$. Then (45) rewrites:

$$\partial_t g + A_{\gamma,\delta} g = (B_\gamma + \delta \mathrm{Id})g + \partial_\xi \mathrm{n} + (a(\xi) - a(\xi+W))\cdot\nabla g - a(\xi+W)\cdot\nabla W\delta_{v=\xi}. \quad (46)$$

By solving (46) and summing over ξ, we are led to the following decomposition of v:

$$v = v^0 + v^\flat + v^\# + P^W + N^W,$$

where

$$\begin{aligned}
v^0(t) &:= \int_{\mathbb{R}} S_{A_{\gamma,\delta}}(t) g(0,\xi) d\xi,\\
v^\flat(t) &:= \int_{\mathbb{R}}\int_0^t S_{A_{\gamma,\delta}}(s)\,(B_\gamma + \delta\mathrm{Id}) g(t-s,\xi) ds d\xi,\\
v^\#(t) &:= \int_{\mathbb{R}}\int_0^t S_{A_{\gamma,\delta}}(s)(a(\xi) - a(\xi+W))\cdot\nabla g(t-s,\xi)\, ds d\xi,\\
\langle P^W(t), \varphi\rangle &:= -\int_{\mathbb{T}^N\times[0,t]} (a(v+W)\cdot\nabla W)(x,s)\left(S^*_{A_{\gamma,\delta}}(t-s)\varphi\right)(x, v(x,s))\, ds dx,\\
\langle N^W(t), \varphi\rangle &:= \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} (t-s) a'(\xi)\cdot\nabla S^*_{A_{\gamma,\delta}}(t-s)\varphi d\mathrm{n}(x,s,\xi),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the duality product between the space of finite Borel measures on $\mathbb{T}^N$ and $C(\mathbb{T}^N)$.

Reproducing the argument of Section 3.2 and 3.3, we obtain the estimates:

$$\int_0^T \|v^0(t)\|^2_{H^{\frac{1}{2}}(\mathbb{T}^N)} dt \le c\,\gamma^{b-1}\,\|u_0\|_{L^1(\mathbb{T}^N)} \quad (47)$$

and

$$\int_0^T \|v^\flat(t)\|^2_{L^2(\mathbb{T}^N)} dt \le c\,\gamma^b \int_0^T \|v(t)\|_{L^1(\mathbb{T}^N)}\, dt. \quad (48)$$

We estimate $v^\#$ similarly. We take the Fourier transform in x (denoted by $\mathcal{F}$ here) and integrate in time. Note that

$$\begin{aligned}
&\mathcal{F}\left[(a(\cdot) - a(\cdot+W))\cdot\nabla g\right](n,\xi,t)\\
&\quad = 2\pi i \mathcal{F}\left[(a(\cdot) - a(\cdot+W))g\right](n,\xi,t)\cdot n + \mathcal{F}\left[(a'(\cdot+W)\cdot\nabla W)g\right](n,\xi,t)
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_0^T |\hat{v}^\#(n,t)|^2 dt\\
&\le c\int_0^T \left|\int_{\mathbb{R}}\int_0^t e^{-i(a(\xi)\cdot n + \gamma|n| + \delta)s}\mathcal{F}\left[(a(\cdot) - a(\cdot+W))g\right](n,\xi,t-s)\cdot n\, ds d\xi\right|^2 dt\\
&+ c\int_0^T \left|\int_{\mathbb{R}}\int_0^t e^{-i(a(\xi)\cdot n + \gamma|n| + \delta)s}\mathcal{F}\left[(a'(\cdot+W)\cdot\nabla W)g\right](n,\xi,t-s)\, ds d\xi\right|^2 dt.
\end{aligned}$$

Denote by A_1 and A_2 the two terms on the right hand side. A_1 is of the form

$$A_1 = \int_0^T \left| \int_0^t e^{-(\gamma|n|+\delta)s} n \cdot \mathcal{G}h(n, ns, t-s) ds \right|^2 dt,$$

where

$$h(n, \xi, t) = \mathcal{F}\left[(a(\cdot) - a(\cdot + W))\, g\right](n, \xi, t)$$

and the integral operator $\mathcal{G}$ has been defined in (25). We proceed as in Section 3.2 and 3.3 (in particular we use Lemma 2.4 of [BD99] for the estimate of the oscillatory integral $\mathcal{G}h$) to obtain

$$A_1 \leq c\gamma^{-2+b} \int_0^T \|h(n, \xi, t)\|^2_{L^2(\xi)}\, dt,$$

and, similarly,

$$A_2 \leq \frac{c\gamma^{-2+b}}{|n|} \int_0^T \|k(n, \xi, t)\|^2_{L^2_\xi} dt$$

with $k(n, \xi, t) = \mathcal{F}\left[(a'(\cdot + W) \cdot \nabla W) g\right](n, \xi, t)$. Summing over $n \neq 0$, we get therefore

