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Stochastic flows and an interface SDE on metric graphs

Hatem Hajri * Olivier Raimond †

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Abstract

This paper consists in the study of a stochastic differential equation on a metric graph, called an interface SDE (ISDE). To each edge of the graph is associated an independent white noise, which drives (ISDE) on this edge. This produces an interface at each vertex of the graph. We first do our study on star graphs with $N \geq 2$ rays. The case $N = 2$ corresponds to the perturbed Tanaka's equation recently studied by Prokaj [18] and Le Jan-Raimond [12] among others. It is proved that (ISDE) has a unique in law solution, which is a Walsh's Brownian motion. This solution is strong if and only if $N = 2$.

Solution flows are also considered. There is a (unique in law) coalescing stochastic flow of mappings φ solving (ISDE). For $N = 2$, it is the only solution flow. For $N \geq 3$, φ is not a strong solution and by filtering φ with respect to the family of white noises, we obtain a (Wiener) stochastic flow of kernels solution of (ISDE). There are no other Wiener solutions. Our previous results [8] in hand, these results are extended to more general metric graphs.

The proofs involve the study of (X, Y) a Brownian motion in a two dimensional quadrant obliquely reflected at the boundary, with time dependent angle of reflection. We prove in particular that, when $(X_0, Y_0) = (1, 0)$ and if S is the first time X hits 0, then Y_S^2 is a beta random variable of the second kind. We also calculate $\mathbb{E}[L_{\sigma_0}]$, where L is the local time accumulated at the boundary, and σ_0 is the first time (X, Y) hits $(0, 0)$.

1 Introduction

In [18], Prokaj proved that pathwise uniqueness holds for the perturbed Tanaka's equation

$$dX_t = \text{sgn}(X_t)dW_t^1 + \lambda dW_t^2 \quad (1)$$

for all $\lambda \neq 0$, where W^1 and W^2 are two independent Brownian motions. When $\lambda = 1$, after rescaling, setting $W^+ = \frac{W^1+W^2}{\sqrt{2}}$ and $W^- = \frac{W^2-W^1}{\sqrt{2}}$, (1) rewrites

$$dX_t = 1_{\{X_t > 0\}}dW_t^+ + 1_{\{X_t \leq 0\}}dW_t^-. \quad (2)$$

Using different techniques, the same result in the case of (2) has also been obtained by Le Jan and Raimond [12] (see also [4, 15]).

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In this paper, an analogous SDE, called an Interface SDE (or (ISDE)), on metric graphs is studied. The first graphs we consider are star graphs with $N \geq 2$ rays $(E_i)_{1 \leq i \leq N}$. Let $(W_i)_{1 \leq i \leq N}$ be N independent Brownian motions. Then a solution to (ISDE) is a Walsh's Brownian motion that follows W^i during an excursion in E_i . Note that (2) is an (ISDE) on a star graph with $N = 2$ rays. Without Tsirelson's results [19], one could have the intuition that for $N \geq 3$, the situation is exactly the same as for (2), i.e. (ISDE) has a unique strong solution. As this intuition is misleading, (ISDE) seems to be an interesting SDE. In a second time, (ISDE) is extended to more general metric graphs with finite number of vertices and edges. It is defined in the same way by attaching to the edges a family of independent Brownian motions.

Stochastic flows (of mappings and kernels) solutions of (ISDE) are also studied. Now, we do not only follow the motion of one particle driven by (ISDE) but of a family of particles. The law of a stochastic flow is determined by a consistent family of n -point motions (see [13]). These n -point motions consist of n solutions of (ISDE), started from eventually n different locations. When $N = 2$, Le Jan and Raimond [12] proved that (2) generates a unique stochastic flow of mappings, which is coalescing in the sense that two solutions of (2) meet in finite time. When $N \geq 3$, we prove that there is still a stochastic coalescing flow of mappings, but it is no longer the unique stochastic flow solution of (ISDE).

To study (ISDE), we establish new results on obliquely reflected Brownian motions (X, Y) on the positive quadrant. These are of independent interest. In particular, when $(X_0, Y_0) = (x, 0)$, we give the law of Y_S , where S is the first time X hits 0. We also give a necessary and sufficient condition on the angles of reflection for the integrability of the local time cumulated at the boundaries before (X, Y) hits $(0, 0)$.

Let us now give some notations that will be used throughout this paper and recall the definition of stochastic flows.

- If M is a locally compact metric space, $C_0(M)$ will denote the set of continuous functions on M vanishing at ∞ . We will denote by $\mathcal{B}(M)$ the Borel σ -field on M and by $\mathcal{P}(M)$ the set of Borel probability measures on M .
- A kernel on M is a measurable mapping $K : M \rightarrow \mathcal{P}(M)$. For $x \in M$, the probability measure $K(x)$ will also be denoted by $K(x, dy)$. We recall that two kernels K_1 and K_2 may be composed by the formula $K_1 K_2(x, dz) = \int_{y \in M} K_1(x, dy) K_2(y, dz)$.
- The two-dimensional quadrant $[0, \infty]^2$ is denoted by \mathcal{Q} . Its boundary is $\partial \mathcal{Q} := \partial_1 \mathcal{Q} \cup \partial_2 \mathcal{Q}$, where $\partial_1 \mathcal{Q} = [0, \infty[\times \{0\}$ and $\partial_2 \mathcal{Q} = \{0\} \times [0, \infty[$. We also set $\mathcal{Q}^* = \mathcal{Q} \setminus \{(0, 0)\}$.
- For X a continuous semimartingale, we will denote by $L_t(X)$ its symmetric local time process at 0, i.e.

$$L_t(X) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s| \leq \epsilon\}} d\langle X \rangle_s.$$

Stochastic flows: Let M be a locally compact metric space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

Definition 1.1. A stochastic flow of mappings (SFM) φ on M is a random family $(\varphi_{s,t})_{s \leq t}$ of measurable mappings on M such that for all $(s, x) \in \mathbb{R} \times M$, $\varphi_{s,s}(x) = x$ and

1. For all $h \in \mathbb{R}$, $s \leq t$, $\varphi_{s+h, t+h}$ is distributed like $\varphi_{s,t}$;

2. For all $s_1 \leq t_1 \leq \dots \leq s_n \leq t_n$, the family $\{\varphi_{s_i, t_i}, 1 \leq i \leq n\}$ is independent;
3. For all $s \leq t \leq u$ and all $x \in M$, a.s. $\varphi_{s, u}(x) = \varphi_{t, u} \circ \varphi_{s, t}(x)$;
4. The mapping $(s, t, x) \mapsto \varphi_{s, t}(x)$ is continuous in probability;
5. For all $s \leq t$, $\varphi_{s, t}(x)$ converges in probability towards ∞ as $x \rightarrow \infty$.

Stochastic flows of kernels (SFK's) K on M are defined as are defined SFM's, i.e. they are random families $(K_{s, t})_{s \leq t}$ of measurable kernels on M such that, for all $(s, x) \in \mathbb{R} \times M$, $K_{s, s}(x) = \delta_x$ and satisfying 1., 2., 3., 4. and 5. in Definition 1.1 with φ replaced by K and $\varphi_{t, u} \circ \varphi_{s, t}$ replaced by $K_{s, t} K_{t, u}$. We refer to Le Jan-Raimond [13] for a more detailed definition and study. Note that if φ is a SFM, then K defined by $K_{s, t}(x) = \delta_{\varphi_{s, t}(x)}$ is a SFK. When it is the case, by misuse of language, we will then say that K is a SFM. All SFM's (resp. all SFK's) considered in the following will be measurable in the sense that the mapping $(s, t, x, \omega) \mapsto \varphi_{s, t}(x, \omega)$ (resp. $K_{s, t}(x, \omega)$) is measurable.

2 Main results

We give here the results proved in this paper, and announced in the introduction.

2.1 The interface SDE on a star graph

2.1.1 Walsh's Brownian motions on star graphs

Let G be a star graph with $N \geq 1$ rays $(E_i)_{1 \leq i \leq N}$ and origin denoted by 0, i.e. $G = \cup_{i=1}^N E_i$ is a locally compact metric space such that

- $E_i \cap E_j = \{0\}$ if $i \neq j$,
- for all i , E_i is isometric to $[0, \infty[$ via a mapping $e_i : [0, \infty[\rightarrow E_i$

and the metric d on G is such that $d(x, y) = |v - u|$ if $x = e_i(u)$, $y = e_i(v)$ for some i, u, v and $d(x, y) = d(x, 0) + d(y, 0)$ otherwise. Set for $x \in G$, $|x| = d(x, 0)$, $G^* = G \setminus \{0\}$ and for $1 \leq i \leq N$, $E_i^* = E_i \setminus \{0\}$.

Let us now define the Brownian motions on G , called Walsh's Brownian motions (abbreviated WBM's). These processes behave like a standard one-dimensional Brownian motion along a ray away from 0 and, when they hit 0, they randomly "choose" a ray E_i .

Suppose given a family of parameters $(p_1, \dots, p_N) \in]0, 1[^N$ such that $\sum_{i=1}^N p_i = 1$.

Definition 2.1. [1] Let $(T_t^+, t \geq 0)$ be the semigroup associated to the reflecting Brownian motion on \mathbb{R}_+ and let $(T_t^0, t \geq 0)$ be the semigroup associated to the Brownian motion on \mathbb{R}_+ killed at 0. Then for $f \in C_0(G)$ and $x \in E_i$, denoting $f_j(r) = f \circ e_j(r)$ for $1 \leq j \leq N$ and $\bar{f}(r) = \sum_{j=1}^N p_j f_j(r)$,

$$P_t f(x) = T_t^+ \bar{f}(|x|) + T_t^0 (f_i - \bar{f})(|x|)$$

defines a Feller semigroup on $C_0(G)$ associated to a diffusion called a Walsh's Brownian motion on G .

If a filtration $(\mathcal{G}_t)_t$ is given, then X is a $(\mathcal{G}_t)_t$ -WBM if X is adapted to $(\mathcal{G}_t)_t$ and if given \mathcal{G}_t , $(X_{t+s}, s \geq 0)$ is a WBM started at X_t .

The family of parameters (p_1, \dots, p_N) being fixed later on, all WBM's considered thereafter will be associated to this family of parameters.

The domain \mathcal{D} : We denote by $C_b^2(G^*)$ the set of all continuous functions $f : G \rightarrow \mathbb{R}$ such that for all $i \in [1, N]$, $f \circ e_i$ is C^2 on $]0, \infty[$ with bounded first and second derivatives both with finite limits at 0. For $f \in C_b^2(G^*)$ and $x = e_i(r) \in G^*$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$. When $x = 0$, set $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$ and $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$. Finally, set

$$\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}. \quad (3)$$

Set $\mathcal{D}_0 = \{f \in \mathcal{D}, f, f'' \in C_0(G)\}$. Then \mathcal{D}_0 is contained in the domain of the generator A of $(P_t)_t$. We have $Af = f''/2$ for $f \in \mathcal{D}_0$ and (A, \mathcal{D}_0) generates the Feller semigroup (P_t) in a unique way.

2.1.2 The interface SDE

Definition 2.2. A solution of the interface SDE (ISDE) on a star graph G with $N \geq 1$ rays $(E_i)_{1 \leq i \leq N}$ is a pair (X, W) of processes defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$ such that

- (i) $W = (W^1, \dots, W^N)$ is a standard (\mathcal{F}_t) -Brownian motion in \mathbb{R}^N ;
- (ii) X is an (\mathcal{F}_t) -adapted continuous process on G ;
- (iii) For all $f \in \mathcal{D}$,

$$f(X_t) = f(X_0) + \sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds. \quad (4)$$

We will say it is a strong solution if X is adapted to the filtration $(\mathcal{F}_t^W)_t$.

In the case where $N = 1$, (ISDE) is easy to study. There is no interface and (ISDE) has a unique (strong) solution, which is a reflected Brownian motion. Thus in the following, we assume $N \geq 2$. It can easily be seen (by choosing for each i a function $f_i \in \mathcal{D}$ such that $f_i(x) = |x|$ if $x \in E_i$) that on E_i , away from 0, X follows the Brownian motion W^i .

In the case $N = 2$, one can assume $E_1 =]-\infty, 0]$ and $E_2 = [0, \infty[$. Applying Itô-Tanaka's formula (or Theorem 4.1 below), we see that (ISDE) is equivalent to the skew Brownian motion version of (2):

$$dX_t = 1_{\{X_t > 0\}} dW_t^+ + 1_{\{X_t \leq 0\}} dW_t^- + (2p_1 - 1) dL_t(X). \quad (5)$$

Note that when $p_1 = 1/2$, (2) and (5) coincide. Our first result is the following

Theorem 2.3. For all $x \in G$,

- (i) There is a, unique in law, solution (X, W) of (ISDE), with $X_0 = x$. Moreover X is a WBM.
- (ii) The solution of (ISDE) is a strong solution if and only if $N = 2$.

To prove (ii), we will check that pathwise uniqueness holds for (ISDE) when $N = 2$. This implies that the solution (X, W) is a strong one. The fact that for each $N \geq 3$, (X, W) is not a strong solution is a consequence of a result of Tsirelson [19] (Theorem 4.6 below) which states that if $N \geq 3$, there does not exist any $(\mathcal{F}_t)_t$ -WBM on G with $(\mathcal{F}_t)_t$ a Brownian filtration (see also [3]).

2.1.3 Stochastic flows solutions of (ISDE)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We define below what is meant by a flow solution of (ISDE).

Definition 2.4. Let φ be a SFM on G and let $\mathcal{W} = (W^i, 1 \leq i \leq N)$ be a family of independent real white noises (see [13, Definition 1.10]). We say that (φ, \mathcal{W}) solves (ISDE) if for all $s \leq t$, $f \in \mathcal{D}$ and $x \in G$, a.s.

$$f(\varphi_{s,t}(x)) = f(x) + \sum_{i=1}^N \int_s^t (1_{E_i} f')(\varphi_{s,u}(x)) dW_u^i + \frac{1}{2} \int_s^t f''(\varphi_{s,u}(x)) du.$$

We will say it is a Wiener solution if for all $s \leq t$, $\mathcal{F}_{s,t}^\varphi \subset \mathcal{F}_{s,t}^\mathcal{W}$.

It will be shown that if (φ, \mathcal{W}) solves (ISDE), then $\mathcal{F}_{s,t}^\mathcal{W} \subset \mathcal{F}_{s,t}^\varphi$ for all $s \leq t$ and thus we may just say φ solves (ISDE). Note that if φ is a Wiener solution, then $\mathcal{F}_{s,t}^\varphi = \mathcal{F}_{s,t}^\mathcal{W}$ for all $s \leq t$.

We will prove the following

Theorem 2.5. (i) There exists a SFM φ solution of (ISDE). This solution is unique in law.

(ii) The SFM φ is coalescing in the sense that for all $s \in \mathbb{R}$ and $(x, y) \in G^2$, a.s.,

$$\inf\{t \geq s : \varphi_{s,t}(x) = \varphi_{s,t}(y)\} < \infty.$$

(iii) The SFM φ is a Wiener solution if and only if $N = 2$.

Note that (iii) in this theorem is a consequence of (ii) in Theorem 2.3. We will also be interested in SFK's solving (ISDE) in the following sense.

Definition 2.6. Let K be a SFK on G and $\mathcal{W} = (W^i, 1 \leq i \leq N)$ be a family of independent real white noises. We say that (K, \mathcal{W}) solves (ISDE) if for all $s \leq t$, $f \in \mathcal{D}$ and $x \in G$, a.s.

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^N \int_s^t K_{s,u}(1_{E_i} f')(x) dW_u^i + \frac{1}{2} \int_s^t K_{s,u} f''(x) du. \quad (6)$$

We will say it is a Wiener solution if for all $s \leq t$, $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^\mathcal{W}$.

Since $\mathcal{F}_{s,t}^\mathcal{W} \subset \mathcal{F}_{s,t}^K$ for all $s \leq t$, we may simply say that K solves (ISDE). Note that when $K = \delta_\varphi$, then K solves (ISDE) if and only if φ also solves (ISDE).

Let (φ, \mathcal{W}) be a solution to (ISDE), with φ a SFM. Then Lemma 3.2 in [13] ensures that there is a SFK $K^\mathcal{W}$ such that: for all $s \leq t$, $x \in G$, a.s. $K_{s,t}^\mathcal{W}(x) = E[\delta_{\varphi_{s,t}(x)} | \mathcal{F}_{s,t}^\mathcal{W}]$. We have the following

Proposition 2.7. $K^\mathcal{W}$ is the unique (up to modification) Wiener solution of (ISDE).

Following [6, Proposition 8] or [12, Proposition 3.1], this proposition can be proved by giving the Wiener chaos expansion of a Wiener solution (see also Proposition 4.2 in [7] for another proof). A consequence of Proposition 2.7 and Theorem 2.5 (iii) is

Corollary 2.8. $K^\mathcal{W}$ is the only SFK solution of (ISDE) if and only if $N = 2$.

Proof. Assume $N = 2$ and let (K, \mathcal{W}) be a solution of (ISDE). Then $\mathbb{E}[K|\mathcal{W}]$ is a Wiener solution of (ISDE). By proposition 2.7 and Theorem 2.5-(iii), $\mathbb{E}[K|\mathcal{W}] = \delta_\varphi$, where φ is the SFM solution of (ISDE) given by Theorem 2.5-(i). This yields $K = \delta_\varphi$. For each $N \geq 3$, we have at least two distinct solutions: φ from Theorem 2.5-(i) and $\mathbb{E}[\delta_\varphi|\mathcal{W}]$. \square

In the case $N \geq 3$, the classification of all laws of flows solutions of (ISDE) is left open.

2.2 Extension to metric graphs

2.2.1 Brownian motions on metric graphs

A metric graph is a locally compact metric space for which there are

- a countable subset $V \subset G$ and
- a countable family $\{E_i; i \in I\}$ of subsets of G that are isometric to an interval $[0, L_i]$ or $[0, \infty[$

such that $G = \cup_{i \in I} E_i$, for all i , the boundary of E_i is contained in V , and for all $i \neq j$, $E_i \cap E_j \subset V$. The sets V and $\{E_i; i \in I\}$ are respectively called the set of vertices and the set of edges of G . For x and y in G , $d(x, y)$ is defined as the length of a shortest continuous path joining x and y . An edge is called a loop if its boundary is reduced to one point.

From now on, G is a metric graph without loops for which I (and thus V) is finite. Star graphs are examples of such metric graphs. For each $i \in I$, let L_i be the length of the edge E_i . Set $J_i = [0, L_i]$ when $L_i < \infty$ and $J_i = [0, \infty[$ when $L_i = \infty$. Then there is an isometry $e_i : J_i \rightarrow E_i$. When $L_i < \infty$, set $\{g_i, d_i\} = \{e_i(0), e_i(L_i)\}$ and when $L_i = \infty$, set $\{g_i, d_i\} = \{e_i(0), \infty\}$. For each $v \in V$, set $I_v^+ = \{i \in I; g_i = v\}$, $I_v^- = \{i \in I; d_i = v\}$ and $I_v = I_v^+ \cup I_v^-$. Note that, since G does not contain any loop, for all $v \in V$, $I_v^+ \cap I_v^- = \emptyset$. Denote by N_v the cardinality of I_v .

To each $v \in V$, we associate a family of parameters $p^v := \{p_i^v; i \in I_v\}$ such that $p_i^v \in]0, 1[$ and $\sum_{i \in I_v} p_i^v = 1$. Let $G^* = G \setminus V$ and denote by $C_b^2(G^*)$ the set of all continuous functions $f : G \rightarrow \mathbb{R}$ such that for all $i \in I$, $f \circ e_i$ is C^2 on the interior of J_i and has bounded first and second derivatives both extendable by continuity to J_i . For $f \in C_b^2(G^*)$ and $x = e_i(r) \in G \setminus V$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$ and for all $v \in V$, set $f'(v) = \bar{f}'(v)$ and $f''(v) = \bar{f}''(v)$ where for $g = f'$ or f'' , $\bar{g}(v)$ is defined by

$$\bar{g}(v) = \sum_{i \in I_v^+} p_i^v (g \circ e_i)(0+) - \sum_{i \in I_v^-} p_i^v (g \circ e_i)(L_i-).$$

Finally set

$$\mathcal{D} = \{f \in C_b^2(G^*) : f'(v) = 0 \text{ for all } v \in V\}.$$

Let A be the operator defined on $\mathcal{D}_0 := \{f \in \mathcal{D}, f, f'' \in C_0(G)\}$ by $Af = f''/2$. Theorem 2.1 in [5] states that (A, \mathcal{D}_0) generates a (unique in law) Feller diffusion on G . This diffusion will be called a Brownian motion (BM) on G .

The family of parameters $p := \{p^v, v \in V\}$ being fixed thereafter, all Brownian motions on G will be associated to this family of parameters.

2.2.2 The interface SDE on G

The different notions of solutions of an interface SDE on G are defined by replacing in Definitions 2.2, 2.4 and 2.6 the set $\{1, \dots, N\}$ by I and by taking for \mathcal{D} the domain of functions defined above. Each of these SDE's will be denoted by (ISDE) and sometimes by $E(G, p)$, when we want to emphasize on G and on the family of parameters p .

Note that if (X, W) solves (ISDE), then up to the first hitting time of two different vertices, (X, W) solves an SDE on a star graph. Using this observation and Theorem 2.3, one can prove that

Theorem 2.9. *For all $x \in G$,*

- There is a, unique in law, solution (X, W) of (ISDE), with $X_0 = x$. Moreover, X is a BM on G .*
- The solution of (ISDE) is strong if and only if $N_v \leq 2$ for all $v \in V$.*

2.2.3 Stochastic flows solutions of (ISDE)

Our purpose now is to construct flows solutions of (ISDE). Our main tools will be Theorems 3.2 and 4.1 in [8]. Let us introduce some more notations.

For each $v \in V$, set $G^v = \{v\} \cup \cup_{i \in I_v} E_i$. Then there is \hat{G}^v a star graph and an isometry i_v from G^v onto a subset of \hat{G}^v such that $i_v(E_i) \subset \hat{E}_i^v$ for all $i \in I_v$, and where $\{\hat{E}_i^v; i \in I_v\}$ is the set of edges of \hat{G}^v . Set, for all $v \in V$, $p^v := (p_i^v : i \in I_v)$.

For $\mathcal{W} := (W_i)_{i \in I}$, a family of independent white noises, and $v \in V$, set $\mathcal{W}^v := (W_i^v)_{i \in I_v}$ the family of independent white noises defined by $W_i^v := W_i$ if $g_i = v$ and by $W_i^v := -W_i$ otherwise.

A family of σ -fields $(\mathcal{F}_{s,t}; s \leq t)$ will be said *independent on disjoint time intervals* (abbreviated: *i.d.i*) as soon as for all $(s_i, t_i)_{1 \leq i \leq n}$ with $s_i \leq t_i \leq s_{i+1}$, the σ -fields $(\mathcal{F}_{s_i, t_i})_{1 \leq i \leq n}$ are independent.

Then, [8, Theorem 3.2] states that to each family of flows $(\hat{K}^v)_{v \in V}$, and to each $\mathcal{W} := (W_i)_{i \in I}$ a family of independent white noises such that

- (i) For all $v \in V$, $(\hat{K}^v, \mathcal{W}^v)$ is a solution of $E(\hat{G}^v, p^v)$ on \hat{G}^v ;
- (ii) The family of flows $(\hat{K}^v)_{v \in V}$ is i.d.i. in the sense that the family $(\bigvee_{v \in V} \mathcal{F}_{s,t}^{\hat{K}^v}, s \leq t)$ is i.d.i;

one can associate a (unique in law) solution (K, \mathcal{W}) of (ISDE).

Conversely [8, Theorem 4.1] states that out of a solution (K, \mathcal{W}) of (ISDE), one can construct a family of flows $(\hat{K}^v)_{v \in V}$ for which (i) and (ii) above are satisfied and such that the law of (K, \mathcal{W}) is uniquely determined by the law of this family. In the following, we will denote by $\hat{\mathbb{P}}^v$ the law of the solution $(\hat{K}^v, \mathcal{W}^v)$. Then $\hat{\mathbb{P}}^v$ is a function of the law of (K, \mathcal{W}) .

The idea behind these two results is that before passing through two distinct vertices, a “global” flow solution of (ISDE) determines (and is determined by) a “local” flow solution of an interface SDE on a star graph (associated to the vertex that has just been visited).

We will prove (see Theorem 6.1) in Section 5 that the i.d.i. condition implies conditional independence with respect to \mathcal{W} of the flows $(\hat{K}^v)_{v \in V}$. This implies the following

Theorem 2.10. *Each family $(\hat{\mathbb{P}}^v)_{v \in V}$, with $\hat{\mathbb{P}}^v$ the law of a solution of $E(\hat{G}^v, p^v)$, is associated to one and only one solution of (ISDE).*

Proof. Suppose we are given $(\hat{\mathbb{P}}^v)_{v \in V}$. Then, on some probability space, it is possible to construct a family of independent white noises $\mathcal{W} = (W^i, i \in I)$ and a family $(\hat{K}^v)_{v \in V}$ of SFK’s respectively on \hat{G}^v such that for all $v \in V$, $(\hat{K}^v, \mathcal{W}^v)$ is a solution of $E(\hat{G}^v, p^v)$ distributed as $\hat{\mathbb{P}}^v$ and such that the flows $(\hat{K}^v)_{v \in V}$ are independent given \mathcal{W} . In other words,

$$\mathcal{L}((\hat{K}^v)_{v \in V} | \mathcal{W}) = \prod_{v \in V} \mathcal{L}(\hat{K}^v | \mathcal{W}^v)$$

where \mathcal{L} stands for the conditional law. This implies in particular that the family $(\hat{K}^v)_{v \in V}$ is i.d.i., and Theorem 3.2 in [8] states that there exists K a SFK on G such that (K, \mathcal{W}) solves (ISDE), with K obtained by well concatenating the flows \hat{K}^v .

The fact that (K, \mathcal{W}) is the only possible (in law) associated solution comes from the fact that the i.d.i. condition implies conditional independence. \square

This theorem and the results we obtained on star graphs imply

Theorem 2.11. (i) *There is a, unique in law, SFM solution of (ISDE).*

(ii) *A SFM solution of (ISDE) is a Wiener solution if and only if $N_v \leq 2$ for all $v \in V$.*

(iii) *There is a unique (up to modification) SFK Wiener solution of (ISDE).*

Proof. Let (K, \mathcal{W}) be a solution of (ISDE), and denote by $(\hat{K}^v)_{v \in V}$ the associated family of flows, respective solutions of $E(\hat{G}^v, p^v)$. Note that K is a SFM if and only if the flows \hat{K}^v are SFM's. Now (i) follows from Theorem 2.10 and Theorem 2.5 (i). Note also that K is a Wiener solution if and only if the flows \hat{K}^v are also Wiener solutions. Thus (ii) follows from Theorem 2.5 (iii), and (iii) follows from Theorem 2.10 and Theorem 2.5 (ii). \square

Let us remark that if (K, \mathcal{W}) is a solution of (ISDE), then the law of (K, \mathcal{W}) depends on the choice of the isometries $(e_i)_{i \in I}$ which define the orientations of the edges of G . However the law of K does not depend on this choice.

2.3 Brownian motions with oblique reflections

To prove Theorems 2.3 and 2.5, we shall study a Brownian motion in the quadrant \mathcal{Q} , obliquely reflected at the boundary and with time dependent angles of reflections. We now give an application of our methods to the obliquely reflected Brownian motion defined by Varadhan and Williams in [20]. Let first (X, Y) be an obliquely reflected Brownian motion on $\mathbb{R} \times \mathbb{R}^+$, with angle of reflection $\theta \in]0, \frac{\pi}{2}[$, started at $(x, 0)$ with $x > 0$. Then there is (B^1, B^2) a two dimensional Brownian motion such that

$$\begin{cases} dX_t &= dB_t^1 - \tan(\theta)dL_t(Y), \\ dY_t &= dB_t^2 + dL_t(Y). \end{cases} \quad (7)$$

The following proposition gives the law of Y_S , where S is the first time the process X hits 0.

Proposition 2.12. *The law of Y_S^2/x^2 is a Beta distribution of the second kind of parameters $(\frac{1}{2} - \frac{\theta}{\pi}, \frac{1}{2} + \frac{\theta}{\pi})$.*

Fix now $\theta_1, \theta_2 \in]0, \frac{\pi}{2}[$ and let (X, Y) be an obliquely reflected Brownian motion in \mathcal{Q} started at $(x, 0)$ with $x > 0$, with angles of reflections on $\partial_1 \mathcal{Q}$ and on $\partial_2 \mathcal{Q}$ respectively given by θ_1 and θ_2 , and killed at the first time σ_0 the process (X, Y) hits $(0, 0)$.

Define the sequence of stopping times $(S_k)_{k \geq 0}$ by $S_0 = 0$ and for $k \geq 0$,

$$\begin{aligned} S_{2k+1} &= \inf\{t \geq S_{2k} : X_t = 0\}, \\ S_{2k+2} &= \inf\{t \geq S_{2k+1} : Y_t = 0\}. \end{aligned}$$

Then $|Z_{S_{2k}}| = X_{S_{2k}}$, $|Z_{S_{2k+1}}| = Y_{S_{2k+1}}$ and Proposition 2.12 implies that

- (i) $(|Z_{S_{k+1}}|/|Z_{S_k}|)_{k \geq 0}$ is a sequence of independent random variables.
- (ii) For all $k \geq 0$, the law of $|Z_{S_{2k+1}}|^2/|Z_{S_{2k}}|^2$ (resp. $|Z_{S_{2k+2}}|^2/|Z_{S_{2k+1}}|^2$) is a Beta distribution of the second kind of parameters $(\frac{1}{2} - \frac{\theta_1}{\pi}, \frac{1}{2} + \frac{\theta_1}{\pi})$ (resp. $(\frac{1}{2} - \frac{\theta_2}{\pi}, \frac{1}{2} + \frac{\theta_2}{\pi})$).

Set $L_t := L_t(X) + L_t(Y)$, the local time accumulated at $\partial \mathcal{Q}$. It is known that σ_0 and L_{σ_0} are finite (see [20, 21]).

Proposition 2.13. *We have that*

- If $\tan(\theta_1) \tan(\theta_2) > 1$, then $\mathbb{E}[L_{\sigma_0}] = \frac{x(\tan(\theta_2)+1)}{\tan(\theta_1) \tan(\theta_2)-1} < \infty$.
- If $\tan(\theta_1) \tan(\theta_2) \leq 1$, then $\mathbb{E}[L_{\sigma_0}] = \infty$.

The assumptions on the wedge and angles considered here are more suitable to our framework but our techniques may be applied to give an expression of $\mathbb{E}[L_{\sigma_0}]$ in other situations.

2.4 Outline of content

In Section 3, obliquely reflected Brownian motions in \mathcal{Q} are studied. In Section 4, Theorem 2.3 is proved. In Section 5, using in particular the results of Section 3, Theorem 2.5 (i) and (ii) are proved. In Section 6, we prove that the i.d.i. condition implies conditional independence thus completing the proof of Theorem 2.10. Finally in Section 7, some extensions are discussed.