$$\begin{aligned} &\int_0^T \|v^{\#,1}(t)\|^2_{L^2(\mathbb{T}^N)} dt \\ &\quad \leq C\gamma^{-2+b} \int_0^T \|\left(a(\cdot) - a(\cdot + W)\right) g\|^2_{L^2_{x,\xi}} + \|(a'(\cdot + W) \cdot \nabla W) g\|^2_{L^2_{x,\xi}}\, dt, \end{aligned}$$

where $v^{\#,1} = v^{\#} - \int_{\mathbb{T}^N} v^{\#} dx$. We then use the hypothesis (11), *i.e.* a sublinear, and the "smallness" of the noise (actually no smallness assumption on η has been done up to now) to write

$$|a(\xi) - a(\xi + W)| \leq c\,|W| \leq c\,\eta, \quad |a'(\xi + W) \cdot \nabla W| \leq c\,\eta$$

and deduce

$$\int_0^T \|v^{\#,1}(t)\|^2_{L^2(\mathbb{T}^N)} dt \leq C\eta^2\gamma^{-2+b} \int_0^T \|v\|_{L^1(\mathbb{T}^N)} dt \tag{49}$$

since $\|g\|^2_{L^2_{x,\xi}} = \|v\|_{L^1(\mathbb{T}^N)}$. The total mass $\int_{\mathbb{T}^N} v^{\#} dx$ will be estimated later, once we have finished to estimate P^W and N^W. Regarding P^W, we have, by Hypothesis (11),

$$\begin{aligned} \langle P^W(t), \varphi \rangle \leq C \int_0^t &(1 + \|v(s)\|_{L^1(\mathbb{T}^N)} + \|W(s)\|_{L^1(\mathbb{T}^N)}) \\ &\times \|\nabla W(s)\|_{L^\infty(\mathbb{T}^N)} \|S^*_{A_\gamma,\delta}(t-s)\varphi\|_{L^\infty(\mathbb{T}^N)}\, ds. \end{aligned}$$

Therefore P^W is more regular than a measure and Lemma 10 implies, for $\eta \leq 1$,

$$\|P^W(t)\|_{L^1(\mathbb{T}^N)} \leq C\eta \int_0^t (1 + \|v(s)\|_{L^1(\mathbb{T}^N)}) e^{-\delta(t-s)} ds.$$

In particular, we have

$$\int_0^T \|P^W(t)\|_{L^1(\mathbb{T}^N)} dt \leq C\frac{\eta}{\delta} \int_0^T (1 + \|v(s)\|_{L^1(\mathbb{T}^N)}) ds. \tag{50}$$

We finally estimate the last term containing the measure n. We write, using again Lemma 10 and the hypothesis (11),

$$\begin{aligned}\langle N^W(t),\varphi\rangle &= \int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} (t-s)a'(\xi)\cdot\nabla S^*_{A_\gamma}(t-s)\varphi d\mathrm{n}(x,s,\xi)\\ &\le c\,\|\varphi\|_{L^\infty(\mathbb{T}^N)}\int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} e^{-\delta(t-s)}d|\mathrm{n}|(x,s,\xi).\end{aligned}$$

Again, we may prove that N^W is more regular than a measure and deduce:

$$\int_0^T \|N^W(t)\|_{L^1(\mathbb{T}^N)}dt \le \frac{C}{\delta}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} d|\mathrm{n}|(x,s,\xi). \tag{51}$$

To complete our estimate, it remains to evaluate the mass of the measure n. We proceed as in Lemma 11 and test equation (46) against ξ to obtain:

$$\begin{aligned}&\frac{1}{2}\|v(t)\|^2_{L^2(\mathbb{T}^N)} + |\mathrm{n}|([0,\mathrm{t}]\times\mathbb{T}^\mathrm{N}\times\mathbb{R})\\ \le&\frac{1}{2}\|u_0\|^2_{L^2(\mathbb{T}^N)} + \left|\int_{[0,t]\times\mathbb{T}^N\times\mathbb{R}} \xi(a(\xi)-a(\xi+W))\cdot\nabla g\,dx\,d\xi\,ds\right|\\ +&\left|\int_{[0,t]\times\mathbb{T}^N} v(x,s)a(v(x,s)+W(x,s))\cdot\nabla W(x,s)dxds\right|.\end{aligned}$$

We use an integration by parts in the second term. By (11) we obtain after easy manipulations:

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathbb{T}^N)}+|\mathrm{n}|([0,t]\times\mathbb{T}^N\times\mathbb{R}) \le \frac{1}{2}\|\tilde{u}_0\|^2_{L^2(\mathbb{T}^N)}+\eta C\int_0^t(\|v(s)\|^2_{L^2(\mathbb{T}^N)}+1)ds.$$

Gronwall lemma then gives $\|v(t)\|^2_{L^2(\mathbb{T}^1)} \le e^{\eta Ct}(\|\tilde{u}_0\|^2_{L^2(\mathbb{T}^1)}+C)$, and therefore

$$|\mathrm{n}|([0,T]\times\mathbb{T}^N\times\mathbb{R}) \le e^{\eta CT}(\|\tilde{u}_0\|^2_{L^2(\mathbb{T}^N)}+C).$$

Since $\|\tilde{u}_0\|_{L^2(\mathbb{T}^N)} \le C\kappa_1\varepsilon^{-N/2}$, it follows by (51) that

$$\int_0^T \|N^W(t)\|_{L^1(\mathbb{T}^N)}dt \le \frac{C}{\delta}e^{\eta CT}(\kappa_1^2\varepsilon^{-N}+1). \tag{52}$$

Finally, since v, v^0 and $v^\flat$ have zero spatial averages, we have

$$\int_{\mathbb{T}^N} v^\# + P^W + N^W dx = 0.$$

Consequently we can estimate the total mass of $v^\#$ by

$$\left|\int_{\mathbb{T}^N} v^\# dx\right| \le \|P^W\|_{L^1(\mathbb{T}^N)} + \|N^W\|_{L^1(\mathbb{T}^N)}.$$

We may now gather all the estimates obtained, *i.e.* (47), (48), (49), (50), (52), and deduce:

$$\begin{aligned}
&\int_0^T \|v^0(t)+v^\flat(t)+v^{\#,1}(t)\|_{L^1(\mathbb{T}^1)}dt \\
&\qquad \le T^{1/2}\left(\int_0^T \|v^0(t)\|^2_{L^2(\mathbb{T}^1)}+\|v^\flat(t)\|^2_{L^2(\mathbb{T}^1)}+\|v^{\#,1}(t)\|^2_{L^2(\mathbb{T}^1)}dt\right)^{1/2} \\
&\qquad \le CT^{1/2}\left(\gamma^{b-1}\kappa_1+(\gamma^b+\eta^2\gamma^{-2+b})\int_0^T \|v(t)\|_{L^1(\mathbb{T}^1)}dt\right)^{1/2} \\
&\qquad \le \frac{1}{4}\int_0^T \|v(t)\|_{L^1(\mathbb{T}^1)}dt + CT(\gamma^b+\eta^2\gamma^{-2+b})+C(T\gamma^{b-1}\kappa_1)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left\| \int_{\mathbb{T}^N} v^{\#}(x)dx + P^W(t)+N(t)\right\|_{L^1(\mathbb{T}^N)} \\
&\qquad \le C\left(\frac{\eta}{\delta}\int_0^T (1+\|v(t)\|_{L^1(\mathbb{T}^N)})dt + \frac{1}{\delta}e^{\eta CT}(\kappa_1^2\varepsilon^{-N}+1)\right).
\end{aligned}$$

Let $r>0$ be a ratio that we will fix later. We choose (in that order), γ, T, δ such that

$$C\gamma^b \le r\varepsilon, \quad C\left(\frac{\gamma^{b-1}\kappa_1}{T}\right)^{1/2} \le r\varepsilon, \quad \frac{C}{\delta}e^{CT}(\kappa_1^2\varepsilon^{-N}+1)\le r\varepsilon,$$

and then η small enough so that

$$C\frac{\eta}{\delta} \le \min(r\varepsilon, \frac{1}{4}), \quad C\eta^2\gamma^{-2+b}\le r\varepsilon.$$

With those choices, and for $r=\frac{1}{40}$, we obtain, if furthermore $\eta \le \frac{\varepsilon}{8}$,

$$\frac{1}{T}\int_0^T \|v(t)\|_{L^1(\mathbb{T}^N)}dt \le \frac{\varepsilon}{4}, \quad \frac{1}{T}\int_0^T \|\tilde{u}(t)\|_{L^1(\mathbb{T}^N)}dt \le \frac{3\varepsilon}{8}$$

and

$$\frac{1}{T}\int_0^T \|u(t)\|_{L^1(\mathbb{T}^N)}dt \le \frac{\varepsilon}{2} < \varepsilon.$$