3 Brownian motion in the quadrant with time dependent angles of reflection

3.1 Brownian motion on the half-plane with oblique reflection

Fix $\theta \in]0, \pi/2[$. Let $Z = (X, Y)$ be the process started from (x, y) in $\mathbb{R} \times \mathbb{R}_+$ obliquely reflected at $\{y = 0\}$, with angle of reflection θ . More precisely, Z satisfies (7) with (B^1, B^2) a two dimensional Brownian motion. Set $S = \inf\{s : X_s = 0\}$. When $y = 0$ and $x > 0$, $Z_s \in \mathcal{Q}$ for all $s \leq S$, and we denote by \mathbb{P}_x^θ the law of $(Z_s; s \leq S)$. Observe that we have the following scaling property :

Proposition 3.1. *For all $x > 0$, if the law of $(Z_s; s \leq S)$ is \mathbb{P}_1^θ , then the law of $(xZ_{x^{-2}s}; s \leq x^2S)$ is \mathbb{P}_x^θ .*

For $z \in \mathbb{C}$, $\arg(z)$, $\mathcal{R}(z)$ and $\mathcal{I}(z)$ will denote respectively the argument, the real part and the imaginary part of z . A complex $z = x + iy$ will be identified with $(x, y) \in \mathbb{R}^2$. So, if f is some mapping on \mathbb{C} , we will write as well $f(z)$ or $f(x, y)$. Following [20], if f is an holomorphic function on an open set U containing \mathcal{Q}^* such that $f(z) \in \mathbb{R}$ for all $z \in]0, \infty[$, then $\phi(z) := \mathcal{R}(f(z)e^{-i\theta})$ is harmonic on U . Moreover,

$$v_1(\theta) \cdot \nabla \phi(x, 0) = 0 \quad \text{for } x > 0 \text{ and where } v_1(\theta) = (-\tan(\theta), 1). \quad (8)$$

Indeed, the condition $f(z) \in \mathbb{R}$ for all $z \in]0, \infty[$ implies that $f'(z) \in \mathbb{R}$ for all $z \in]0, \infty[$. Thus

$$\nabla \phi(x, 0) = (\mathcal{R}(f'(x)e^{-i\theta}), \mathcal{R}(if'(x)e^{-i\theta})) = f'(x)(\cos(\theta), \sin(\theta))$$

and (8) follows. These properties imply in particular that $(\phi(Z_{s \wedge S}))_s$ is a local martingale. For $b \in \mathbb{R}$ and $f(z) = z^b$, the function ϕ defined above will be denoted by ϕ_b .

Lemma 3.2. *Let $(Z_s; s \leq S)$ be a process of law \mathbb{P}_x^θ , with $x > 0$.*

- (i) *For $a > x$ and $0 < b < 1 + 2\theta/\pi$, set $c_b = 1$ if $b \leq 4\theta/\pi$ and $c_b = \cos(\theta)/\cos(b\pi/2 - \theta)$ otherwise. Then*

$$\mathbb{P} \left(\sup_{s \leq S} |Z_s| > a \right) \leq c_b \left(\frac{x}{a} \right)^b.$$

(ii) For $a < x$ and $0 < b < 1 - 2\theta/\pi$, set $c_b = \cos(\theta)/\cos(b\pi/2 + \theta)$. Then

$$\mathbb{P}\left(\inf_{s \leq S} |Z_s| < a\right) \leq c_b \left(\frac{a}{x}\right)^b.$$

Proof. Using the scaling property, we may take $x = 1$. For $a \geq 0$, set $\sigma_a = \inf\{s : |Z_s| = a\}$.

Proof of (i): Fix $a > 1$ and $0 < b < 1 + 2\theta/\pi$. For $c_b^0 = \inf\{\cos(\theta), \cos(b\pi/2 - \theta)\}$ and $s \leq S$, we have

$$c_b^0 |Z_s|^b \leq \phi_b(Z_s) \leq |Z_s|^b.$$

Moreover

$$\mathbb{P}(\sup_{s \leq S} |Z_s| > a) = \mathbb{P}(\sigma_a < S).$$

Since $(\phi_b(Z_{s \wedge \sigma_a \wedge S}))_s$ is a true martingale, for all $s \geq 0$,

$$\cos(\theta) = \phi_b(1) = \mathbb{E}[\phi_b(Z_{s \wedge \sigma_a \wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_b(Z_{s \wedge \sigma_a})1_{\{\sigma_a < S\}}].$$

As $s \rightarrow \infty$, this last term converges using dominated convergence to

$$\mathbb{E}[\phi_b(Z_{\sigma_a})1_{\{\sigma_a < S\}}] \geq c_b^0 a^b \mathbb{P}(\sigma_a < S).$$

This easily implies (i).

Proof of (ii): Fix $a < 1$ and $0 < b < 1 - 2\theta/\pi$. For $c_b^1 = \cos(b\pi/2 + \theta)$ and $t \leq S$,

$$c_b^1 |Z_s|^{-b} \leq \phi_{-b}(Z_s) \leq |Z_s|^{-b}.$$

We also have that

$$\mathbb{P}(\inf_{s \leq S} |Z_s| < a) = \mathbb{P}(\sigma_a < S).$$

By the martingale property, for all $s \geq 0$,

$$\cos(\theta) = \phi_{-b}(1) = \mathbb{E}[\phi_{-b}(Z_{s \wedge \sigma_a \wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_{-b}(Z_{t \wedge \sigma_a})1_{\{\sigma_a < S\}}]$$

and this converges as $s \rightarrow \infty$ to

$$\mathbb{E}[\phi_{-b}(Z_{\sigma_a})1_{\{\sigma_a < S\}}] \geq c_b^1 a^{-b} \mathbb{P}(\sigma_a < S).$$

This easily implies (ii). □

Corollary 3.3. Let $(Z_s; s \leq S)$ be distributed as \mathbb{P}_x^θ . Then, if $|b - 2\theta/\pi| < 1$,

$$\mathbb{E}(\sup_{s \leq S} |Z_s|^b) < \infty.$$

Proof. To simplify, assume $x = 1$. Let b and b' be such that $0 < b < b' < 1 + 2\theta/\pi$. Then

$$\mathbb{E}(\sup_{s \leq S} |Z_s|^b) = \int_0^\infty \mathbb{P}[\sup_{s \leq S} |Z_s| > a^{1/b}] da \leq 1 + c_b \int_1^\infty a^{-b'/b} da < \infty.$$

Let b and b' be such that $-1 + 2\theta/\pi < b' < b < 0$. Then

$$\mathbb{E}(\sup_{s \leq S} |Z_s|^b) = \int_0^\infty \mathbb{P}[\inf_{s \leq S} |Z_s| < a^{1/b}] da \leq 1 + c_b \int_1^\infty a^{-b'/b} da < \infty.$$

□

Corollary 3.4. Let $(Z_s = (X_s, Y_s); s \leq S)$ be distributed as \mathbb{P}_x^θ . Let f be an holomorphic function on an open set containing \mathcal{Q}^* for which $f(z) \in \mathbb{R}$ for all $z \in]0, \infty[$. Assume there exist $C > 0$, $b_- \in]0, 1 - 2\theta/\pi[$ and $b_+ \in]0, 1 + 2\theta/\pi[$ such that

$$|f(z)| \leq C(|z|^{-b_-} + |z|^{b_+}) \text{ for all } z \in \mathcal{Q}^*.$$

Then setting $\phi(x, y) = \mathcal{R}(f(x + iy)e^{-i\theta})$, we have

$$E[\phi(iY_S)] = \cos(\theta)f(x).$$

Proof. Recall that $(\phi(Z_{t \wedge S}))_t$ is a local martingale (stopped at time S). Using Corollary 3.3, it is a uniformly integrable martingale. And we conclude using the martingale property. □

Note that the functions $f(z) = z^b$, for $b \in]-1 + 2\theta/\pi, 1 + 2\theta/\pi[$, $f(z) = \log(z)^\ell$ for $\ell > 0$ satisfy the assumptions of Corollary 3.4.

Corollary 3.5. Let $(Z_s = (X_s, Y_s); s \leq S)$ be distributed as \mathbb{P}_x^θ . Then

- $E[Y_S^b] = x^b \frac{\cos(\theta)}{\cos(\theta - b\pi/2)}$ for $b \in]-1 + 2\theta/\pi, 1 + 2\theta/\pi[$,
- $\mathbb{E}[\log(Y_S)] = \log(x) - \frac{\pi}{2} \tan(\theta)$,
- $\mathbb{E}[(\log(x^{-1}Y_S))^2] = \frac{\pi^2}{4} (1 + 2 \tan^2(\theta))$.

Proof. The calculation of $\mathbb{E}[Y_S^b]$ is immediate. Using the scaling property one only needs to do the next calculations when $x = 1$. Now, for $x = 1$ and all $\ell > 0$,

$$\mathbb{E}[\mathcal{R}((\log(Y_S) + i\pi/2)^\ell e^{-i\theta})] = 0.$$

Applying this identity for $\ell = 1$, we get the value of $\mathbb{E}[\log(Y_S)]$. For $\ell = 2$, we get

$$\mathbb{E}[(\log(Y_S))^2 - (\pi/2)^2] \cos(\theta) + \pi \log(Y_S) \sin(\theta) = 0.$$

The calculation of $\mathbb{E}[(\log(Y_S))^2]$ easily follows. □

Proof of Proposition 2.12. Denote by $\beta'(\alpha, \beta)$ the Beta distribution of the second kind with positive parameters α and β . Recall it is a distribution on \mathbb{R}^+ with density given by

$$h(x) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha, \beta)}, \quad x > 0$$

where $B(\alpha, \beta)$ is the usual Beta function. Let X be distributed as $\beta'(\alpha, \beta)$. Then for $-\alpha < t < \beta$,

$$\mathbb{E}[X^t] = \frac{B(\alpha+t, \beta-t)}{B(\alpha, \beta)}.$$

Suppose that $\beta = 1 - \alpha$, then $\mathbb{E}[X^t] = \frac{B(\alpha+t, 1-\alpha-t)}{B(\alpha, 1-\alpha)}$. Using the fact that $B(z, 1-z) = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we get

$$\mathbb{E}[X^t] = \frac{\sin(\alpha\pi)}{\sin((\alpha+t)\pi)}.$$

Thus, when $(\alpha, \beta) = (\frac{1}{2} - \frac{\theta}{\pi}, \frac{1}{2} + \frac{\theta}{\pi})$, we have for $-\frac{1}{2} + \frac{\theta}{\pi} < t < \frac{1}{2} + \frac{\theta}{\pi}$,

$$\mathbb{E}[X^t] = \frac{\sin(\pi/2 - \theta)}{\sin(\pi/2 - \theta + t\pi)} = \frac{\cos(\theta)}{\cos(\theta - t\pi)}.$$

Therefore, taking $x = 1$, we have that for all $-\frac{1}{2} + \frac{\theta}{\pi} < t < \frac{1}{2} + \frac{\theta}{\pi}$, (replacing b with $2t$ in Proposition 3.4) $\mathbb{E}[X^t] = \mathbb{E}[Y_S^{2t}]$. This implies Proposition 2.12. \square

3.2 Brownian motion on the quadrant with time dependent reflections

Our purpose in this section and in Section 3.3 is to construct a Brownian motion in \mathcal{Q} started at $z = (x, 0)$ with $x > 0$, and stopped at its first hitting time of $(0, 0)$, with time dependent reflections.

Suppose we are given a sequence of random variables $(\Theta_n)_{n \geq 0}$ and a sequence of processes $(Z^n)_{n \geq 1}$, with $Z^n = (Z_t^n = (X_t^n, Y_t^n); t \leq S_n)$, such that:

- (i) There is $[\theta_-, \theta_+] \subset]0, \pi/2[$ such that with probability 1, for all $n \geq 0$, $\Theta_n \in [\theta_-, \theta_+]$.
- (ii) Set $U_0 = x$ and for $n \geq 1$, $U_n = Y_{S_n}^n$. Set also for $n \geq 0$, $\mathcal{G}_n = \sigma((\Theta_k, Z^k); 1 \leq k \leq n) \vee \sigma(\Theta_0)$. Then given \mathcal{G}_n , Z^{n+1} is distributed as $\mathbb{P}_{U_n}^{\Theta_n}$ (recall the definition of \mathbb{P}_u^θ given in Section 3.1).

For $\theta \in]0, \pi/2[$, set

$$v_1(\theta) = (-\tan(\theta), 1) \quad \text{and} \quad v_2(\theta) = (1, -\tan(\theta)).$$

Set $T_0 = 0$ and $T_n = \sum_{k=1}^n S_k$ for $n \geq 1$. For $n \geq 0$, set

$$\begin{aligned} Z_t &= (X_{t-T_{2n}}^{2n+1}, Y_{t-T_{2n}}^{2n+1}) \quad \text{for all } t \in [T_{2n}, T_{2n+1}[, \\ Z_t &= (Y_{t-T_{2n+1}}^{2n+2}, X_{t-T_{2n+1}}^{2n+2}) \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[. \end{aligned}$$

Thus, we have defined a process $(Z_t; t < T_\infty)$, where $T_\infty = \lim_{n \rightarrow \infty} T_n$. For $t \geq T_\infty$, set $Z_t = (0, 0)$. Since for all $t < T_\infty$, $Z_t \neq (0, 0)$, we have that $T_\infty = \sigma_0 := \inf\{t : Z_t = (0, 0)\}$. However, it is not obvious that Z is a continuous at T_∞ (see Corollary 3.7).

Note that there exists B , a two-dimensional Brownian motion, such that for $n \geq 0$,

$$\begin{cases} dZ_t = dB_t + v_1(\Theta_{2n})dL_t^1 & \text{for all } t \in [T_{2n}, T_{2n+1}[, \\ dZ_t = dB_t + v_2(\Theta_{2n+1})dL_t^2 & \text{for all } t \in [T_{2n+1}, T_{2n+2}[, \end{cases}$$

with L^1 and L^2 respectively the local times processes of X and Y . Define $(v_t; t < \sigma_0)$ by: for $n \geq 0$

$$\begin{aligned} v_t &= v_1(\Theta_{2n}) & \text{for all } t \in [T_{2n}, T_{2n+1}[, \\ v_t &= v_2(\Theta_{2n+1}) & \text{for all } t \in [T_{2n+1}, T_{2n+2}[. \end{aligned}$$

Then for all $t < \sigma_0$,

$$Z_t = Z_0 + B_t + \int_0^t v_s dL_s \quad (9)$$

where $Z_0 = (x, 0)$ and $L = L^1 + L^2$ is the accumulated local time at ∂Q until t .

3.3 The corner is reached

For $a \geq 0$, set $\sigma_a := \inf\{t : |Z_t| = a\}$. Following [20], we first prove that $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$ for all $K > x$. This is the major difficulty we encountered here although the proof when the angles of reflections remain constant on each boundary is quite easy [20, Lemma 2.1]. The main idea is inspired from [2]. Define for $n \geq 1$, $V_n = U_n/U_{n-1}$. Then using the scaling property (Proposition 3.1) and the strong Markov property, we see that, for all $n \geq 0$, given \mathcal{G}_n , V_{n+1} is distributed as $\tilde{Y}_{\tilde{S}}$, where $((\tilde{X}_t, \tilde{Y}_t); t \leq \tilde{S})$ has law $\mathbb{P}_1^{\Theta^n}$.

Lemma 3.6. *With probability 1, $\sum_{n \geq 0} U_n$ is finite.*

Proof. For $n \geq 1$, write

$$U_n = x \exp\left(\sum_{k=1}^n \log(V_k)\right).$$

We denote by $\mathbb{E}_{\mathcal{G}_k}$ the conditional expectation with respect to \mathcal{G}_k . By Corollary 3.5, for all $k \geq 1$, $\mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)] = -\frac{\pi}{2} \tan(\Theta_{k-1})$ and $\mathbb{E}_{\mathcal{G}_{k-1}}[(\log(V_k))^2] = \frac{\pi^2}{4}(1 + 2 \tan^2(\Theta_{k-1}))$. Note now that

$$\sum_{k=1}^n \log(V_k) = M_n + \sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)]$$

where $M_n := \sum_{k=1}^n (\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)])$ is a martingale. Denote by $\langle M \rangle_n$ its quadratic variation given by

$$\sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{k-1}}[(\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)])^2] = \sum_{k=1}^n \frac{\pi^2}{4}(1 + \tan^2(\Theta_{k-1})).$$

Thus $\langle M \rangle_\infty = \infty$ and so $\lim_{n \rightarrow \infty} M_n / \langle M \rangle_n = 0$. Since $\Theta_k \in [\theta_-, \theta_+]$, this easily implies the lemma. \square

A first consequence of Lemma 3.6 is

Corollary 3.7. *With probability 1, $\lim_{t \uparrow \sigma_0} Z_t = (0, 0)$.*

Proof. For $\epsilon > 0$ and $n \geq 0$, set

$$A_n^\epsilon = \left\{ \sup_{s \in [T_n, T_{n+1}]} |Z_s| > \epsilon \right\}.$$

Lemma 3.2 (i), with $b = 1$, implies that for all $n \geq 0$,

$$\mathbb{P}(A_n^\epsilon | \mathcal{G}_n) \leq \epsilon^{-1} U_n \cotan(\theta_-) + 1_{\{U_n \geq \epsilon\}}.$$

Thus, by Lemma 3.6, $\sum_n \mathbb{P}(A_n^\epsilon | \mathcal{G}_n) < \infty$ a.s. for all $\epsilon > 0$ and the corollary follows by applying the conditional Borel-Cantelli lemma. \square

Lemma 3.6 will also be used to prove

Lemma 3.8. *For all $K > x$, $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$.*

Proof. For all $n \geq 0$ and $t \in [0, S_{n+1}]$, set

$$W_t^{n+1} = \cos(\Theta_n)(X_t^{n+1} - U_n) + \sin(\Theta_n)Y_t^{n+1}$$

Recall $\sigma_0 = \lim_{n \rightarrow \infty} T_n$. Define the continuous process $(W_t; t \leq \sigma_0)$ such that $W_0 = 0$ and for $n \geq 0$ and $t \in [T_n, T_{n+1}]$, $W_t = W_{t-T_n}^{n+1} + W_{T_n}$. Then, it is straightforward to check that $(W_t; t \leq \sigma_0)$ is a Brownian motion stopped at σ_0 . Since for all $n \geq 0$, $U_n \geq 0$ and $\Theta_n \in]0, \pi/2[$, we get that on the event $\{\sigma_K \geq T_{n+1}\}$,

$$\sup_{t \in [T_n, T_{n+1}]} W_t \leq 2K + W_{T_n}.$$

Thus, on $\{\sigma_K = \infty\}$, $\sup_{t \leq \sigma_0} W_t \leq 2K + \sup_{n \geq 0} W_{T_n}$. Now for all $n \geq 0$, $W_{S_{n+1}}^{n+1} = \sin(\Theta_n)U_{n+1} - \cos(\Theta_n)U_n \leq U_{n+1}$. Note that for all $n \geq 0$,

$$W_{T_{n+1}} - W_{T_n} = W_{S_{n+1}}^{n+1}.$$

This implies that on the event $\{\sigma_K = \infty\}$, $\sup_{t \leq \sigma_0} W_t \leq 2K + \sum_{n \geq 0} U_n$, which is a.s. finite using Lemma 3.6. This shows that a.s. $\{\sigma_K = \infty\} \subset \{\sigma_0 < \infty\}$ and finishes the proof. \square

And following [20], we prove

Theorem 3.9. *With probability 1, we have $\sigma_0 < \infty$.*

Proof. Set $b = \frac{4\theta_-}{\pi}$. Let $\phi(x, y) = \mathcal{R}((x + iy)^b e^{-i\theta_-})$, then ϕ is harmonic on some open set U containing \mathcal{Q}^* and

$$\begin{aligned} \nabla \phi(x, 0) &= bx^{b-1}(\cos(\theta_-), \sin(\theta_-)), \\ \nabla \phi(0, y) &= by^{b-1}(\sin(\theta_-), \cos(\theta_-)). \end{aligned}$$

Thus for all $t < \sigma_0$ such that $Z_t \in \partial \mathcal{Q}$, we have $v_t \cdot \nabla \phi(Z_t) \leq 0$. It follows from (9) and Itô's formula that for all $0 < \epsilon < x < K$ and $t \geq 0$,

$$\mathbb{E}[\phi(Z_{t \wedge \sigma_\epsilon \wedge \sigma_K})] \leq \phi(x, 0).$$

Letting $t \rightarrow \infty$ and using dominated convergence, we deduce

$$\mathbb{E}[\phi(Z_{\sigma_\epsilon \wedge \sigma_K})] \leq \phi(x, 0).$$

Obviously $\phi(z) \geq \cos(\theta_-)|z|^b$ for all $z \in \mathcal{Q}$. Setting $p_{\epsilon, K} = \mathbb{P}(\sigma_\epsilon < \sigma_K)$, we get

$$\cos(\theta_-)(\epsilon^b p_{\epsilon, K} + K^b(1 - p_{\epsilon, K})) \leq x^b.$$

From this, we deduce

$$p_{\epsilon, K} \geq \frac{(K^b - x^b / \cos(\theta_-))}{K^b - \epsilon^b}.$$

As in [20], since $\sigma_0 \wedge \sigma_K < \infty$, $\lim_{\epsilon \rightarrow 0} p_{\epsilon, K} = \mathbb{P}(\sigma_0 < \sigma_K)$, this yields

$$\mathbb{P}(\sigma_0 < \sigma_K) \geq 1 - \frac{x^b}{K^b \cos(\theta_-)}. \quad (10)$$

Letting $K \rightarrow \infty$, it comes that $\mathbb{P}(\sigma_0 < \infty) = 1$. \square

Remark 3.10. Using the inclusion $\{\sup_{t < \sigma_0} |Z_t| > \epsilon\} \subset \{\sigma_\epsilon < \sigma_0\}$ and (10), we deduce that for all $\epsilon > 0$,

$$\lim_{x \rightarrow 0^+} \mathbb{P}(\sup_{t < \sigma_0} |Z_t| > \epsilon) = 0. \quad (11)$$

This fact will be used in Section 3.

3.4 The local time process

Following Williams [21], we prove in this section that

Theorem 3.11. *With probability 1, $L_{\sigma_0} := \lim_{t \uparrow \sigma_0} L_t$ is finite.*

Proof. In what follows, we refer to the proof of Theorem 1 in [21] for more details. Let $\tilde{\theta} \in]0, \theta_- \wedge \pi/4[$ and set $\tilde{b} = \frac{4\tilde{\theta}}{\pi}$. Let $\tilde{\phi}$ be defined as the function ϕ in the proof of Theorem 3.9, with the parameters (b, θ_-) replaced by $(\tilde{b}, \tilde{\theta})$. Then there exists $c > 0$ such that for all t for which $Z_t \in \partial\mathcal{Q}$, we have $v_t \cdot \nabla \tilde{\phi}(Z_t) \leq -c|Z_t|^{\tilde{b}-1}$. For each $\gamma > 0$, define $f_\gamma = e^{-\gamma \tilde{\phi}}$. Then f_γ is twice continuously differentiable in \mathcal{Q}^* and

$$\Delta f_\gamma(z) = \gamma^2 f_\gamma(z) (\tilde{b}|z|^{\tilde{b}-1})^2 \text{ for } z \in \mathcal{Q}^*.$$

Moreover for all t such that $Z_t \in \partial\mathcal{Q}$,

$$v_t \cdot \nabla f_\gamma(Z_t) = -\gamma f_\gamma(Z_t) (v_t \cdot \nabla \tilde{\phi}(Z_t)).$$

For $t < \sigma_0$, set

$$A_t = -\gamma \int_0^t (v_s \cdot \nabla \tilde{\phi}(Z_s)) dL_s + \frac{\gamma^2}{2} \int_0^t (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds.$$

and $A_{\sigma_0} = \lim_{t \uparrow \sigma_0} A_t$. Then

$$A_{\sigma_0} \geq c\gamma \int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s + \frac{\gamma^2}{2} \int_0^{\sigma_0} (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds \geq c\gamma \int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s.$$

Itô's formula implies that for $t < \sigma_0$,

$$f_\gamma(Z_t)e^{-At} = f_\gamma(Z_0) + \int_0^t e^{-As} (\nabla f_\gamma(Z_s) \cdot dB_s).$$

Taking the expectation, we get

$$\mathbb{E} \left[\exp \left(-c\gamma \int_0^{\sigma_0} |Z_s|^{\bar{b}-1} dL_s \right) \right] \geq f_\gamma(Z_0).$$

This easily implies that for all $r > 0$,

$$\mathbb{E} \left[\exp \left(-\gamma cr^{\bar{b}-1} \int_0^{\sigma_0} 1_{\{|Z_s| \leq r\}} dL_s \right) \right] \geq f_\gamma(Z_0).$$

Letting $\gamma \downarrow 0$, we get that a.s.

$$\int_0^{\sigma_0} 1_{\{|Z_s| \leq r\}} dL_s < \infty. \quad (12)$$

Let $S_r = \sup\{t \geq 0 : |Z_t| > r\}$, then by the continuity of Z , $S_r < \sigma_0$ and thus $L_{S_r} < \infty$. By combining this with (12), we get $L_{\sigma_0} < \infty$. \square

3.5 On the integrability of L_{σ_0}

In this section, Proposition 2.13 is proved. We use the notations of Section 3.2. Note that $L_{\sigma_0} = \sum_{n=1}^{\infty} L_{S_n}^n$, where L^n is the local time at 0 of Y^n and where $Z^n = (X^n, Y^n)$. Recall that for $n \geq 0$, given \mathcal{G}_n , the law of Z^{n+1} is $\mathbb{P}_{U_n}^{\Theta_n}$, where $U_0 = x$ and $U_n = Y_{S_n}^n$ for $n \geq 1$.

Let $Z^0 = (X_t^0, Y_t^0)_{t \leq S^0}$ be a process of law \mathbb{P}_x^θ . Then, if $L_t^0 = L_t(Y^0)$, for all $t \geq 0$, $Y_{t \wedge S^0}^0 = B_{t \wedge S^0}^2 + L_{t \wedge S^0}^0$ where $(B_{t \wedge S^0}^2)_t$ is a Brownian motion stopped at time S^0 . Thus $\mathbb{E}[Y_{t \wedge S^0}^0] = \mathbb{E}[L_{t \wedge S^0}^0]$. Taking the limit as $t \rightarrow \infty$ and using Corollary 3.3 leads to $\mathbb{E}[L_{S^0}^0] = \mathbb{E}[Y_{S^0}^0]$. But $\mathbb{E}[Y_{S^0}^0] = x \cotan(\theta)$ by Corollary 3.5 and this implies that

$$\mathbb{E}[L_{S_{n+1}}^{n+1} | \mathcal{G}_n] = U_n \cotan(\Theta_n).$$

Consequently

$$\mathbb{E}[L_{\sigma_0}] = \sum_{n \geq 0} \mathbb{E}[U_n \cotan(\Theta_n)].$$

If for all n , U_n and Θ_n are independent, then

$$\mathbb{E}[U_n \cotan(\Theta_n)] = \mathbb{E}[\cotan(\Theta_n)] \mathbb{E}[U_n] = \dots = x \prod_{k=0}^n \mathbb{E}[\cotan(\Theta_k)].$$

If for all n , $\Theta_{2n} = \theta_1 \in]0, \pi/2[$ and $\Theta_{2n+1} = \theta_2 \in]0, \pi/2[$, then setting $c_1 = \cotan(\theta_1)$ and $c_2 = \cotan(\theta_2)$,

$$\begin{aligned} \mathbb{E}[L_{\sigma_0}] &= x(c_1 + c_1 c_2 + c_1^2 c_2 + c_1^2 c_2^2 + \dots) \\ &= x c_1 (1 + c_2 + c_1 c_2 + c_1 c_2^2 + \dots) \\ &= x c_1 ((1 + c_2) + (1 + c_2) c_1 c_2 + \dots) \end{aligned}$$

which is finite if and only if $c_1 c_2 < 1$. In this case, we have $\mathbb{E}[L_{\sigma_0}] = \frac{x c_1 (1 + c_2)}{1 - c_1 c_2}$ and Proposition 2.13 is proved. Note that, if $\theta_1 = \theta_2 = \theta$, then $\mathbb{E}[L_{\sigma_0}] < \infty$ if and only if $\theta \in]\pi/4, \pi/2[$ and in this case $\mathbb{E}[L_{\sigma_0}] = \frac{x}{\tan(\theta) - 1}$.

4 Proof of Theorem 2.3

Theorem 2.3 (i) is proved in Section 4.1. For the construction of a solution, we use the Freidlin-Sheu formula for WBM (see Theorem 4.1 below). The uniqueness in law of the solutions of (ISDE) follows from the fact that WBM is the unique solution of a martingale problem.

Theorem 2.3 (ii) is proved in Section 4.2. To prove pathwise uniqueness for (ISDE) when $N = 2$, we proceed as in [4] using the local times techniques introduced in [11, 17]. The fact that the solution of (ISDE) is not strong when $N \geq 3$ is a consequence of a theorem by Tsirelson (see Theorem 4.6 below).

We prove Theorem 2.3 only for $x = 0$, the case $x \neq 0$ following easily.

4.1 Proof of Theorem 2.3 (i)

Let us recall the Freidlin-Sheu formula (see [5] and also [6, Theorem 3]).

Theorem 4.1. [5] *Let $(X_t)_{t \geq 0}$ be a Walsh's Brownian motion on G and $B_t^X = |X_t| - |X_0| - L_t(|X|)$. Then B^X is a Brownian motion and for all $f \in C_b^2(G^*)$, we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dB_s^X + \frac{1}{2} \int_0^t f''(X_s) ds + f'(0) L_t(|X|).$$

We call B^X the Brownian motion associated to X .

Remark that in this formula the local martingale part of $f(X_t)$ is always a stochastic integral with respect to B^X . This is an expected fact since B^X has the martingale representation property for $(\mathcal{F}_t^X)_t$ ([1, Theorem 4.1]). This martingale representation property will be used to prove the uniqueness in law of the solutions to (ISDE).

4.1.1 Construction of a solution of (ISDE)

Let X be a WBM with $X_0 = 0$ and let B^X be the Brownian motion associated to X . Take an N -dimensional Brownian motion $V = (V^1, \dots, V^N)$ independent of X . Let (\mathcal{F}_t) denote the filtration generated by X and V . For $i \in [1, N]$, define

$$W_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \int_0^t 1_{\{X_s \notin E_i\}} dV_s^i.$$

Then $W := (W^1, \dots, W^N)$ is an N -dimensional (\mathcal{F}_t) -Brownian motion by Lévy's theorem and

$$B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW_s^i.$$

Then, using Theorem 4.1, (X, W) solves (ISDE). Denote by μ the law of (X, W) .

4.1.2 Uniqueness in law

To prove the uniqueness in law, we apply two lemmas. The first lemma states that the WBM is the unique solution of a martingale problem. The second lemma gives conditions that ensure that a Walsh's Brownian motion is independent of a given family of Brownian motions.

Lemma 4.2. Let (\mathcal{F}_t) be a filtration and X be a G -valued (\mathcal{F}_t) -adapted continuous process such that for all $f \in \mathcal{D}$,

$$M_t^f := f(X_t) - f(x) - \frac{1}{2} \int_0^t f''(X_s) ds. \quad (13)$$

is a martingale with respect to (\mathcal{F}_t) . Then X is an (\mathcal{F}_t) -WBM.