This is the desired conclusion. ■

4.3 Conclusion

We assume that (11) holds. Let u_0^1, u_0^2 be in $L^1(\mathbb{T}^N)$. Let $\varepsilon>0$, we take $\tilde{u}_0^1$, $\tilde{u}_0^2$ in $L^3(\mathbb{T}^N)$ such that $\|u_0^i-\tilde{u}_0^i\|_{L^1(\mathbb{T}^N)} \le \frac{\varepsilon}{4}$. . We denote by u^1, u $\tilde{u}^1$, $\tilde{u}^2$ the corresponding solutions. We associate to $\tilde{u}_0^1$, $\tilde{u}_0^2$ the sequence of stopping times constructed in section 4.1. We choose T and η given by Proposition 12 and

obtain thanks to the L^1-contraction (16),

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{T}\int_{\tau_\ell}^{\tau_\ell+T}\|u^1(s)-u^2(s)\|_{L^1(\mathbb{T}^N)}ds\le\varepsilon\,\Big|\,\mathcal{F}_{\tau_\ell}\Big)\\ \ge\mathbb{P}\Big(\frac{1}{T}\int_{\tau_\ell}^{\tau_\ell+T}\|\tilde{u}^1(s)-\tilde{u}^2(s)\|_{L^1(\mathbb{T}^N)}ds\ \le\frac{\varepsilon}{2}\,\Big|\,\mathcal{F}_{\tau_\ell}\Big)\\ \ge\mathbb{P}\Big(\sup_{[\tau_\ell,\tau_\ell+T]}\|W(t)-W(\tau_\ell)\|_{W^{1,\infty}(\mathbb{T}^N)}\le\eta\,\Big|\,\mathcal{F}_{\tau_\ell}\Big).\end{aligned}$$

By the strong Markov property, the right hand side is non random and independent on ℓ. It is clearly positive. We denote it by λ. For $\ell_0, k\in\mathbb{N}$ we then have

$$\mathbb{P}\left(\frac{1}{T}\int_{\tau_\ell}^{\tau_\ell+T}\|u^1(s)-u^2(s)\|_{L^1(\mathbb{T}^N)}ds\ge\varepsilon,\ \text{ for } \ell=\ell_0,\dots,\ell_0+k\right)\le(1-\lambda)^k,$$

and therefore

$$\begin{aligned}\mathbb{P}\Big(\lim_{\ell\to\infty}\frac{1}{T}\int_{\tau_\ell}^{\tau_\ell+T}\|u^1(s)-u^2(s)\|_{L^1(\mathbb{T}^N)}ds\ge\varepsilon\Big)\\ =\mathbb{P}\Big(\exists\ell_0\in\mathbb{N}\,;\,\frac{1}{T}\int_{\tau_\ell}^{\tau_\ell+T}\|u^1(s)-u^2(s)\|_{L^1(\mathbb{T}^N)}ds\ge\varepsilon,\ \text{ for } \ell\ge\ell_0\Big)\\ =0.\end{aligned}\tag{53}$$

Note the limit exists since, by (16), $t\mapsto\|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^N}$ is a.s. non-increasing. This latter property is again used to deduce from (53) that

$$\mathbb{P}\left(\lim_{t\to\infty}\|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^N)}\ge\varepsilon\right)=0.$$

We have thus proved:

$$\lim_{t\to\infty}\|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^N)}=0,\ a.s.$$

This easily implies the second statement (uniqueness part) of Theorem 1.

A Solutions in L^1

In this section, we develop the L^1-theory for the Cauchy Problem associated to (1). In the deterministic framework, this has been done in Bénilan, Carrillo, Wittbold [BCW00] and in [CP03]. In [BCW00], a concept of *renormalized entropy solution* is introduced. In [CP03] it is noticed that the kinetic formulation for scalar (parabolic degenerate here) conservation laws can be expanded from the L^∞ to the L^1 framework with quite minor adaptations, and this is the approach we will follow here: although, for $u\in L^1$, it may be impossible to give a sense to $\mathrm{div}(A(u))$, still we will show that the problem is well-posed. Note that the basic reason for this is that (on the whole space at least) non-uniqueness for (1), in the case $g_k=0$, requires some growth at infinity: this fact is illustrated by the counter-examples constructed in Goritskii, Panov [GP02] in particular.