Proof. We exactly follow the proof of [1, Theorem 3.2] and only check that with our conventions $f'(0) = f''(0) = 0$ for $f \in \mathcal{D}$, we avoid all trivial solutions to the previous martingale problem (with the hypothesis of Theorem 3.2 of [1], the trivial process $X_t = 0$ is a possible solution of the martingale problem (3.3) in [1]). For $i \in [1, N]$, set $q_i = 1 - p_i$ and let f_i and g_i be defined by

$$\begin{aligned} f_i(x) &= q_i |x| 1_{\{x \in E_i\}} - p_i |x| 1_{\{x \notin E_i\}}, \\ g_i(x) &= (f_i(x))^2 = q_i^2 |x|^2 1_{\{x \in E_i\}} + p_i^2 |x|^2 1_{\{x \notin E_i\}}. \end{aligned}$$

Then f_i and g_i are C^2 on G^* . We have $f_i'(x) = q_i$ for $x \in E_i^*$, $f_i'(x) = -p_i$ for $x \notin E_i$ and $f_i'(0) = 0$. Moreover, $f_i''(x) = 0$ for $x \in G$. We also have $g_i'(x) = 2q_i^2 |x|$ for $x \in E_i^*$, $g_i'(x) = 2p_i^2 |x|$ for $x \notin E_i$ and $g_i'(0) = 0$. Moreover, $g_i''(x) = 2q_i^2$ for $x \in E_i^*$, $g_i''(x) = 2p_i^2$ for $x \notin E_i$ and $g_i''(0) = 2p_i q_i$. Set $Y_t^i := f_i(Z_t)$. Although f_i is not bounded, by a localization argument, we have that Y_t^i is a local martingale. We also have that $(Y_t^i)^2 - \frac{1}{2} \int_0^t g_i''(Z_s) ds$ is a local martingale. Thus

$$\langle Y^i \rangle_t = \int_0^t (q_i^2 1_{\{Z_s \in E_i^*\}} + p_i^2 1_{\{Z_s \notin E_i\}} + p_i q_i 1_{\{Z_s = 0\}}) ds.$$

Set

$$U_t^i = \int_0^t (q_i^{-1} 1_{\{Y_s^i > 0\}} + p_i^{-1} 1_{\{Y_s^i < 0\}} + (p_i q_i)^{-1/2} 1_{\{Y_s^i = 0\}}) dY_s^i.$$

Then U_t^i is a local martingale with $\langle U^i \rangle_t = t$; that is U_t^i is a Brownian motion. Let $\phi(y) = q_i 1_{\{y > 0\}} + p_i 1_{\{y < 0\}} + \sqrt{p_i q_i} 1_{\{y = 0\}}$. Then Y^i is a solution of the stochastic differential equation

$$Y_t^i = Y_0^i + \int_0^t \phi(Y_s^i) dU_s^i.$$

As in [1], the solution of this SDE is pathwise unique and following the end of the proof of [1, Theorem 3.2], we arrive at

$$\mathbb{E}[f(Z_t) | \mathcal{F}_s] = P_{t-s} f(Z_s)$$

for all $s \leq t$ and $f \in C_0(G)$, and where P_t is the semigroup of the Walsh's Brownian motion. \square

Lemma 4.3. Let (\mathcal{G}_t) be a filtration, X be a (\mathcal{G}_t) -WBM and $B = (B^1, \dots, B^d)$ be a (\mathcal{G}_t) -Brownian motion in \mathbb{R}^d , with $d \geq 1$. Denote by B^X the Brownian motion associated to X . Then B^X and B are independent if and only if X and B are independent.

Proof. Clearly, if X and B are independent, then B^X and B are independent. Let us prove the converse. Let U be a bounded $\sigma(B)$ -measurable random variable. Then

$$U = \mathbb{E}[U] + \sum_{i=1}^d \int_0^\infty H_s^i dB_s^i$$

with H^i predictable for the filtration \mathcal{F}^B and $E[\int_0^\infty (H_s^i)^2 ds] < \infty$. Let U' be a bounded $\sigma(X)$ -measurable random variable. Since B^X has the martingale representation property for \mathcal{F}^X [1, Theorem 4.1], we deduce that

$$U' = \mathbb{E}[U'] + \int_0^\infty H_s dB_s^X$$

with H predictable for \mathcal{F}^X and $\mathbb{E}[\int_0^\infty (H_s)^2 ds] < \infty$. Then H and $(H^i)_{1 \leq i \leq d}$ are also predictable for (\mathcal{G}_t) . It is also easy to check that B^X is a (\mathcal{G}_t) -Brownian motion. Now

$$\begin{aligned} \mathbb{E}[UU'] &= \mathbb{E}[U]\mathbb{E}[U'] + \mathbb{E}\left[\sum_{i=1}^d \int_0^\infty H_s^i dB_s^i \int_0^\infty H_s dB_s^X\right] \\ &= \mathbb{E}[U]\mathbb{E}[U'] + \sum_{i=1}^d \mathbb{E}\left[\int_0^\infty H_s^i H_s d\langle B^i, B^X \rangle_s\right] \\ &= \mathbb{E}[U]\mathbb{E}[U']. \end{aligned}$$

□

Let (X, W) be a solution of (ISDE) defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ and such that $X_0 = 0$. Without loss of generality, we can assume that $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$. For all $f \in \mathcal{D}$, $\sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i$ is a martingale and therefore X is a solution to the martingale problem of Lemma 4.2. Thus X is a WBM. Let B be a Brownian motion independent of (X, W) , denote by B^X the Brownian motion associated to X and set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^B$. Note that B^X is a (\mathcal{G}_t) -Brownian motion. For $i \in [1, N]$, define

$$V_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s + \int_0^t 1_{\{X_s \notin E_i\}} dW_s^i.$$

Then $V := (V^1, \dots, V^N)$ is an N -dimensional (\mathcal{G}_t) -Brownian motion independent of B^X . By the previous lemma V is also independent of X . It is easy to check that for all $i \in [1, N]$,

$$W_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \int_0^t 1_{\{X_s \notin E_i\}} dV_s^i.$$

This proves that the law of (X, W) is μ .

4.2 Proof of Theorem 2.3 (ii)

4.2.1 The case $N = 2$

To prove that the solution is a strong one, it suffices to prove that pathwise uniqueness holds for (ISDE). Fix $p \in]0, 1[$, and set $\beta = \frac{1-p}{p}$.

Lemma 4.4. *Let X be a continuous process, B^+ and B^- be two independent Brownian motions. Set $Y_t = \beta X_t 1_{\{X_t \geq 0\}} + X_t 1_{\{X_t < 0\}}$. Then (X, B^+, B^-) is a solution to (ISDE) or equivalently of*

$$dX_t = 1_{\{X_t > 0\}} dB_t^+ + 1_{\{X_t \leq 0\}} dB_t^- + (2p - 1)dL_t(X) \quad (14)$$

if and only if (Y, B^+, B^-) is a solution of the following SDE

$$dY_t = \beta 1_{\{Y_t > 0\}} dB_t^+ + 1_{\{Y_t \leq 0\}} dB_t^-. \quad (15)$$

Proof. Suppose (X, B^+, B^-) solves (14). Set $B_t = \int_0^t 1_{\{X_s > 0\}} dB_s^+ + \int_0^t 1_{\{X_s \leq 0\}} dB_s^-$. Then B_t is a Brownian motion and (X, B) is a solution of the SDE $dX_t = dB_t + (2p-1)dL_t(X)$. It is well known (see for example Section 5.2 in the survey [14]) that (Y, B) solves

$$dY_t = \beta 1_{\{Y_t > 0\}} dB_t + 1_{\{Y_t \leq 0\}} dB_t$$

and thus that (Y, B^+, B^-) solves (15). The converse can be proved in the same way. \square

Proposition 4.5. *Pathwise uniqueness holds for (ISDE).*

Proof. Lemma 4.4 implies that the proposition holds if pathwise uniqueness holds for (15). Let (Y, B^+, B^-) and (Y', B^+, B^-) be two solutions of (15) with $Y_0 = Y'_0 = 0$. Set $\text{sgn}(y) = 1_{\{y > 0\}} - 1_{\{y < 0\}}$. We shall use the same techniques as in [4] (see also [11, 17]) and first prove that a.s.

$$\int_{]0, +\infty[} L_t^a(Y - Y') \frac{da}{a} < \infty. \quad (16)$$

By the occupation times formula

$$\int_{]0, +\infty[} L_t^a(Y - Y') \frac{da}{a} = \int_0^t 1_{\{Y_s - Y'_s > 0\}} \frac{d\langle Y - Y' \rangle_s}{Y_s - Y'_s}.$$

It is easily verified that

$$d\langle Y - Y' \rangle_s \leq C |\text{sgn}(Y_s) - \text{sgn}(Y'_s)| ds$$

where $C = (1 + \beta^2)/2$. Let $(f_n)_n \subset C^1(\mathbb{R})$ such that $f_n \rightarrow \text{sgn}$ pointwise and $(f_n)_n$ is uniformly bounded in total variation. By Fatou's lemma, we get

$$\begin{aligned} \int_{]0, +\infty[} L_t^a(Y - Y') \frac{da}{a} &\leq C \liminf_n \int_0^t 1_{\{Y_s - Y'_s > 0\}} \frac{|f_n(Y_s) - f_n(Y'_s)|}{Y_s - Y'_s} ds \\ &\leq C \liminf_n \int_0^t 1_{\{Y_s - Y'_s > 0\}} \left| \int_0^1 f'_n(Z_s^u) du \right| ds \end{aligned}$$

where

$$Z_s^u = (1 - u)Y_s + uY'_s.$$

It is easy to check the existence of a constant $A > 0$ such that for all $s \geq 0$ and $u \in [0, 1]$, $\frac{d}{du} \langle Z^u \rangle_s \geq A^{-1}$. Hence, setting $C' = A \times C$, we have

$$\begin{aligned} \int_{]0, +\infty[} L_t^a(Y - Y') \frac{da}{a} &\leq C' \liminf_n \int_0^1 \int_0^t |f'_n(Z_s^u)| d\langle Z^u \rangle_s du \\ &\leq C' \liminf_n \int_0^1 \int_{\mathbb{R}} |f'_n(a)| L_t^a(Z^u) da du. \end{aligned}$$

Now taking the expectation and using Fatou's lemma, we get

$$\mathbb{E} \left[\int_{]0, +\infty[} L_t^a(Y - Y') \frac{da}{a} \right] \leq C' \liminf_n \int_{\mathbb{R}} |f'_n(a)| da \sup_{a \in \mathbb{R}, u \in [0, 1]} \mathbb{E} [L_t^a(Z^u)].$$

It remains to prove that $\sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}[L_t^a(Z^u)] < \infty$. By Tanaka's formula, we have

$$\begin{aligned} \mathbb{E}[L_t^a(Z^u)] &= \mathbb{E}[|Z_t^u - a|] - \mathbb{E}[|Z_0^u - a|] - \mathbb{E}\left[\int_0^t \text{sgn}(Z_s^u - a) dZ_s^u\right] \\ &\leq E[|Z_t^u - Z_0^u|]. \end{aligned}$$

It is easy to check that the right-hand side is uniformly bounded with respect to (a, u) which permits to deduce (16). Consequently, since $\lim_{a \downarrow 0} L^a(Y - Y') = L^0(Y - Y')$, (16) implies that $L_t^0(Y - Y') = 0$ and thus by Tanaka's formula, $|Y - Y'|$ is a local martingale which is also a nonnegative supermartingale, with $|Y_0 - Y'_0| = 0$ and finally Y and Y' are indistinguishable. \square

4.2.2 The case $N \geq 3$

Let (X, W) be a solution to (ISDE). Then X is an (\mathcal{F}_t) -WBM, where $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$. If (X, W) is a strong solution, we thus have that X is an (\mathcal{F}_t^W) -WBM, which is impossible when $N \geq 3$ because of the following Tsirelson's theorem:

Theorem 4.6. [19] *There does not exist any $(\mathcal{G}_t)_t$ -Walsh's Brownian motion on a star graph with three or more rays with $(\mathcal{G}_t)_t$ a Brownian filtration.*

5 Proof of Theorem 2.5

In this section, we prove the assertions (i) and (ii) of Theorem 2.5. We first construct a coalescing SFM solution of (ISDE). To construct this SFM, we will use the following

Theorem 5.1. [13] *Let $(P^{(n)}, n \geq 1)$ be a consistent family of Feller semigroups acting respectively on $C_0(M^n)$ where M is a locally compact metric space such that*

$$P_t^{(2)} f^{\otimes 2}(x, x) = P_t^{(1)} f^2(x) \text{ for all } f \in C_0(M), x \in M, t \geq 0. \quad (17)$$

Then there is a (unique in law) SFM $\varphi = (\varphi_{s,t})_{s \leq t}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$P_t^{(n)} f(x) = \mathbb{E}[f(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n))]$$

for all $n \geq 1, t \geq 0, f \in C_0(M^n)$ and $x \in M^n$.

To apply this theorem, we construct a consistent family of n -point motions (i.e. the Markov process associated to $P^{(n)}$) up to their first coalescing times in Section 5.1. After associating to the two-point motion an obliquely reflected Brownian motion in \mathcal{Q} in Section 5.2, we prove the coalescing property in Section 5.3 and the Feller property in Section 5.4. It is then possible to apply Theorem 5.1 and as a result we get a flow φ . In Section 5.4, we show that φ solves (ISDE). Finally, we prove in Section 5.5 that φ is the unique SFM solving (ISDE).

Note finally that in the case of Le Jan and Raimond [12], all the angles of reflection of the obliquely reflected Brownian motion associated to the two-point motion are equal to $\pi/4$. This simplifies greatly the study of Section 3.

5.1 Construction of the n -point motion up to the first coalescing time

Fix $x = (x_1, \dots, x_n) \in G^n \setminus \Delta_n$, where $\Delta_n := \{x \in G^n : \exists i \neq j, x_i = x_j\}$. Let (Y, W) be a solution of (ISDE), with $Y_0 = 0$. For $t \geq 0$ and $j \leq n$, set

$$X_t^{j,0} = \begin{cases} Y_t & \text{if } x_j = 0, \\ e_i(|x_j| + W_t^i) & \text{if } x_j \neq 0 \text{ and } i \text{ is such that } x_j \in E_i^*. \end{cases}$$

Set $\tau_0 = 0, r = \inf\{|x_j| : 1 \leq j \leq n\}$ and

$$\tau_1 = \begin{cases} \inf\{t \geq 0 : \exists j \text{ such that } X_t^{j,0} = 0\}, & \text{if } r > 0, \\ \inf\{t \geq 0 : \exists j \neq j_0 \text{ such that } X_t^{j,0} = 0\}, & \text{if } r = 0 \text{ and where } j_0 \text{ is such that } x_{j_0} = 0. \end{cases}$$

For $t \leq \tau_1$, set $X_t^j = X_t^{j,0}$ and let $X_t^{(n)} = (X_t^1, \dots, X_t^n)$. Note that a.s. $X_{\tau_1}^{(n)} \notin \Delta_n$.

Assume now that $(\tau_k)_{k \leq \ell}$ and $(X_t^{(n)} = (X_t^1, \dots, X_t^n))_{t \leq \tau_\ell}$ have been defined such that a.s. for all $1 \leq k \leq \ell$, $X_{\tau_k}^{(n)} \notin \Delta_n$ and there is a unique integer j_k such that $X_{\tau_k}^{j_k} = 0$. Now introduce an independent solution (Y^ℓ, W^ℓ) of (ISDE), with $Y_0^\ell = 0$, and define $(X_t^{(n)})_{t \in [\tau_\ell, \tau_{\ell+1}]}$ by analogy with the construction of $(X_t^{(n)})_{t \in [0, \tau_1]}$ by replacing x with $X_{\tau_\ell}^{(n)}$. Thus, we have defined $X_t^{(n)}$ for all $t < \tau_\infty$, where $\tau_\infty := \lim_{n \rightarrow \infty} \tau_n$. Moreover, by construction, $(\tau_n)_{n \geq 0}$ is an increasing sequence of stopping times with respect to the filtration associated to $X^{(n)}$.