A.1 Generalized solutions

In [DV10], [DV], the notion of solution is extended to a notion of generalized solution. The notion of solution given by Definition 4 can be extended accordingly. The proof of Theorem 5 is a rather straightforward extension of the L^∞ result. Only the decay of the kinetic measure and Lemma 7 in [DV], needed to prove time continuity, have to be revised. We will do this below in sections A.2 and A.3

Definition 13 (Young measure)**.** *Let (X, λ) be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$. We say that a map $\nu\colon X \to \mathcal{P}_1(\mathbb{R})$ is a Young measure on X if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from X to $\mathbb{R}$ is measurable. We say that a Young measure ν vanishes at infinity if*

$$\int_X \int_{\mathbb{R}} |\xi| d\nu_z(\xi) d\lambda(z) < +\infty. \tag{54}$$

Definition 14 (Kinetic function)**.** *Let (X, λ) be a finite measure space. A measurable function $f\colon X \times \mathbb{R} \to [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ-a.e. $z \in X$, for all $\xi \in \mathbb{R}$,*

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

We say that f is an equilibrium *if there exists a measurable function $u\colon X \to \mathbb{R}$ such that $f(z, \xi) = \mathbf{1}_{z>\xi}$ a.e., or, equivalently, $\nu_z = \delta_{u(z)}$ for a.e. $z \in X$.*

Definition 15 (Generalized solution)**.** *Let $f_0\colon \Omega \times \mathbb{T}^N \times \mathbb{R} \to [0, 1]$ be a kinetic function. A measurable function $f\colon \Omega \times \mathbb{T}^N \times [0, T] \times \mathbb{R} \to [0, 1]$ is said to be a generalized solution to (1) with initial datum f_0 if $(f(t))$ is predictable and is a kinetic function such that $\nu := -\partial_\xi f$ satisfies*

$$\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi| d\nu_{x,t}(\xi) dx\right) < +\infty, \tag{55}$$

and such that there exists a kinetic measure m such that for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T) \times \mathbb{R})$,

$$\begin{aligned}
&\int_0^T \langle f(t), \partial_t \varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla\varphi(t)\rangle dt \\
&= -\sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi)\varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\
&-\frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}^2(x, \xi) d\nu_{(x,t)}(\xi) dx dt + m(\partial_\xi \varphi), \ \ a.s.
\end{aligned} \tag{56}$$

What differ mostly between Definition 3 of kinetic measure and the corresponding definition in [DV10] is the condition at infinity *2*. This condition is for example crucial to show the equivalence between Definition 4 and the notion of renormalized solution (what we will not do here since we will not need the latter). As we will see, this decay condition is related to the decay of $k \mapsto (u - k)^\pm$. We explore this link and then use it to prove that solutions in the sense of Definition 4 lead to a well-posed problem.

A.2 Decay of the kinetic measure

Proposition 16. *Let $u_0 \in L^1(\mathbb{T}^N)$. Let a measurable function $u\colon \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ be a solution to (1) according to Definition 4. Let $T > 0$. There exists a decreasing function $\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{k\to+\infty} \varepsilon(k) = 0$ depending on T and on the functions*

$$k \mapsto \|(u_0 - k)^+\|_{L^1(\mathbb{T}^N)}, \quad k \mapsto \|(u_0 - k)^-\|_{L^1(\mathbb{T}^N)}$$

only such that, for all $k \geq 1$,

$$\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|(u(t) - k)^\pm\|_{L^1(\mathbb{T}^N)}\right) + \mathbb{E}m(A_k) \leq \varepsilon(k),$$

where $A_k = \mathbb{T}^N \times [0,T] \times \{\xi \in \mathbb{R}, k \leq |\xi| \leq k+1\}$.

Proof. Step 1. For $k \geq 0$, set

$$\theta_k(u) = \mathbf{1}_{k<u<k+1}, \quad \Theta_k(u) = \int_0^u \int_0^r \theta_k(s)dsdr.$$

Let $\gamma \in C_c^1([0,T))$ be non-negative and satisfy $\gamma(0) = 1$, $\gamma' \leq 0$. After a preliminary step of approximation that uses the monotone convergence Theorem, we take $\varphi(x,t,\xi) = \gamma(t)\Theta_k'(\xi)$ in (14) to obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^N} \Theta_k(u(x,t))|\gamma'(t)|dxdt + \int_{A_k^+} \gamma(t)dm(x,t,\xi)\\ = \frac{1}{2}\int_0^T \int_{\mathbb{T}^N} \gamma(t)\mathbf{G}^2(x)\theta_k(u(x,t))dxdt + \int_{\mathbb{T}^N} \Theta_k(u_0(x))dx\\ + \sum_{j\geq 1}\int_0^T \int_{\mathbb{T}^N} g_j(x)\Theta_k'(u(x,t))\gamma(t)dxd\beta_k(t), \quad (57)\end{aligned}$$

where $A_k^+ = \mathbb{T}^N \times [0,T] \times \{\xi \in \mathbb{R}, k \leq \xi \leq k+1\}$. Taking then expectation, we have

$$\begin{aligned}\mathbb{E}\int_0^T \int_{\mathbb{T}^N} \Theta_k(u(x,t))|\gamma'(t)|dxdt + \mathbb{E}\int_{A_k^+} \gamma(t)dm(x,t,\xi)\\ = \frac{1}{2}\mathbb{E}\int_0^T \int_{\mathbb{T}^N} \gamma(t)\mathbf{G}^2(x)\theta_k(u(x,t))dxdt + \int_{\mathbb{T}^N} \Theta_k(u_0(x))dx. \quad (58)\end{aligned}$$

Note that

$$(u - (k+1))^+ \leq \Theta_k(u) \leq (u-k)^+, \quad (59)$$

for all $k \geq 0$, $u \in \mathbb{R}$. Note also

$$\theta_{k_2}(u) \leq \mathbf{1}_{k_2 \leq u} \leq \frac{(u-k_1)^+}{k_2 - k_1}, \quad 0 \leq k_1 < k_2.$$

In particular, using that $m \geq 0$, (58), using (2) and taking $k_1 = k$, $k_2 = k_1 + \theta - 1$ where $\theta > 1$ in (58) gives

$$\begin{aligned}\mathbb{E}\int_0^T \int_{\mathbb{T}^N} (u(x,t) - (k+\theta))^+|\gamma'(t)|dxdt\\ \leq \frac{D_0}{2(\theta-1)}\mathbb{E}\int_0^T \int_{\mathbb{T}^N} (u(x,t)-k)^+\gamma(t)dxdt + \int_{\mathbb{T}^N} (u_0(x)-k)^+dx. \quad (60)\end{aligned}$$

Choose θ large enough so that $\frac{D_0}{2(\theta-1)} = \alpha < 1$. Denote by $\psi_n(t)$ the function

$$\psi_n(t) = \mathbb{E}\int_{\mathbb{T}^N} (u(x,t) - n\theta)^+ dx$$

and let $I_n \subset [0,T]$ be the set of Lebesgue points of ψ_n. Then $I = \cap_{n\in\mathbb{N}} I_n$ is of full measure. For $t \in I$, $t < T$, we take $k = n\theta$, $\gamma(s) = \min(1, \frac{1}{\varepsilon}(s - (t+\varepsilon))^-)$ for $\varepsilon < T - t$ in (60) and let $\varepsilon \to 0$. This yields the following inequality:

$$\psi_{n+1}(t) \le \alpha \int_0^t \psi_n(s) ds + \psi_n(0). \tag{61}$$

For $n = 0$, using the bound

$$\psi_0(t) \le \psi_1(t) + \theta T,$$

(61) and Gronwall's Lemma, we obtain a bound

$$\psi_0(t) = \mathbb{E}\int_{\mathbb{T}^N} (u(x,t))^+ dx \le M := C(T, \|u_0\|_{L^1(\mathbb{T}^N)}), \tag{62}$$

for $t \in I$. From (61) we also show recursively

$$\psi_{n+1}(t) \le \alpha^{n+1}\frac{t^{n+1}}{(n+1)!}M + \sum_{j=0}^{n} \alpha^j \frac{t^j}{j!}\psi_{n-j}(0).$$

We deduce $\psi_{n+1}(t) \le (M+1)e^T\delta(n+1)$, where

$$\delta(n) = \alpha^n + \sum_{j=0}^{n-1} \alpha^j \psi_{n-1-j}(0).$$

Then $\lim_{n\to+\infty}\delta(n) = 0$ since $\lim_{k\to+\infty}\psi_k(0) = 0$. Set $\varepsilon^+(k) = \delta([\frac{k}{\theta}])$, then since the left hand side below is a decreasing function of k,

$$\operatorname{ess\,sup}_{t\in[0,T]} \mathbb{E}\int_{\mathbb{T}^N} (u(x,t) - k)^+ dx \le \varepsilon^+(k), \tag{63}$$

and $\varepsilon^+ \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function depending on T and on the function $k \mapsto \|(u_0 - k)^+\|_{L^1(\mathbb{T}^N)}$ only such that $\lim_{k\to+\infty}\varepsilon^+(k) = 0$.