We denote by $\mathbb{P}_x^{(n),0}$ the law of $(X_t^{(n)})_{t < \tau_\infty}$. Notice that for all i and all ℓ , $(X_{t \wedge \tau_\ell}^i)$ is a WBM stopped at time τ_ℓ . Thus a.s. on the event $\{\tau_\infty < \infty\}$, $X_{\tau_\infty}^{(n)} := \lim_{t \uparrow \tau_\infty} X_t^{(n)}$ exists (this is a well known result for the standard Brownian motion, see [16, Proposition 3.3], which can easily be extended to WBM). Note also that there exist $i \neq j$ such that, a.s. on the event $\{\tau_\infty < \infty\}$, $X_{\tau_\ell}^i = X_{\tau_{\ell+1}}^j = 0$ for infinitely many ℓ 's and thus that $\lim_{t \uparrow \tau_\infty} X_t^i = \lim_{t \uparrow \tau_\infty} X_t^j = 0$. Therefore, $X_{\tau_\infty}^{(n)} \in \Delta_n$ a.s. By construction, this implies that

$$\tau_\infty = T_{\Delta_n} := \inf\{t \geq 0 : X_t^{(n)} \in \Delta_n\}. \quad (18)$$

We will prove in Section 5.3 that $\tau_\infty < \infty$ a.s.

5.2 An obliquely reflected Brownian motion associated to the 2-point motion

Fix $x \in G^*$. Let (X, Y) be the process with law $\mathbb{P}_{(x,0)}^{(2),0}$ constructed in Section 5.1. Then for all $n \geq 0$,

$$\begin{aligned} \tau_{2n+1} &= \inf\{t \geq \tau_{2n} : X_t = 0\}, \\ \tau_{2n+2} &= \inf\{t \geq \tau_{2n+1} : Y_t = 0\}. \end{aligned}$$

Letting, for all $n \geq 0$, i_{2n} and i_{2n+1} in $\{1, \dots, N\}$ be such that $X_{\tau_{2n}} \in E_{i_{2n}}$ and $Y_{\tau_{2n+1}} \in E_{i_{2n+1}}$, we have $X_t \in E_{i_{2n}}^*$ for all $t \in [\tau_{2n}, \tau_{2n+1}[$ and $Y_t \in E_{i_{2n+1}}^*$ for all $t \in [\tau_{2n+1}, \tau_{2n+2}[$.

Define, for $i \in [1, N]$, $f^i : G \rightarrow \mathbb{R}$ by

$$f^i(x) = -|x| \text{ if } x \in E_i \quad \text{and} \quad f^i(x) = |x| \text{ if not.}$$

Define now $(U_t, V_t)_{t < \tau_\infty}$ such that for $n \geq 0$

$$(U_t, V_t) = \begin{cases} (|X_t|, f^{i_{2n}}(Y_t)) & \text{for } t \in [\tau_{2n}, \tau_{2n+1}[; \\ (f^{i_{2n+1}}(X_t), |Y_t|) & \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}[. \end{cases}$$

Remark that $(U_t, V_t)_{t < \tau_\infty}$ is a continuous process with values in $\{(u, v) \in \mathbb{R}^2 : u + v > 0\}$ and such that for all $n \geq 0$, $U_{\tau_{2n}} > 0$, $V_{\tau_{2n}} = 0$, $U_{\tau_{2n+1}} = 0$ and $V_{\tau_{2n+1}} > 0$. Note that the excursions of this process outside of \mathcal{Q} occur on straight lines parallel to $\{y = -x\}$.

Let, for $n \geq 0$,

$$\Theta_n = \arctan\left(\frac{p_{i_n}}{1 - p_{i_n}}\right).$$

Define for $t < \tau_\infty$, $A(t)$ the amount of time where X and Y do not both belong to the same ray before time t . Note that $A(t) = \int_0^t \mathbf{1}_{\{(U_s, V_s) \in \mathcal{Q}\}} ds$.

Set $\gamma(t) = \inf\{s \geq 0 : A(s) > t\}$. Set for $n \geq 0$, $T_n = A(\tau_n)$ and $S_{n+1} = T_{n+1} - T_n$. Define for $t < T_\infty := \lim_{n \rightarrow \infty} T_n$,

$$(U_t^r, V_t^r) = (U_{\gamma(t)}, V_{\gamma(t)})$$

and for $t \geq T_\infty$, $(U_t^r, V_t^r) = (0, 0)$. Note that $T_{2n+1} = \inf\{t \geq T_{2n} : V_t^r = 0\}$, $T_{2n+2} = \inf\{t \geq T_{2n+1} : U_t^r = 0\}$ and that $\gamma(T_n) = \tau_n$.

Lemma 5.2. *Given Θ_0 , the law of $(U_t^r, V_t^r)_{t \leq S_1}$ is $\mathbb{P}_{|x|}^{\Theta_0}$.*

The proof of this lemma is given at the end of this section. Define the sequence of processes $(Z^n)_{n \geq 1}$ such that for $n \geq 0$,

$$\begin{aligned} Z^{2n+1} &= (U_{t+T_{2n}}^r, V_{t+T_{2n}}^r)_{t \leq S_{2n+1}}, \\ Z^{2n+2} &= (V_{t+T_{2n+1}}^r, U_{t+T_{2n+1}}^r)_{t \leq S_{2n+2}}. \end{aligned}$$

Set also for $n \geq 0$, $U_{2n} = U_{T_{2n}}^r$ and $U_{2n+1} = V_{T_{2n+1}}^r$.

Applying Lemma 5.2 and using the strong Markov property at the stopping times τ_n , with the fact that if (X, Y) is distributed as $\mathbb{P}_{(x,y)}^{(2),0}$, then (Y, X) is distributed as $\mathbb{P}_{(y,x)}^{(2),0}$, one has the following

Lemma 5.3. *For all $n \geq 0$, given \mathcal{F}_{τ_n} , the law of Z^{n+1} is $\mathbb{P}_{U_n}^{\Theta_n}$.*

This lemma shows that the sequences $(\Theta_n)_{n \geq 0}$ and $(Z^n)_{n \geq 1}$ satisfy (i) and (ii) in the beginning of Section 3.2 since for all $n \geq 0$,

$$\mathcal{G}_n = \sigma((\Theta_k, Z^k); 1 \leq k \leq n) \vee \sigma(\Theta_0) \subset \mathcal{F}_{\tau_n}.$$

Thus $(U_t^r, V_t^r)_{t < T_\infty}$ is a Brownian motion in \mathcal{Q}^* started from $(|x|, 0)$, with time dependent angle of reflections at the boundaries given by $(\Theta_n)_{n \geq 0}$ and stopped when it hits $(0, 0)$, as defined in Section 2. In particular, (U^r, V^r) is a continuous process and $\lim_{t \uparrow T_\infty} (U_t^r, V_t^r) = (0, 0)$.

Remark 5.4. *Note that $(i_n)_{n \geq 0}$ is an homogeneous Markov chain started from $i_0 = 1$ with transition matrix $(P_{i,j})$ given by : for $(i, j) \in [1, N]^2$, $P_{i,j} = \frac{p_i}{\sum_{k \neq i} p_k}$. Remark also that given \mathcal{G}_n , Z^{n+1} and i_{n+1} are independent and a fortiori Z^{n+1} and Θ_{n+1} are also independent.*

Proof of Lemma 5.2. Set for $t \geq 0$, $(\hat{U}_t, \hat{V}_t) := (|x| + W_t^{i_0}, f^{i_0}(Y_t))$. Then, for $t \leq \tau_1$, $(\hat{U}_t, \hat{V}_t) = (U_t, V_t)$. Since Y is a WBM started at 0, it is well known that \hat{V} is a skew Brownian motion with parameter $1 - p_{i_0}$. This can be seen using Freidlin-Sheu formula, which shows that

$$\hat{V}_t = \int_0^t (\mathbf{1}_{\{\hat{V}_s > 0\}} - \mathbf{1}_{\{\hat{V}_s \leq 0\}}) dB_s^Y + (1 - 2p_{i_0})L_t(\hat{V}). \quad (19)$$

Define $\hat{A}(t) = \int_0^t \mathbf{1}_{\{\hat{V}_s \geq 0\}} ds = \int_0^t \mathbf{1}_{\{Y_s \notin E_{i_0}\}} ds$ and $\hat{\gamma}(t) = \inf\{s \geq 0 : \hat{A}(s) > t\}$. Note that $\hat{A}(t) = A(t)$ for all $t \leq \tau_1$ and thus $\hat{\gamma}(t) = \gamma(t)$ for all $t \leq S_1 = A(\tau_1)$. It is also well known that $\hat{V}_t^r := \hat{V}_{\hat{\gamma}(t)}$ is a reflecting Brownian motion on \mathbb{R}_+ . Set $M_t = \int_0^t \mathbf{1}_{\{\hat{V}_s > 0\}} d\hat{V}_s = \int_0^t \mathbf{1}_{\{Y_s \notin E_i\}} dB_s^Y$. Then $B_t^2 := M_{\hat{\gamma}(t)}$ is a Brownian motion. We also have $\hat{V}_t \vee 0 = M_t + (1 - p_{i_0})L_t(\hat{V})$, which implies that $\hat{V}_t^r = B_t^2 + (1 - p_{i_0})L_{\hat{\gamma}(t)}(\hat{V})$ and therefore $L_t(\hat{V}^r) = (1 - p_{i_0})L_{\hat{\gamma}(t)}(\hat{V})$. Note finally that $L(\hat{V}) = L(|Y|)$.

Set for $t \geq 0$, $B_t^1 = \int_0^{\hat{\gamma}(t)} \mathbf{1}_{\{\hat{V}_s > 0\}} dW_s^{i_0}$. By Lévy's theorem B^1 and B^2 are two independent Brownian motions. Finally, set $\hat{U}_t^r = \hat{U}_{\hat{\gamma}(t)}$. Then, for all $t \leq S_1$, $(\hat{U}_t^r, \hat{V}_t^r) = (U_t^r, V_t^r)$.

Lemma 5.2 is a direct consequence of the following

Lemma 5.5. *For all $t \geq 0$,*

$$\begin{aligned}\hat{U}_t^r &= |x| + B_t^1 - \frac{p_i}{1 - p_i} L_t(\hat{V}^r) \\ \hat{V}_t^r &= B_t^2 + L_t(\hat{V}^r).\end{aligned}$$

Proof. We closely follow the proof of Lemma 4.3 [12]. Let $\epsilon > 0$ and define the sequences of stopping times σ_k^ϵ and τ_k^ϵ such that $\tau_0^\epsilon = 0$ and for $k \geq 0$,

$$\begin{aligned}\sigma_k^\epsilon &= \inf\{t \geq \tau_k^\epsilon; \hat{V}_t = -\epsilon\}, \\ \tau_{k+1}^\epsilon &= \inf\{t \geq \sigma_k^\epsilon; \hat{V}_t = 0\}.\end{aligned}$$

Note first that (19) implies that

$$\sum_{k \geq 0} (\hat{V}_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)} - \hat{V}_{\tau_k^\epsilon \wedge \hat{\gamma}(t)})$$

converges in probability as $\epsilon \rightarrow 0$ towards $B_t^2 + (1 - 2p_i)L_{\hat{\gamma}(t)}(\hat{V})$. Since $\hat{V}_t^r = B_t^2 + (1 - p_i)L_{\hat{\gamma}(t)}(\hat{V})$, setting

$$L_t^{\epsilon, r} = \sum_{k \geq 0} (\hat{V}_{\tau_{k+1}^\epsilon \wedge \hat{\gamma}(t)} - \hat{V}_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)}),$$

$L_t^{\epsilon, r}$ converges towards $p_i L_{\hat{\gamma}(t)}(\hat{V})$ in probability as $\epsilon \rightarrow 0$. Now for $t > 0$,

$$\hat{U}_t^r = |x| + \sum_{k \geq 0} (\hat{U}_{\tau_{k+1}^\epsilon \wedge \hat{\gamma}(t)} - \hat{U}_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)}).$$

Set for $t \geq 0$,

$$B_t^{\epsilon, 1} = \sum_{k \geq 0} (W_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)}^i - W_{\tau_k^\epsilon \wedge \hat{\gamma}(t)}^i).$$

Note that $d(\hat{U}_s + \hat{V}_s) = \sum_{j \neq i} \mathbf{1}_{\{Y_s \in E_j\}} dW_s^j$ and thus when $Y_s \in E_i^*$ (i.e. \hat{V}_s is negative), $\hat{U}_s + \hat{V}_s$ remains constant, and we have

$$\begin{aligned}\hat{U}_t^r &= |x| + \sum_{k \geq 0} (\hat{U}_{\tau_{k+1}^\epsilon \wedge \hat{\gamma}(t)} - \hat{U}_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)}) + \sum_{k \geq 0} (\hat{U}_{\sigma_k^\epsilon \wedge \hat{\gamma}(t)} - \hat{U}_{\tau_k^\epsilon \wedge \hat{\gamma}(t)}) \\ &= |x| - L_t^{\epsilon, r} + B_t^{\epsilon, 1}.\end{aligned}$$

Since $B_t^{\epsilon,1}$ converges in probability towards B_t^1 , we get

$$\hat{U}_t^r = |x| + B_t^1 - p_{i_0} L_{\hat{\gamma}(t)}(\hat{V}).$$

And we conclude using that $L_t(\hat{V}^r) = (1 - p_{i_0}) L_{\hat{\gamma}(t)}(\hat{V})$. \square

5.3 Coalescing property

Our purpose in this section is to prove that the consistent family defined in Section 5.1 has the coalescing property, that is

Proposition 5.6. *With probability 1, $\tau_\infty < \infty$.*

Proof. By symmetry and the strong Markov property, it suffices to prove that $\tau_\infty < \infty$ a.s. only for $n = 2$ and $(X_0, Y_0) = (x, 0)$ for some $x \in G^*$. We use the notations of Section 5.2.

Since $(|X_t|, t \leq \tau_\infty)$ is a reflected Brownian motion stopped at time τ_∞ , it suffices to prove that a.s. $L_{\tau_\infty}(|X|) < \infty$. Denote by L_t^1 and L_t^2 the local times accumulated by Z respectively on $\{u = 0\}$ and $\{v = 0\}$ up to t and $L_t = L_t^1 + L_t^2$. First, note that for $t \leq S_1$, $L_t(V^r) = (1 - p_i) L_{\gamma(t)}(V) = (1 - p_i) L_{\gamma(t)}(|Y|)$. Thus $L_{\tau_1}(|Y|) = \frac{L_{S_1}(V^r)}{1 - p_i}$. Note also that $L_{\tau_1}(|X|) = 0$. Thus

$$L_{\tau_1}(|X|) + L_{\tau_1}(|Y|) = \frac{L_{S_1}}{1 - p_i}.$$

Set $C = \sup_{\{1 \leq i \leq N\}} (1 - p_i)^{-1}$. By induction, we get that

$$L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) = \sum_{n \geq 0} \frac{L_{T_{n+1}} - L_{T_n}}{1 - p_{i_n}} \leq C L_{T_\infty}$$

By Theorem 3.11 a.s. $L_{T_\infty} < \infty$, and so $L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) < \infty$. \square

5.4 Construction of φ

Let $(P^{(n)}, n \geq 1)$ be the unique consistent family of Markovian semigroups such that

- $P^{(1)}$ is the semigroup of the WBM on G .
- The n -point motion of $P^{(n)}$ started from $x \in G^n$ up to its entrance time in Δ_n is distributed as $\mathbb{P}_x^{(n),0}$.
- The n -point motion (X^1, \dots, X^n) of $P^{(n)}$ is such that if $X_s^i = X_s^j$ then $X_t^i = X_t^j$ for all $t \geq s$.

We will prove that $P^{(n)}$ is Feller for all n and that (17) holds. By [13, Lemma 1.11], this amounts to check the following condition.