Step 2. The estimate on $m(A_k)$ follows from (58). Therefore, to conclude, we need to show an estimate on $\mathbb{E}(\operatorname{ess\,sup}_{t\in[0,T]}\int_{\mathbb{T}^N}(u(x,t) - n\theta)^+ dx)$. This is the classical argument for semi-martingales that we will use. We will merely focus on the martingale term in (57). We first let γ approach $\mathbf{1}_{(0,t)}$ as in Step 1. Then, by the Burkholder - Davis - Gundy inequality, we have, for a constant $C > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\operatorname{ess\,sup}_{t\in[0,T]}\left|\sum_{j\ge1}\int_0^t\int_{\mathbb{T}^N} g_j(x)\Theta_k'(u(x,t))dx d\beta_k(t)\right|\right] \\
&\qquad \le C\mathbb{E}\left[\int_0^T \sum_{j\ge1}\left(\int_{\mathbb{T}^N}|g_j(x)||\Theta_k'(u(x,t))|dx\right)^2\right]^{1/2} \\
&\qquad\qquad \le CD_0^{1/2}\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^N}|\Theta_k'(u(x,t))|^2 dx\right]^{1/2},
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and (2) in the last line. Since

$$|\Theta_k'(u)|^2 \leq \mathbf{1}_{k\leq u} \leq (u-(k+1))^+$$

and $\mathbb{E}X^{1/2} \leq (\mathbb{E}X)^{1/2}$ by Jensen's inequality, we obtain by (63)

$$\mathbb{E}\left[\operatorname{ess\,sup}_{t\in[0,T]}\left|\sum_{j\geq 1}\int_0^t\int_{\mathbb{T}^N} g_j(x)\Theta_k'(u(x,t))dxd\beta_k(t)\right|\right] \leq CD_0^{1/2}T\varepsilon^+(k+1),$$

as desired. This concludes the proof of the Proposition. ■

A.3 Uniqueness

The condition at infinity *2.* in Definition 3, although weaker than the condition in [DV10] is enough to perform the proof of uniqueness, and of reduction of generalized solutions to solutions (*cf.* Theorem 15. in [DV]). What is more problematic in the L^1 framework is the proof of pathwise continuity of solutions, *cf.* Corollary 16. in [DV]. Indeed the proof of this result uses the equivalence between convergence to an equilibrium at the kinetic level and strong convergence at the level of functions. This is Lemma 7 in [DV], that we adapt here in the following way.

Lemma 17 (Convergence to an equilibrium)**.** *Let (X,λ) be a finite measure space. Let (f_n) be a sequence of kinetic functions on $X\times\mathbb{R}$: $f_n(z,\xi) = \nu_z^n(\xi,+\infty)$ where ν^n are Young measures on X satisfying*

$$\sup_n \int_X\int_{\mathbb{R}} |\xi| d\nu_z(\xi)d\lambda(z) < +\infty.$$

Let f be a kinetic function on $X\times\mathbb{R}$ such that $f_n \rightharpoonup f$ in $L^\infty(X\times\mathbb{R})$ weak-. Assume that f_n and f are equilibria:*

$$f_n(z,\xi) = \mathbf{1}_{u_n(z)>\xi}, \quad f(z,\xi) = \mathbf{1}_{u(z)>\xi}$$

and assume that the following equi-integrability condition is satisfied:

$$\sup_n \|(u_n-k)^\pm\|_{L^1(X)} = o(1) \quad [k\to+\infty]. \tag{64}$$

Then $u_n \to u$ in $L^1(X)$ strong.

Proof. We simply give the sketch of the proof of Lemma 17. Assume first

$$R = \sup_n \|u_n\|_{L^\infty(X)} < +\infty. \tag{65}$$

By testing the convergence $f_n \rightharpoonup f$ against $\varphi(z)\mathbf{1}_{\xi>-R}$, where $\varphi\in L^2(X)$, we obtain $u_n \rightharpoonup u$ in $L^2(X)$-weak. By testing the convergence $f_n \rightharpoonup f$ against $\xi\mathbf{1}_{\xi>-R}$, we obtain the convergence in norm $\|u_n\|^2_{L^2(X)} \to \|u\|^2_{L^2(X)}$ and since $L^2(X)$ is a Hilbert space, we conclude to $u_n \to u$ in $L^2(X)$ strong, and in particular in $L^1(X)$ strong. In the general case, we observe that the above arguments can be applied to $T_k(u_n)$, where T_k is the truncation operator

$$T_k(u) = \max(-k,\min(u,k)). \tag{66}$$

We then use (64) to conclude to the convergence $u_n \to u$ in $L^1(X)$. ■

Proposition 16 yields the equi-integrability estimate that we need. Therefore, as in [DV], we obtain the following two results.