Lemma 5.7. *Let (X, Y) be the two point motion associated to $P^{(2)}$, then for all positive ϵ*

$$\lim_{d(x,y) \rightarrow 0} \mathbb{P}_{(x,y)}^{(2),0}[d(X_t, Y_t) > \epsilon] = 0.$$

Proof. As in the proof of Proposition 5.6, we take $y = 0$. Then using the same notations, for all positive ϵ , $\{d(X_t, Y_t) > \epsilon\} \subset \{\sup_{t < \sigma_0} |Z_t| > \epsilon\}$. Now the lemma follows from Remark 3.10. \square

By Theorem 5.1, a SFM φ can be associated to $(P^{(n)})_n$.

Proposition 5.8. *Let φ be a SFM associated to $(P^{(n)})_n$. Then there exists a family of independent white noises $\mathcal{W} = (W^i, 1 \leq i \leq N)$ such that*

- (i) $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^{\varphi}$ for all $s \leq t$ and
- (ii) (φ, \mathcal{W}) solves (ISDE).

Proof. Let $V_{s,\cdot}(x)$ be the Brownian motion associated to $\varphi_{s,\cdot}(x)$. For all $i \in [1, N]$ and $s \leq t$, set

$$W_{s,t}^i = \lim_{|x| \rightarrow \infty, x \in E_i, |x| \in \mathbb{Q}} V_{s,t}(x).$$

For all $i \in [1, N]$ and $s \leq t$, with probability 1, this limit exists. Indeed if $x, y \in E_i$ are such that $|x| \leq |y|$, then a.s. $V_{s,t}(x) = V_{s,t}(y)$ for all $s \leq t \leq \tau_s^x = \inf\{u \geq s; \varphi_{s,u}(x) = 0\}$. Moreover $W^i = (W_{s,t}^i, s < t)$ is a real white noise. Indeed, W^i is centered and Gaussian, and by the flow property of φ and using $\varphi_{s,u}(x) = e_i(|x| + W_{s,u}^i)$ if $s \leq u \leq \tau_s^x$ and $x \in E_i$, we have $W_{s,u}^i = W_{s,t}^i + W_{t,u}^i$. It is also clear that W^i has independent increments with respect to (s, t) . Thus, W^i is a real white noise. The fact that $\mathcal{W} = (W^i, 1 \leq i \leq N)$ is a family of independent real white noises easily holds.

For $x \in G$ and $t \geq 0$,

$$\langle W_{s,\cdot}^i, V_{s,\cdot}(x) \rangle_t = \lim_{|y| \rightarrow \infty, y \in E_i, |y| \in \mathbb{Q}} \langle V_{s,\cdot}(y), V_{s,\cdot}(x) \rangle_t = \int_s^t 1_{\{\varphi_{s,u}(x) \in E_i\}} du.$$

This yields

$$V_{s,t}(x) = \sum_{i=1}^N \int_s^t 1_{\{\varphi_{s,u}(x) \in E_i\}} dW_u^i.$$

By Theorem 4.1, we deduce that (φ, \mathcal{W}) solves (ISDE). \square

Denote by \mathbb{P}_E the law of (φ, \mathcal{W}) .

5.5 Uniqueness in law of a SFM solution of (ISDE)

In this section, we show that the SFM φ constructed in Section 5.4 is the only SFM solution of (ISDE). More precisely, we show

Proposition 5.9. *Let (φ, \mathcal{W}) be a solution of (ISDE), with φ a SFM. Then the law of (φ, \mathcal{W}) is \mathbb{P}_E .*

Proof. We start by showing

Lemma 5.10. *For all $x = e_i(r) \in G$, we have $\varphi_{s,t}(x) = e_i(r + W_{s,t}^i)$ for all $s \leq t \leq \tau_s^x = \inf\{t \geq s : \varphi_{s,t}(x) = 0\}$. In particular for all $1 \leq i \leq N$, $s \leq t$, we have $\mathcal{F}_{s,t}^{W^i} \subset \mathcal{F}_{s,t}^{\varphi}$.*

Proof. Fix $i \in [1, N]$. Let $f \in \mathcal{D}$ be such that $f(x) = |x|$ for all $x \in E_i$. By applying f in (ISDE), we deduce the first claim. The second claim is then an immediate consequence by taking a sequence $(x_k)_k \subset E_i$ converging to ∞ . \square

With this lemma and Theorem 2.3 we prove the following

Lemma 5.11. *Let $x = (x_1, \dots, x_n) \in G^n$. Let $S = \inf\{t \geq 0 : (\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n)) \in \Delta_n\}$. Then $(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n))_{t \leq S}$ is distributed like $\mathbb{P}_x^{(n),0}$.*

Proof. Let $x \in G^n \setminus \Delta_n$ such that $x_{j_0} = 0$ for some j_0 . For $j \in [1, n]$, set $Y_t^j = \varphi_{0,t}(x_j)$ and $Y_t^{(n)} = (Y_t^1, \dots, Y_t^n)$. Set for $i \in [1, N]$, $W_t^i = W_{0,t}^i$ and $W_t = (W_t^1, \dots, W_t^n)$. Note that for all $j \in [1, n]$, (Y^j, W) is a solution of (ISDE). Set

$$\sigma_1 = \inf\{t \geq 0 : \exists j \neq j_0 \text{ such that } Y_t^j = 0\}$$

and for $\ell \geq 1$,

$$\sigma_{\ell+1} = \inf\{t \geq \sigma_\ell : \exists j \text{ such that } Y_t^j = 0 \text{ and } Y_{\sigma_\ell}^j \neq 0\}.$$

Let $\sigma_\infty = \lim_{\ell \rightarrow \infty} \sigma_\ell$, then $\sigma_\infty = \inf\{t : Y_t^{(n)} \in \Delta_n\}$. By Theorem 2.3, the law of (Y^{j_0}, W) is uniquely determined. Now, for $j \neq j_0$, we have that for $t \leq \sigma_1$, $Y_t^j = e_i(|x_j| + W_t^i)$ where i is such that $x_j \in E_i^*$. This shows that $(Y_t^{(n)})_{t \leq \sigma_1}$ is distributed as $(X_t^{(n)})_{t \leq \tau_1}$, constructed in Subsection 5.1. Adapting the previous argument on the time interval $[\sigma_\ell, \sigma_{\ell+1}]$, we show that for all $\ell \geq 1$, $(Y_t^{(n)})_{t \leq \sigma_\ell}$ is distributed as $(X_t^{(n)})_{t \leq \tau_\ell}$. The Lemma easily follows. \square

Lemma 5.11 permits to conclude the proof of Proposition 5.9. Indeed, the law of a SFM is uniquely determined by its family of n -point motions $X^{(n)}$. Using the fact that Δ_n is an absorbing set for $X^{(n)}$, the strong Markov property at time $T^n = \inf\{t : X_t^{(n)} \in \Delta_n\}$ and the consistency of the family of n -point motions, we see that the law of a SFM is uniquely determined by its family of n -point motions stopped at its first entrance time in Δ_n . \square

6 Extension to metric graphs

Let $(I_k)_{1 \leq k \leq K}$ be a family of finite sets such that $I_k \cap I_\ell \cap I_m = \emptyset$ for all $1 \leq k < \ell < m \leq K$. Let $(G_k)_{1 \leq k \leq K}$ be a family of star graphs such that for each k , $G_k = \cup_{i \in I_k} E_k^i$, where $\{E_k^i, i \in I_k\}$ is the set of edges of G_k . Set $I = \cup_{i=1}^K I_k$ and $\mathcal{I} = \cup_{k=1}^K I_k \times \{k\}$. For each k , let $p_k := (p_k^i)_{i \in I_k}$ be a family of parameters such that $0 < p_k^i < 1$ and $\sum_{i \in I_k} p_k^i = 1$.

Let $\mathcal{W} = \{W_k^i, (i, k) \in \mathcal{I}\}$ be a family of white noises such that

- For all $k \neq \ell$ and $i \in I_k \cap I_\ell$, then $W_k^i + W_\ell^i = 0$.
- For all family $\{(i_1, k_1), \dots, (i_n, k_n)\} \subset \mathcal{I}$ such that i_1, \dots, i_n are n distinct indices, $\{W_{k_1}^{i_1}, \dots, W_{k_n}^{i_n}\}$ is an independent family of white noises.

Notice that the law of \mathcal{W} is uniquely described by these two properties and that \mathcal{W} can be constructed out of a family of $|\mathcal{I}|$ independent white noises. Let us also remark that the second property implies that for each k , $\{W_k^i, i \in I_k\}$ is a family of independent white noises.

Theorem 6.1. *Let $(K_k)_{1 \leq k \leq K}$ be a family of SFK's respectively defined on G_k . Assume that*

- For all k , (K_k, W_k) solves $E(G_k, p_k)$,
- $(\mathcal{F}_{s,t} := \vee_k \mathcal{F}_{s,t}^{K_k})_{s \leq t}$ is i.d.i.

Then, the flows $(K_k)_{1 \leq k \leq K}$ are independent given \mathcal{W} .

The rest of this section will consist in proving Theorem 6.1. For a SFK K and a white noise W , $K(t)$ denotes $K_{0,t}$ and $W(t)$ denotes $W_{0,t}$.

6.1 Feller semigroups

Let $n := (n_k)_{1 \leq k \leq K}$ be a family of nonnegative integers and set $G^{(n)} := \prod_k G_k^{n_k}$. For $t \geq 0$, $x := (x_k)_{1 \leq k \leq K} \in G^{(n)}$ and $w \in \mathbb{R}^{|I|}$, set for $f \in C_0(G^{(n)})$ and $g \in C_0(\mathbb{R}^{|I|})$,

$$\mathbb{Q}_t^{(n)}(f \otimes g)(x, w) = \mathbb{E}[(\otimes_k (K_k(t))^{\otimes n_k})f(x)g(w + W(t))].$$

Note that the i.d.i property implies that $\mathbb{Q}^{(n)}$ defines a Feller semigroup on $G^{(n)} \times \mathbb{R}^{|I|}$. Denote by $\mathbb{Q}_{(x,w)}^{(n)}$ the law of the diffusion started at (x, w) associated to this semigroup.

Define also for all k , $\mathbb{Q}_t^{(k, n_k)}$ the Feller semigroup on $G_k^{n_k} \times \mathbb{R}^{|I|}$ such that for $f_k \in C_0(G_k^{n_k})$ and $g \in C_0(\mathbb{R}^{|I|})$,

$$\mathbb{Q}_t^{(k, n_k)}(f_k \otimes g)(x_k, w) = \mathbb{E}[(K_k(t))^{\otimes n_k} f_k(x_k)g(w + W(t))].$$

Denote as above by $\mathbb{Q}_{(x_k, w)}^{(k, n_k)}$ the law of the diffusion started at (x_k, w) associated to this semigroup.

Let (X, W) be a diffusion of law $\mathbb{Q}_{(x,0)}^{(n)}$, then for all (i, k) , (X_k^i, W) is a diffusion of law $\mathbb{Q}_{(x_k^i, 0)}^{(k, 1)}$ and (X_k^i, W_k) is a solution of $E(G_k, p_k)$ with $X_k^i(0) = x_k^i$. This fact can easily be seen as a consequence of

Lemma 6.2. For all $k \in \{1, \dots, K\}$ and all $x \in G_k$, if (X, W) is a diffusion of law $\mathbb{Q}_{(x,0)}^{(k, 1)}$, then (X, W) is a solution of $E(G_k, p_k)$.

Proof. In the following, set $G = G_k$, $p = p_k$, $\mathbb{Q}_t = \mathbb{Q}_t^{(k, 1)}$ and $N = |I_k|$. It is obvious that W is an N -dimensional Brownian motion. It is also clear that X is a WBM. Denote by B^X the Brownian motion associated to X . Then by Freidlin-Sheu formula, (X, W) solves $E(G, p)$ as soon as $B_t^X = \sum_i \int_0^t 1_{\{X_s \in E_i\}} dW_s^i$. It is enough to prove that $\langle B^X, W^i \rangle_t = \int_0^t 1_{\{X_s \in E_i\}} ds$ for all i .

Recall the definition of \mathcal{D} from (3) and set $\mathcal{D}_1 = \{f \in \mathcal{D} : f, f', f'' \in C_0(G)\}$. Denote by A the generator of \mathbb{Q}_t and $\mathcal{D}(A)$ its domain, then $\mathcal{D}_1 \otimes C_0^2(\mathbb{R}^N) \subset \mathcal{D}(A)$ and for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$,

$$A(f \otimes g)(x, w) = \frac{1}{2}f(x)\Delta g(w) + \frac{1}{2}f''(x)g(w) + \sum_{i=1}^N (f'1_{E_i})(x) \frac{\partial g}{\partial w^i}(w).$$

Thus for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$,

$$f(X_t)g(W_t) - \int_0^t A(f \otimes g)(X_s, W_s) ds \text{ is a martingale.} \quad (20)$$

Applying Freidlin-Sheu formula for $f(X_t)$, and then Itô's formula for $f(X_t)g(W_t)$, we get that, using (20)

$$\sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s) ds = \sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s) d\langle B^X, W_i \rangle_s.$$

Since this holds for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$, we get $\langle B^X, W^i \rangle_t = \int_0^t 1_{\{X_s \in E_i\}} ds$ for all i . \square

6.2 A sufficient condition for conditional independence

For $x = (x_k)_k \in G^{(n)}$ and $w \in \mathbb{R}^{|I|}$, let $\mathbb{P}_{(x,w)}^{(n)}$ be the law of (X_1, \dots, X_K, W) such that X_1, \dots, X_K are independent given W and for all k , (X_k, W) is distributed as $\mathbb{Q}_{(x_k,w)}^{(k,n_k)}$. Denote by $\mathbb{E}_{(x,w)}^{(n)}$ the expectation with respect to $\mathbb{P}_{(x,w)}^{(n)}$. Denote also by $\mathbb{E}_{(x_k,w)}^{(k,n_k)}$ the expectation with respect to $\mathbb{P}_{(x_k,w)}^{(k,n_k)}$. For Z , a $\sigma(W)$ -measurable random variable, we simply denote $\mathbb{E}_{(x,0)}^{(n)}[Z]$ and $\mathbb{E}_{(x_k,w)}^{(k,n_k)}[Z]$ by $\mathbb{E}[Z]$.

Proposition 6.3. *If for all $n := (n_k)_k$ and all $x := (x_k)_k \in G^{(n)}$,*

$$\mathbb{Q}_{(x,0)}^{(n)} = \mathbb{P}_{(x,0)}^{(n)}, \quad (21)$$

then the flows $(K_k)_k$ are independent given W .

Proof. Fix $n = (n_k)_k$, $x = (x_k)_k$ and $f = \otimes_k f_k$, with $f_k \in C_0(G_k^{n_k})$, then

$$\begin{aligned} \mathbb{E}\left[\prod_k (K_k(t))^{\otimes n_k} f_k(x_k)\right] &= \mathbb{Q}_t^{(n)}(f \otimes 1)(x, 0) \\ &= \mathbb{E}_{(x,0)}^{(n)}\left[\prod_k f_k(X_k(t))\right] \\ &= \mathbb{E}\left[\prod_k \mathbb{E}_{(x_k,0)}^{(k,n_k)}[f_k(X_k(t))|W]\right]. \end{aligned}$$

Then the proposition follows from the fact that

$$\mathbb{E}_{(x_k,0)}^{(k,n_k)}[f_k(X_k(t))|W] = \mathbb{E}[(K_k(t))^{\otimes n_k} f_k(x_k)|W]. \quad (22)$$

Let us check (22). For this, take g_0, \dots, g_J in $C_0(\mathbb{R}^{|I|})$ and fix $0 = t_0 < \dots < t_J = t$. Then setting, for all $g \in C_0(\mathbb{R}^{|I|})$, $h \in C_0(G_k^{n_k} \times \mathbb{R}^{|I|})$ and all $t \geq 0$, $\mathbb{Q}_t^g h(x, w) = g(w) \mathbb{Q}_t^{(k,n_k)} h(x, w)$, one has (to lighten the notation below, f_k, x_k, X_k and $(K_k(t))^{\otimes n_k}$ are denoted by f, x, X and K_t)

$$\begin{aligned} \mathbb{E}\left[K_t f(x) \prod_{0 \leq j \leq J} g_j(W(t_j))\right] &= \mathbb{Q}_{t_1}^{g_0} \cdots \mathbb{Q}_{t_J - t_{J-1}}^{g_{J-1}}(f \otimes g_J)(x, 0) \\ &= \mathbb{E}_{(x,0)}^{(k,n_k)}\left[f_k(X(t)) \prod_{0 \leq j \leq J} g_j(W(t_j))\right] \\ &= \mathbb{E}\left[\mathbb{E}_{(x,0)}^{(k,n_k)}[f(X(t))|W] \prod_{0 \leq j \leq J} g_j(W(t_j))\right], \end{aligned}$$

which suffices to deduce (22). □

Note that, using the Feller property, (21) is satisfied for all n and all x as soon as it is satisfied for all n and all x in a dense subset of $G^{(n)}$.