Theorem 18 (Uniqueness, Reduction)**.** *Let $u_0 \in L^1(\mathbb{T}^N)$. Assume (2)-(3). Then, there is at most one solution in the sense of Definition 4 with initial datum u_0 to (1). Besides, any generalized solution f is actually a solution, i.e. if f is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0>\xi}$, then there exists a solution u to (1) with initial datum u_0 such that $f(x,t,\xi) = \mathbf{1}_{u(x,t)>\xi}$ a.s., for a.e. (x,t,ξ).*

Theorem 19 (Continuity in time)**.** *Let $u_0 \in L^1(\mathbb{T}^N)$. Assume (2)-(3) are satisfied. Then the solution u to (1) with initial datum u_0 has a representative in the space $L^1(\Omega; L^\infty(0,T; L^1(\mathbb{T}^N)))$ with almost sure continuous trajectories in $L^1(\mathbb{T}^N)$.*

A.4 Existence

Theorem 20 (Existence)**.** *Let $u_0 \in L^1(\mathbb{T}^N)$. Assume (2)-(3). Then, there exists a solution in the sense of Definition 4 with initial datum u_0 to (1).*

Proof. We approach u_0 by $u_0^n := T_n(u_0)$, where the truncation operator is defined in (66). By [DV10] this defines a sequence of solutions (u_n) and by the contraction property in L^1 it is a Cauchy sequence, hence converges to a given u in $L^1(\Omega \times \mathbb{T}^N \times (0,T))$. This u is predictable. Let now m^n be the kinetic measure associated to u^n. We can assume, up to further extraction, that $u_n \to u$ is converging a.s. Taking the limit in (56) written for $u = u_n$, we see that the quantity $\mathbb{E}\left[\mathbf{1}_A m_n(\partial_\xi \varphi)\right]$ is converging to a certain quantity $M(A,\varphi)$, for all $A \in \mathcal{F}$, $\varphi \in C_c^1(\mathbb{T}^N \times [0,T) \times \mathbb{R})$. There remains to identify $M(A,\varphi)$. Let $r > 0$ and let $K_r = \mathbb{T}^N \times [0,T] \times [-r,r]$. We consider the restriction m_r^n of m^n to K_r. By Proposition 16, we have $\mathbb{E} m_r^n(K_r) \le C_r$, where C_r is independent on n. Viewing m_r^n as a Young measure $\Omega \to \mathcal{M}^+(K_r)$, we deduce the existence of a subsequence still denoted (m_r^n) and of a Young measure μ_r such that

$$\mathbb{E}\left[\mathbf{1}_A m_r^n(\psi)\right] \to \mu_r(\mathbf{1}_A \times \psi), \tag{67}$$

for all $A \in \mathcal{F}$, for all ψ continuous and bounded on K_r. We apply Theorem 4.3.5 p.92 and Theorem 2.1.3, G), p.25 in [1]. Disintegration ([1] p.19) of the measure μ_r give a measure m_r on K_r such that

$$\mathbb{E}\left[\mathbf{1}_A m_r^n(\psi)\right] \to \mathbb{E}\left[\mathbf{1}_A m_r(\psi)\right]. \tag{68}$$

We consider a sequence $(r_n) \uparrow +\infty$ and use an argument of diagonal extraction, to obtain a sequence still denoted (m_n) and a measurable application m in the sense of Item *1.* of Definition 3 such that

$$\mathbb{E}\left[\mathbf{1}_A m^n(\psi)\right] \to \mathbb{E}\left[\mathbf{1}_A m(\psi)\right], \tag{69}$$

for all $\psi \in C_c(\mathbb{T}^N \times [0,T] \times \mathbb{R})$, for all $A \in \mathcal{F}$. By Proposition 16, it follows that m satisfies also Item 2 of Definition 3. This concludes the proof, since $M(A,\varphi) = \mathbb{E}\left[\mathbf{1}_A m(\partial_\xi \varphi)\right]$, where m is a kinetic measure. ■

We may also prove the existence of solution by convergence of the parabolic approximation as in [DV10]. We will not give the details of the proof. Our final result is the following one.

Theorem 21 (Resolution of (1) in L^1)**.** *Let $u_0 \in L^1(\mathbb{T}^N)$. There exists a unique measurable $u\colon \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ solution to (1) with initial datum u_0 in the sense of Definition 4. Besides, u has almost surely continuous trajectories in $L^1(\mathbb{T}^N)$ and u is the a.s. limit in $L^1(\mathbb{T}^N \times (0,T))$ of the parabolic approximation to (1). Moreover, given u_0^1 and $u_0^2 \in L^1(\mathbb{T}^N)$, the following holds:*

$$\|u^1(t) - u^2(t)\|_{L^1(\mathbb{T}^N)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{T}^N)}, \ a.s.$$

Remark 22 (Multiplicative noise)**.** *Here we have developed the L^1 theory for an additive noise (the functions g_k are independent on u) since this was assumed from the start, but all the statement above remain true if the noise is multiplicative with the same hypotheses as in [DV10] for example.*

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