6.3 Uniqueness up to the first meeting time at 0

A point $x_k \in G_k^{n_k}$, will be denoted by $(x_k^i)_{1 \leq i \leq n_k}$. Take $n = (n_k)_k$ as in Section 6.1 and choose $x = (x_k)_k \in G^{(n)}$ such that for all $k \neq \ell$, $0 \notin \{x_k^i, 1 \leq i \leq n_k\} \cap \{x_\ell^j, 1 \leq j \leq n_\ell\}$. In the following,

(X, W) will be distributed as $\mathbb{P}_{(x,0)}^{(n)}$ or as $\mathbb{Q}_{(x,0)}^{(n)}$ with $X = (X_1, \dots, X_K)$, and $(\mathcal{F}_t)_{t \geq 0}$ denotes the filtration generated by (X, W) . For $t \geq 0$, let $R_k(t) := \{X_k^i(t) \mid 1 \leq i \leq n_k\}$. Define the sequence of stopping times $(\sigma_j)_{j \geq 0}$ such that $\sigma_0 = 0$ and for all $j \geq 0$,

$$\sigma_{j+1} = \inf\{t \geq \sigma_j \mid \exists k, 0 \in R_k(t) \text{ and } 0 \notin R_k(\sigma_j)\}. \quad (23)$$

Using the strong Markov property, it is easy to see that for all $j \geq 1$, there is only one k such that $0 \in R_k(\sigma_j)$ and that the sequence $(\sigma_j)_{j \geq 1}$ is strictly increasing. Denote by $\sigma_\infty = \lim_{j \rightarrow \infty} \sigma_j$.

Proposition 6.4. *The law of $(X(t), W(t))_{t < \sigma_\infty}$ is the same under $\mathbb{Q}_{(x,0)}^{(n)}$ and under $\mathbb{P}_{(x,0)}^{(n)}$.*

Proof. Let (X, W) be distributed as $\mathbb{Q}_{(x,0)}^{(n)}$. Without loss of generality, assume there exists ℓ such that $0 \in R_\ell(0)$ and $0 \notin \cup_{k \neq \ell} R_k(0)$. Then $\{(X_k(t), t \leq \sigma_1), k \neq \ell\}$ is $\sigma(W)$ -measurable and therefore $\{(X_k(t), t \leq \sigma_1), 1 \leq k \leq K\}$ is a family of independent random variables given W . So the conditional law of this family given W is the same as the conditional law of $\{(X_k(t), t \leq \sigma_1), 1 \leq k \leq K\}$ given W whenever (X, W) is distributed as $\mathbb{P}_{(x,0)}^{(n)}$. Denote this law by $\mu(x, W)$. Using the strong Markov property at time σ_n , we get that given \mathcal{F}_{σ_n} and W , the law of $\{(X_k(t + \sigma_n), t \leq \sigma_{n+1} - \sigma_n), 1 \leq k \leq K\}$ is $\mu(X(\sigma_n), W(\cdot + \sigma_n) - W(\sigma_n))$. Since this characterizes the law of $(X(t), W(t))_{t < \sigma_\infty}$, the proposition is proved. \square

This proposition implies in particular that if $\mathbb{P}_{(x,0)}^{(n)}(\sigma_\infty = \infty) = 1$, then $\mathbb{Q}_{(x,0)}^{(n)} = \mathbb{P}_{(x,0)}^{(n)}$.

6.4 The meeting time at 0 is infinite

Our purpose here is to prove the following

Proposition 6.5. *For all $n = (n_k)$ and $x = (x_k) \in G^{(n)}$ such that $0 \notin \{x_k^i, 1 \leq i \leq n_k\} \cap \{x_l^j, 1 \leq j \leq n_l\}$, for all $k \neq \ell$, we have*

$$\mathbb{P}_{(x,0)}^{(n)}(\sigma_\infty = \infty) = 1. \quad (24)$$

Proof. Assume $K = 2$, $n_1 = n_2 = 1$ and take $x = (x_1, 0)$ with $x_1 \neq 0$. It is easy to see that if (24) holds in this particular case, then it also holds in the general case. We use in the following the notations of Subsection 6.3. Note that for $k \in \{1, 2\}$, (X_k, W_k) is a solution of $E(G_k, p_k)$ and that

$$\sigma_\infty = \inf\{t \geq 0 : X_1(t) = X_2(t) = 0\}.$$

Set $I^c = I_1 \cap I_2$. For $k \in \{1, 2\}$ and $i \in I_k$, set $\theta_k^i = \arctan\left(\frac{p_k^i}{1-p_k^i}\right)$ if $i \in I^c$ and $\theta_k^i = 0$ otherwise. Recall the definition of $(\sigma_n)_n$: $\sigma_0 = 0$ and for all $\ell \geq 0$,

$$\begin{aligned} \sigma_{2\ell+1} &= \inf\{t \geq \sigma_{2\ell} : X_1(t) = 0\}, \\ \sigma_{2\ell+2} &= \inf\{t \geq \sigma_{2\ell+1} : X_2(t) = 0\}. \end{aligned}$$

For all $\ell \geq 0$, set $U_{2\ell}^0 = |X_1(\sigma_{2\ell})|$ and $U_{2\ell+1}^0 = |X_2(\sigma_{2\ell+1})|$. Let $i_{2\ell} \in I_1$ and $i_{2\ell+1} \in I_2$ be such that $X_1(\sigma_{2\ell}) \in E_1^{i_{2\ell}}$ and $X_2(\sigma_{2\ell+1}) \in E_2^{i_{2\ell+1}}$. Set $\Theta^n = -\theta_2^{i_n}$ if n is even and $\Theta^n = -\theta_1^{i_n}$ if n is odd.

Say $x_1 \sim x_2$ if there is $i \in I^c$ such that $x_1 \in E_1^i$ and $x_2 \in E_2^i$ and say $x_1 \not\sim x_2$ otherwise. Set $A_t = \int_0^t 1_{\{X_1(s) \not\sim X_2(s)\}} ds$, $\gamma_t = \inf\{s \geq 0 : A_s > t\}$ and $\mathcal{G}_t = \mathcal{F}_{\gamma_t}$. Set $S_n = A_{\sigma_n}$ and define $\Theta_t = \Theta^n$ for $t \in [S_n, S_{n+1}]$. Note that $S_\infty := \lim_{n \rightarrow \infty} S_n = A_{\sigma_\infty}$.

Our purpose now is to define a càdlàg process $(U_t, V_t)_{t < S_\infty}$, such that for all $n \geq 0$, conditionally on \mathcal{G}_{S_n} , $(U_t, V_t)_{t \in [S_n, S_{n+1})}$ is a Brownian motion in the quadrant \mathcal{Q} , obliquely reflected on $\partial_1 \mathcal{Q}$ with angle of reflection Θ^n , such that

- $(U_{S_n}, V_{S_n}) = (U_n^0, 0)$;
- $U_t > 0$ for all $t \in [S_n, S_{n+1})$

and such that (using the notation $H_{t-} = \lim_{s \uparrow t} H_s$ for H a càdlàg process)

- when n is odd, $V_{S_{n+1}-} = 0$ if $X_1(\sigma_{n+1}) \in E_1^{i_n}$ and $U_{S_{n+1}-} = 0$ if not,
- when n is even, $V_{S_{n+1}-} = 0$ if $X_2(\sigma_{n+1}) \in E_2^{i_n}$ and $U_{S_{n+1}-} = 0$ if not.

The construction being exactly the same for all $n \geq 0$, we just do it for $n = 0$. Note that $X_1(t) \in E_1^{i_0}$ and $|X_1(t)| = |x_1| + W_1^{i_0}(t)$ for $t \leq \sigma_1$. We then have two cases:

First case: $i_0 \in I_1 \setminus I^c$. For all $t < \sigma_1$, we have $X_2(t) \not\sim X_1(t)$ and $A(t) = t$. We also have $S_1 = \sigma_1$. Set for $t < S_1$, $U_t = |X_1(t)|$ and $V_t = |X_2(t)|$. Then $(U_t, V_t)_{t < S_1}$ is a normally reflected Brownian motion in the quadrant \mathcal{Q} started at $(|x_1|, 0)$ and killed when hitting $\{x = 0\}$.

Second case: $i_0 \in I^c$. For $t \leq \sigma_1$, set $X_t = |X_1(t)|$ and

$$Y_t = |X_2(t)| (1_{\{X_2(t) \not\sim X_1(t)\}} - 1_{\{X_2(t) \sim X_1(t)\}}).$$

Note that for $t \leq \sigma_1$, $A_t = \int_0^t 1_{\{Y_s > 0\}} ds$ and $1_{\{Y_t < 0\}} d(X_t + Y_t) = 0$. The process $(X_t, Y_t)_{t \leq \sigma_1}$ behaves as a two dimensional Brownian motion in \mathcal{Q} and evolves on straight lines parallel to $\{y = x\}$ outside \mathcal{Q} . Finally set $(U_t, V_t) = (X_{\gamma_t}, Y_{\gamma_t})$, for $t < S_1$. Following the proof of Lemma 5.5, we check that $(U_t, V_t)_{t < S_1}$ is an obliquely reflected Brownian motion, with angle of reflection Θ^0 .

Define the sequence $(T_j)_{j \geq 0}$ by $T_0 = 0$ and for all $j \geq 0$,

$$T_{j+1} = \inf\{S_n : S_n > T_j \text{ and } U_{S_n-} = 0\}.$$

Set $T_\infty = \lim_{j \rightarrow \infty} T_j = S_\infty = A_{\sigma_\infty}$. Then $(U_t, V_t)_{t < S_\infty}$ is continuous except at the times T_j , $j \geq 1$. Moreover, we have the following

Lemma 6.6. *The process (U_t, V_t) is $(\mathcal{G}_t)_t$ -adapted. There is a two-dimensional $(\mathcal{G}_t)_t$ -Brownian motion (B^1, B^2) , such that for all $j \geq 0$ and all $t \in [T_j, T_{j+1}[$,*

$$\begin{aligned} U_t &= U_{T_j} + \int_{T_j}^t (dB_s^1 - \tan(\Theta_s) dL_s(V)), \\ V_t &= \int_{T_j}^t (dB_s^2 + dL_s(V)). \end{aligned}$$

Moreover $U_{T_{j+1}} = V_{T_{j+1}-}, U_{T_{j+1}-} = 0$ and $U_t > 0$ for all $t < S_\infty$.

Out of the noncontinuous process (U, V) , we construct a continuous process $(Z_t^r = (X_t^r, Y_t^r), t < T_\infty)$ by: For all $n \geq 0$, $(X_t^r, Y_t^r) = (U_t, V_t)$ for $t \in [T_{2n}, T_{2n+1})$ and $(X_t^r, Y_t^r) = (V_t, U_t)$ for $t \in [T_{2n+1}, T_{2n+2}[$. Then, putting everything together, we get the following

Lemma 6.7. *The process $(Z_t^r, t < T_\infty)$ is an obliquely reflected Brownian motion on \mathcal{Q} started from $(|x_1|, 0)$, and we have, for all $n \geq 0$,*

$$\begin{aligned} T_{2n+1} &= \inf\{t \geq T_{2n} : Y_t^r = 0\}, \\ T_{2n+2} &= \inf\{t \geq T_{2n+1} : X_t^r = 0\}, \end{aligned}$$

$\lim_{t \uparrow T_\infty} Z_t^r = (0, 0)$ and for all $t < T_\infty$,

$$\begin{aligned} dX_t^r &= dB_t^1 + dL_t(X^r) - \tan(\Theta_t) dL_t(Y^r), \\ dY_t^r &= dB_t^2 - \tan(\Theta_t) dL_t(X^r) + dL_t(Y^r). \end{aligned}$$

Moreover $T_\infty = \infty$ implies $\sigma_\infty = \infty$.

To conclude the proof of Proposition 6.5, it remains to prove that a.s. $T_\infty = \infty$. We exactly follow [10, Page 161]. For $a \geq 0$, define

$$\tau_a = \inf\{t \geq 0 : |Z_t^r| = a\}.$$

Take $\epsilon < |x_1| < A$ and set $\tau_{\epsilon, A} = \tau_\epsilon \wedge \tau_A$. Then, by Itô's formula, setting $R_t = |Z_t^r|$, we have that, for all $t \geq 0$,

$$\log(R_{t \wedge \tau_{\epsilon, A}}) = \log(|x_1|) + M_t + C_t$$

where M is a martingale started from 0 and C is a nonnegative nondecreasing process (using that $\Theta_t \leq 0$). Thus by letting $t \rightarrow \infty$, we get $E[\log(R_{\tau_{\epsilon, A}})] \geq \log(|x_1|)$. So

$$\log(\epsilon)\mathbb{P}(\tau_\epsilon < \tau_A) + \log(A)(1 - \mathbb{P}(\tau_\epsilon < \tau_A)) \geq \log(|x_1|)$$

and consequently

$$\mathbb{P}(\tau_\epsilon < \tau_A) \leq \frac{\log(A) - \log(|x_1|)}{\log(A) - \log(\epsilon)}.$$

Replacing ϵ with $\epsilon(A) = A^{-A}$, yields

$$\mathbb{P}(T_\infty < \infty) = \lim_{A \rightarrow \infty} \mathbb{P}(T_\infty < \tau_A) \leq \lim_{A \rightarrow \infty} \mathbb{P}(\tau_{\epsilon(A)} < \tau_A) = 0.$$

□

7 Final remarks

It would be interesting to extend the framework of the present paper to the case of a star graphs with an infinite number of rays $G = \cup_{n \in \mathbb{N}} E_n$. Suppose we are given a family $p = (p_n)_{n \in \mathbb{N}} \subset]0, 1[$ such that $\sum_n p_n = 1$. Then the WBM associated to p can still be defined via its semigroup (as in the introduction). It satisfies also a Freidlin-Sheu formula similar to the finite case (see [9]):

$$df(Z_t) = f'(Z_t)dB_t^Z + \frac{1}{2}f''(Z_t)dt$$

where B^Z is again the martingale part of $|Z|$ and f runs over an appropriate domain of functions \mathcal{D} . Now suppose given a family $(W^n)_{n \in \mathbb{N}}$ of independent Brownian motions, then the natural extension of (ISDE) associated to p is the following

$$df(Z_t) = \sum_n (f' 1_{E_n})(Z_t) dW_t^n + \frac{1}{2} f''(Z_t) dt, \quad f \in \mathcal{D}$$

which we denote again by (ISDE). The Brownian motion B^Z has also the martingale representation property for $(\mathcal{F}^Z)_t$ [3, Proposition 19 (ii)]. Thus following our arguments, under some conditions on Z , the law of any solution $(Z, W^n, n \in \mathbb{N})$ to (ISDE) is unique. One could also investigate stochastic flows solutions of (ISDE). However, in contrast to the finite case, here we have

$$\inf \left\{ \arctan \left(\frac{p_n}{1-p_n} \right) : n \in \mathbb{N} \right\} = 0.$$

This is the new difficulty with respect to the present paper. We leave the question of existence of a SFM in this case open.

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